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Progressive participation

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A single seller faces a sequence of buyers with unit demand. The buyers are forward-looking and long-lived. Each buyer has private information about his arrival time *and* valuation where the latter evolves according to a geometric Brownian motion. Any incentive-compatible mechanism has to induce truth-telling about the arrival time and the evolution of the valuation.

We establish that the optimal stationary allocation policy can be implemented by a simple posted price. The truth-telling constraint regarding the arrival time can be represented as an optimal stopping problem that determines the first time at which the buyer participates in the mechanism. The optimal mechanism thus induces progressive participation by each buyer: he either participates immediately or at a future random time.

Keywords. Dynamic mechanism design, observable arrival, unobservable arrival, repeated sales, interim incentive constraints, interim participation constraints, stopping problem, option value, progressive participation.

JEL classification. D44, D82, D83.

1. Introduction

We consider a classic mechanism design problem in a dynamic environment. The seller (she) seeks to offer a good (or service) to the buyers for recurring consumption. The willingness to pay of each buyer (he) evolves according to a geometric Brownian motion over time and is private information. The arrival time (and departure time) of each buyer is random and private information as well. The buyer can choose the time at which he enters into a contract with the seller. While he can participate in a contract with the seller immediately upon arrival, he has the option to postpone his participation until a future date. Thus, participation may be progressive over time. The objective

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of the seller is to find a stationary revenue-maximizing mechanism in this dynamic environment. The design of the contract or mechanism is unrestricted and may consist of leasing contracts, sale contracts, or any other form of dynamic contract.

We depart from much of the earlier analysis of dynamic mechanisms in our treatment of the participation decision of the buyer, with the notable exception of Garrett (2016), which we will discuss shortly. Each buyer has the option to wait and sign any contract after he has received additional information about his willingness to pay. In particular, he can time the acceptance of a contract until he has a sufficiently high willingness to pay. Thus, both the participation constraints that are in place *before* the buyer has signed the contract and the incentive constraints *after* the buyer has signed the contract are fully responsive to the arrival of new information, and are consequently represented as sequential constraints.

Our main result (Theorem 3) is that a single sale price is indeed an optimal progressive mechanism in the class of all possible stationary mechanisms. More precisely, a single posted price is a simple indirect implementation of a direct mechanism that achieves the revenue-maximizing optimum. The main challenge in establishing this result is the sequential participation constraints. These new constraints state that the value function of the buyer must be the solution to an optimal stopping problem, which itself involves the value function. We relax this problem by restricting the buyer to a small set of deviations, namely threshold strategies. This relaxation has the advantage that the buyer's participation strategies can be mapped into the real line, which allows us to reduce the problem into a static mechanism design problem. This static problem is a variant of the classical nonlinear pricing problem of Mussa and Rosen (1978), with the nonstandard feature that each buyer can (deterministically) increase his type at the cost of multiplicatively decreasing his interim utility. This additional constraint leads to a failure of the first-order approach. We show that the resulting mathematical program can be expressed as a Pontryagin control problem with contact constraints (Theorem 2) and develop a verification result for such problems that might be of independent interest (Theorem 1).

In earlier work on dynamic mechanism design, the seller is typically (i) assumed to know the arrival time of the buyer, and (ii) has the ability to commit herself to make a single and once-and-for-all offer to the buyer at the moment of arrival. In particular, the seller can commit herself to *never* make another offer to the buyer in any future period. The dynamic revenue-maximizing mechanism—in the absence of sequential participation constraints—sells the object with probability 1 and forever at fixed price (see Bergemann and Strack (2015)). Thus, the object is sold to all buyers who have an initial willingness to pay above a certain threshold. Conversely, all buyers whose initial value is below this threshold would not buy the object, neither at the beginning of time nor anytime thereafter. Thus, both in the dynamic as well as in the progressive mechanism, the optimal allocation can be implemented by an indirect mechanism in the form of a posted price. But in the dynamic mechanism the posted price is offered as part of a "buy now or never" sale, whereas in the progressive mechanism the posted price is offered as part of a "buy whenever" sale. Thus, the posted price leads to very distinct allocations in case an immediate sale does not occur.

We view the relaxation of the two above-mentioned restrictions as a significant step to bring the design of dynamic mechanisms closer to many interesting economic applications. For example, the consumer clearly has a choice of when to sign up for a mobile phone contract, a gym membership, or a service contract for a kitchen appliance. Importantly, as the consumer waits, he may receive more information about his willingness to pay for the product. To the extent that these restrictions impose additional constraints on the seller, they directly weaken the power of dynamic mechanism design. The additional constraints for the seller are reflected in a larger set of reporting strategies for the buyers. A buyer can misreport both his willingness to pay as well as his arrival time. This creates an option value for the buyer, as instead of choosing a contract immediately, he can wait and enter into a contract with the seller when it is most favorable for him to do so. Given the menu of contracts offered by the seller, the buyer thus solves an optimal stopping problem to determine when to enter into a contractual relationship with the seller. Subject to the (random) evolution of his type and his willingness to pay, he can choose when to enter into an agreement with the seller. This suggests that the buyer will receive a larger information rent than in the standard dynamic mechanism design framework where the buyer has to sign a contract with the seller immediately.

The analysis of revenue-maximizing mechanisms in an environment where the buyer's private information changes over time started with Baron and Besanko (1984) and Besanko (1985). Since these early contributions, recent notable contributions are by Courty and Li (2000), Battaglini (2005), Eső and Szentes (2007), and Pavan, Segal, and Toikka (2014).[1] These papers derive in increasing generality the dynamic revenue-maximizing mechanism. The analyses in these contributions have in common the same set of constraints on the choice of mechanism. The seller has to satisfy all of the sequential incentive constraints, but only a single ex ante participation constraint. In earlier work (Bergemann and Strack (2015)), we considered the same set of constraints in a continuous-time setting, which permits for additional and explicit results regarding the nature of the optimal allocation policy—an outcome unattainable in the discrete-time setting. Here, we again use the continuous-time setting for very similar reasons.

Garrett (2016) offers a notable exception by considering unobservable arrival and stochastic values in which the private value of each buyer is governed by a Markov process with binary values, low and high. The set of allocation policies is restricted to any deterministic time-dependent sales price policy. The seller maximizes the expected discounted revenue. Garrett (2016) then provides conditions under which a time-invariant price path is optimal within the class of deterministic price paths. He further observes that an optimal policy in the class of all dynamic direct mechanisms—one that does not restrict attention to deterministic sale price paths (and implied restrictions on reporting types)—may lead to very different results and implications. By contrast, we establish that a sale price policy is optimal in the class of all stationary mechanisms, including leasing and buy-back policies. Thus, the restrictions on mechanisms in Garrett (2016)

[1] Bergemann and Välimäki (2019) provide a survey into the recent developments of dynamic mechanism design.

and our paper differ in two dimensions. Whereas Garrett (2016) only allows deterministic sale price policies, we allow all possible allocation policies. However, we restrict attention to stationary policies, whereas he allows time-dependent policies such as price cycles. Thus, the set of feasible policies is not nested in either direction.[2]

The importance of a privately observed arrival time is also investigated in Deb (2014) and Garrett (2017). In contrast to the present work, these papers do not investigate a stationary environment. Instead, while the mechanism starts at time 0, the buyer may arrive at a later time. The main concern therefore is how to encourage the early arrival to contract early. In a setting with either a durable or nondurable good, respectively, these authors find that the optimal mechanism treats early-arriving participants more favorably than late-arriving participants in terms of prices *and* quantities.

A separate literature analyzes the optimal sales of a durable good with the recurrent entry of new consumers, and is directly concerned with the timing of the purchase decision of the buyers. This literature, beginning with Conlisk, Gerstner, and Sobel (1984), Sobel (1991), and, more recently, Board (2008), restricts attention to (i) a sequence of prices rather than general allocation mechanisms and (ii) perfectly persistent values. There are related concerns with the emphasis on the ex ante participation constraints in the literature on dynamic mechanism design that pursue different directions from the one presented here. Skreta (2006, 2015), Deb and Said (2015), and Lobel and Paes Leme (2019) investigate the sequential screening model under limited commitment by the seller and pursue the equilibrium implications. A more radical departure from the ex ante or interim participation constraint to ex post participation constraints is suggested in recent work by Krähmer and Strausz (2015), Ashlagi, Daskalakis, and Haghpanah (2016), and Bergemann, Castro, and Weintraub (2020) in the context of the sequential screening model of Courty and Li (2000).

The remainder of the paper proceeds as follows. Section 2 introduces the model and the mechanism design problem. Section 3 shows how the progressive mechanism design problem can be related to an auxiliary static problem. Section 4 reviews the optimal mechanism in an environment with observable arrival. Section 5 derives the optimal progressive mechanism. Section 6 discusses how the arguments developed generalize beyond geometric Brownian motion and stationary mechanisms. Section 7 concludes.

2. Model

Payoffs and allocation We consider a stationary model with a single seller (she) and a single representative buyer (he). Time is continuous and indexed by $t \in [0, \infty)$. The seller and the buyer discount the future at the same rate $r > 0$. At each time t, the buyer demands one unit of the good. The buyer departs and gets replaced with a newly arriving buyer at rate $\gamma \geq 0$.[3] The first buyer arrives at time 0 and we denote by i the buyer who arrived ith to the market. We denote the random arrival time of buyer i by $\alpha_i \in \mathbb{R}_+$,

[2]We consider a continuum of values rather than binary values. We conjecture that a sale price would also remain the optimal stationary policy with binary values generated by a Markov process as in Garrett (2016).

