Efficiency with Endogenous Population Growth. Do Children have too Many Rights?*

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Abstract

Fertility rates are declining in many countries. But are fertility rates *inefficiently* low? This paper addresses this question by exploring the efficiency properties of equilibria in an overlapping generations setting with endogenous fertility and dynastic parental altruism, using the notion of $\mathcal{P}-$efficiency proposed by Golosov, Jones and Tertilt (2007). In contrast with Schoonbroodt and Tertilt (2014), who show that any equilibrium for which non-negativity constraints on intergenerational transfers are binding is $\mathcal{A}-$inefficient (and, under the assumption that new lives always increase social welfare, also $\mathcal{P}-$inefficient), I characterize symmetric, $\mathcal{P}-$efficient allocations as the equilibria arising from different distribution of rights among the agents, and show that many equilibria exhibiting binding constraints on transfers are $\mathcal{P}-$efficient. To be more precise, except for dynamically inefficient equilibria, there is no need to alter children's rights in order to achieve efficiency.

Key words: Efficiency, Optimal Population, Population Principles, Endogenous Fertility, $\mathcal{A}-$efficiency, $\mathcal{P}-$efficiency, Millian efficiency, Property Rights.

JEL: D91, H21, H5, E62, J13

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1 Introduction

Low fertility rates seem to concern the governments of many countries, and some of them are implementing pro-natalist policies such as tax breaks, direct transfers or fertility-dependant pension schemes. But are the fertility rates in these countries *inefficiently* low? Schoonbroodt and Tertilt (2014) –hereafter, ST– have addressed this question in an overlapping generations environment in which parental preferences exhibit dynastic altruism, using the notions of $\mathcal{A}-$ and $\mathcal{P}-$efficiency proposed by Golosov, Jones and Tertilt (2007). ST show that, when non-negativity constraints on voluntary intergenerational transfers are binding, equilibrium fertility rates are $\mathcal{A}-$ inefficient. Furthermore, under an assumption imposing that potential, unborn agents obtain lower utility than the utility obtained by any living agent, then equilibrium outcomes are also $\mathcal{P}-$inefficient.

In this paper, I focus on $\mathcal{P}-$efficiency. As in the general formulation by Golosov, Jones and Tertilt (2007) –hereafter, GJT–, the notion of $\mathcal{P}-$efficiency used here is based on a preliminary assumption about the utility *attributed to* potential agents in those allocations in which they are not born. Unlike GJT or ST, who both regard said utility as that *obtained* by every potential agent when unborn, I regard the utility attributed to potential agents as the minimum utility that they *must* obtain, if alive, to ensure that their addition to a population does not reduce social welfare. Moreover, I assume that the utility attributed to unborn agents in any allocation is bounded from below by the utility obtained by the living agents of the same generation, an assumption in line with principles such as "New lives reduce social welfare if the agents living those lives are strictly worse off than those agents of the same generation born before them". With this assumption, equilibria may exhibit dynamic inefficiencies associated with an over-accumulation of capital and may, therefore, be $\mathcal{P}$-inefficient, but only when the non-negativity constraints on intergenerational transfers are *permanently* binding and, in the long run, costs of rearing children do not exceed the present value of future labour incomes. Thus, in contrast with $\mathcal{A}-$efficiency, that may require parents to appropriate a fraction of their children's labour income, the way property rights are initially distributed among the agents does not necessarily alter the $\mathcal{P}-$efficiency of the final allocation of resources arising from said initial distribution of rights.

The *rationale* behind the notions of $\mathcal{A}-$ and $\mathcal{P}-$efficiency is the following. The Pareto criterion, on which the notion of efficiency relies, can be used to rank allocations in which the set of agents is fixed, but not to rank allocations with different fertility choices. Even if we knew the preferences of all the living agents in an economy, there is no way of knowing whether or not the agents alive in one allocation are better off than they would be unborn. To overcome this problem, GJT propose two extensions of the Pareto criterion for the ranking of allocations with different fertility choices. The first, referred to as the $\mathcal{A}-$dominance criterion, ranks any two allocations according to the utility profiles of the agents living in both of them. The second, referred to as the $\mathcal{P}-$dominance criterion, is constructed from a preliminary assumption regarding the utility "obtained" by potential agents if unborn. Complemented by the utility functions of the living agents, this assumption provides a complete description of the "preferences" of all potential agents across social states, so that extending the Pareto criterion using these preferences is straightforward. The two extensions of the Pareto criterion give rise to two notions of efficiency, respectively referred to as $\mathcal{A}-$ and $\mathcal{P}-$efficiency, applicable to settings with endogenous fertility. Although GJT are aware that the utility obtained by the unborn is not observable, they show, through different examples, that the $\mathcal{P}-$efficiency of many allocations does not depend on the particular assumption made on the utility obtained by the unborn. With dynastic altruism, for example, $\mathcal{A}-$efficiency implies $\mathcal{P}-$efficiency.

The argument used by ST to establish the $\mathcal{A}-$inefficiency of equilibria with binding constraints on transfers is the following. If, in a given equilibrium, children are endowed with more resources than their parents are willing to provide, then parents may obtain net welfare gains by having more children and providing them with fewer resources. These "new children" become "cheaper" than their living siblings, and bring net social welfare gains because their preferences are not taken into account when an allocation in which they are born is compared, using the $\mathcal{A}-$dominance criterion, with one in which they are not. The argument remains valid for $\mathcal{P}-$efficiency, provided we assume –as ST do– that the utility obtained by potential, unborn agents is lower than the utility obtained by living agents.

However, $\mathcal{A}-$ and $\mathcal{P}-$efficiency may have somewhat unexpected consequences. For example, Cordoba and Ripoll (2016) argue that non-negativity constraints on voluntary transfers from parents to children might be responsible for the observed negative relationship between fertility rates and average income across countries, which seems paradoxical because it means that those countries with inefficiently low fertility rates may exhibit higher average income levels than those with $\mathcal{A}-$ and $\mathcal{P}-$efficient fertility. Córdoba and Liu (2021) explore the properties of $\mathcal{A}-$ and $\mathcal{P}-$efficiency with fixed resources, showing that both notions of efficiency drive the economy to *Malthusian Stagnation.* ST themselves show that, if parents are not altruistic towards their descendants and simply value children as a consumption good, then in the only $\mathcal{A}-$ and $\mathcal{P}-$efficient equilibrium, "the initial old expropriate all income from their children, who consume zero, and no children are born". More recently, Pérez-Nievas, Conde-Ruiz and Giménez (2019) show that, if parental altruism extends to a finite number of descendants, then a positive measure of living agents of every generation must obtain arbitrarily low incomes *in every* $\mathcal{A}-$efficient allocation.

These results originate in the fact that when two allocations are ranked with the $\mathcal{A}-$dominance criterion, it is implicitly assumed that the addition of potential agents to a population increases social welfare as long as some of their parents or their already living siblings are better off and none of them is worse off. Thus, the addition of a potential agent to a *utility unaffected* population leaves social welfare unaltered, irrespectively of the living conditions of said agent. A similar implicit assumption underlies the $\mathcal{P}-$dominance criterion, provided ST's assumption on the utility obtained by the unborn is satisfied. In this case, the addition of a potential agent to a utility unaffected population always increases social welfare. In this paper, I focus on the general notion of $\mathcal{P}-$efficiency, although I differ from both GJT and ST in two respects:

First of all, I explicitly regard the utility attributed to potential agents, that accompanies the notion of $\mathcal{P}-$dominance, not as the utility that they obtain when they are not born, but as a utility threshold determining whether their addition (or substraction) to the population born before them increases, decreases or leaves social welfare unaltered. To be more precise, the utility attributed to a potential agent is the minimum utility that they must obtain, if alive, in order to ensure that, if added to the population formed by all living agents without decreasing their utilities, then social welfare does not decrease.

Secondly, I allow that the utility thresholds attributed to potential agents depend on the utility actually obtained by the living agents in every allocation. Without imposing a particular specification on the function determining these thresholds, I explore the properties of $\mathcal{P}-$efficiency when the utility attributed to potential, unborn agents in any allocation is bounded from below by the utility obtained by any agent of the same generation alive in the same allocation.

With this assumption, increasing fertility rates and providing the newborn with fewer resources than those available to other members of the same generation reduces social welfare. Thus, for the notion of $\mathcal{P}-$efficiency proposed here, ST's arguments do not apply, and many equilibria are actually $\mathcal{P}-$efficient. To state this formally, I explore the properties of the equilibria arising from different distributions of rights between parents and their descendants, in a general framework that includes that studied by ST and many others. I focus on decentralized markets where agents may claim a (possibly negative) pre-specified amount of resources from their parents and are free to provide their descendants with gifts that exceed the amount of resources that they are entitled to receive. I mainly concentrate on economies where dynasty heads receive the same initial endowment and equilibria are *symmetric* –that is, every agent of the same generation takes the same decisions. When an allocation is symmetric and the utility attributed to the unborn is bounded from below by the minimum utility obtained by the living agents of the same generation, the unborn agents are attributed at least the same utility as the living agents of the same generation. In Theorem 1, I use this property of symmetric allocations to establish I) that the allocation corresponding to a symmetric equilibrium is $\mathcal{P}-$efficient if it is Millian efficient,[1] that is, it cannot be $\mathcal{A}-$dominated by any other symmetric allocation; and II) that every symmetric, $\mathcal{P}-$efficient allocation can be characterized as the equilibrium of a decentralized mechanism associated with a particular distribution of rights between parents and their descendants. As in other overlapping generation economies, some distributions of rights may give rise to equilibria in which capital is over-accumulated and are only *statically* Millian efficient (and, hence, $\mathcal{P}-$inefficient), that is, they cannot be $\mathcal{A}-$dominated by a symmetric reallocation of resources involving a finite number of intergenerational transfers.

The paper also provides sufficient conditions for full (or *dynamic*) Millian efficiency (and, hence, for $\mathcal{P}-$efficiency) of symmetric equilibria. If non-negativity constraints on transfers are only *temporarily* binding, $\mathcal{P}-$efficiency is easily demonstrated. When they are *permanently* binding, a difficulty arises because costs of rearing n_{t+1} children and providing each with e_{t+1} units of resources are given by $n_{t+1}e_{t+1}$, a non-convex function of two endogenous variables. Taking non-convexities into account, the standard condition ensuring the dynamic Pareto efficiency of a growth path –which requires that, in the long run, the interest rate exceeds the rate at which the economy grows–, may not be sufficient for $\mathcal{P}-$efficiency. To get round this, I offer first a technical condition based on the minimum reduction in the income transferred to each descendant that would compensate parents for a reduction in their own income. Then I provide a sufficient condition for $\mathcal{P}-$efficiency that ensures that said technical condition holds: in the long run, costs of raising children must be higher than the present value of future labour incomes. The paper contains a detailed exploration of a model that extends that by Razin and Ben Zion (1975), showing that many $\mathcal{A}-$inefficient equilibria are $\mathcal{P}-$efficient.

In my view, these results suggest that we should take ST's claims for a market failure with caution. Leaving aside any normative concerns about the $\mathcal{A}-$dominance criterion in non-dynastic environments, the notion of $\mathcal{P}-$efficiency proposed here is based on a principle according to which new lives should not be considered as a social improvement when the agents living those lives are worse off than those living members of the same generation. Replacing a $\mathcal{P}-$efficient allocation by another requires sacrificing that principle. Moreover, since $\mathcal{A}-$efficient allocations can also be characterized as the equilibria arising from a distribution of rights, restoring $\mathcal{A}-$efficiency in a welfare improving way (in the sense given by the $\mathcal{A}-$dominance criterion) requires that some agents enjoy fewer rights than their

[1] See Conde-Ruiz et al. (2010).

siblings. Furthermore, if we implement policies that restore $\mathcal{A}-$efficiency from an initial $\mathcal{P}-$efficient allocation but maintain a symmetric distribution of rights among the agents, then we may make some (possibly, most) generations worse off than they would be in the original allocation.

The paper is organized as follows. In Section 2, I describe the framework, discuss the notion of efficiency proposed in the paper and define formally the notion of an equilibrium generated by a sequence of intergenerational transfers. In Section 3, I provide a version of the two Welfare Theorems for symmetric $\mathcal{P}-$efficiency. I also provide sufficient conditions ensuring the $\mathcal{P}-$efficiency of equilibria, and parametric examples showing that many equilibria with binding constraints on intergenerational transfers are $\mathcal{P}-$efficient. In Section 4, I present the main conclusions and discuss several possibilities for further research. All proofs are included in the Appendix.

2 The setting. Assumptions and definitions

The paper focuses on a class of infinite horizon, overlapping generations economies analogous to that described by ST. Each individual in an economy lives for at most three periods, and individuals living at $t = 0, 1, 2, ...$are referred to as *children, middle-aged adults* or *old adults* depending on whether it is their first, second or third period of life. As in GJT, there is a set of unit measure $\mathsf{D}^0 \equiv [0, 1]$ formed by all *dynasty heads* alive when the economy starts. For each $t \geq 1$, every middle-aged adult *potentially alive* at $t \geq 1$ is identified by a by a vector $i^t = (i^{t-1}, i_t) \in [0, 1] \times [0, \overline{n}]^t$, where i_t specifies the the agent's position in the siblings birth order and $i^{t-1} = (i_0, i_1..., i_{t-1})$ identifies the agent's parent. The set of agents of generation t (i.e., in their middle age at t) potentially alive at t will be denoted by $\mathcal{P}^t \equiv [0, 1] \times [0, \overline{n}]^t$.

A *fertility plan* n is a sequence of functions $\mathsf{n} = \{\mathsf{n}_{t+1} : \mathcal{P}^t \longrightarrow [0, \overline{n}]\}_{t \geq 0}$ determining, for each $t \geq 0$ and each potential agent $i^t \in \mathcal{P}^t$, the fertility choice $\mathsf{n}_{t+1}(i^t)$ selected by i^t. For each t and every $i^t = (i^{t-1}, i_t) \in \mathcal{P}^t$, agent i^t is *alive with* n if agent i^{t-1} is also alive and $i_t \leq \mathsf{n}_t(i^{t-1})$ is satisfied. For every $t \geq 0$, the set of agents of generation t alive with a fertility plan n is denoted by D^t.