[3]The model allows for the special case of a single agent who is not replaced by setting $\gamma = 0$ and this case is sufficient to understand many of the trade-offs in the choice of the optimal progressive mechanism.

and the random departure time of buyer i equals the random arrival time $\alpha_{i+1} \in \mathbb{R}_+$ of buyer $i+1$.[4]

The flow valuation of buyer i at time $t \in [\alpha_i, \alpha_{i+1}]$ is denoted by $\theta_t^i \in \mathbb{R}_+$, and the quantity allocated to buyer i at time t is $x_t^i \in [0, 1]$. The flow preferences of the buyer are represented by a (quasi-)linear utility function,

$$u_t^i = \theta_t^i x_t^i - p_t^i, \tag{1}$$

and $p_t^i \in \mathbb{R}$ is the flow payment from the buyer to the seller.

The arrival time α_i (and the departure time α_{i+1}) as well as the flow valuations $(\theta_s^i)_{s\in[\alpha_i,t]}$ are private information held by buyer i at time t.[5] The arrival and departure times of each buyer are assumed to be independent of his valuation process. The valuation of buyer i, $\theta_t^i \in \mathbb{R}_+$, at the time of his arrival $t = \alpha_i$ is distributed according to a cumulative distribution function $F : [0, \overline{\theta}] \to \mathbb{R}$, with strictly positive, bounded density $f \equiv F' > 0$ on the support. The prior distribution F is the same for every buyer i and every arrival time α_i.

The valuation of each buyer evolves randomly over time, independent of the valuation of other buyers. We assume that each buyer's valuation $(\theta_t^i)_{t\in[\alpha_i,\infty)}$ follows a geometric Brownian motion,[6]

$$\mathrm{d}\theta_t^i = \sigma\theta_t^i\, \mathrm{d}W_t, \tag{2}$$

where $(W_t)_{t\in\mathbb{R}_+}$ is a Brownian motion and $\sigma \in \mathbb{R}_+$ is the volatility that measures the speed of information arrival. The geometric Brownian motion forms a martingale and, consequently, the buyer's best estimate of his valuation at any future time is his current valuation, i.e., for all $s \geq t$: $\mathbb{E}_t[\theta_s^i] = \theta_t^i$. Furthermore, θ_t^i takes only positive values, and so the buyer's valuation for the good is always positive. The flow of allocations (x_t^i) and payments (p_t^i) will depend on the reports of the buyer to the seller, to which we turn next.

Stationary mechanism A mechanism specifies after each history a set of messages for each buyer and the allocation as a function of the complete history of messages sent by this buyer. Throughout this paper, we impose that the allocation—quantity and monetary transfer—are independent of the identity of buyer i. The quantity process (x_t)

[4]An equivalent formulation would consist of a continuum of buyers where each buyer arrives and departs with rate γ. The average behavior of such a continuum of buyers will match the expected behavior of a single representative buyer. The main advantage of the representative buyer model is that it avoids technical issues due to integration over a continuum of independent random variables, which is formally not well defined in standard probability theory; see, e.g., Judd (1985).

[5]We note that as the arrival time is private information to buyer i, the departure time has to be private information as well, or else the departure of buyer i would be informative of the arrival time of buyer $i+1$.

[6]It is without loss of generality to assume that the drift of the geometric Brownian motion of the buyer's valuation is 0. We could allow for any positive drift μ as long as μ is smaller than the discount rate. If the value would solve the stochastic differential equation $\mathrm{d}\theta_t^i = \theta_t^i(\mu\, \mathrm{d}t + \sigma\, \mathrm{d}W_t)$, then the adjusted process $\tilde{\theta}_t^i = e^{-\mu t}\theta_t^i$ would be a martingale. Thus, the buyer's value for the object at time t is the same as if his value were $\tilde{\theta}_t^i$ and his discount rate were $r - \mu$. The fact that the buyer discounts the transfer at a different rate relative to the allocation is irrelevant as he still discounts the transfers at the same rate as the seller.

specifies whether or not the buyer consumes the good at any time. We assume that the assignment of the object is reversible, i.e., the seller can give the buyer an object for some time and then take it away later.

Definition 1 (Mechanism). A mechanism (x, p) specifies at every time $t \in \mathbb{R}_+$, where some buyer i is active $t \in [\alpha_i, \alpha_{i+1})$, the allocation $x_t((m_s^i)_{\alpha_i \leq s \leq t})$ as well as the transfer $p_t((m_s^i)_{\alpha_i \leq s \leq t})$ as a function of the messages $(m_s^i)_{\alpha_i \leq s \leq t}$ sent by this buyer prior to time t.

A direct mechanism is a mechanism where the messages of the buyer are his reported arrival time and his reported flow valuations. We denote the reported arrival time $\hat{\alpha}$ and reported valuations $(\hat{\theta}_s)_{\hat{\alpha} \leq s \leq t}$ by the circumflex to distinguish true and reported times and valuations.

Definition 2 (Direct Mechanism). A direct mechanism (x, p) specifies at every time $t \in \mathbb{R}_+$, where buyer i is reported present $t \in [\hat{\alpha}_i, \hat{\alpha}_{i+1})$, the allocation $x_t(\hat{\alpha}_i, (\hat{\theta}_s^i)_{\hat{\alpha}_i \leq s \leq t})$ and the transfer $p_t(\hat{\alpha}_i, (\hat{\theta}_s^i)_{\hat{\alpha}_i \leq s \leq t})$ as a function of the reported arrival time $\hat{\alpha}_i$ and reported valuations $(\hat{\theta}_s^i)_{\hat{\alpha}_i \leq s \leq t}$.

As the payoff environment is stationary, we restrict attention to stationary mechanisms where the allocations are independent of the arrival time of the buyer. More formally, we require that a buyer who arrives at time α and whose valuations follows the path (θ_s), receives the same allocation as a buyer who arrives at a different time α' and whose valuations follow the same path of valuations shifted by the difference in arrival times, i.e., $\theta'_s = \theta_{s+(\alpha-\alpha')}$. Thus, the seller cannot discriminate against the buyer based on his arrival time.

Definition 3 (Stationary Direct Mechanism). A direct mechanism (x, p) is stationary if for all arrival times α, α' and valuation paths θ,

$$x_t\big(\alpha, (\theta_s)_{\alpha \leq s \leq t}\big) = x_{t+(\alpha'-\alpha)}\big(\alpha', (\theta_s)_{\alpha' \leq s \leq t+(\alpha'-\alpha)}\big)$$
$$p_t\big(\alpha, (\theta_s)_{\alpha \leq s \leq t}\big) = p_{t+(\alpha'-\alpha)}\big(\alpha', (\theta_s)_{\alpha' \leq s \leq t+(\alpha'-\alpha)}\big).$$

We should emphasize that in a stationary mechanism, the allocation can depend on the tenure of the agent, thus, in particular, the length and entire contingent history of the relationship, but not on the arrival time of the agent. The stationarity of the mechanism leads to the distinct interpretation of a sales price alluded to in the Introduction. A stationary sales price is a buy-whenever price, whereas the sales price in the presence of observable arrival is a buy-now-or-never price.[7]

[7]A literal interpretation of the representative agent model might suggest additional revelation of information. For example, if the single departing buyer could inform the seller about his departure, then within the current representative agent model, the seller would also be informed about the arrival of a new buyer and, hence, turn the private information regarding the arrival time into public information available to the seller. We do not adopt this literal interpretation as we take the representative agent to represent a large number of departing and arriving agents. Formally, the definition of a stationary mechanism does not allow the mechanism to be contingent on the departure (or arrival) time of the buyer.

Progressive mechanism By the revelation principle we can, without loss of generality, restrict attention to *direct mechanisms* where it is optimal for the buyer to report his arrival time α and his valuation θ_t truthfully at every time t. Each buyer i seeks to maximize his discounted expected net utility given his valuation $\theta^i_{\alpha_i}$ at his arrival time α_i:

$$\mathbb{E}\left[\int_{\alpha_i}^{\alpha_{i+1}} e^{-r(t-\alpha_i)}\big(\theta^i_t x^i_t - p^i_t\big)\,\mathrm{d}t \,\Big|\, \alpha_i, \theta^i_{\alpha_i}\right].$$

The seller seeks to maximize the expected discounted net revenue collected from her interaction with the sequence of all buyers:

$$\mathbb{E}\left[\sum_{i=0}^{\infty}\int_{\alpha_i}^{\alpha_{i+1}} e^{-rt} p^i_t\,\mathrm{d}t\right]. \tag{3}$$

Define the indirect utility $V_\alpha : \mathbb{R}_+ \to \mathbb{R}$ of a buyer who arrives at time α with a value of θ_α and reports his arrival and valuations $(\theta_s)_{\alpha\le s\le t}$ truthfully by

$$\begin{aligned} V_\alpha(\theta_\alpha) &= \mathbb{E}\left[\int_{\alpha_i}^{\alpha_{i+1}} e^{-r(t-\alpha_i)}\big(\theta^i_t x^i_t - p^i_t\big)\,\mathrm{d}t \,\Big|\, \alpha_i=\alpha, \theta^i_{\alpha_i}=\theta_\alpha\right] \\ &= \mathbb{E}\left[\int_{\alpha_i}^{\infty} e^{-(r+\gamma)(t-\alpha_i)}\big(\theta^i_t x^i_t - p^i_t\big)\,\mathrm{d}t \,\Big|\, \alpha_i=\alpha, \theta^i_{\alpha_i}=\theta_\alpha\right]. \end{aligned}$$

The second equality follows immediately from the law of iterated expectations and the fact that the departure time α_{i+1} of the buyer is independent of the arrival time α_i and the valuation process θ^i, and, hence, of x^i_t, p^i_t.