In addition to children, there is only one homogenous good produced at every period $t \geq 1$ using, as inputs, an amount K_t of the same good invested in the previous period as physical capital and an amount of labour N_t provided by middle-aged adults. That is, $Y_t \leq F_t(K_t, N_t)$, where Y_t is total output and $F_t : \mathbb{R}^2_+ \longrightarrow \mathbb{R}_+$ is non-decreasing, concave, and linearly homogeneous. For each $t \geq 0$, rearing n_{t+1} children at time t costs

$b_t n_{t+1}$ units of the homogeneous good, with $b_t > 0$.[2] Fertility and consumption plans of potential agents are represented by a fertility plan n and a sequence of integrable functions $\mathsf{c} = \left\{(\mathsf{c}_t^m, \mathsf{c}_{t+1}^o) : \mathcal{P}^t \longrightarrow \mathbb{R}_+^2\right\}_{t\geq 0}$ that determines, for each $t \geq 0$ and each $i^t \in \mathcal{P}^t$, the consumption vector $(\mathsf{c}_t^m(i^t), \mathsf{c}_{t+1}^o(i^t))$ chosen by agent i^t through his/her life cycle. Thus, children do not take consumption decisions.

At time $t = 0$, the amount of resources available to finance consumption $(\mathsf{c}_0^m(i^0))$, fertility $(\mathsf{n}_1(i^0))$ and investment decisions $(\mathsf{k}_1^o(i^0))$ of each dynasty head is bounded from above by her initial endowment $\bar{\mathsf{e}}_0(i^0)$. Thus, total resources available to all dynasty heads is bounded by the initial total –or average– endowment $\bar{e}_0 = \int_0^1 \bar{\bar{\mathsf{e}}}_0(i^0)di^0$, that is,

$$\int_0^1 \left[\mathsf{c}_0^m(i^0) + b_0\mathsf{n}_1(i^0) + \mathsf{k}_1^o(i^0)\right] di^0 \leq \int_0^1 \bar{\bar{\mathsf{e}}}_0(i^0)di^0 \equiv \bar{e}_0. \tag{1}$$

For each period $t \geq 0$, each living agent is endowed with 1 unit of labour time when they reach middle age. Then, labour is supplied inelastically, so that labour supply at any given period coincides with the measure of middle-aged agents alive at t, that is, $N_t = \int_{\mathsf{D}^t} di^t$. The resource constraint at each date $t \geq 1$ is

$$\int_{\mathsf{D}^{t-1}} \mathsf{c}_t^o(i^{t-1})di^{t-1} + \int_{\mathsf{D}^t} \left[\mathsf{c}_t^m(i^t) + b_t\mathsf{n}_{t+1}(i^t)\right] di^t + K_{t+1} \leq F_t(K_t, N_t),$$

which, by writing $\mathsf{k}_{t+1}^o(i^t)$ for capital investment attributed to i^t , that is,

$$\mathsf{k}_{t+1}^o(i^t) = \mathsf{n}_{t+1}(i^t)\left(K_{t+1}/N_{t+1}\right);$$

and $\mathsf{e}_t(i^t)$ for the amount of resources –or *income*– available to finance consumption, fertility and investment decisions of agent i^t, that is,

$$\mathsf{e}_t(i^t) = \mathsf{c}_t^m(i^t) + b_t\mathsf{n}_{t+1}(i^t) + \mathsf{k}_{t+1}^o(i^t);$$

can be equivalently represented as

$$\int_{\mathsf{D}^{t-1}} \mathsf{c}_t^o(i^{t-1})di^{t-1} + \int_{\mathsf{D}^t} \mathsf{e}_t(i^t)di^t \leq \int_{\mathsf{D}^{t-1}} F_t(\mathsf{k}_t^o(i^{t-1}), \mathsf{n}_t(i^{t-1}))di^{t-1}. \tag{2}$$

In what follows, an allocation $\mathsf{a} = (\mathsf{x}, \mathsf{k}^o)$ is a fertility-consumption plan

$$\mathsf{x} = \left\{\mathsf{x}_t = (\mathsf{c}_t^m, \mathsf{c}_{t+1}^o, \mathsf{n}_{t+1}) : \mathcal{P}^t \longrightarrow \mathbb{R}_+^2 \times [0, \bar{n}]\right\}_{t\geq 0},$$

and an investment plan $\mathsf{k}^o = \left\{\mathsf{k}_{t+1}^o : \mathcal{P}^t \longrightarrow \mathbb{R}_+\right\}_{t\geq 0}$ determining the choices of every potential agent –if alive– in every period. An allocation a is feasible if it satisfies the initial

[2] The model can be easily adapted to study the $\mathcal{P}-$efficiency properties of the equilibria arising in economies with time costs.

condition in (1) and the resource constraint in (2) for $t \geq 1$. The set formed by all feasible allocations is denoted by $\mathcal{F}$.

Let $\mathbb{R}^*$ be the set of extended real numbers $\mathbb{R}^* \equiv \{-\infty\} \cup \mathbb{R}$. Preferences of potential agents of generation t, if alive, are represented by a utility function $\mathcal{U}_t : \mathcal{F} \times \mathcal{P}^t \to \mathbb{R}^*$ satisfying, for every $\mathsf{a} \in \mathcal{F}$ and every $i^t \in \mathsf{D}^t$,

$$\mathcal{U}_t(\mathsf{a}, i^t) = U\left(\mathsf{x}_t(i^t), \frac{\int_0^{\mathsf{n}_{t+1}(i^t)} \mathcal{U}_{t+1}(\mathsf{a}, i^t, i_{t+1}) di_{t+1}}{\mathsf{n}_{t+1}(i^t)}\right); \tag{PR}$$

where $U : \mathbb{R}_+^2 \times [0, \overline{n}] \times \mathbb{R}^* \to \mathbb{R}^*$ is non-decreasing and concave on the interior of its domain. The framework covers, as a particular case, that studied by ST, for whom

$$U(x_t, U_{t+1}) = v(c_t^m) + \gamma^o v(c_{t+1}^o) + \Psi(n_{t+1}, U_{t+1}), \tag{STPR}$$

with Ψ being either additively separable –as Razin and Ben Zion (1975)– or homogeneous –as in Barro and Becker (1989)[3].

2.1 $\mathcal{A}-$efficiency and dynastic optima

Following GJT, an allocation $\mathsf{a} \in \mathcal{F}$ $\mathcal{A}-$dominates an allocation $\widetilde{\mathsf{a}}$ if the inequality $\mathcal{U}_t(\mathsf{a}, i^t) \geq \mathcal{U}_t(\widetilde{\mathsf{a}}, i^t)$ holds for every $t \geq 0$ and every $i^t \in \mathsf{D}^t \cap \widetilde{\mathsf{D}}^t$, and holds as a strict inequality for some $t \geq 0$ and a set $\mathsf{B}^t \subseteq \mathsf{D}^t \cap \widetilde{\mathsf{D}}^t$ of positive measure. An allocation is $\mathcal{A}-$efficient if it is not $\mathcal{A}-$dominated by any other feasible allocation. Since dynasty heads are alive in every feasible allocation, a class of allocations that are trivially $\mathcal{A}-$efficient allocations are *dynastic optima.* An allocation $\mathsf{a}^* \in \mathcal{F}$ is a dynastic optimum if it solves

$$\max_{\mathsf{a} \in \mathcal{F}} \left\{ \int_0^1 \mathcal{U}_0(\mathsf{a}, i^0) di^0 : \int_0^1 \mathsf{e}_0(i^0) di^0 = \overline{e}_0 \right\} \equiv \mathcal{V}_0(\overline{e}_0).$$

2.2 $\mathcal{P}-$efficiency

As in GJT, the notions of $\mathcal{P}-$dominance and $\mathcal{P}-$efficiency used in the paper are associated to a sequence of functions $\mathcal{U}^N = \left\{\mathcal{U}_t^N : \mathcal{F} \times \mathcal{P}^t \to \mathbb{R}^*\right\}_{t \geq 1}$ that determines, for every t and every $i^t \in \mathcal{P}^t$, the utility $\mathcal{U}_t^N(\mathsf{a}, i^t)$ obtained by –or attributed to– i^t, in the allocation a, when i^t is not born. Complemented with the utility functions that represent the preferences of the agents when they are alive, the sequence $\mathcal{U}^N$ allows one to represent the "preferences" of every potential agent of generation t with the function $\mathcal{U}_t^P : \mathcal{F} \times \mathcal{P}^t \to \mathbb{R}^*$ defined, for

[3] In the original formulation of these models, agents live for one period and $\gamma^o = 0$ holds.

every allocation $\mathsf{a} \in \mathcal{F}$ and every potential agent i^t, by

$$\mathcal{U}_t^P(\mathsf{a}, i^t) = \begin{cases} \mathcal{U}_t(\mathsf{a}, i^t), & \text{if } i^t \in \mathcal{D}^t; \\ \mathcal{U}_t^N(\mathsf{a}, i^t), & \text{otherwise.} \end{cases}$$

An allocation a $\mathcal{P}-$dominates an allocation $\widetilde{\mathsf{a}}$ if the inequality $\mathcal{U}_t^P(\mathsf{a}, i^t) \geq \mathcal{U}_t^P(\widetilde{\mathsf{a}}, i^t)$ holds for every $t \geq 0$ and every potential agent $i^t \in \mathcal{P}^t$, and holds a strict inequality for some $t \geq 0$ and a set $\mathsf{B}^t \subseteq \mathcal{P}^t$ of positive measure. A feasible allocation is $\mathcal{P}-$efficient if it is not $\mathcal{P}-$dominated by any other feasible allocation.

Although GJT explicitly assume that the preferences of all potential agents are represented by well defined utility functions and distinguish these preferences from societal preferences, any extension of the Pareto criterion always involves a value judgment about how the individuals forming a society should rank any two allocations with different populations, using information on their individual preferences. Both GJT (in some of their applications) and ST[4] assume that the utility attributed to any potential agent i^t, if unborn, satisfies

$$\mathcal{U}_t^N(\mathsf{a}, i^t) = u_N \equiv \inf \left\{ \mathcal{U}_t(\mathsf{a}, \widetilde{i}^t) : \widetilde{i}^t \in \mathsf{D}^t; t \geq 1; \mathsf{a} \in \mathcal{F} \right\}. \quad (3)$$

With this assumption, they implicitly assume that increasing fertility rates increases social welfare if it does not make the already living agents worse off; while decreasing fertility rates does not increase social welfare even if it benefits those agents that are still alive after said reduction in fertility rates takes place. Value judgements of this kind are especially difficult when fertility is endogenous, not only because "preferences of the unborn agents are inherently impossible to observe", as GJT put it, but also because, even if we could reach a consensus on the conditions under which a life is worth living for the individual living that life, it is not clear that these conditions should be the only –or even the most important– information to determine whether or not altering fertility rates increases or decreases social welfare.

Although formally identical to that of GJT, the approach followed here explicitly regards the utility attributed to the unborn as an assumption on social preferences, an interpretation closely analogous to how *critical levels* are interpreted in the *Social Choice* literature.[5] There, a critical level $\underline{u}$ is defined as the utility level for which the addition, to a utility-unaffected population, of a person obtaining exactly that utility leaves social welfare unaltered. Here, I regard the utility $\mathcal{U}_t^N(\mathsf{a}, i^t)$ attributed to a potential agent i^t in a given allocation a as the minimum utility that i^t should obtain so that the addition of i^t to

[4]See, respectively, Assumption 4 in Golosov et al. (2007) and Assumption 5 in Schoonbroodt and Tertilt (2014).

[5]See, e.g., Blackorby, Bossert and Donaldson (2005).

the population formed by all other agents alive in a –without affecting their utilities– either increases or leaves unaltered social welfare. As in the different versions of *Critical Level Utilitarianism*, the critical level attributed to the unborn may not coincide with the utility level representing *neutrality*, that is, the utility level above which a life is worth living.

As in *Average Utilitarianism* or in the recent literature on *Variable-Value* population principles[6], I assume that the utility attributed to a potential agent –if unborn– in a given allocation depends on the utility obtained by the agents alive in that allocation. To be more precise, I assume that the following holds:

*A*1 For every allocation $\mathsf{a} \in \mathcal{F}$, every $t \geq 1$ for which $\mathsf{D}^t \neq \varnothing$ and every $i^t \in \mathcal{P}^t$, the utility attributed to i^t –if unborn– in a satisfies

$$\mathcal{U}_t^N(\mathsf{a}, i^t) \geq \min\left\{\mathcal{U}_t(\mathsf{a}, i) : \, i \in \mathsf{D}^t\right\}.$$

Assumption *A*1 is compatible with many different specifications of the sequence $\mathcal{U}^N$, that is, with many different principles restricting social preferences. For example, the principle "New lives increase social welfare only when these lives make the already living agents better off and the agents living said lives are not worse off than their already living *siblings*" would be represented by a utility function satisfying, for every allocation a, every $t \geq 1$ for which $\mathsf{D}^t \neq \varnothing$ and every $i^t = (i^{t-1}, i_t) \in \mathcal{P}^t \backslash \mathsf{D}^t$,

$$\mathcal{U}_t^N(\mathsf{a}, i^t) = \min\left\{\mathcal{U}_t(\mathsf{a}, i^{t-1}, i_t') : \, i_t' \leq \mathsf{n}_t(i^{t-1})\right\} \tag{4}$$

while the weaker principle "New lives increase social welfare only when these lives make the already living agents better off and the agents living said lives are not worse off than any living member of the same generation" would be represented by a utility function satisfying, for every allocation a, every $t \geq 1$ for which $\mathsf{D}^t \neq \varnothing$ and each $i^t \in \mathcal{P}^t \backslash \mathsf{D}^t$,

$$\mathcal{U}_t^N(\mathsf{a}, i^t) = \min\left\{\mathcal{U}_t(\mathsf{a}, \widetilde{i}^t) : \, \widetilde{i}^t \in \mathsf{D}^t\right\}. \tag{5}$$

Observe that $\mathcal{P}-$efficiency does not impose that parents must treat their immediate descendants equally. As we shall see in Section 3, such an equal treatment of all descendants may or may not occur in the equilibrium outcome arising when these descendants have the same rights and duties, but it is neither necessary nor sufficient for $\mathcal{P}-$efficiency. Suppose, for example, that, starting from an allocation a, dynasty heads obtain utility gains by selecting an allocation a' in which they have more children. If some of the children living in a' but not in a obtain lower utility than any of the children living in a, then a'

[6]See, e.g., Bossert, Cato and Kamaga (2021) and the references therein.

neither $\mathcal{P}-$dominates nor is $\mathcal{P}-$dominated by the allocation a, since dynasty heads –who are alive in every allocation– are better off in a' than they are in a. Unlike the social preferences usually explored in the Social Choice literature, the social preferences induced by the $\mathcal{P}-$dominance criterion are, with *any* specification of $\mathcal{U}^N$, *incomplete*. Therefore, there are, in general, many $\mathcal{P}-$efficient allocations, none of which is $\mathcal{P}-$better than any other from the point of view of social preferences. In particular, since the conditions required to establish $\mathcal{P}-$dominance are stronger than those required to establish $\mathcal{A}-$dominance, every $\mathcal{A}-$efficient allocation is also $\mathcal{P}-$efficient under Assumption $A1$ or under any other specification of $\mathcal{U}^N$.