It is optimal for the buyer to report truthfully if

$$V_\alpha(\theta_\alpha) \ge \sup_{\hat\alpha\ge\alpha_i,(\hat\theta_t)} \mathbb{E}\left[\int_{\hat\alpha}^{\infty} e^{-(r+\gamma)(t-\alpha_i)}\big(\theta^i_t \hat x^i_t - \hat p^i_t\big)\,\mathrm{d}t \,\Big|\, \alpha_i=\alpha, \theta^i_{\alpha_i}=\theta_\alpha\right], \tag{IC}$$

where allocation $\hat x^i_t = x_t(\hat\alpha, (\hat\theta_s)_{\hat\alpha\le s\le t})$ as well as payment $\hat p^i_t = p_t(\hat\alpha, (\hat\theta_s)_{\hat\alpha\le s\le t})$ are functions of the reported arrival time $\hat\alpha$ as well as all subsequently reported valuations $(\hat\theta_s)_{\hat\alpha\le s\le t}$. We note here that the supremum in (IC) is taken over stopping times $\hat\alpha$ as the buyer can condition his reported arrival on his current (and past) valuation θ_t.

We restrict attention to mechanisms where the buyer participates voluntarily, i.e., for all arrival times α and all initial values θ_α, the buyer's expected utility from participating in the mechanism is nonnegative:

$$V_\alpha(\theta_\alpha) \ge 0. \tag{PC}$$

While imposing incentive compatibility constraints (IC) as well as participation constraints (PC) is standard in the literature on (dynamic) mechanism design, we note that the incentive compatibility constraint (IC) imposed here is stronger than the one usually imposed in the literature. As the arrival time α is not observable, the seller has to provide incentives for the buyer to report his arrival truthfully. In fact, the incentive constraint (IC) directly implies the participation constraint (PC), as the buyer can always decide to never report his arrival, $\hat{\alpha} = \infty$. We denote by $\mathcal{M}$ the set of all incentive-compatible stationary mechanisms where every buyer participates voluntarily.

The seller seeks to maximize her revenue subject to the incentive and participation constraints, and we refer to this as the *progressive mechanism* design problem.

3. Aggregation and revenue equivalence

As a first and significant step in the analysis, we establish that the progressive mechanism design problem can be related to an auxiliary static problem. The static formulation aggregates the progressive problem over time with suitable weights into a static problem. In the new static problem, the buyer reports only his initial valuation, and the seller chooses an expected and discounted aggregate quantity $q \in \mathbb{R}_+$ to allocate to the buyer. We establish that in any incentive-compatible progressive mechanism, both the value of the buyer as well as the revenue of the seller are only a function of this aggregate quantity.

Toward this end, we first rewrite the revenue of the seller from the sequence of buyers, given by (3), in terms of the revenue collected from the interaction with a *single* buyer i only. After all, in a stationary direct mechanism, the allocation and transfer depend only on the time elapsed since the arrival time of buyer i.

Lemma 1 (Expected Revenue). *The expected discounted revenue in the optimal mechanism equals*

$$\frac{r+\gamma}{r} \max_{(x,p)\in\mathcal{M}} \mathbb{E}\left[\int_{\alpha_i}^{\alpha_{i+1}} e^{-r(t-\alpha_i)} p_t^i \, \mathrm{d}t\right], \tag{4}$$

where i is an arbitrary buyer.

The proofs of some of the auxiliary results are relegated to the Appendix. This result follows directly from the independence of values across the buyers. The term $(r+\gamma)/r$ results from the geometric series of discounting and replacement, and can be interpreted as the discounted number of generations. We can therefore, without loss of generality, assume that the representative buyer arrives at time zero, $\alpha_i = 0$, to determine the revenue the seller derives from her interaction with all the buyers. With the focus on a single instance of buyer i, we can therefore drop the index i indicating his identity i and his arrival time α_i, and denote by $V(\theta_0)$ the indirect utility of the buyer who arrived at time $t = 0$ with initial valuation θ_0.

We now define an "aggregate quantity" $q : \Theta \to \mathbb{R}_+$ that is allocated to a buyer with initial valuation θ_0 by

$$q(\theta_0) \triangleq \mathbb{E}\left[\int_0^\infty e^{-(r+\gamma)t} x_t \frac{\mathrm{d}\theta_t}{\mathrm{d}\theta_0} \, \mathrm{d}t \;\middle|\; \theta_0\right]. \tag{5}$$

The aggregate quantity $q(\theta_0)$ is the expected discounted integral over the flow quantities (x_t). The flow quantity x_t is weighted by a term that represents the information rent in period t due to the initial private information θ_0, as we explain next.

The first term inside the integral is simply the discounted quantity in period t, $e^{-(r+\gamma)t}x_t$. The second term is the derivative of the valuation θ_t in period t with respect to the initial value θ_0. We can now use the fact that the geometric Brownian motion can be explicitly represented as

$$\theta_t = \theta_0 \exp\left(-\frac{\sigma^2}{2}t + \sigma W_t\right),$$

and, thus, the derivative is given by

$$\frac{\mathrm{d}\theta_t}{\mathrm{d}\theta_0} = \exp\left(-\frac{\sigma^2}{2}t + \sigma W_t\right).$$

The above derivative represents the influence that the initial value θ_0 has on the future state θ_t. We note that the impulse response in the case of the geometric Brownian motion is independent of the initial state θ_0. In Bergemann and Strack (2015), we referred to it as *stochastic flow* (see Kunita (1997)), and it is the analogue of the impulse response function in a discrete time dynamic mechanism (see Pavan, Segal, and Toikka (2014, Definition 3)). We can therefore write the aggregate quantity $q(\theta_0)$ more explicitly as

$$q(\theta_0) \triangleq \mathbb{E}\left[\int_0^\infty e^{-(r+\gamma)t}x_t \exp\left(-\frac{\sigma^2}{2}t + \sigma W_t\right)\mathrm{d}t \,\middle|\, \theta_0\right]. \tag{6}$$

The expected "aggregate quantity" $q(\theta_0)$ thus weighs the discounted quantity with the corresponding stochastic flow, or information rent, that originates from the initial value θ_0. As the quantity x_t is bounded between 0 and 1, and the exponential term is a martingale, it follows that the aggregate quantity is bounded as well; i.e., for all $\theta_0 \in [0, \overline{\theta}]$,

$$0 \le q(\theta_0) \le \frac{1}{r+\gamma}. \tag{7}$$

We complete the description of the static auxiliary problem with the virtual value at time $t = 0$,

$$J(\theta_0) \triangleq \theta_0 - \frac{1-F(\theta_0)}{f(\theta_0)}, \tag{8}$$

the "virtual flow value" of the buyer upon arrival to the mechanism. As in the discrete time setting, the stochastic flow enters the dynamic version of virtual utility as established in Theorem 1 of Bergemann and Strack (2015):

$$J_t(\theta_t) \triangleq \theta_t - \frac{1-F(\theta_0)}{f(\theta_0)}\frac{\mathrm{d}\theta_t}{\mathrm{d}\theta_0}. \tag{9}$$

We denote by

$$\theta^\circ \triangleq \inf\{\theta_0 : J(\theta_0) \ge 0\} \tag{10}$$

the lowest type with a nonnegative virtual value. We assume that the distribution of initial valuations is such that $\theta \mapsto \min\{0, f(\theta)J(\theta)\}$ is nondecreasing.[8]

The expected quantity q and the virtual utility J are useful as they completely summarize the expected discounted revenue of the seller and the value of the buyer.

Proposition 1 (Aggregation and Revenue Equivalence). *In any incentive-compatible mechanism, the following statements hold:*

(a) The value of the buyer with initial valuation θ is

$$V(\theta) = \int_0^\theta q(z)\,\mathrm{d}z + V(0) \tag{11}$$

and the expected discounted revenue of the seller is

$$\mathbb{E}\left[\int_0^\infty e^{-(r+\gamma)t} p_t\,\mathrm{d}t\right] = \int_0^{\overline{\theta}} J(\theta)q(\theta)\,\mathrm{d}F(\theta) - V(0). \tag{12}$$

(b) The aggregate quantity $q(\theta_0)$ increases in θ_0.

Proposition 1 gives expressions of the objective functions of the buyer and seller in terms of the discounted quantities q only. In earlier work, we obtained a revenue equivalence result for dynamic allocation problems (see Theorem 1 in Bergemann and Strack (2015)). The new and important insight of Proposition 1 is that we can aggregate the intertemporal allocation (x_t) into a single static quantity $q(\theta_0)$ that serves as a sufficient statistic for the determination of the indirect utility and the discounted revenue at the same time. In the presence of the geometric Brownian motion and unit demand, Proposition 1 asserts that there is a particularly transparent reduction given by (5). We should emphasize that the reduction to an auxiliary static program can be extended to a wide class of stochastic processes and allocation problems. We discuss these generalizations in detail in Section 6.

Proposition 1 follows from the the truth-telling constraint at time zero. We emphasize that the conditions of Proposition 1 provide only necessary conditions for incentive compatibility and optimality of the mechanisms as they omit both

(i) the possibility to misreport the arrival time

(ii) the buyer's truth-telling constraints after time zero.

Indeed we will show in Section 4 that the monotonicity of q is not a sufficient condition for incentive compatibility under *unobservable* arrival. We find that there are further restrictions on the shape of the aggregate quantity $q(\theta_0)$ beyond monotonicity that are

[8]This is a weak technical assumption that is satisfied for most standard distributions like the uniform distribution, the exponential distribution, or the log-normal distribution. For example, for the uniform distribution $U([0, \overline{\theta}])$, we have that $f(\theta)J(\theta) = (2\theta - \overline{\theta})/\overline{\theta}$, which is increasing in θ. For the exponential distribution with mean $\mu > 0$, we have that $\min\{0, f(\theta)J(\theta)\} = \min\{0, (\theta/\mu - 1)\exp(-\theta/\mu)\}$, which is also increasing in θ.

due to the above intertemporal incentive constraints (i) and (ii). These additional restrictions will impose upper bounds on the derivative of aggregate quantity $q(\theta_0)$. In consequence, the revenue problem given by (12) is transformed from what looks like a standard unit demand problem with extremal solutions to an optimal control problem.

We will derive the revenue-maximizing mechanism for the seller when she does not observe the arrival time of the buyer in Section 5. As a point of reference, it will be instructive for us to first understand what the seller would do if the (individual) arrival time of each buyer would be observable by the seller.

4. Failure of local incentive compatibility with unobservable arrival

With observable arrival, the optimal direct mechanism can be implemented by a simple sales contract. We first review these results and then investigate how this specific sales contract performs in an environment with unobservable arrival.

With observable arrival time by the buyer, we are in the canonical dynamic mechanism design environment. In Bergemann and Strack (2015), we derived the revenue-maximizing mechanism for the current problem of interest: unit demand with values governed by a geometric Brownian motion. The optimal mechanism can be implemented by an indirect mechanism that offers the product for sale at an optimally determined price P (see Proposition 8 of Bergemann and Strack (2015)). The optimal mechanism awards the object to the buyer if and only if his virtual value is positive upon arrival: $J(\theta_0) \geq 0$, where the critical value threshold θ° is determined by $J(\theta^\circ) = 0$.

The buyer thus receives the object *forever* whenever his *initial valuation* θ_0 is above the threshold value θ°. With observable arrival, this allocation can be implemented in a sales contract where the seller charges a sales price of $\theta^\circ/(r+\gamma)$, which entitles the buyer to ownership and continued consumption at all future times. A revenue-equivalent implementation would be to sell the good at time $t = 0$ and then charge the buyer a constant flow price of $p^\circ = \theta^\circ$ in all future periods, independent of his future value θ_s for all $s \geq 0$. Thus, the indirect utility of the buyer when his arrival is observable equals

$$V(\theta_0) = \max\left\{0, \frac{\theta_0 - \theta^\circ}{r+\gamma}\right\}.$$

We now abandon the restrictive informational assumption of observable arrival and let the arrival time be private information to each buyer. We ask what would happen if the seller were to maintain the above sales policy at the optimal observable price p° as a stationary contract. Now, any newly arriving buyers with value close to p° would conclude that rather than buy immediately, he should wait until he learns more about his value, and purchase the object if and only if he learned that he has a sufficiently high valuation for the object. Thus, the sale would occur (i) later and (ii) to fewer buyers. Thus, the sale price contract fails to remain incentive-compatible in an environment with unobservable arrival times.

Still, we can ask how the buyer would behave when faced with a stationary mechanism that offers him the object for sale at flow price p. In the presence of unobservable arrival, the buyer can determine the optimal purchase time by an optimal stopping problem. We denote by T the random time at which the buyer leaves the market. If the buyer acquires the good at time t with valuation θ_t at any given price $p > 0$, whether it is $p = p^\circ$ or not, then his expected continuation utility is

$$\mathbb{E}_t\left[\int_t^T e^{-r(s-t)}(\theta_s - p)\,\mathrm{d}s\right] = (\theta_t - p)\mathbb{E}_t\left[\int_t^T e^{-r(s-t)}\,\mathrm{d}s\right] = \frac{\theta_t - p}{r+\gamma}.$$

The first equality above follows from the fact that θ is a martingale (independent of T); the second equality follows as the time T at which the buyer leaves the market is exponentially distributed with mean $t + 1/\gamma$ (from a time t perspective). The time τ at which the buyer optimally purchases the good thus solves the stopping problem:

$$\sup_\tau \frac{1}{r+\gamma}\mathbb{E}\big[e^{-r\tau}\mathbf{1}_{\{\tau<T\}}(\theta_\tau - p)\big].$$

As the buyer leaves the market with rate γ, this problem is equivalent to the problem where the discount rate is given by $(r+\gamma)$, i.e., the buyer solves the stopping problem

$$\sup_\tau \frac{1}{r+\gamma}\mathbb{E}\big[e^{-(r+\gamma)\tau}(\theta_\tau - p)\big]. \tag{13}$$

The stopping problem given in (13) is the classic irreversible investment problem analyzed in Dixit and Pindyck (1994, Chapter 5, p. 135ff). For a given sales price p, it leads to a determination of a threshold $w(p)$ that the buyer's valuation θ_τ needs to reach at the stopping time τ.[9]

To simplify notation, we define a constant β that summarizes the discount rate r, the renewal rate γ, and the variance σ^2 in a manner relevant for the stopping problem:

$$\beta \triangleq \frac{1}{2} + \frac{1}{2}\sqrt{1 + 8\frac{r+\gamma}{\sigma^2}} > 1. \tag{14}$$

Proposition 2 (Sales Contract). *In a sales contract with flow price p, the buyer acquires the object once his valuation θ reaches a time-independent threshold $w(p)$ given by*

$$w(p) = \frac{\beta}{\beta - 1}p. \tag{15}$$

The buyer's value in this sales contract is given by

$$V(\theta) = \begin{cases} \dfrac{1}{r+\gamma}\left(\dfrac{\theta}{w(p)}\right)^\beta \big(w(p) - p\big), & \textit{if } \theta < w(p) \\ \dfrac{1}{r+\gamma}\big(w(p) - p\big), & \textit{if } \theta \geq w(p). \end{cases}$$

[9] Dixit and Pindyck (1994) consider an investment problem with a real asset. There, the geometric Brownian motion may have a growth rate, thus a positive drift, $\alpha > 0$. The positive quadratic root in their equation (16) becomes (14) after setting the growth rate α, the drift of the geometric Brownian, to zero. The discount rate ρ in Dixit and Pindyck (1994) is in our setting the sum of the discount and renewal rates; thus, $\rho = r + \gamma$. The difference between the discount and growth rate, $\delta = \rho - \alpha$, is then simply the adjusted discount rate.

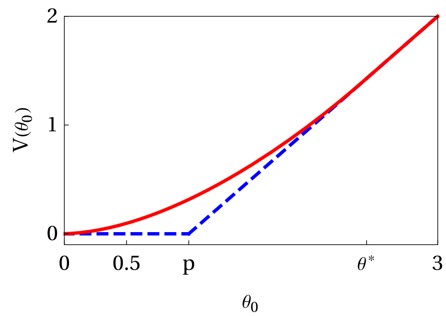

Figure 1. The value function of the buyer in a sales contract with flow price $p = 1$ when he has to participate immediately (dashed) and when he can delay his arrival (solid), given $\beta = 1.7$.

The result follows from Dixit and Pindyck (1994, Section 5.2).

We can now illustrate the payoff consequences due to the private information regarding the arrival time. In Figure 1, we display the value functions of the buyer across these two informational environments. The dashed lines depict the value function for the buyer in the setting with observable arrival time. The value is zero for all values below the threshold θ° and then a linear function of the initial value. Notably, the value function has a kink at the threshold level θ°. The solid curve depicts the value function when the sales contract is offered at the above terms as a stationary contract.

As shown in Proposition 2, the buyer reacts to this contract by reporting his arrival only once his value exceeds

$$w(p^\circ) = w(\theta^\circ) = \frac{\beta}{\beta - 1}\theta^\circ > \theta^\circ.$$

Now the value function is smooth everywhere and coincides with the blue curve whenever the initial value weakly exceeds the critical type $w(p^\circ)$. Importantly, for all values θ_0 below $w(p^\circ)$, the solid curve is above the dashed lines, which depicts the option value as expressed by (15). Notably, the value is strictly positive for all initial values, which expresses the fact that the option value guarantees every value θ_0 an information rent, quite distinct from the environment with observable arrival. Hence all buyers with low valuations would deviate by not reporting their arrival immediately, and the optimal contract with observable arrival cannot be implemented with unobservable arrival. Thus, the optimal sales contract under observable arrival fails to remain incentive-compatible in an environment with unobservable arrival.

5. The optimal progressive mechanism

The discussion in the previous section establishes that the first-order approach fails once the buyer can misreport his arrival time. To identify the optimal policy that satisfies the arrival incentive constraint we will employ the following strategy. First, we identify tractable necessary conditions for the truthful reporting of arrival by considering a specific class of deviations in the arrival time dimension. We then find the optimal mechanisms for the relaxed problem where we impose only these necessary conditions using a novel result on optimization theory we develop. Finally, we verify that in this mechanism it is indeed optimal to report the arrival time truthfully.

5.1 *Truthful reporting of arrivals*

In the first step we find a necessary condition such that the buyer wants to report his arrival immediately. Observe that if it were optimal for the buyer to reveal his presence to the mechanism immediately, then the value from revealing his presence at any stopping time $\hat{\alpha}$ must be smaller than revealing his presence at time 0. As the buyer can condition the time at which he reports his arrival to the mechanism on his past valuations, the following constraint must hold for all stopping times $\hat{\alpha}$, which may depend on the buyer's valuation path $(\theta_t)_t$:[10]

$$V(\theta_0) \geq \sup_{\hat{\alpha}} \mathbb{E}\big[e^{-(r+\gamma)\hat{\alpha}} V(\theta_{\hat{\alpha}}) \mid \theta_0\big]. \tag{IC-A}$$

We first show that the buyer's value function V in any incentive-compatible mechanism must be continuously differentiable and convex.