To conclude, observe that, when an allocation a that is $\mathcal{P}-$efficient under Assumption $A1$ is replaced by another allocation a' with more individuals, the principles underlying Assumption $A1$ must be sacrificed, that is, either those living in both a and a' –i.e., parents and their older children living already in a– are worse off in the allocation a'; or some people who live in a' but not in a –i.e., the youngest children living in a'– obtain lower utilities than any member of the same generation alive in both a and a'. Thus, the allocation a' may $\mathcal{A}-$dominate a, but selecting a' instead of a would give rise to an unequal treatment between siblings (or, more generally, between the oldest and the youngest people of some generations) that was not present in a. Societies must decide whether or not this type of unequal treatment is desirable.

2.2.1 Symmetric allocations: $\mathcal{P}-$efficiency and Millian efficiency Throughout most of the paper, I focus on *symmetric* allocations, that is, on allocations for which any two agents of the same generation take the same decisions. An allocation $\mathsf{a} \in \mathcal{F}$ is symmetric if, for each $t \geq 0$ and each $i^t \in \mathsf{D}^t$, one has $(\mathsf{x}_t(i^t), \mathsf{k}^o_{t+1}(i^t)) = (x_t, k_{t+1})$ for some $(x_t, k^o_{t+1}) \in \mathbb{R}^2_+ \times [0, \overline{n}] \times \mathbb{R}_+$. The set of symmetric allocations, denoted by $\mathcal{S} \subset \mathcal{F}$, can be represented by the set formed by all sequences $a \equiv \left\{(x_t, k^o_{t+1})\right\}_{t=0,1,2...}$ satisfying the initial condition $c^m_0 + b_0 n_1 + k^o_1 \leq \overline{e}_0$, as well as the feasibility constraint

$$c^o_t + n_{t+1}\left[c^m_t + b_t n_{t+1} + k^o_{t+1}\right] \leq F_t(k^o_t, n_t) \tag{6}$$

for each $t \geq 1$. Whenever an allocation $\mathsf{a} \in \mathcal{F}$ is symmetric, I shall write $\mathsf{a} = a$. Notice that the preferences of the agents of generation t on symmetric allocations can be represented by a utility function $U_t : \mathcal{S} \to \mathbb{R}$ recursively defined, for each $a \in \mathcal{S}$, by

$$U_t(a) = U(x_t, U_{t+1}(a)).$$

Suppose that the function determining the utility obtained by the unborn satisfies either (4) or (5). In any of these cases, if an allocation $\mathsf{a} = a$ is symmetric and exhibits positive fertility rates at every period, then one has, for every $i^t \in \mathcal{P}^t$,

$$\mathcal{U}_t^N(\mathsf{a}; i^t) = U_t(a). \tag{7}$$

In Pérez-Nievas et al. (2019), it is shown that, if the utility attributed to the unborn satisfies (7) for every symmetric allocation a and value functions associated to $\mathcal{P}-$efficiency are concave, said symmetric allocation a is $\mathcal{P}-$efficient if, and only if, it is *Millian efficient*.[7] A Millian efficient allocation is a symmetric allocation $a \in \mathcal{S}$ for which there does not an alternative, symmetric allocation a' that makes all generations of agents better off without making any generation worse off, and can be equivalently described as a symmetric allocation that is not $\mathcal{A}-$dominated by any other symmetric allocation.[8] When a dynastic optimum is symmetric, it is both $\mathcal{A}-$ and Millian efficient. However, said dynastic optimum is the only symmetric allocation exhibiting these two properties.

Unfortunately, determining whether or not a symmetric allocation is Millian efficient (and, hence, $\mathcal{P}-$efficient) is not straightforward. Although obtaining a set of necessary conditions for Millian efficiency is relatively simple, those necessary conditions only ensure that an allocation satisfying these conditions satisfies a weaker notion of efficiency, that Conde-Ruiz et al. (2010) refer to as *static Millian efficiency*, a notion of efficiency closely related to the notion of static *-or short run-* Pareto efficiency introduced by Balasko and Shell (1980). A symmetric allocation $\widehat{a}$ is statically Millian efficient if there does not exist a finite period T and an alternative symmetric allocation a such that *i)* $x_t = \widehat{x}_t$ holds for every $t \geq T$; and *ii)* a $\mathcal{A}-$dominates $\widehat{a}$. Thus, a statically efficient allocation cannot be $\mathcal{A}-$dominated by a symmetric reallocation of resources that involves only a finite number of generations. As in the literature with exogenous fertility, a statically efficient allocation that fails to be fully Millian efficient is referred to as *dynamically* inefficient.

2.3 Equilibria and Rights

To complete the Section, I explore how the agents would act in a decentralized, market mechanism complemented with a system of intergenerational transfers enforced by the government. As in the First Welfare Theorem or in Coase Theorem, the objective is to explore whether or not the way property rights are initially distributed among the participants affects the $\mathcal{P}-$efficiency in the final allocation of resources arising from said distribution of rights. The objective is, therefore, analogous to that pursued by ST, although I extend the

[7]See Theorem 3 in Pérez-Nievas et al. (2019).

[8]Lang (2005) and Michel and Wigniolle (2007) explore the same notion of efficiency under different names.

analysis to $\mathcal{P}-$efficiency under Assumption $A1$. I assume that labour time available at each period $t \geq 1$ is owned by workers and that the property (and the income stream) of the only capital good available at t belongs to those who accumulated capital in the previous period $t-1$. I allow, however, for different initial distributions of rights (that is, on the authority to decide) and duties over the use of the consumption good between the agents and their descendants. To be more precise, all agents of generation $t \geq 1$ are obliged to provide with $\underline{g}_{t+1}$ units of the consumption good to each of their immediate descendants. Each $\underline{g}_{t+1}$ may be negative, in which case it is the agents of generation $t+1$ who are obliged to provide their parents with $\underline{\tau}_{t+1} = -\underline{g}_{t+1}$ units of the consumption good. As in Coase Theorem, I don´t presuppose that those generating an alleged positive (or, respectively, negative) externality on others should be subsidized (or, respectively, taxed) by those affected by the externality. Note that the distribution of rights and duties induced by a sequence $\underline{g} = \left\{\underline{g}\right\}_{t\geq 1}$ can be regarded as symmetric, in the sense that rights and duties are the same for any two agents of the same generation. The equilibrium arising when $\underline{g} \equiv 0$ will be referred to as the *laissez faire* equilibrium.

Given the initial distribution of rights, the allocation of resources is determined by market transactions and voluntary transfers from parents to their descendants. There are two markets operating at each date $t \geq 0$: a financial market, that allows agents of generation t to lend an arbitrary amount k^o_{t+1} of the homogeneous good in period t and obtain a return equal to $R_{t+1}k^o_{t+1}$ in period $t+1$; and a job market, in which labour is exchanged against the consumption good at a price w_t. Since the agents of generation $t \geq 0$ are altruistic towards their descendants, they might be willing to transfer, at period $t+1$, an amount $\mathsf{g}_{t+1}(i^t, i_{t+1}) \geq \underline{g}_{t+1}$ of the *numeraire* to each immediate descendant $i^{t+1} = (i^t, i_{t+1})$ when said descendant reaches middle age. By choosing their gifts, parents determine the *income scheme*

$$\mathsf{e}_{t+1}(i^t, i_{t+1}) = w_{t+1} + \mathsf{g}_{t+1}(i^t, i_{t+1}) \geq w_{t+1} + \underline{g}_{t+1}$$

and, therefore, the *income distribution* $\mathsf{E}_{t+1}(\cdot, i^t)$ available to their descendants, defined, for each $e_{t+1} \geq w_{t+1} + \underline{g}_{t+1}$ by

$$\mathsf{E}_{t+1}(e_{t+1}, i^t) = \frac{1}{\mathsf{n}_{t+1}(i^t)} \int_{i_{t+1}:\mathsf{e}_t(i^t, i_{t+1})\leq e_{t+1}} di_{t+1}.$$

Parents are allowed to choose not to treat their descendants equally. That is, they are not restricted to choose a degenerate distribution taking only a single value in $\mathbb{R}_+$. For any allocation a, each period t and each agent $i^t \in \mathsf{D}^t$, write $\mathsf{e}^A_{t+1}(i^t)$ for the average income obtained by agent i^t's immediate descendants, and $\mathcal{U}^A_{t+1}(\mathsf{a}; i^t)$ for the average utility obtained by all alive descendants of i^t, that is,

$$\mathsf{e}_{t+1}^A(i^t) = \frac{1}{\mathsf{n}_{t+1}(i^t)} \int_0^{\mathsf{n}_{t+1}(i^t)} \mathsf{e}_t(i^t, i_{t+1}) di_{t+1} = \int_{\mathbb{R}_+} ed\mathsf{E}_{t+1}(e, i^t);$$

and

$$\mathcal{U}_{t+1}^A(\mathsf{a}; i^t) = \frac{1}{\mathsf{n}_{t+1}(i^t)} \int_0^{\mathsf{n}_{t+1}(i^t)} \mathcal{U}_{t+1}(\mathsf{a}; i^t, i_{t+1}) di_{t+1}.$$

If the agents of generation t hold correct expectations on future prices –represented by a sequence $p^{-t} \equiv \{w_\tau, R_\tau\}_{\tau \geq t+1}$– and on their descendants' future decisions, then any of these agents will choose their consumption-fertility bundle $\widehat{\mathsf{x}}_t(i^t)$ and their savings $\widehat{\mathsf{k}}_{t+1}(i^t)$ to solve

$$\max_{(x_t, k_{t+1}) \in \mathbb{R}_+^2 \times [0,\overline{n}] \times \mathbb{R}_+} \Big\{ U\left(x_t, \mathcal{U}_{t+1}^A(\mathsf{a}; i^t)\right) : c_t^m + b_t n_{t+1} + k_{t+1}^o = \mathsf{e}_t(i^t);$$

$$c_{t+1}^o = R_{t+1} k_{t+1}^o - n_{t+1}\left(\mathsf{e}_{t+1}^A(i^t) - w_t\right) \Big\} \equiv W_t^p\left(\mathsf{e}_t(i^t), \mathsf{e}_{t+1}^A(i^t), \mathcal{U}_{t+1}^A(\mathsf{a}; i^t)\right).$$

Proceeding recursively, it follows that the consumption, investment and fertility decisions taken by an agent of generation t are completely determined by the agent's income $\mathsf{e}_t(i^t)$, the sequence of income distributions $\mathsf{E}^{-t} \equiv \left\{\mathsf{E}_{t+\tau} : [0,\overline{n}]^{t+\tau} \to \mathbb{R}_+\right\}_{\tau=1,2...}$ chosen by the agent and his/her descendants, and the sequence of prices p^{-t}.

Given a sequence of transfers $\underline{g}$, write $\underline{e}$ for the sequence $\underline{e} = \{\underline{e}_t\}_{t \geq 1}$ specifying, for each $t \geq 0$, the minimum income $\underline{e}_t = w_t + \underline{g}_t$ available to the agents of generation t. Assume that gift strategies depend only on time and on the income available to the agents after their parents have chosen their gifts. For any $x \in \mathbb{R}$, write $\mathcal{D}[x, \infty)$ for the set of continuous from the right, non-decreasing distribution functions with support on the set $[x, \infty)$. If strategies are symmetric[9] and any two agents of the same generation receiving the same income choose the same income scheme for their immediate descendants, then the income distribution arising with a symmetric strategy of an agent of generation t can be represented by a mapping $E_{t+1} : [\underline{e}_t, \infty) \to \mathcal{D}\left[\underline{e}_{t+1}, \infty\right)$ that determines, for any $e_t \geq w_t + \underline{g}_t = \underline{e}_t$, the distribution of income $E_{t+1}(\cdot / e_t) \in \mathcal{D}\left[\underline{e}_{t+1}, \infty\right)$ arising from the gift strategy of the agent. The utility payoffs $B_t^p(e_t, E^{-t})$ obtained by each agent of generation t with the sequence of strategies $E^{-t} = \left\{E_s : [\underline{e}_s, \infty) \to \mathcal{D}\left[\underline{e}_{s+1}, \infty\right)\right\}_{s>t}$ if they receive an income e_t can be recursively defined, for $t \geq 0$, by

$$B_t^p(e_t, E^{-t}) = W_t^p\left(e_t, \int_{\underline{e}_{t+1}}^{\infty} edE_{t+1}(e/e_t), \int_{\underline{e}_{t+1}}^{\infty} B_{t+1}^p(e, E^{-(t+1)})dE_{t+1}(e/e_t)\right).$$

There are, in general, many symmetric strategies arising as a Subgame Perfect Equilibria of the voluntary transfers game played within families. Among these, I select the sequence

[9] Note that a sequence of symmetric strategies may not give rise to a symmetric allocation.