PROPOSITION 3 (Differentiability and Convexity of Value Function). *The value function in any incentive-compatible mechanism is continuously differentiable and convex.*

PROOF. It follows from the envelope theorem that the value function is continuous and convex in any mechanism where truthfully reporting the initial valuation is incentive-compatible. Furthermore, the envelope theorem implies that V is absolutely continuous; thus, any nondifferentiability must take the form of a convex kink. As it is never optimal to stop in a convex kink, it follows that V is differentiable. □

If the buyer's arrival time were observable, then the indirect utility need not be continuously differentiable as illustrated in Section 4. By contrast, with unobservable arrival, if the buyer finds it optimal to report his arrival immediately, then (IC-A) implies that there cannot be kinks in the value function. This is because a kink in the value function would imply a first-order gain for the buyer through the information he would get by waiting to report his arrival time. As the cost of waiting due to discounting is second-order, this implies that a mechanism with a kinked indirect utility cannot be incentive-compatible. Thus, Proposition 3 strengthens Proposition 1 by guaranteeing differentiability of the value function.

In the next step, we will relax the problem by restricting the buyer to a small class of deviations in reporting his arrival. The class of deviations we are going to consider is to have the buyer report his arrival the first time his valuation crosses a time-independent cutoff $x > \theta_0$:

$$\tau_x = \inf\{t \geq 0 \colon \theta_t \geq x\}.$$

[10]This is a version of the revelation principle as the seller can replicate every outcome where the buyer does not report his arrival immediately in a contract where the buyer reveals his arrival immediately, but never gets the object before he would have revealed his arrival in the original contract.

Note that the optimal deviation of the buyer will not (necessarily) be of this form for every direct mechanism. By restricting to deviations of this form we hope that in the optimal mechanism the optimal deviation will be of this form and the restriction will be nonbinding. With the geometric Brownian motion, we can explicitly compute the expected discounted time for a buyer with initial value θ_0 to hit any arbitrary valuation threshold x.

LEMMA 2 (Expected Discounted Time). *The expected discounted time $\tau_x = \inf\{t\colon \theta_t \geq x\}$ until a buyer's valuation exceeds a threshold x conditional on the initial valuation θ_0 is given by*

$$\mathbb{E}\big[e^{-(r+\gamma)\tau_x} \mid \theta_0\big] = \min\left\{\left(\frac{\theta_0}{x}\right)^{\beta}, 1\right\}. \tag{16}$$

Thus, if the initial value θ_0 exceeds the threshold x, then the expected discounted time is simply 1. In other words, there is no waiting at all. By contrast, if the initial value θ_0 is below the threshold x, then the expected discounted time is smaller when the gap between the initial value θ_0 and the target threshold x is larger. The magnitude of the discounting is again determined entirely by the constant β, which summarizes the primitives of the dynamic environment, namely r, γ, and σ^2, as defined earlier in (14).

5.2 *Information rents associated with unobservable arrival*

We established in Lemma 2 that the payoff from deviating to τ_x when reporting the arrival time, while continuing to report values truthfully, is given by

$$\mathbb{E}\big[e^{-(r+\gamma)\tau_x} V(v_{\tau_x}) \mid \theta_0\big] = V(x)\left(\frac{\theta_0}{x}\right)^{\beta}.$$

The term

$$\left(\frac{\theta_0}{x}\right)^{\beta}$$

captures the discount factor caused by the time the buyer has to wait to reach a value of x before participating in the mechanism. When the buyer then participates in the mechanism, he receives the indirect utility $V(x)$ of a buyer whose initial value equals x. Now, in any mechanism where (IC-A) is satisfied, the buyer does not want to deviate to the strategy τ_x and we must have

$$V(\theta_0) \geq V(x)\left(\frac{\theta_0}{x}\right)^{\beta} \quad \Leftrightarrow \quad V(x)x^{-\beta} \leq V(\theta_0)\theta_0^{-\beta}. \tag{17}$$

As (17) holds for all θ_0 and $x > \theta_0$, we have that the buyer does not want to deviate to any reporting strategy $(\tau_x)_{x>\theta_0}$ if and only if $V(x)x^{-\beta}$ is decreasing. Taking derivatives yields that this is the case whenever[11]

$$V'(x) \leq \beta \frac{V(x)}{x}. \tag{18}$$

By the earlier revenue equivalence result (see Proposition 1), the derivative of the value function $V(\theta)$ is equal to the aggregate quantity $q(\theta)$; thus,

$$q(\theta_0) = V'(\theta_0).$$

We therefore obtain a necessary condition for the aggregate quantity $q(\theta_0)$ to ensure that the buyer is reporting his arrival truthfully.

Proposition 4 (Upper Bound on Discounted Quantities). *The aggregate quantity is bounded from above by*

$$q(\theta_0) \leq \beta \frac{V(\theta_0)}{\theta_0} \tag{19}$$

in any mechanism where it is optimal to report arrival truthfully, i.e., that satisfies (IC-A).

Intuitively, (19) bounds the discounted quantity a buyer of initial type θ_0 can receive. Note that (19) is always satisfied if the value function of all initial values θ_0 of the buyer from participating in the mechanism is sufficiently high. Intuitively, due to discounting, the buyer does not want to delay reporting his arrival when the value from participating is high. As we can always increase the value to all types of the buyer by possibly offering a subsidy to the lowest type, we can reformulate (19) as a lower bound on the value $V(0)$ of the lowest type $\theta_0 = 0$.

Proposition 5 (Lower Bound on Information Rent). *In any mechanism that satisfies (IC-A), we have that*

$$V(0) \geq \sup_{\theta \in \Theta} \left(\frac{\theta q(\theta)}{\beta} - \int_0^\theta q(z)\, \mathrm{d}z \right).$$

Proof. By Propositions 1 and 4, we have that (IC-A) implies that for all θ,

$$\begin{aligned} \frac{\theta q(\theta)}{\beta} &\leq V(\theta) = V(0) + \int_0^\theta q(z)\, \mathrm{d}z \\ \Leftrightarrow \quad \frac{\theta q(\theta)}{\beta} &- \int_0^\theta q(z)\, \mathrm{d}z \leq V(0). \end{aligned}$$

Taking the supremum over θ yields the result. □

The above result establishes a lower bound on the cost of providing the buyer with incentives to report his arrival time truthfully. This lower bound depends only on the allocation q. Intuitively, the seller may need to pay subsidies independent of the buyer's

[11] $0 \geq V'(x)x^{-\beta} - \beta x^{-\beta-1} V(x) \Rightarrow V'(x) \leq \beta \frac{V(x)}{x}$.

type to provide incentives for the buyer to report his arrival time truthfully if the quantity q is too convex and the option value of waiting is thus too high.[12] The subsidy would correspond to a payment made to the buyer upon arrival and independent of his reported value θ_0. Such a scheme makes delaying the arrival costly to the buyer due to discounting, and it is potentially very costly as it requires the seller to pay the buyer just for "showing up." We will show that in the optimal mechanism, this issue will not be relevant, as the optimal mechanism does not reward the buyer merely for arriving.[13]

As a consequence of Proposition 5, we get an upper bound on the revenue in any incentive-compatible mechanism.

Corollary 1 (Revenue Bound). *An upper bound on the revenue in any incentive-compatible mechanism is given by*

$$\int_0^{\overline{\theta}} q(z)J(z)\,\mathrm{d}F(z) - \max_{\theta\in[0,\overline{\theta}]}\left(\frac{\theta q(\theta)}{\beta} - \int_0^{\theta} q(z)\,\mathrm{d}z\right). \tag{20}$$

The upper bound on revenue in (20) is obtained by considering only a small class of deviations. In particular, the buyer is only allowed to misreport his arrival via simple threshold strategies where he enters the mechanism once his valuation is sufficiently high. Economically,

$$V(0) = \max_{\theta\in[0,\overline{\theta}]}\left(\frac{\theta q(\theta)}{\beta} - \int_0^{\theta} q(z)\,\mathrm{d}z\right)$$

is a lower bound on the information rent the buyer must receive to ensure that he reports his arrival truthfully in a mechanism that implements the allocation q. As discussed before, this information rent is paid to the buyer in the form of a transfer that is independent of his consumption and, thus, is even received by those types who never consume the object. We note that due to the maximum, this information rent cannot be rewritten as an expectation and, thus, is fundamentally different from the classical information rent term. As a consequence, pointwise maximization cannot be used to find the optimal contract even in the relaxed problem. We next develop the mathematical tools to deal with this type of nonstandard maximization problem.

5.3 *Optimal control and comparison principle*

We now characterize the optimal mechanism. To do so we proceed by first finding the allocation q that maximizes the upper bound on revenue (20). Second, we construct an incentive-compatible mechanism that implements this allocation. As (20) is an upper

[12] An immediate corollary from this formula is that it is infinitely costly to implement a policy that leads to a value function V that admits a convex kink and thus has an infinite derivative $V' = q$ at some point, as argued in Proposition 3.

[13] Such subsidy schemes were discussed in Gershkov, Moldovanu, and Strack (2015, 2018) in a context where the buyer's value does not evolve over time. In Gershkov, Moldovanu, and Strack (2015) such subsidies are sometimes necessary in order to to incentivize the buyer to report his arrival time truthfully.

bound on revenue in any incentive-compatible mechanism, we then find a revenue-maximizing mechanism.

A mathematical challenge is that due to the information rent from arrival, the relaxed problem (20) is nonlocal and nonlinear in the quantity q. A change of the quantity for one type can affect the surplus extracted from all higher and lower types. Consider the relaxed problem of finding the revenue-maximizing mechanism such that the buyer never wants to misreport his arrival using a cut-off stopping time. By Proposition 4, the indirect utility V of the buyer in this mechanism solves the optimization problem

$$\max_{V} \int_{\underline{\theta}}^{\overline{\theta}} V'(z)J(z)f(z)\,\mathrm{d}z - V(0) \tag{21}$$

subject to

$$V'(\theta) \in \left[0, \frac{1}{r+\gamma}\right] \quad \text{for all } \theta \tag{22}$$

$$V \text{ is convex} \tag{23}$$

$$V'(\theta) \leq \beta \frac{V(\theta)}{\theta} \quad \text{for all } \theta \in (0, 1). \tag{24}$$

We will further relax the problem by initially ignoring the convexity constraint (23) and later verifying that the relaxed solution indeed satisfies the convexity condition. By the revenue equivalence result, Proposition 1, we can restate the allocation problem in terms of the indirect utility of the buyer. The novel and important restriction is given by the inequality (24) that states that the information rent of the buyer cannot grow too fast. The inequality thus presents an upper bound on the allocated quantity $q(\theta) = V'(\theta)$.[14]

We could approach the above problem as an optimal control problem where $V(\theta)$ is the state variable and $V'(\theta)$ is the control variable. The presence of the derivative constraint (24), which combines, in an inequality, the state and the control variable, renders this problem intractable. In particular, to the best of our knowledge the current problem is not directly covered by any standard result in optimization theory.[15] In particular, while a nonstandard version of the Pontryagin maximum principle with state-dependent control constraints could, in principle, be used to deal with the derivative constraint (24),[16] this approach would lead to a description of the optimal policy in terms of a multidimensional ordinary differential equation (ODE). And significantly, we cannot infer the optimal policy from the resulting multidimensional ODE.

To avoid these issues, we adopt a proof technique that has proved useful in stochastic optimal control, as established by Peng (1992); see also El Karoui, Peng, and Quenez

[14]At this point, we skip a complete formulation of the original problem as we later directly verify that the solution to the relaxed problem is implementable. We could state the original problem as a calculus of variations problem where the condition (24) would have to be replaced by $V''(\theta)\frac{\sigma^2\theta^2}{2} \leq (r+\gamma)V(\theta)$ under a suitable generalized notion of the second derivative.

[15]This constraint is fundamentally different from the Border constraint appearing in multi-buyer mechanism design problems, which is a (weak) majorization constraint.

[16]See, for example, Evans (1983) for a detailed introduction into the Pontryagin maximum principle.

(1997) for a wide range of applications of this technique. A comparison principle asserts a specific property of a differential inequality if an auxiliary inequality has a certain property. An important comparison result is Gronwall's inequality that allows us to bound a function that is known to satisfy a certain differential inequality by the solution of the corresponding differential equation. Following standard arguments in the literature on comparison principles, we use Gronwall's inequality to establish the following lemma.

Lemma 3 (Comparison Principle). *Let $g, h : [0, \overline{\theta}] \to \mathbb{R}$ be absolutely continuous and satisfy $g'(\theta) \le \Phi(g(\theta), \theta)$ and $h'(\theta) \ge \Phi(h(\theta), \theta)$, where $\Phi : \mathbb{R} \times [0, \overline{\theta}] \to \mathbb{R}$ is uniformly Lipschitz continuous in the first variable. If $g(0) \le h(0)$, we have that $g(\theta) \le h(\theta)$ for all $\theta \in [0, \overline{\theta}]$.*

We can then use the comparison principle to show that in optimization problems with a differential inequality constraint, one can increase the objective by making the constraint binding. This yields the following characterization of the optimal solution.

Theorem 1 (State-Dependent Control Constraints). *Let $\Phi : \mathbb{R} \times [0, \overline{\theta}] \to \mathbb{R}_+$ be increasing and uniformly Lipschitz continuous in the first variable as well as continuous in the second on every interval $[a, \overline{\theta}]$ for $a > 0$.*[17] *Let $\mathcal{J} : [0, \overline{\theta}] \to \mathbb{R}$ be continuous, satisfy $\mathcal{J}(0) = -1$, and let $z \mapsto \min\{\mathcal{J}(z), 0\}$ be nondecreasing. Consider the maximization problem*

$$\max_{w} \int_0^{\overline{\theta}} \mathcal{J}(\theta) w'(\theta) \, \mathrm{d}\theta - w(0) \tag{25}$$

over all differentiable functions $w : [0, \overline{\theta}] \to \mathbb{R}$ that satisfy $w'(\theta) \le \Phi(w(\theta), \theta)$. There exists an optimal policy w to this problem such that for all $\theta \in (0, \overline{\theta}]$,

$$w'(\theta) = \Phi\big(w(\theta), \theta\big). \tag{26}$$

Intuitively, keeping the quantity $w'(\theta)$ given to the type θ with $\mathcal{J}(\theta) = 0$ fixed and making the constraint bind will increase the quantity w' given to higher types and decrease the quantity given to lower types, which is beneficial. To apply Theorem 1 to the optimization problem given by (21), (22), and (24), we define

$$\mathcal{J}(\theta) \triangleq f(\theta) J(\theta)$$

and

$$\Phi(v, \theta) \triangleq \min\left\{\beta \frac{v}{\theta}, \frac{1}{r + \gamma}\right\}.$$

An immediate observation is that $\mathcal{J}(0) = -1$. Applying Proposition 1 to the optimization problem (21)–(24) by ex post verifying that the solution is nonnegative and convex, and, hence, feasible, yields the following characterization of the relaxed optimal mechanism.

[17] This means that for every $a > 0$, there exists a constant $L_a < \infty$ such that $|\Phi(v, \theta) - \Phi(w, \theta)| \le L_a \cdot |v - w|$ for all $\theta \in [a, \overline{\theta}]$.

Theorem 2 (Optimal Control). *There exists a $\theta^\star \in [0, \overline{\theta}]$ such that a solution to the control problem (21)–(24) is given by*

$$V(\theta) = \begin{cases} \left(\dfrac{\theta}{\theta^\star}\right)^\beta \dfrac{\theta^\star/\beta}{r+\gamma} & \text{for } \theta \le \theta^\star \\ \dfrac{\theta^\star/\beta}{r+\gamma} + \dfrac{\theta - \theta^\star}{r+\gamma} & \text{for } \theta^\star \le \theta. \end{cases} \tag{27}$$

We arrived at the optimization problem (21)–(24) by relaxing the original mechanism design problem in two ways. First, we allowed the buyer to misreport his arrival only using cutoff stopping times. Second, we ignored the monotonicity constraint associated with truthful reporting of the initial value. Next, we establish that the solution of Theorem 2 can be implemented by a simple indirect mechanism, namely a posted price.

5.4 *Posted price as an optimal progressive mechanism*

Using the characterization of the purchase behavior of the buyer in Proposition 2 and standard stochastic calculus arguments, we can completely describe the seller's average revenue for a given sales contract.

Theorem 3 (Sales Contracts are Revenue Maximizing). *The revenue-maximizing mechanism can be implemented by a posted price $p^\star$ that maximizes the flow revenue:*

$$\frac{p}{r}\int_0^\infty \min\left\{\left(\frac{\beta-1}{\beta}\frac{\theta}{p}\right)^\beta, 1\right\} f(\theta)\,\mathrm{d}\theta. \tag{28}$$

Proof. By Proposition 2, the buyer acquires the object once his valuation exceeds $\theta^\star = \frac{\beta}{\beta-1}p$. By Lemma 1, the expected revenue that the seller generates from a buyer with initial valuation θ_0 is given by

$$\begin{aligned} \frac{r+\gamma}{r}\mathbb{E}\left[\int_{\tau_{\theta^\star}}^\infty e^{-(r+\gamma)t} p\,\mathrm{d}t \;\middle|\; \theta_0\right] &= \frac{1}{r}\mathbb{E}\left[e^{-(r+\gamma)\tau_{\theta^\star}} p \mid \theta_0\right] = \frac{p}{r}\mathbb{E}\left[e^{-(r+\gamma)\tau_{\theta^\star}} \mid \theta_0\right] \\ &= \frac{p}{r}\min\left\{\left(\frac{\theta_0}{\theta^\star}\right)^\beta, 1\right\} = \frac{p}{r}\min\left\{\left(\frac{\beta-1}{\beta}\frac{\theta_0}{p}\right)^\beta, 1\right\}. \end{aligned}$$

Consequently, the expected discounted revenue from a buyer with random initial valuation distributed according to F is given by

$$\frac{p}{r}\int_0^\infty \min\left\{\left(\frac{\beta-1}{\beta}\frac{\theta}{p}\right)^\beta, 1\right\} f(\theta)\,\mathrm{d}\theta.$$

Consider the sales contract where the object is sold at a flow price of $p = \frac{\beta-1}{\beta}\theta^\star$. Proposition 2 yields that the buyer's value in a sales contract, if he reports his arrival optimally, is given by

$$V(\theta)=\begin{cases}\dfrac{1}{r+\gamma}\left(\dfrac{\theta}{\theta^\star}\right)^{\beta}\dfrac{1}{\beta}\theta^\star & \text{for } \theta\le\theta^\star\\ \dfrac{1}{r+\gamma}\left(\theta-\dfrac{\beta-1}{\beta}\theta^\star\right) & \text{for } \theta\ge\theta^\star\end{cases}$$

and thus satisfies (27), establishing that the sales contract is revenue-maximizing. □

Equation (28) reduces the problem of finding an optimal sales contract to a single-dimensional maximization problem over the price. It is worth noting that the revenue up to a linear scaling depends on r, γ, and σ only through β, which implies that the optimal sales price is only a function of β and the distribution of initial valuations F.

The expression inside the integral of (28) represents the expected quantity to be sold to a buyer with initial value θ. In contrast to a standard revenue function under unit demand, the realized quantities are not merely 0 or 1. Rather, the seller offers a positive quantity to all buyers, namely

$$\min\left\{\left(\frac{\beta-1}{\beta}\frac{\theta}{p}\right)^{\beta},1\right\}. \tag{29}$$

This expression reflects the expected discount until the object is consumed by those buyers who have an initial value below the optimal purchase threshold

$$w(p)=\frac{\beta}{\beta-1}p,$$

derived in Proposition 2. The indirect utility given in (27) is then implemented by a sales contract with a flow price of

$$p^\star=\frac{\beta-1}{\beta}\theta^\star.$$

The complete expression (29) then follows from Lemma 2 as the expected discounted probability of a sale to a buyer with initial value θ. Thus, an increase in the sales price p uniformly lowers the probability of a sale for every value θ. The problem for the seller with unobservable arrival is therefore how to respond to slower and more selective sales. As the revenue with relaxed incentive constraints is an upper bound on the revenue in the original problem, and this upper bound is achieved by some sales contract, it follows that a sales contract is a revenue-maximizing mechanism.