$\widehat{E}$ that dynasty heads (or all parents) would choose if they were able to control the gift schemes selected by their descendants; so that, for each $t \geq 0$ and each $e_t \geq \underline{e}_t$, the sequence $\widehat{E}^{-t}$ solves

$$\mathcal{V}_t^p(e_t; \underline{e}^{-t}) = \max_{E^{-t}} \left\{ B_t^p(e_t, E^{-t}) \right\}, \tag{8}$$

among all sequences

$$E^{-t} = \left\{ E_s : [\underline{e}_s, \infty) \to \mathcal{D}\left[\underline{e}_{s+1}, \infty\right) \right\}_{s>t}.$$

Therefore, the sequence of value functions $\{\mathcal{V}_t^p\}_{t\geq 0}$ satisfies, for each $t \geq 0$ and each $e_t \geq \underline{e}_t$,

$$\mathcal{V}_t^p(e_t, \underline{e}^{-t}) = \max_{E_{t+1} \in \mathcal{D}[\underline{e}_{t+1}, \infty)} \left\{ W_t^p\left(e_t, \int_{\underline{e}_{t+1}}^{\infty} e dE_{t+1}(e), \int_{\underline{e}_{t+1}}^{\infty} \mathcal{V}_{t+1}^p(e, \underline{e}^{-(t+1)}) dE_{t+1}(e)\right)\right\}. \tag{9}$$

Thus, with the sequence $\widehat{E}$, each agent plays a best response to both their parents' and their descendants' strategies, which implies that $\widehat{E}$ forms a subgame perfect equilibrium of the game and maximizes the utility of dynasty heads among all symmetric strategies forming a subgame perfect equilibrium of the game. This notion of equilibrium coincides with that explored by ST, except that they simply assume, without proof, that the agents will not treat their descendants asymmetrically. Furthermore, if the non-negativity constraints on bequests are never binding and the agents live for one period, the notion of equilibrium proposed here coincides with that studied by GJT, who do not assume that bequests are symmetric.

Although the notion of equilibrium described above does not impose that parents treat their children equally, it is straightforward to show that they will choose to do so if the sequence of value functions $\{\mathcal{V}_t^p(\cdot, \underline{e}^{-t})\}_{t\geq 0}$ is concave[10]. If, in addition, all dynasty heads receive the same initial endowment and $\mathsf{e}(i^o) = \overline{e}_0$ is satisfied for each $i^0 \in [0,1]$, the interaction of markets and families gives rise to a symmetric allocation $\widehat{a}$ and a sequence of prices p that I shall refer to as a *symmetric equilibrium (* or, simply, as an equilibrium*) associated to a sequence of intergenerational transfers* $\underline{g}$. Said equilibrium is defined formally as follows:

Definition 1 *A symmetric allocation* $\widehat{a}$ *and a sequence of prices* $p \equiv \{w_t, R_t\}_{t\geq 1}$ *constitute an equilibrium associated to a sequence of intergenerational transfers* $\underline{g}$ *if,*
i) aggregate capital $\widehat{K}_{t+1}^o$ *and labour* $\widehat{N}_{t+1}$ *chosen by firms maximize profits, that is,*

$$D_1 F_{t+1}(\widehat{K}_{t+1}, \widehat{N}_{t+1}) = R_{t+1} \text{ and } D_2 F_{t+1}(\widehat{K}_{t+1}, \widehat{N}_{t+1}) = w_{t+1} \text{ for } t \geq 0;$$

ii) the agents use their income to maximize their utility, given the income and utility obtained by their immediate descendants; that is,

$$U_t(\widehat{a}) = W_t^p\left(\widehat{e}_t, \widehat{e}_{t+1}, U_{t+1}(\widehat{a})\right) \text{ for } t \geq 0;$$

[10]Although in the general, non-convex setting studied here, value functions may not be concave, they are concave under certain conditions, discussed in Section 3.3.

iii) *capital and labor markets clear, that is,*

$$\widehat{K}_{t+1} = \widehat{N}_t \widehat{k}^o_{t+1} \text{ and } \widehat{N}_{t+1} = \widehat{N}_t \widehat{n}_{t+1} \text{ for } t \geq 0;$$

iv) *the sequence* $\{\widehat{e}_t\}_{t\geq 0}$ *is the sequence of gifts corresponding to the sequence of symmetric strategies that maximizes the utility of the dynasty head among all possible subgame perfect equilibria of the game of voluntary transfers played within families; that is, for each* $t \geq 0$, *the function* $\mathcal{V}^p_t(\cdot; w^{-t} + \underline{g}^{-t})$ *is concave and satisfies*

$$U_t(\widehat{a}) = \mathcal{V}^p_t(\widehat{e}_t; w^{-t} + \underline{g}^{-t}).$$

To summarize, the equilibrium generated by a sequence of transfers $\underline{g}$ is, therefore, a symmetric allocation $\widehat{a}$ and a sequence of prices p satisfying, for $t \geq 0$,

$$\begin{aligned} U_t(\widehat{a}) &= \mathcal{V}^p_t(\widehat{e}_t; w^{-t} + \underline{g}^{-t}) = \mathcal{V}^p_t(\widehat{e}_t; \widehat{e}^{-t}) \\ &= \max_{e_{t+1} \geq w_{t+1} + \underline{g}_{t+1}} \left\{ W^p_t(\widehat{e}_t, e_{t+1}, \mathcal{V}^p_t(e_{t+1}; \widehat{e}^{-(t+1)}) \right\}; \end{aligned}$$

as well as the profit maximization and market clearing conditions, which reduce to

$$R_{t+1} = D_1 F_t(\widehat{k}_{t+1}, \widehat{n}_{t+1});\ w_{t+1} = D_2 F_t(\widehat{k}^o_{t+1}, \widehat{n}_{t+1}). \tag{10}$$

3 Equilibria and $\mathcal{P}-$efficiency

3.1 Two Welfare Theorems

In Theorem 1 below, I provide a version of the two Welfare Theorems and characterize symmetric, $\mathcal{P}-$efficient allocations as the equilibria associated to different distributions of rights, represented by sequences of intergenerational transfers.

Theorem 1

I) *The allocation* $\widehat{a}$ *corresponding to an equilibrium* $(\widehat{a}, p)$ *associated to a sequence* $\underline{g}$ *of intergenerational transfers is statically Millian efficient. Furthermore, if* $\widehat{a}$ *is also fully Millian efficient, then it is also* $\mathcal{P}-$*efficient.*

II) *Let* $\widehat{\mathsf{a}} = \widehat{a}$ *be a symmetric,* $\mathcal{P}-$*efficient allocation and let* $p = \{R_t, w_t\}_{t\geq 1}$ *be a sequence of prices satisfying (10) for each* $t \geq 1$.*Then there exists a sequence* $\underline{g}$ *of intergenerational transfers such that* $(\widehat{a}, p)$ *is an equilibrium associated to* $\underline{g}$.

Thus, as stated in Theorem 1.I), a version of the First Welfare Theorem holds only for the static notion of Millian efficiency. As for the notion of Pareto efficiency in settings with exogenous fertility, not every initial distribution of rights ensures that the allocation corresponding to said distribution is $\mathcal{P}-$efficient. Still, when the equilibrium is fully Millian efficient, then it is also $\mathcal{P}-$efficient.

In turn, Theorem 1.II) can be regarded as a version of the Second Welfare Theorem, since it states that every symmetric, $\mathcal{P}-$efficient allocation can be characterized as the equilibrium arising from an initial distribution of rights and duties represented by a sequence of intergenerational transfers. In contrast with the Second Welfare Theorem, in the equilibrium that decentralizes a symmetric, $\mathcal{P}-$efficient allocation, intergenerational transfers must be lump sum for children but not for parents, as the latter must receive (or provide with) a total amount of resources from (or to) their descendants that depends on their fertility decisions. A consequence of this result is that a standard, pay-as-you-go pension program by which each old adult is entitled to receive an amount P_t of resources as a pension and every middle aged agent is obliged to contribute with $\tau_t = -\underline{g}_t$ units of resources to finance the program may be $\mathcal{P}-$inefficient, unless pensions are explicitly tied to fertility choices and $P_t = n_t \tau_t$ is satisfied. Conde-Ruiz et al. (2010)[11] and ST themselves have established, respectively, the Millian and the $\mathcal{A}-$inefficiency of standard, pay-as-you-go pension schemes in which pensions are not linked with fertility choices. If a system of intergenerational transfers to the old is implemented, children can be regarded as an investment good, since they are, in a sense, being used to provide the agents of the previous generation with resources when they are old. But if these transfers are not tied to fertility decisions, children become a public investment good and are underprovided, from the point of view of both $\mathcal{A}-$ and $\mathcal{P}-$efficiency.

3.2 $\mathcal{P}-$efficiency of equilibria. Sufficient conditions

The range of distributions of rights that give rise to Millian efficient paths and, in view of Theorem 1, to $\mathcal{P}-$efficient allocations, is much larger than the range of distributions that generate dynastic optima, i.e., $\mathcal{A}-$efficient allocations. A rather obvious case in which a symmetric equilibrium $(\widehat{a}, p)$ is $\mathcal{P}-$efficient but not $\mathcal{A}-$efficient is that in which the non-negativity constraints on voluntary transfers are not binding from some period T on, that is, the case in which $\widehat{e}_t > w_t + \underline{g}_t$ for $t \geq T$. In an allocation satisfying this property, parents choosing after T are providing their children with more resources than the amount of resources $w_t + \underline{g}_t$ that these children are entitled to receive. Thus, parents would not obtain

[11]In this case, in a setting without altruism.

utility gains if $\underline{g}_t$ is reduced. But a symmetric equilibrium may be $\mathcal{P}-$efficient even when non-negativity constraints on transfers are *permanently* binding. In the proof of Theorem 2 below, I show that the allocation $\widehat{a}$ corresponding to a symmetric equilibrium $(\widehat{a}, p)$ must also satisfy, for each $t \geq 0$,

$$U_t(\widehat{a}) = W_t^p\left(\widehat{e}_t, \widehat{e}_{t+1}, U_{t+1}(\widehat{a})\right) = W_t\left(\widehat{e}_t, \widehat{e}_{t+1}, U_{t+1}(\widehat{a})\right),$$

where the indirect utility function $W_t : \mathbb{R}_+^2 \times \mathbb{R}^* \rightarrow \mathbb{R}^*$ is defined, for each $(e_t, e_{t+1}, U_{t+1}) \in \mathbb{R}_+^2 \times \mathbb{R}^*$, by

$$\begin{aligned} W_t\left(e_t, e_{t+1}, U_{t+1}\right) \quad = \quad & \max_{(x_t, k_{t+1}) \in \mathbb{R}_+^2 \times [0, \overline{n}] \times \mathbb{R}_+} \Bigg\{ U\left(x_t, U_t\right) : c_t^m + b_t n_{t+1} + k_{t+1}^o = e_t; \\ & c_{t+1}^o = F_t(k_{t+1}^o, n_{t+1}) - n_{t+1} e_{t+1} \Bigg\}. \end{aligned}$$

That is, $W_t\left(e_t, e_{t+1}, U_{t+1}\right)$ determines the maximum utility that an agent can obtain with a symmetric, feasible allocation, if said agent is endowed with e_t units of income and is constrained to provide with e_{t+1} units of income to each of their descendants, each of whom obtains a utility equal to U_{t+1}. As shown in the proof of Theorem 2, the allocation $\widehat{a}$ corresponding to an equilibrium is dynamically inefficient if there exists a sequence $\{e_t\}_{t\geq 1}$ satisfying

$$e_t < \widehat{e}_t \tag{11}$$

and

$$W_t\left(e_t, e_{t+1}, U_{t+1}(\widehat{a})\right) = W_t\left(\widehat{e}_t, \widehat{e}_{t+1}, U_{t+1}(\widehat{a})\right) \tag{12}$$

for every $t \geq 1$. Indeed, if such a sequence exists, a representative dynasty head may obtain more utility than the utility she obtains with $\widehat{a}$. She simply needs to select the symmetric allocation a for which $U_t(a) = W_t\left(e_t, e_{t+1}, U_{t+1}(\widehat{a})\right)$ is satisfied for each $t \geq 1$. Since the function $W_0(\overline{e}_0, \cdot, U_1(\widehat{a}))$ is strictly decreasing in e_1, the dynasty head obtains $U_0(a) = W_0(\overline{e}_0, e_1, U_1(\widehat{a})) > W_0(\overline{e}_0, \widehat{e}_1, U_1(\widehat{a}))$ and all other agents obtain the same utility as the utility that they obtain in $\widehat{a}$, which implies that $\widehat{a}$ is $\mathcal{P}-$inefficient.

Thus, dynamic inefficiencies arise when the income available to every generation of agents is too high, so that increasing intergenerational transfers from every generation to the previous one leaves the welfare obtained by every generation unaltered and makes dynasty heads better off. Still, many equilibria for which voluntary transfers from parents to children are zero may correspond to $\mathcal{P}-$efficient paths. In Theorem 2 below, I provide sufficient conditions ensuring that a sequence $\{e_t\}_{t\geq 1}$ satisfying *(11)* and *(12)* cannot exist and, hence, ensuring the $\mathcal{P}-$efficiency of equilibria for which the non-negativity constraints on transfers are binding.

Some new notation is required. For each $t \geq 1$, let $\pi_t : \mathbb{R}^2_+ \times \mathbb{R} \to \mathbb{R}$ be defined, for each $(\widehat{e}_t, \widehat{e}_{t+1}, U_{t+1}) \in \mathbb{R}^2_+ \times \mathbb{R}^*$, by

$$\pi_t\left(\widehat{e}_t, \widehat{e}_{t+1}, U_{t+1}\right) = \inf_{(e_t, e_{t+1}) << (\widehat{e}_t, \widehat{e}_{t+1})} \left\{ \frac{\widehat{e}_{t+1} - e_{t+1}}{\widehat{e}_t - e_t} : W_t\left(e_t, e_{t+1}, U_{t+1}\right) \geq W_t\left(\widehat{e}_t, \widehat{e}_{t+1}, U_{t+1}\right) \right\}.$$

That is, $\pi_t\left(\widehat{e}_t, \widehat{e}_{t+1}, U_{t+1}\right)$ determines the minimum reduction $\nabla e_{t+1} = \widehat{e}_{t+1} - e_{t+1}$ in e_{t+1} needed to compensate the agents of generation t for a reduction $\nabla e_t = \widehat{e}_t - e_t$ in e_t without decreasing the indirect utility that they obtain with $(\widehat{e}_t, \widehat{e}_{t+1}, U_{t+1})$. For a symmetric equilibrium $(\widehat{a}, p)$ one has

$$\pi_t(e^*_t, e^*_{t+1}, U_{t+1}(a^*)) \leq -\frac{D_1 W_t(\widehat{e}_t, \widehat{e}_{t+1}, U_{t+1}(\widehat{a}))}{D_2 W_t(\widehat{e}_t, \widehat{e}_{t+1}, U_{t+1}(\widehat{a}))} = \frac{R_{t+1}}{\widehat{n}_{t+1}};$$

with $\pi_t(\widehat{e}_t, \widehat{e}_{t+1}, U_{t+1}(\widehat{a})) = \frac{R_{t+1}}{\widehat{n}_{t+1}}$ when the indirect utility function W_t is quasiconcave. Since, in the non-convex studied here, W_t is not, in general, quasiconcave, the standard criteria for dynamic efficiency (based on the ratio $\frac{R_{t+1}}{\widehat{G}_{t+1}} = \frac{R_{t+1}}{\left(\frac{\widehat{e}_{t+1}}{\widehat{e}_t}\right)\widehat{n}_{t+1}}$) may not be sufficient to identify $\mathcal{P}-$efficient paths. In Theorem 2, I extend the arguments obtained –for Millian efficiency– in a setting without altruism by Conde-Ruiz et al. (2010) to the setting studied here and provide sufficient conditions ensuring the $\mathcal{P}-$efficiency of equilibria.

Theorem 2 *Let $(\widehat{a}, p)$ be the equilibrium generated by a sequence of transfers $\underline{g} = \left\{\underline{g}_t\right\}_{t \geq 1}$.*

I) Suppose that there exists $T > 0$ such that $\widehat{e}_\tau > w_\tau + \underline{g}_\tau$ is satisfied for each $\tau \geq T$. Then $\widehat{a}$ is $\mathcal{P}$-efficient.

II) Suppose that $\widehat{a}$ satisfies

$$\liminf_{T\to\infty} \left(\frac{\widehat{e}_{T+1}}{\prod_{t=1}^{T} \pi_t(\widehat{e}_t, \widehat{e}_{t+1}, U_{t+1}(\widehat{a}))} \right) = \liminf_{T\to\infty} \prod_{t=1}^{T} \left(\frac{\left(\frac{\widehat{e}_{t+1}}{\widehat{e}_t}\right)}{\pi_t(\widehat{e}_t, \widehat{e}_{t+1}, U_{t+1}(\widehat{a}))} \right) = 0. \quad (13)$$

Then $\widehat{a}$ is $\mathcal{P}$-efficient.

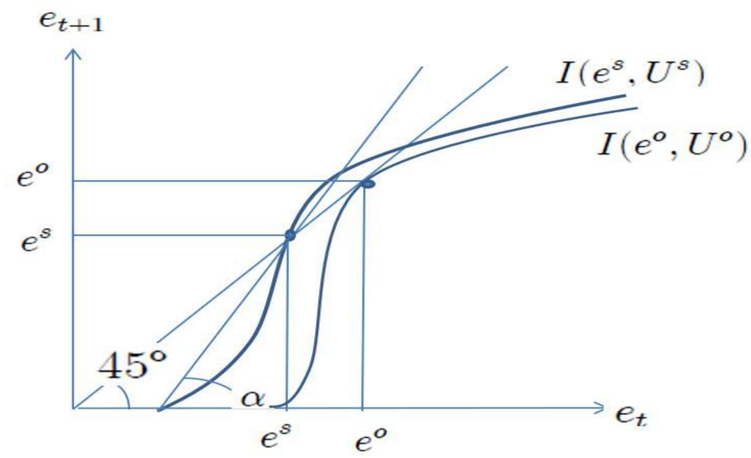
III) A sufficient condition ensuring that $\widehat{a}$ satisfies *(13) and, hence, that it is $\mathcal{P}-$efficient, is that the sequence of prices satisfies*

$$\lim_{t\to\infty} \left(b_t - \frac{w_{t+1}}{R_{t+1}} \right) > 0. \quad (14)$$

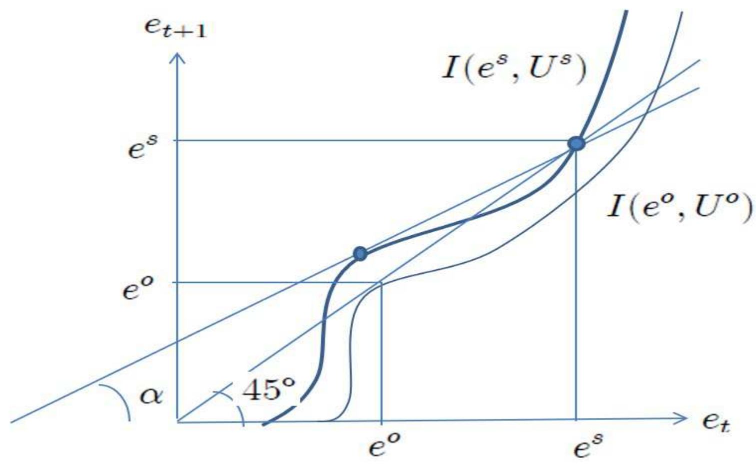
To obtain an intuition of the sufficient conditions in Theorem 2, it is useful to consider a steady state $(a^s, p^s) = (x^s, k^{os}, R^s, w^s)$ of an economy with time invariant technologies, in which indirect utility functions satisfy $W_t \equiv W$ for each $t \geq 0$, all variables and prices are constant and the economy grows at a rate $G^s = n^s$. In a steady state, the ratio $n^s/R^s =$

$-\frac{D_1W(e^s,e^s)}{D_2W(e^s,e^s)}$ measures the ratio of marginal benefits (n^s) to marginal costs (R^s) of introducing a compulsory system of intergenerational transfers to the old or, equivalently, of reducing the amount of resources that every generation is entitled to receive from the previous generation. In an exogenous fertility setting, indirect utility functions are quasiconcave and the ratio of marginal benefits to marginal costs decreases as the size of these transfers increases and e^s decreases. Therefore, if marginal costs exceed marginal benefits and $n^s/R^s < 1$ is satisfied, then increasing transfers to the old cannot be beneficial for the middle aged, no matter how much we increase these transfers. But things change when fertility is endogenous. As we move away from the margin and increase transfers to the old, both n^s and R^s vary, and marginal benefits of further transfers may turn to be higher than costs for a sufficiently high increase in these transfers. Thus, the standard condition $n^s/R^s < 1$ applicable to steady states is replaced by a condition imposing that $\pi(e^s, e^s, U^s) > 1$ holds, which ensures that increasing transfers to the old cannot be beneficial for the middle aged beyond the margin.

More specifically, the sufficient condition $\pi(e^s, e^s, U^s) > 1$ guarantees that the indifference curve $I(e^s, U^s)$, defined by $W(e_t, e_{t+1}, U^s) = W(e^s, e^s, U^s)$, does not cross the line $e_{t+1} = e_t$ at a point $(\widetilde{e}^s, \widetilde{e}^s) << (e^s, e^s)$, as Figure 1 illustrates. In Figure 1(a) and Figure 1(b), I represent $\pi(e^s, e^s, U^s)$ in two steady states that satisfy the standard condition $n^s/R^s > 1$ but that differ in their dynamic efficiency properties. In the steady state in Figure 1(a), $\pi(e^s, e^s, U^s)$ is the slope of the line that joins the point (e^s, e^s) with the point at which the indifference curve $I(e^s, U^s)$, defined by $W(e_t, e_{t+1}, U^s) = W(e^s, e^s, U^s)$, crosses the e_t axes. Thus, $\pi(e^s, e^s, U^s) = tg\alpha > 1$ holds, which implies that there is no other steady state $\widetilde{a}^s$ such that the pair $(\widetilde{e}^s, \widetilde{e}^s)$ belongs to the indifference curve $I(e^s, U^s)$ and satisfies $\widetilde{e}^s < e^s$. The steady state a^s is, therefore, $\mathcal{P}-$efficient. In contrast, in the steady state in Figure 1(b), $\pi(e^s, e^s, U^s)$ is the lowest slope of the indifference curve $I(e^s, U^s)$ on the set $\{(e_t, e_{t+1}) : (e_t, e_{t+1}) \leq (e^s, e^s)\}$. Therefore, $\pi(e^s, e^s, U^s) = tg\alpha < 1$ holds, which implies that there exists an alternative steady state $\widetilde{a}^s$ for which the pair $(\widetilde{e}^s, \widetilde{e}^s)$ belongs to the indifference curve $I(e^s, U^s)$ and satisfies $\widetilde{e}^s < e^s$. Said steady state $\widetilde{a}^s$ $\mathcal{P}-$dominates the steady state a^s, which implies that a^s is $\mathcal{P}-$inefficient. Both figures represent also the so called *golden rule* steady state a^o for which $n^o/R^o = 1$ is satisfied. In figure 1(a), the golden rule maximizes the utility obtained by a representative agent among all stationary allocations and the fertility rate n^o. In contrast, in the economy represented in Figure 1(b), a steady state for which $n^o/R^o = 1$ is satisfied simply identifies either a local maximum –as the steady state e^o in Figure 1(b)– or a local minimum.



(a) A $\mathcal{P}-$efficient steady state $\pi(e^s, e^s, U^s) > 1$



(b) A $\mathcal{P}-$efficient steady state $\pi(e^s, e^s, U^s) < 1$

Figure 1:

In the proof of Theorem 2, I show that if the sequence of prices corresponding to an equilibrium $(\widehat{a}, p)$ satisfies $b_t > \frac{w_{t+1}}{R_{t+1}}$ for some $t \geq 0$, then

$$\left(\frac{\left(\frac{\widehat{e}_{t+1}}{\widehat{e}_t}\right)}{\pi_t(\widehat{e}_t, \widehat{e}_{t+1}, U_{t+1}(\widehat{a}))} \right) \leq \left(1 - \left(b_t - \frac{w_{t+1}}{R_{t+1}} \right) \frac{\widehat{n}_{t+1}}{\widehat{e}_t} \right) < 1.$$

Thus, if the allocation corresponding to an equilibrium $(\widehat{a}, p)$ converges to a steady state and $b^s < w^s/R^s$ is satisfied, the allocation $\widehat{a}$ is $\mathcal{P}-$efficient. In a steady state, marginal costs of increasing transfers to the old, measured by R^s, exceed the maximal return –for parents– of increasing said transfers, measured by w^s/b, under the assumption that wages and interest rates remain constant as we increase intergenerational transfers to the old. Using the terminology coined by Becker and Barro (1988), condition (14) requires that, asymptotically, children constitute a *financial burden* for parents.

Whether or not (14) is likely to be satisfied in real economies is an empirical question which is out of the scope of this paper. However, if we regard the model described above as a reduced form specification of a structural model in which average expenditure on each child is determined endogenously and varies with parental income, some straightforward computations suggest that (14) might be *currently* satisfied in some economies. To see this, write N_t for the number of middle-aged adults alive at t (or the number of children born at $t-1$), Y_t for total income at t and $G_{t+1} = Y_{t+1}/Y_t$ for the rate at which the economy grows in period $t+1$. If labour incomes represent 60% of total income and total expenditure in children represents 25% of total income[12] we have (if we abstract from time costs of raising children),

$$\frac{w_{t+1}}{b_t} = \frac{\left(\frac{0.60Y_{t+1}}{N_{t+1}}\right)}{\left(\frac{0.25Y_t}{N_{t+1}}\right)} \simeq 2.4G_{t+1}.$$

Thus, if raising children takes 25 years, the rate of return to assets is 6% per year and the rate of growth of the economy is 2% per year we have

$$\frac{R_{t+1}}{G_{t+1}} = \left(\frac{1.06}{1.02}\right)^{25} \simeq 2.616 > 2.4;$$

which implies that $b_t > w_{t+1}/R_{t+1}$ is satisfied at t. [13] However, we should be cautious. If, for example, the rate of growth of the economy is 2,5% per year, then we have $\frac{R_{t+1}}{G_{t+1}} = 2.31$, which yields $b_t < w_{t+1}/R_{t+1}$. Also, we should take into account that part of the observed expenditures on children are compulsory, and therefore we might not be measuring what children would cost if expenditures on each child were chosen in a $\mathcal{P}-$efficient way. In any case, condition (14 is just a *sufficient* condition for $\mathcal{P}-$efficiency, and the following examples show that many symmetric allocations are $\mathcal{P}-$efficient even when condition (14) is not satisfied.

[12] According to a Report to the European Commission, children account for between 20% and 30% of the budget of those households with children up to the age of 14. See Letablier et al. (2009).

[13] Córdoba and Ripoll (2016) claim that condition (14) is not currently satisfied, for *low-income* households, in the U.S. economy. However, in their computations, they exclude college expenditures and assume that both costs of raising children and wages differ among income groups.