6. Discussion

6.1 *An example: The uniform prior*

We illustrate the results now for the case of the uniform prior on the unit interval $[0,1]$. We compute the value threshold $\theta^\star$ and the associated flow price $p^\star$ in the optimal progressive mechanism from the revenue formula (28):

$$\theta^\star=\frac{1}{2}\frac{1+\beta}{\beta},\qquad p^\star=\frac{1}{2}\frac{\beta^2-1}{\beta^2}.$$

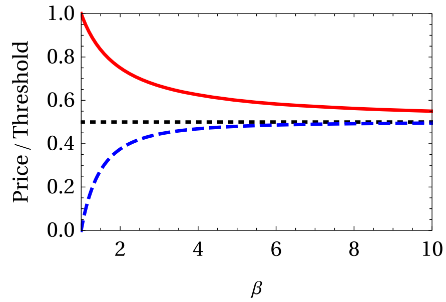

FIGURE 2. Progressive threshold (solid), dynamic threshold and price (dotted), and progressive price (dashed). The initial value θ_0 is uniformly distributed on $[0, 1]$.

In the dynamic mechanism, the value threshold and the associated price are determined exclusively by the virtual value at $t = 0$, and thus under the uniform distribution, the corresponding threshold and flow price are given by

$$\theta^\circ = p^\circ = \frac{1}{2}.$$

Thus, the price in the progressive mechanism is below the dynamic mechanism, whereas the threshold of the progressive mechanism is above the dynamic mechanism:

$$p^\star < p^\circ = \theta^\circ < \theta^\star. \tag{30}$$

In Figure 2, we display the behavior of the thresholds and the prices as a function of $\beta \in (1, \infty)$. If the discount and/or the renewal rate decrease, the buyer becomes more forward-looking, and the gap between the value threshold $\theta^\star$ and the price p^* increases. Intuitively, as the option value becomes more significant, the buyer chooses to wait until his value has reached a higher threshold and thus he will wait longer to enter into a relationship with the seller. Faced with a more forward-looking buyer, the seller decreases the flow price as β decreases. Yet, the decrease in the flow price only partially offsets the option value, and the buyer still waits longer to enter into the relationship with the seller. In contrast, the threshold value and the price in the dynamic mechanism, θ° and p°, respectively, remain invariant with respect to the patience β of the buyer.

An important aspect of the progressive mechanism is that the buyer enters the relationship gradually rather than once and for all, as in the dynamic mechanism. In Figure 3(a) we plot the probability that an initial type drawn from the uniform distribution consumes the object as a function of the time since his arrival. In the dynamic mechanism, this probability is constant over time. As all values θ_0 above $\theta^\circ = 1/2$ buy the object and all those with initial values $\theta_0 < \theta^\circ = 1/2$ never buy the object, the probability of consumption does not change over time and is always equal to $1/2$. By contrast, in the progressive mechanism, the probability of participation is progressing over time and, thus, the probability of consumption is increasing over time. The geometric Brownian motion displays sufficient variance, so that eventually every buyer purchases the product.

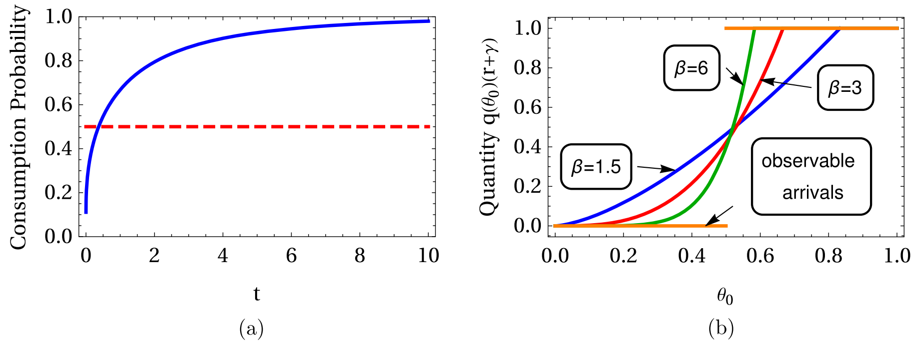

FIGURE 3. (a) Consumption probability over time: progressive (solid); dynamic (dashed). The initial valuation is uniformly distributed on $[0, 1]$, $\beta = 1.5$, and $\sigma = 1$. (b) Quantities assigned in progressive mechanism for: $\beta = 1.5$, $\beta = 3$, and, $\beta = 6$, observable arrival in horizontal lines. The initial valuation is uniformly distributed on $[0, 1]$.

We now zoom in on the purchase behavior of the initial types θ_0. Figure 3(b) illustrates the discounted expected consumption quantity $q(\theta_0)$ as a function of the initial valuation θ_0 for various values of β. In the dynamic mechanism there is a sharp distinction in the consumption quantities between the initial values below and above the threshold of $\theta^\circ = 1/2$. By contrast, in the progressive mechanism, the consumption quantity is continuous and monotone increasing in the initial value θ_0. As the buyer becomes more patient, and hence as β decreases, the slope of the consumption quantity flattens outs and the threshold $\theta^\star$ upon which consumption occurs is immediately increasing.

6.2 *Positive production cost*

We assumed so far that the marginal cost of production for the seller is constant and equal to zero. We now discuss the case of positive production cost, $c > 0$. The value of the buyer from consumption is unchanged and can still be expressed completely in terms of the aggregate quantity q defined in (6). It follows from the expression (9) that the virtual value of the buyer at time t is given by $J_t(\theta_t) = \theta_t J(\theta_0)$.

If the cost of production is given by a fixed cost that the seller has to incur once and only once, then our earlier analysis remains essentially unchanged. The only modification arises in the determination of the optimal threshold $\theta^\star$, which is now influenced by the cost of production. The more challenging situation is the case of the positive flow cost $c > 0$ that has to be incurred every time the good is consumed. Here we can define the indirect production cost of the seller $C : \mathbb{R} \to \mathbb{R}$ as the minimal cost necessary for providing the agent with a given aggregate quantity q, i.e.,

$$C(q) = \min_x \mathbb{E}\left[\int_0^\infty e^{-(r+\gamma)t} x_t c \, \mathrm{d}t\right]$$

$$\text{subject to } q = \mathbb{E}\left[\int_0^\infty e^{-(r+\gamma)t} x_t \exp\left(-\frac{\sigma^2}{2}t + \sigma W_t\right) \mathrm{d}t\right].$$

Taking the Lagrangian yields that the optimal policy in the above problem is to allocate the object to the agent if and only if $\theta_t/\theta_0 = \exp(-\frac{\sigma^2}{2}t + \sigma W_t)$ exceeds some threshold ϕ. This allocation rule is implementable as it is monotone in θ_t (see Bergemann and Strack (2015)). The indirect cost thus simplifies to

$$C(q) = c\int_0^\infty e^{-(r+\gamma)t}\mathbb{P}\left[\exp\left(-\frac{\sigma^2}{2}t + \sigma W_t\right) \geq \phi\right] \mathrm{d}t \tag{31}$$

$$\text{subject to } q = \mathbb{E}\left[\int_0^\infty e^{-(r+\gamma)t}\mathbf{1}_{\{\exp(-\frac{\sigma^2}{2}t+\sigma W_t)\geq\phi\}}\exp\left(-\frac{\sigma^2}{2}t + \sigma W_t\right) \mathrm{d}t\right].$$

The optimal contract thus has the form that it offers a menu of time-independent flow prices for consuming the good coupled with upfront payments. A higher upfront payment leads to a lower price of consuming the good later. To derive the exact optimal menu, the seller now needs to solve the analogue of problem (21)–(24) with production cost C:

$$\max_V \int_{\underline{\theta}}^{\overline{\theta}} \left[V'(z)J(z) - C\left(V'(z)\right)\right] f(z)\,\mathrm{d}z - V(0)$$

subject to

$$V'(\theta) \in \left[0, \frac{1}{r+\gamma}\right] \quad \text{for all } \theta$$

V is convex

$$V'(\theta) \leq \beta \frac{V(\theta)}{\theta} \quad \text{for all } \theta \in (0, 1).$$

As this problem is nonlinear in V', we can no longer appeal to a version of the comparison result that we developed in Proposition 1. As discussed earlier in Section 5.3, we could pursue a nonstandard version of the Pontryagin maximum principle with state-dependent control constraints to address this problem, but this would require a distinct approach. Here, we restrict ourselves to a numerical solution of the problem and display a typical solution in Figure 4. The progressive solution has the same qualitative form as the corresponding optimal dynamic mechanism (see Proposition 9 in Bergemann and Strack (2015)). In particular, high initial types face a lower threshold for receiving the good at later times (see the bottom left illustration of Figure 4). In contrast to the case without production cost, no agent now consumes the object forever and the seller never provides the object when the value falls below the marginal cost of production. Low types are again indifferent between entering the contract immediately and delaying. In equilibrium, they enter the contract immediately upon arrival and consume in the future above a threshold ϕ that exceeds their initial value. Intuitively, by having the buyers enter the contract immediately, the seller can provide them with the same expected utility at a lower cost, as she only provides them with the object when their value is sufficiently high.