In all the examples, I assume that preferences extend those in Razin and Ben Zion (1975)[14] and are represented by utility functions of the form

$$U(x_t, U_{t+1}) = u_\theta(h(x_t)) + \beta U_{t+1},$$

where $\beta \in (0,1)$, $h : \mathbb{R}^3_+ \to \mathbb{R}$ is positive valued, non-decreasing, concave and linearly homogeneous and $u_\theta : \mathbb{R}_+ \to \mathbb{R}^*$ is strictly increasing and concave. For the numerical examples, I also assume that,

$$u_\theta(h) = \begin{cases} \frac{1}{\theta}(h)^\theta, & \text{if } 0 \neq \theta < 1; \\ \ln h, & \text{if } \theta = 0. \end{cases}$$

With these assumptions, each indirect utility function W_t and each function π_t adopt, respectively, the separable forms

$$W_t(e_t, e_{t+1}, U_{t+1}) = W_t(e_t, e_{t+1}) + \beta U_{t+1};$$

and

$$\pi_t(\widehat{e}_t, \widehat{e}_{t+1}, U_{t+1}) = \pi_t(\widehat{e}_t, \widehat{e}_{t+1}) = \inf_{(e_t, e_{t+1}) << (\widehat{e}_t, \widehat{e}_{t+1})} \left\{ \frac{\widehat{e}_{t+1} - e_{t+1}}{\widehat{e}_t - e_t} : W_t(e_t, e_{t+1}) \geq W_t(\widehat{e}_t, \widehat{e}_{t+1}) \right\}.$$

Moreover, when a dynastic optimum $\mathsf{a}^* = a^*$is symmetric one must have, for each $t \geq 0$,

$$U_t(a^*) = \mathcal{V}_t(e_t^*) = \max_{e_{t+1} \in \mathbb{R}_+} \left\{ W_t(e_t^*, e_{t+1}) + \beta \mathcal{V}_{t+1}(e_{t+1}) \right\}.$$

For each $t \geq 0$, write k^m_{t+1} and $y_{t+1} = f_{t+1}(k^m_{t+1})$, respectively, for capital *per* worker ($k^m_{t+1} = \frac{k^o_{t+1}}{n_{t+1}}$) and output *per* worker ($y_{t+1} = f_{t+1}(k^m_{t+1}) = F_t(k^m_{t+1}, 1)$) at period $t+1$ and assume the following holds:

$$\lim_{k^m_{t+1} \longrightarrow 0} \left[\left(b_t + k^m_{t+1} \right) f_t'(k^m_{t+1}) - f_t(k^m_{t+1}) \right] > 0. \tag{15}$$

With this assumption, the homotheticity of preferences implies that the solution

$$\left[x_t(e_t, e_{t+1}), k^o_{t+1}(e_t, e_{t+1}) \right]$$

to the optimization problem in the definition of $W(e_t, e_{t+1})$ satisfies $x_t(e_t, e_{t+1}) = e_t x_t(1, e_{t+1})$ and $k^o_{t+1}(e_t, e_{t+1}) = e_t k^o_{t+1}(1, e_{t+1}) \geq 0$. Thus, the implicit prices –given respectively, by

[14] A draft studying the properties of equilibria in an extension of Barro and Becker's (1989) model is also available from the author upon request.

marginal productivity of capital and labour– associated to said solution are completely determined by e_{t+1}.More precisely, it is straightforward to write each indirect utility function W_t as the function corresponding to a standard consumer problem

$$W_t(e_t, e_{t+1}) = W_t^C(e_t, e_{t+1}; R_{t+1}(e_{t+1}), w_t(e_{t+1})),$$

where

$$W_t^C(e_t, e_{t+1}; R_{t+1}, w_t) = \max_{(x_t)\in\mathbb{R}^3_+} \left\{ \left\{ u_\theta\left(h(x_t)\right) : c_t^m + \frac{c_{t+1}^o}{R_{t+1}} + \left(b - \frac{w_{t+1} - e_{t+1}}{R_{t+1}}\right) n_{t+1} \le e_t \right\} \right\}, \tag{16}$$

while the function $(k^m_{t+1}, R_{t+1}, w_{t+1}) : \mathbb{R}_+ \to \mathbb{R}^3_+$ is implicitly defined, for each e_{t+1}, by

$$f_{t+1}(k^m_{t+1}(e_{t+1})) = \frac{c^o_{t+1}(1, e_{t+1})}{n_{t+1}(1, e_{t+1})} + e_{t+1};$$

$$R_{t+1}(e_{t+1}) = f'_{t+1}(k^m_{t+1}(e_{t+1}))$$

and

$$w_{t+1}(e_{t+1}) = f_{t+1}(k^m_{t+1}(e_{t+1})) - k^m_{t+1}(e_{t+1}) f'_{t+1}(k^m_{t+1}(e_{t+1})).$$

Having this in mind, consider the Hicksian expenditure function defined by

$$E^H(e_{t+1}, R_{t+1}, w_{t+1}) = \min_{x_0\in\mathbb{R}^3_+} \left\{ c_t^m + \frac{c^o_{t+1}}{R_{t+1}} + \left(b_t - \frac{w_{t+1} - e_{t+1}}{R_{t+1}}\right) n_{t+1} \ : \ h(x_t) \ge 1 \right\}$$

and write $\mathcal{E}_{t+1}(e_{t+1})$ for $\mathcal{E}_{t+1}(e_{t+1}) = E^H(e_{t+1}, R_{t+1}(e_{t+1}), w_{t+1}(e_{t+1}))$. With this notation, indirect utility functions can also be written as

$$W_t(e_t, e_{t+1}) = u_\theta\left(\frac{e_t}{\mathcal{E}_{t+1}(e_{t+1})}\right).$$

3.3.1 Existence of equilibria For each $t \ge 0$, let $m_t : \mathbb{R}_+ \to \mathbb{R}_+$ be defined, for each $e \ge 0$, by

$$m_{t+1}(e) = -\frac{D_1 W_t(e, e)}{D_2 W_t(e, e)} = \frac{R_{t+1}(e)}{n_{t+1}(e, e)} = \frac{e\mathcal{E}'_{t+1}(e)}{\mathcal{E}_{t+1}(e)},$$

that is, $m_{t+1}(e)$ is the slope of an indifference curve of the form $W_t(e_t, e_{t+1}) = W_t(e, e)$ at the point $(e_t, e_{t+1}) = (e, e)$. Note that for every (e_t, e_{t+1}) one has

$$-\frac{D_1 W_t(e_t, e_{t+1})}{D_2 W_t(e_t, e_{t+1})} = \left(\frac{e_{t+1}}{e_t}\right) m_{t+1}(e_{t+1}).$$

Thus, if each function m_{t+1} is strictly decreasing, then the slopes of the indifference curves defined by $W_t(e_t, e_{t+1}) = W_t(\widehat{e}_t, \widehat{e}_{t+1})$ are lower as we move away from the origin along any ray for which $\widehat{e}_{t+1} = c\widehat{e}_t$ holds, as in Figure 1(a). It can also be shown[15] that, if each function m_{t+1} is strictly decreasing, then value functions characterizing equilibria are concave, which implies that parents will choose to treat their children equally in the game of voluntary transfers played within families, as required in the definition of an equilibrium. For utility and production functions of the form

$$h(x_t) = \begin{cases} \left[\gamma^m (c_t^m)^\sigma + \gamma^o \left(c_{t+1}^o\right)^\sigma + \gamma^n (n_{t+1})^\sigma\right]^{\frac{1}{\sigma}}, & \text{if } 0 \neq \sigma < 1, \\ (c_t^m)^{\gamma^m} \left(c_{t+1}^o\right)^{\gamma^o} (n_{t+1})^{\gamma^n}, & \text{if } \sigma = 0, \end{cases}$$

with $\gamma^m + \gamma^o + \gamma^n = 1$; and

$$F_t(K, L) = \begin{cases} \left[A_t (K)^\rho + A_t (L)^\rho\right]^{\frac{1}{\rho}}, & \text{if } 0 \neq \rho < 1, \\ A_t (K)^\alpha (L)^{(1-\alpha)}, & \text{if } \rho = 0; \end{cases}$$

the monotonicity properties of m_{t+1} depend on the parameters σ and ρ. A sufficient condition ensuring that m_{t+1} is decreasing is that $\sigma \leq 0$ and $\rho \leq 0$ hold. Therefore, if both utility and production functions belong to the Cobb-Douglas family, then m_{t+1} is strictly decreasing.

3.3.2 Equilibria with time invariant technologies When technologies are time invariant –that is, $F_t \equiv F$ and $b_t \equiv b$ $\forall t \geq 0$–, indirect utility and value functions characterizing an equilibrium and a dynastic optimum are also time invariant, and so are the functions R_t, w_t, m_t and π_t. If m is strictly decreasing, then the equilibrium generated by a (constant) sequence of transfers $\underline{g}$ –that is, a sequence $\underline{g}$ such that $\underline{g}_t = \underline{g}^c$ for each $t \geq 1$– exists and it is unique. Its properties can be explored easily using the *policy function* associated associated to a dynastic optimum a^*, that is, the function $P : \mathbb{R}_+ \to \mathbb{R}_+$ for which

$$\mathcal{V}(e) = W(e, P(e)) + \beta\mathcal{V}(P(e))$$

is satisfied. To be more precise, suppose that $w(0) + \underline{g}^c \geq 0$ is satisfied and suppose also that there exists a unique $\underline{e}^c$ such that $w(e^c) + \underline{g}^c = e^c$ holds. In this case, the growth path $\{\widehat{e}_t\}_{t\geq 0}$ corresponding to the symmetric equilibrium $(\widehat{a}, p)$ generated by $\underline{g}$ is characterized by the equation

$$\widehat{e}_{t+1} = P_{\underline{g}}(\widehat{e}_t),$$

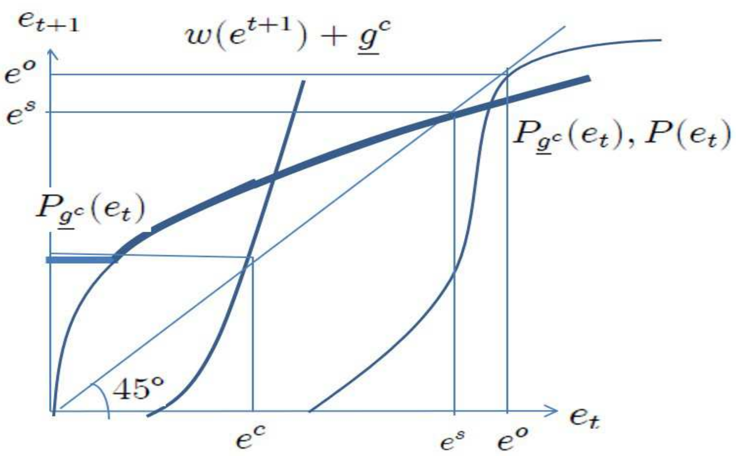
and the initial condition $\widehat{e}_0 = \overline{e}_0$; where $P_{\underline{g}} : \mathbb{R}_+ \to \mathbb{R}_+$ is defined, for each $e \in \mathbb{R}_+$, by

$$P_{\underline{g}}(e) = \begin{cases} e^c = w(e^c) + \underline{g}^c, & \text{if } e < P^{-1}(e^c), \\ P(e), & \text{otherwise.} \end{cases}$$

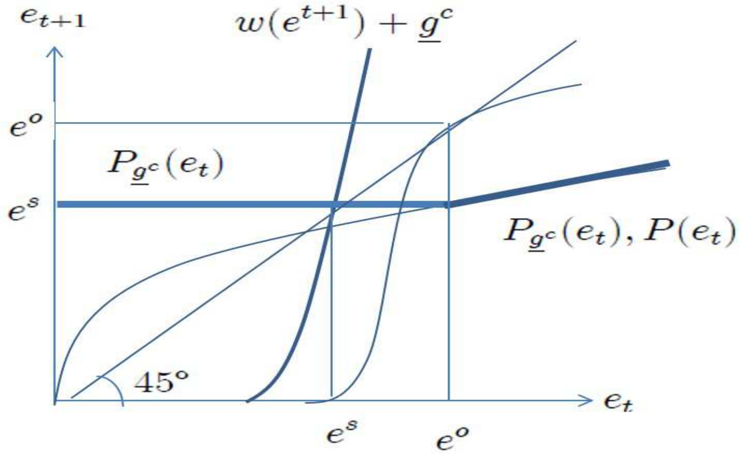
[15] A formal proof of this statement is available from the author upon request.

Figure 2(a) and Figure 2(b) represent the function $P_{\underline{g}}$ characterizing the equilibrium dynamics of an economy for which m is strictly decreasing, e^c is well defined and $\theta < 0$ holds. A negative θ implies that richer parents are willing to transfer more income to each of their descendants, and –if m is strictly decreasing– ensures that the policy function P is strictly increasing and concave. Although, in both figures, the wage function $w(\cdot)$ corresponds to an economy with Cobb-Douglas preferences and production functions, the argument can be easily extended to other economies for which there exists a unique e^c for which $e^c = w(e^c) + \underline{g}^c$ is satisfied.

Figure 2(a) represents the equilibrium policy function $P_{\underline{g}}$ when e^c is lower than the steady state $e^s = e^d$ corresponding to a dynastic optimum, while Figure 2(b) represents a case in which $e^s = e^c > e^d$ holds. In the first case, the equilibrium dynamics are analogous to those characterizing a dynastic optimum, except for those economies with low initial incomes, that grow at higher rates when constraints on intergenerational transfers are binding. By Theorem 2.I), the equilibrium represented in Figure 2(a) is $\mathcal{P}-$efficient. In contrast, in the equilibrium represented in Figure 2(b), the economy reaches its steady state $e^s = e^c$ at $t = 1$, and the non-negativity constraint on voluntary transfers become permanently binding after that period. Still, the equilibrium may be $\mathcal{P}-$efficient, as Figure 2(b) illustrates. Since the economy converges to the steady state $e^s = e^c$, the sufficient condition for dynamic efficiency in Theorem 2.II) reduces to $\pi(e^c.e^c) > 1$. Furthermore, since m is decreasing, it is straightforward to show that $\pi(e^c.e^c) > 1$ holds whenever $R(e^c) > n(e^c, e^c)$ holds or, equivalently, whenever $e^c < e^o$ holds. The income level e^o corresponds to the *golden rule* steady state, that satisfies $\frac{R(e^o)}{n(e^o,e^o)} = 1 = \pi(e^o.e^o)$.



(a) An equilibrium for which $e^s > e^c$



(b) An equilibrium for which $e^s = e^c$

Figure 2: $\mathcal{P}-$efficient symmetric equilibria.

The equilibrium dynamics change if either $\theta > 0$ or $\theta = 0$ is satisfied. If $\theta > 0$ holds, then the policy function P is strictly decreasing and the path $\{\widehat{e}_t\}_{t \geq 0}$ corresponding to an equilibrium may oscillate. With logarithmic preferences ($\theta = 0$), the dynastic optimum reaches a steady state a^d, characterized by $m(e^d) = \frac{1}{\beta}$, with no transitional dynamics. In this case, there are only two possibilities: either the non-negativity constraints on transfers are never binding or they are permanently binding. The equilibrium dynamics with logarithmic preferences are exactly analogous to those arising with Barro-Becker preferences without time costs, as those studied by Alvarez (1999).

Time invariant economies. Numerical examples Suppose that $\theta < 0$ and $\rho = \sigma = 0$. With Cobb-Douglas preferences and production functions, the parameters of the model can be selected in such a way that the steady-state values of the *laissez faire* equilibrium are consistent with the data observed in real economies. To be more precise, suppose that $\alpha = 0.4$; $\beta = \left(\frac{1.02}{1.06}\right)^{25} \approx 0.38; b = 67.48$; $\gamma^n = \frac{0.25}{0.60} \approx 0.416; \gamma^o = \left(\frac{1.02}{1.06}\right)^{25}\left(\frac{0.4}{0.6}\right) \approx 0.254$ and $\gamma^m = 1 - \gamma^n - \gamma^o$. Then, in the steady state a^s corresponding to the *laissez faire* equilibrium, we have $R^s = (1.06)^{25} \approx 4.29; G^s = n^s = (1.02)^{25} \simeq 1.64; w^s = 0.60y^s$ and $bn^s \approx 0.25y^s$. The parameters of the model have been selected in such a way that $e^s = e^d = w^s$ holds, that is, the equilibrium steady state coincides with that corresponding to a dynastic optimum (which ensures that the equilibrium is $\mathcal{P}-$efficient) and, at the same time, intergenerational transfers from parents to children are zero. Slight variations in the parameters γ^n and γ^o do not affect the equilibrium, steady-state values for $R^s, G^s, \frac{w^s}{y^s}$ and $\frac{bn^s}{y^s}$, although they may determine whether $e^s = e^d > w^s$ or $e^s = w^s > e^d$ holds. For example, if we replace γ^n and γ^o, respectively, by $\widetilde{\gamma}^n = 0,410$ and $\widetilde{\gamma}^n = 0,22$, then the equilibrium values for $R^s, G^s, \frac{w^s}{y^s}$ and $\frac{bn^s}{y^s}\gamma^o$ remain unaltered, although $e^s = e^d > w^s$ is satisfied. Note that, even when $e^s = w^s > e^d$ is satisfied, the fact that m is decreasing implies that the equilibrium is $\mathcal{P}-$efficient, as $R^s > G^s$ implies that $\pi(e^s, e^s) > 1$ is satisfied.

Suppose now that we replace β by $\beta' = \left(\frac{1.025}{1.06}\right)^{25} \approx 0.43$ and b by $b' = 57.48$. Then, in the steady state corresponding to a *laissez faire* equilibrium we have $R^s = (1.06)^{25}; G^s = n^s = (1.025)^{25} \simeq 1.85; w^s = 0.60y^s$ and $bn^s \simeq 0.25y^s$. As in the previous case, $e^s = e^d = w^s$ holds and the *laissez faire* equilibrium is, therefore, $\mathcal{P}-$efficient, even though, in this case, the sufficient condition for $\mathcal{P}-$efficiency in Theorem 2.III) is not satisfied, as $b^s < w^s/R^s$ holds.

3.3.3 Time-varying technologies. Exogenous growth With time invariant technologies, an economy grows only if $n^s > 1$ is satisfied. With time-varying production and cost functions (more precisely, if $F_t(K_t,L_t) = AK_t^\alpha\left(\xi^t L_t\right)^{1-\alpha}$ and $b_{t-1} = \xi^{t-1}b$ are satisfied for each $t \geq 1$ and some $\xi > 1$) it can be shown that the *laissez faire* equilibrium $\widehat{a}$ arising in an economy with Cobb-Douglas preferences converges to a balanced growth path with constant fertility rates in which

$$\frac{\widehat{c}^m_{t+1}}{\widehat{c}^m_t} = \frac{\widehat{c}^o_{t+1}}{\widehat{c}^o_t} = \frac{\widehat{k}^m_{t+1}}{\widehat{k}^m_t} = \frac{\widehat{e}_{t+1}}{\widehat{e}_t} = \frac{f_{t+1}(\widehat{k}_{t+1})}{f_t(\widehat{k}_t)} = \xi;$$

and

$$\frac{R_{t+1}}{R_t} = \frac{\widehat{n}_{t+1}}{\widehat{n}_t} = 1$$

are satisfied. With logarithmic preferences, the dynastic optimum a^* must satisfy $m_{t+1}(e^*_{t+1}) = \frac{1}{\beta}$, for each $t \geq 0$, and therefore follows also a balanced growth path, with no transitional dynamics, from $t = 1$.

Time-varying technologies. A numerical example Suppose that $\theta = 0$ and $\rho = \sigma = 0$. As with time invariant technologies, the parameters of the model can be selected in such a way that the *laissez faire* equilibrium values for $R^s, G^s, \frac{w_t}{y_t}$ and $\frac{b_t n^s}{y_t}$ –all of which are constant in a balanced growth path– are consistent with the data observed in real economies. To be more precise, suppose $\alpha = 0.4; \beta = \frac{1.096}{(1.06)^{25}} \approx 0,25; \xi = 1.5; b = 100.66;\ \ \gamma^n = \frac{0.25}{0.60}$ and $\gamma^o = \left(\frac{1.02}{1.06}\right)^{25}\left(\frac{0.4}{0.6}\right)$. Then, in the balanced growth path corresponding to the equilibrium we have $R^s = (1.06)^{25} \approx 4,29; n^s = 1.09; G^s = \xi n^s = (1.02)^{25} \simeq 1.64; w_t = 0.60 y_t$ and $b_t n^s \approx 0.25 y_t$. In this case, said parameters have been selected in such a way that the *laissez faire* equilibrium corresponding to those parameters coincides with the dynastic optimum a^* and, at the same time, transfers from parents to children are zero. But even if these parameters vary slightly and the balanced growth path corresponding to the laissez faire equilibrium satisfies $\widehat{e}_t = w_t > e^*_t$ for every $t \geq 1$, it is $\mathcal{P}-$efficient. Since each function m_t is strictly decreasing and $R^s > \xi n^s = \frac{\widehat{e}_{t+1}}{\widehat{e}_t} n^s$ holds through the balanced growth path, we have $\frac{\widehat{e}_{t+1}}{\widehat{e}_t} < \pi_t(\widehat{e}_t, \widehat{e}_{t+1})$, which ensures that the equilibrium is $\mathcal{P}-$efficient.

3.4 Equilibria and $\mathcal{P}-$efficiency. Concluding Remarks

To conclude the Section, the following remarks are in order.

Remark 1. Suppose that the agents' preferences are represented by utility functions of the form

$$\mathcal{U}_t(\mathsf{a}, i^t) = u_\theta(h(\mathsf{x}_t(i^t))) + \beta \min \left\{ \mathcal{U}_{t+1}(\mathsf{a}, i^t, i_{t+1}) : i_{t+1} \leq \mathsf{n}_{t+1}(i^t) \right\}. \quad (17)$$

The equilibria arising with such preferences are described by the same equations than those arising with the specification of parental preferences given in *(PR)* and explored in this Section. However, if parental preferences are represented as in *(17),* then dynamically efficient equilibria are both $\mathcal{A}-$ and $\mathcal{P}-$efficient, independently of the utility attributed to the unborn.

Remark 2. Suppose that the utility attributed to the unborn satisfies, for each $t \geq 0$, each $\mathsf{a} \in \mathcal{F}$ and each potential agent i^t,

$$\mathcal{U}^N_t(\mathsf{a}, i^t) = \min \left\{ \mathcal{U}_\tau(\mathsf{a}, \widetilde{i}^t) : \widetilde{i}^t \in \mathsf{D}^\tau : \tau \leq t \right\}; \quad (18)$$

or

$$\mathcal{U}^N_t(\mathsf{a}, i^t) = \underline{u}; \quad (19)$$

with $\underline{u} > \underline{u}_N$. In these cases (in the latter case, if the utility threshold $\underline{u}$ is higher than the utility obtained, in an equilibrium, by the agents born at $t = 0$), equilibria are still $\mathcal{P}-$efficient in the time invariant economies studied in the examples, independently of whether the non-negativity constraints on transfers are permanently binding or they are binding, only temporarily, at $t = 1$. In any equilibrium, children born at $t = 0$ obtain the lowest utilities amongst any agent alive (with the only possible exception of dynasty heads), and dynasty heads cannot increase their utility by having more children at $t = 0$ if they are constrained to provide the newborn with the same income as their siblings. If the non-negativity constraints on transfers are permanently binding, then all parents will face the same constraints as dynasty heads; and if they are binding only at $t = 1$, then the agents born at $t > 1$ obtain, in the equilibrium allocation, the maximum utility they can obtain with their resources. Therefore, they cannot obtain utility gains by discriminating their children either and said equilibrium is $\mathcal{P}-$efficient.

The argument can be extended to economies with time varying technologies if the non-negativity constraints on transfers are binding only for $t = 1$, but not if they are permanently binding. In any case, recall that implementing an allocation with higher fertility rates that $\mathcal{P}-$dominates (in the sense given by any alternative specification to the utilities obtained by the unborn) an equilibrium that is $\mathcal{P}-$efficient under Assumption $A1$ would still require providing some agents with fewer resources than those available to their living siblings.

Remark 3. In all the examples explored throughout the Section, achieving a symmetric, $\mathcal{A}-$efficient allocation (that is, a dynastic optimum) without compensating the middle aged agents living in a $\mathcal{P}-$efficient equilibrium only benefits one generation of agents, the dynasty heads, and makes all other generations of agents worse off than they are in the initial equilibrium.

Remark 4. The results obtained in the paper can be extended to economies in which dynasty heads receive different initial endowments or parents are willing to treat their descendants asymmetrically, which gives rise to non-symmetric equilibria. Although, to save on space, I do not explore non-symmetric equilibria here, it can be shown[16] that a non-symmetric equilibrium is $\mathcal{P}-$efficient under exactly analogous conditions as those in Theorem 2, particularly, if the non-negativity constraints on transfers are binding only temporarily and also if, in the long run, costs of rearing children are higher than the present value of future wages. In non symmetric equilibria, those agents obtaining an income $\mathsf{e}_t^*(i^t) = w_t + \underline{g}_t$ are also the ones obtaining the lowest utility among all alive people of the same generation, independently of the dynasty to which they belong, which drives the results. Observe that, while in symmetric equilibria, the utility attributed to the unborn is,

[16] A formal proof of this statement is available from the author upon request.

under Assumption $A1$, at least as high as the utility obtained by the living agents, that is not necessarily the case in non-symmetric equilibria. If, for example, the utility attributed to the unborn satisfies(5), then the utility attributed to potential agents in a non-symmetric equilibrium may be lower than the utility obtained by many living agents.

4 Conclusions

In this paper, I explore the $\mathcal{P}-$efficiency properties of the allocations arising as the equilibria corresponding to different distributions of rights in an environment with endogenous fertility and dynastic altruism. Although, as proved by ST, most of these equilibria are $\mathcal{A}-$inefficient, they may be $\mathcal{P}-$efficient if the utility attributed to the unborn is bounded from below by the minimum utility obtained by the living agents of the same generation. Thus, an important qualitative conclusion of GJT prevails: in the absence of externalities, missing financial markets or dynamic efficiency problems, the fact that fertility decisions are endogenous does not mean that markets fail to deliver efficient allocations.

The paper should not be taken as an amendment to ST. In a $\mathcal{P}-$efficient, *laissez faire* equilibrium with binding constraints on transfers, encouraging fertility rates is $\mathcal{A}-$dominant, but it gives rise to an asymmetry between the oldest and the youngest people of some generations that was not present in the original equilibrium. However, in real economies, resources are not allocated through a laissez faire, market mechanism. If, for example, governments enforce a standard, pay-as-you-go public pension program, then the equilibrium allocation is $\mathcal{P}-$inefficient because intergenerational transfers are not tied to fertility choices. In this case, replacing the pension program by a fertility-dependant pension scheme is not only $\mathcal{P}-$dominant, it may also implement a dynastic optimum without treating any two agents of the same generation differently. We should be cautious because, in real economies, there may exist other intergenerational transfers enforced by governments that are not tied to fertility choices. Public education, for example, requires intergenerational transfers from the middle aged to the following generation. Therefore, incorporating human capital accumulation into the setting studied in the paper may be worth exploring.

There are several other directions that may be worth exploring. The possibility that non symmetric equilibria are $\mathcal{P}-$efficient, together with the examples, suggests that constraints on voluntary transfers made within families may affect poor families rather than rich families. In this case, relaxing these constraints may induce a higher increase in fertility rates among poor families, which might give rise to a persistent inequality. Another direction would be to extend the results to environments with uncertainty in which the agents may differ in their randomly determined characteristics. Finally, another direction would be to explore the consequences of different fertility policies in environments in which other potential market failures, such as pollution problems or public goods, are present.

5 Appendix: Proofs

Proof of Theorem 1. Let $(\widehat{a}, p)$ be an equilibrium associated to a sequence of transfers $\underline{g}$.

I.1) To prove that $\widehat{a}$ is statically Millian efficient, observe from the definition of an equilibrium that $\widehat{a}$ maximizes the utility of the dynasty head among all symmetric allocations satisfying $e_t \geq \widehat{e}_t$ for each $t \geq 1$, in the economy for which production functions adopt the linear form $F_t(K_t, N_t) = R_t K_t + w_t N_t$ for $t \geq 1$. Also, profit maximization conditions with constant returns to scale imply that, for each $t \geq 1$ and each $(k_t^o, n_t) \neq (\widehat{k}, \widehat{n}_t)$, one must have

$$0 = F_t(\widehat{k}_t^o, \widehat{n}_t) - R_t \widehat{k}_t^o - w_t \widehat{n}_t \geq F_t(k_t^o, n_t) - R_t k_t^o - w_t n_t.$$

Therefore,

$$F_t(k_t^o, n_t) \leq R_t k_t^o + w_t n_t; \tag{20}$$

which implies that $\widehat{a}$ maximizes the utility of dynasty heads among all symmetric allocations satisfying $e_t \geq \widehat{e}_t$ in the economy whose production functions are described by the sequence $\{F_t\}_{t \geq 1}$. Otherwise, there must exist a symmetric allocation a satisfying $e_t \geq \widehat{e}_t$ for each $t \geq 1$ that gives dynasty heads higher utility than the utility that they obtain with $\widehat{a}$. But if such allocation a is feasible, then it must be feasible also in the economy in which production functions adopt the linear form implicitly defined by prices, which contradicts the assumption imposing that $\widehat{a}$ maximizes the utility of dynasty heads among all symmetric allocations satisfying $e_t \geq \widehat{e}_t$ in the linear economy. Therefore,

$$U_t(\widehat{a}) = \max_{a \in S} \left\{ U_t(a) : e_t = \widehat{e}_t;\ e^{-t} \geq \widehat{e^{-t}} \right\} \equiv \mathcal{V}_t^C(\widehat{e}_t, \widehat{e^{-t}}) \text{ for } t \geq 1. \tag{21}$$

Observe now that the sequence $\left\{ \mathcal{V}_t^C(\widehat{e}_t, \widehat{e^{-t}}) \right\}_{t \geq 0}$ satisfies, for each t,

$$\mathcal{V}_t^C(\widehat{e}_t, \widehat{e^{-t}}) = \max \left\{ W_t(\widehat{e}_t, e_{t+1}, \mathcal{V}_{t+1}^C(e_{t+1}, \widehat{e^{-(t+1)}}) : e_{t+1} \geq \widehat{e}_{t+1} \right\}.$$

Therefore, on the range of income streams e satisfying $e \geq \widehat{e}$, each function $\mathcal{V}_t^C$ must be strictly increasing in e_t and strictly decreasing in e^{-t}.Taking this into account, the static Millian efficiency of $\widehat{a}$ follows easily: if $\widehat{a}$ is dominated by a symmetric allocation a that satisfies also the necessary conditions for Millian efficiency in (21), there must exist a period $t_1 > 0$ for which $e_{t_1} < \widehat{e}_{t_1}$ is satisfied, or otherwise the dynasty head must be worse off with the symmetric allocation a than she is with $\widehat{a}$. Since $\mathcal{V}_t^C$ must be strictly increasing in e_t and strictly decreasing on e^{-t}, there must exist a period $t_2 > t_1$ for which $e_{t_2} < \widehat{e}_{t_2}$ holds, and so on. Proceeding recursively, there must exist an infinite sequence $\mathcal{T} = \{t_1, t_2, t_3, .., \}$ such that $e_t < \widehat{e}_t$ holds for every $t \in \mathcal{T}$, which establishes that $\widehat{a}$ must be statically Millian efficient.

I.2) To show that $\widehat{\mathsf{a}} = \widehat{a}$ must be $\mathcal{P}-$efficient if it is fully Millian efficient, observe, from the definitions, that all agents living in $\widehat{a}$ are not restricted to treat their descendants equally when taking their consumption, fertility, investment and gift decisions. Therefore, they take these decisions to maximize the utility they can obtain with their incomes, if they have to provide all their descendants with (at least) the same income than the income they receive in $\widehat{a}$. Thus, for any allocation a that $\mathcal{P}-$dominates $\widehat{a}$, every $t \geq 1$ and every $i^t \in \mathsf{D}^t$ one must have $\mathsf{n}_{t+1}(i^t) \geq \widehat{n}_{t+1}$, or otherwise some of the agents living in both a and $\widehat{\mathsf{a}}$ would be worse off than they are in $\widehat{\mathsf{a}}$. But, by Assumption $A1$, an allocation with more individuals than the number of individuals in $\widehat{\mathsf{a}}$ cannot $\mathcal{P}-$dominate $\widehat{\mathsf{a}}$, because all those agents living in a but not in $\widehat{\mathsf{a}}$ must receive (at least) the same income as the income received by those agents alive at $\widehat{\mathsf{a}}$, or otherwise they would obtain lower utility. Therefore, a non symmetric allocation $\mathcal{P}-$dominating $\widehat{\mathsf{a}}$ does not exist, which completes the proof of the I) statement in Theorem 1.

To prove II), let $\widehat{\mathsf{a}}$=$\widehat{a}$ be a symmetric, $\mathcal{P}-$efficient allocation and let $p = \{(R_t, w_t)\}_{t\geq 1}$ be the sequence of prices defined, for each t, by R_t= $D_1F(\widehat{k}^o_t,\widehat{n}_t)$ and w_t= $D_1F(\widehat{k}^o_t,\widehat{n}_t)$. Also, Let $\underline{g} = \left\{\underline{g}_t\right\}_{t\geq 1}$ be defined, for each $t \geq 1$, by $\underline{g}_t$=$\widehat{e}_t$−w_t. Note that, since $\widehat{a}$ is feasible in the economy with linear production functions defined by prices, it must also be $\mathcal{P}-$efficient in said economy. But this implies that $U_t(\widehat{a}) = \mathcal{V}^p_t(\widehat{e}_t, w^{-t}+\underline{g}^{-t})$ must be satisfied for each t, therefore establishing that $(\widehat{a}, p)$ is an equilibrium associated to $\underline{g}$, which completes the proof of Theorem 1II). □

Proof of Theorem 2 Let $(\widehat{a}, p)$ be an equilibrium associated to $\underline{g}$, To prove I), proceed as in the proof of Theorem 1 to show that, if constraints on transfers are not binding from some T on, the agents born after T is reached maximize the utility they can obtain –with their incomes– with any feasible allocation. Thus, a symmetric allocation a $\mathcal{A}-$dominating $\widehat{a}$ cannot exist: if e_t<$\widehat{e}_t$ is satisfied for some $t > T$, then the agents of generation t must be worse off with a than they are with $\widehat{a}$, a contradiction; and if $e_t > \widehat{e}_t$ is satisfied for some $t > T$, then the agents born before t must be worse off with a than they are in $\widehat{a}$, also a contradiction that, taking Theorem 1.I) into account, establishes Theorem 2.I).

To prove II), suppose that $\widehat{a}$ satisfies (13) but is dynamically inefficient. Assume, without loss of generality, that the symmetric allocation a that $\mathcal{A}-$dominates the allocation $\widehat{a}$ satisfies e_t< e^*_t and $U_t(a) = U_t(\widehat{a})$ for $t \geq 1$. Now observe that, since $\widehat{a}$ satisfies (13), there must exist a sufficiently large T^* and a subsequence $\mathcal{T}$ such that, for each $T \in \mathcal{T}$ such that

$T \geq T^*$ one has

$$\left(\frac{\widehat{e}_{T+1}}{\prod_{t=0}^{T} \pi_t\left(\widehat{e}_t, \widehat{e}_{t+1}, U_t(\widehat{a})\right)}\right) < \epsilon = \widehat{e}_1 - e_1. \tag{22}$$

Use now the definition of π_t to obtain the chain of inequalities

$$\begin{aligned}
0 &< \widehat{e}_1 - e_1 = \epsilon \leq \frac{(\widehat{e}_2 - e_2)}{\pi_1\left(\widehat{e}_1, \widehat{e}_2, U_2(\widehat{a})\right)} \leq \frac{\widehat{e}_3 - e_3}{\pi_1\left(\widehat{e}_1, \widehat{e}_2, U_2(\widehat{a})\right)\pi_2\left(\widehat{e}_2, \widehat{e}_3, U_3(\widehat{a})\right)} \leq \\
&\leq \ldots \leq \\
&\leq \frac{e^*_{T+1} - e_{T+1}}{\prod_{t=1}^{T} \pi_t\left(\widehat{e}_t, \widehat{e}_{t+1}, U_t(\widehat{a})\right)} < \frac{e^*_{T+1}}{\prod_{t=1}^{T} \pi_t\left(\widehat{e}_t, \widehat{e}_{t+1}, U_t(\widehat{a})\right)},
\end{aligned}$$

which contradicts (22) and, therefore, establishes that $\widehat{a}^*$ is $\mathcal{P}-$efficient.

To prove III), for a given sequence of prices p and each $t \geq 0$, let $\left\{\pi^p_t : \mathbb{R}^2_+ \times \mathbb{R}^* \to \mathbb{R}_+\right\}$ be the sequence of functions defined, for every $t \geq 0$ and every $(\widehat{e}_t, \widehat{e}_t, U_{t+1}) \in \mathbb{R}^2_+ \times \mathbb{R}^*$, by

$$\pi^p_t\left(\widehat{e}_t, \widehat{e}_{t+1}, U_{t+1}\right) = \inf_{(e_t, e_{t+1}) << (\widehat{e}_t, \widehat{e}_{t+1})} \left\{\frac{\widehat{e}_{t+1} - e_{t+1}}{\widehat{e}_t - e_t} : W^p_t\left(e_t, e_{t+1}, U_{t+1}\right) \geq W^p_t\left(\widehat{e}_t, \widehat{e}_{t+1}, U_{t+1}\right)\right\}.$$

That is, $\pi^p_t\left(\widehat{e}_t, \widehat{e}_{t+1}, U_{t+1}\right)$ is the function determining the minimum reduction in e_{t+1} needed to compensate an agent of generation t for a reduction in e_t in the economy in which production functions adopt the linear form $F_t(K_t, L_t) = R_t K_t + w_t N_t$ implicitly defined by the equilibrium prices. Recall from the definitions and the proof of Theorem 1, that, for each $t \geq 0$, the pair $(\widehat{x}_t, \widehat{k}^o_{t+1})$ solves the optimization problems in the definitions of both W^p_t and W_t, that is,

$$U_t(\widehat{a}) = W_t\left(\widehat{e}_t, \widehat{e}_{t+1}, U_{t+1}(\widehat{a})\right) = W^p_t\left(\widehat{e}_t, \widehat{e}_{t+1}, U_{t+1}(\widehat{a})\right).$$

Also, profit maximizing conditions imply that for every $t \geq 0$ and every $(k^o_{t+1}, n_{t+1}) \in \mathbb{R}^2_+$, (20) must hold. It follows that, for every (e_t, e_{t+1}) the solution to the optimization problem in the definition of $W_t\left(e_t, e_{t+1}, U_t(\widehat{a})\right)$ is also feasible in the optimization problem in the definition of $W^p_t\left(e_t, e_{t+1}, U_t(\widehat{a})\right)$. Therefore,

$$W^p_t\left(e_t, e_{t+1}, U_{t+1}(\widehat{a})\right) \geq W_t\left(e_t, e_{t+1}, U_{t+1}(\widehat{a})\right),$$

and

$$\pi^p_t\left(e_t, e_{t+1}, U_{t+1}(\widehat{a})\right) \geq \pi_t\left(e_t, e_{t+1}, U_{t+1}(\widehat{a})\right),$$

must be satisfied for every $t \geq 0$ and every $(e_t, e_{t+1}) << (\widehat{e}_t, \widehat{e}_{t+1})$. Thus, a sufficient condition for dynamic efficiency can be obtained easily by replacing each term $\pi_t\left(\widehat{e}_t, \widehat{e}_{t+1}, U_t(\widehat{a})\right)$ in (13) by $\pi^p_t\left(\widehat{e}_t, \widehat{e}_{t+1}, U_t(\widehat{a})\right)$. Bearing this in mind, and taking into account that the optimization problem in the definition of W^p_t is a standard consumer problem, the section of the indifference curve defined by

$$I(\widehat{e}_t, \widehat{e}_{t+1}, U_{t+1}(\widehat{a})) = \left\{(e_t, e_{t+1}) << (\widehat{e}_t, \widehat{e}_{t+1}) : W^p_t(e_t, e_{t+1}, U_{t+1}(\widehat{a})) = W^p_t(\widehat{e}_t, \widehat{e}_{t+1}, U_{t+1}(\widehat{a}))\right\}$$

can be equivalently represented as

$$I(\widehat{e}_t,\widehat{e}_{t+1},U_{t+1}(\widehat{a})) = \{e_{t+1} << \widehat{e}_{t+1} : E_t^p(e_{t+1}, W_t^p(\widehat{e}_t,\widehat{e}_{t+1},U_{t+1}(\widehat{a})), U_{t+1}(\widehat{a})) = e_t\},$$

with $E_t^p(e_{t+1}, U_t, U_{t+1})$ being the expenditure function defined, for each $(e_{t+1}, U_t, U_{t+1}) \in \mathbb{R}_+ \times \mathbb{R}^* \times \mathbb{R}^*$, by

$$E_t^p(e_{t+1}, U_t, U_{t+1}) = \min_{x_t \geq 0} \left\{ c_t^m + \frac{c_{t+1}^o}{R_{t+1}} + \left[b_t - \frac{w_{t+1} - e_{t+1}}{R_{t+1}} \right] n_{t+1} : U(x_t, U_{t+1}) \geq U_t \right\}.$$

Since the expenditure function is concave in prices, the indifference curve $I(\widehat{e}_t,\widehat{e}_{t+1},U_{t+1}(\widehat{a}))$ defines e_t as a concave function of e_{t+1}; which, in turn, implies that the indirect utility function W_t^p is quasiconvex. Hence, $\pi_t^p(\widehat{e}_t,\widehat{e}_{t+1},U_t(\widehat{a}))$ must be equal to the line joining the point $(\widehat{e}_t,\widehat{e}_{t+1})$ with the line $e_{t+1} = 0$, that is,

$$\pi_t^p(\widehat{e}_t,\widehat{e}_{t+1},U_t(\widehat{a})) = \frac{\widehat{e}_{t+1}}{\widehat{e}_t - E_t^p(0, U_t(\widehat{a}), U_{t+1}(\widehat{a}))} = \frac{\widehat{e}_{t+1}/\widehat{e}_t}{1 - \frac{E_t^p(0,U_t(\widehat{a}),U_{t+1}(\widehat{a}))}{\widehat{e}_t}}$$

Also, if $\lim_{t\to\infty} \left(b_t - \frac{w_{t+1}}{R_{t+1}} \right) > 0$ is satisfied, from a sufficiently high t one must have

$$\widehat{e}_t > E_t^p(0, U_t(\widehat{a}), U_{t+1}(\widehat{a})) > \left(b_t - \frac{w_{t+1}}{R_{t+1}} \right) \widehat{n}_{t+1} > 0.$$

Hence,

$$\lim_{T\to\infty}\inf \prod_{t=1}^{T} \left(\frac{\widehat{e}_{t+1}/\widehat{e}_t}{\pi_t^p(\widehat{e}_t,\widehat{e}_{t+1},U_{t+1}(\widehat{a}))} \right) = A_\tau \lim_{T\to\infty} \prod_{t=\tau}^{T} \left(1 - \left(b_t - \frac{w_{t+1}}{R_{t+1}} \right) \frac{\widehat{n}_{t+1}}{\widehat{e}_t} \right) = 0. \quad (23)$$

Therefore, the allocation $\widehat{a}$ satisfies the sufficient condition for dynamic efficiency in (13), which establishes that $\widehat{a}$ is Millian and $\mathcal{P}-$efficient and, therefore, completes the proof of Theorem 2.□

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