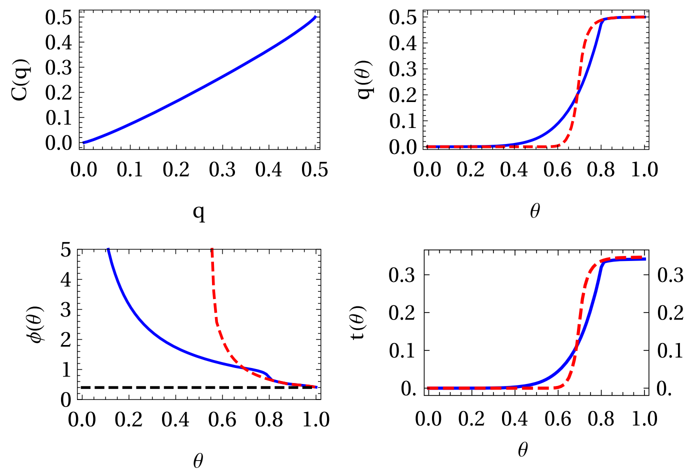

Figure 4. The indirect cost as defined in (31) (upper left). The optimal policy $q(\theta) = V'(\theta)$ (upper right). The threshold above which the agent consumes (bottom left) and the upfront transfer (bottom right). The dashed curve corresponds to the optimal contract with observable arrival, the solid curve corresponds to the optimal contract under our progressive participation constraint, F is uniform on $[0, 1]$, $\sigma = 0.3$, $r + \gamma = 0.2$, and $c = 0.4$.

7. Concluding remarks

We considered a dynamic mechanism problem where each buyer is described by two dimensions of private information: his willingness to pay (which may change over time) *and* his arrival time. We considered a stationary environment—in which the buyers arrive and depart at random—and a stationary contract. In this arguably more realistic setting for revenue management, the seller has to guarantee both interim incentives and interim participation constraints. As the buyer has the valuable option of delaying his participation, the mechanism has to offer incentives to enter into the relationship.

The central challenge in our environment is that the first-order approach and other standard methods fail as global incentive constraints bind in the optimal contract. We were able to solve this multidimensional incentive problem by rephrasing the participation decision of the buyer as a stopping problem and then solving a new optimal control problem. More precisely, we decomposed the progressive mechanism problem into an intertemporal participation (entry) problem and an intertemporal incentive problem. Given the separability between these two problems, our approach can possibly be extended to allocation problems beyond the unit demand problem considered here. There are (at least) two natural directions to extend the analysis.

First, the stochastic evolution of the value was governed by the geometric Brownian motion, and clearly other stochastic processes could be considered. What changes for more general stochastic processes is the expected revenue as a function of the value of the buyer given in (20) and (21). The particularly simple multiplicative structure of the virtual value is a consequence of the geometric Brownian motion. For other processes, such as the arithmetic Brownian motion or the mean-reverting Ornstein–Uhlenbeck process, the corresponding virtual value is obtained in Bergemann and Strack (2015).

Using these virtual values and adjusting the value for deviating to a threshold strategy, one obtains a relaxed program that is analogous to (21)–(24). Notably, this provides a reduction of our original dynamic problem into a completely static problem without any incentive constraints.

For general processes or models, the resulting problem will not admit the same simple multiplicatively separable structure. As a consequence, we could not use our Proposition 1 to solve for the optimal mechanism, but would have to rely on other methods such as the Pontryagin principle. Yet, whenever the solution to this relaxed general program is implementable, i.e., monotone, and the progressive participation constraint binds below a threshold, it will constitute an optimal mechanism. As we illustrated in the convex cost case in Section 6.2, this can be checked numerically.

By contrast, if the virtual value were to react too strongly to the type, then the progressive participation constraint (18) may be binding at several disconnected intervals. This would imply that there is not a single and always lower interval at which the agent would wait, but rather a collection of disconnected intervals. In each one of these intervals, the agent would wait until his valuation leaves the interval, either below or above. In consequence, the optimal strategy for the agent could no longer be expressed in terms of a simple threshold strategy as it is in the current setting.

Second, we derived the incentive constraint (IC-A) for stationary mechanisms when the buyer has private information about his arrival time and his current value. We motivated the restriction to a stationary mechanism as a standing offer by the seller in a market with renewal among consumers. In other words, time 0 is not economically meaningful as new buyers are constantly arriving and, thus, it is always time 0 for someone. We did not analyze whether, in the steady state of the environment, a stationary mechanism is optimal in a larger class of feasible mechanisms, which can be offered in a time- and state-dependent fashion.

Suppose we would consider an optimal nonstationary mechanism. In particular, the mechanism could then depend on the calendar time t. Let $V(t,\theta)$ be the value of a buyer who arrives at time t with a valuation of θ. Now, a generalized version of the progressive participation constraint (IC-A) would be equivalent to a partial differential equation:

$$V_t(t,\theta)+V_{\theta\theta}(t,\theta)\frac{\sigma^2\theta^2}{2}\leq(r+\gamma)V(t,\theta). \tag{32}$$

Thus, while the nature and description of the optimal mechanism have not changed conceptually, the optimal control problem is now subject to a constraint in the form of a partial differential equation. In general, it is not known how to obtain a solution of an equation of this form. On a fundamental level, it is not known how to compute the buyer's optimal stopping time as a best response to an arbitrary nonstationary, and, thus, time inhomogenous, policy by the seller; see Peskir and Shiryaev (2006).

However, in special cases of our model, we know that a stationary policy is optimal. For example, when there is only initial private information, and, thus, the variance σ of the geometric Brownian motion is 0, Board (2008) shows that a stationary sale price constitutes the optimal commitment policy. We suspect that as either the variance or the discounting is small, the stationary solution will remain the uniquely optimal solution.

A complete analysis will require additional arguments to address a host of additional challenges.

Appendix

Proof of Lemma 1. As each buyer's allocation is only a function of his own reports, and the willingness to pay is independent across different buyers, the law of iterated expectations implies that the revenue can be rewritten as

$$\mathbb{E}\left[\sum_{i=0}^{\infty}\int_{\alpha_i}^{\alpha_{i+1}} e^{-rt} p_t^i \, \mathrm{d}t\right] = \mathbb{E}\left[\sum_{i=0}^{\infty} e^{-r\alpha_i}\mathbb{E}\left[\int_{\alpha_i}^{\alpha_{i+1}} e^{-r(t-\alpha_i)} p_t^i \, \mathrm{d}t\right]\right].$$

As buyers are ex ante identical, we can without loss assume that they are treated the same in the optimal mechanism, which yields that the revenue equals

$$\max_{(x,p)\in\mathcal{M}} \mathbb{E}\left[\sum_{i=0}^{\infty}\int_{\alpha_i}^{\alpha_{i+1}} e^{-rt} p_t^i \, \mathrm{d}t\right] = \max_{(x,p)\in\mathcal{M}} \mathbb{E}\left[\int_{\alpha_i}^{\alpha_{i+1}} e^{-r(t-\alpha_i)} p_t^i \, \mathrm{d}t\right]\mathbb{E}\left[\sum_{i=0}^{\infty} e^{-r\alpha_i}\right].$$

Note that the survival time of each generation i, $\alpha_{i+1}-\alpha_i$ are independently and identically exponentially distributed with rate γ. From this and the fact that $\alpha_0 = 0$, it follows that

$$\begin{aligned}\mathbb{E}\left[\sum_{i=0}^{\infty} e^{-r\alpha_i}\right] &= \mathbb{E}\left[\sum_{i=0}^{\infty} e^{-r\alpha_0}\prod_{j=0}^{i-1} e^{-r(\alpha_{j+1}-\alpha_j)}\right] = \mathbb{E}\left[e^{-r\alpha_0}\right]\sum_{i=0}^{\infty}\prod_{j=0}^{i-1}\mathbb{E}\left[e^{-r(\alpha_{j+1}-\alpha_j)}\right] \\ &= \sum_{i=0}^{\infty}\mathbb{E}\left[e^{-r(\alpha_{j+1}-\alpha_j)}\right]^i = \sum_{i=0}^{\infty}\left(\frac{\gamma}{r+\gamma}\right)^i = \frac{r+\gamma}{r}.\end{aligned}$$

This yields the result. □

Proof of Proposition 1. The first part of the proposition follows by applying the envelope theorem. By the hypothesis of the proposition, it is optimal for the buyer to report his initial value θ_0 truthfully. Therefore, we can compute the derivative of the buyer's indirect utility by treating the allocation (x, p) as independent of the buyer's report:

$$\begin{aligned}V'(\theta_0) &= \frac{\partial}{\partial\theta_0}\mathbb{E}\left[\int_0^{\infty} e^{-(r+\gamma)t}\{x_t\theta_t - p_t\}\,\mathrm{d}t \,\middle|\, \theta_0\right] \\ &= \mathbb{E}\left[\int_0^{\infty} e^{-(r+\gamma)t} x_t\left(\frac{\partial}{\partial\theta_0}\theta_t\right)\mathrm{d}t \,\middle|\, \theta_0\right]. \end{aligned} \tag{33}$$

As $(\theta_t)_{t\geq 0}$ is a geometric Brownian motion, the evolution of θ_t can be explicitly represented as

$$\theta_t = \theta_0 \exp\left(-\frac{\sigma^2}{2}t + \sigma W_t\right). \tag{34}$$

We can then insert the derivative $\partial\theta_t/\partial\theta_0$ and obtain

$$\begin{aligned}V'(\theta_0) &= \mathbb{E}\left[\int_0^\infty e^{-(r+\gamma)t} x_t \left(\frac{\partial}{\partial\theta_0}\theta_t\right) \mathrm{d}t \,\middle|\, \theta_0\right] \\ &= \mathbb{E}\left[\int_0^\infty e^{-(r+\gamma)t} x_t \left(\frac{\partial}{\partial\theta_0}\left\{\theta_0 \cdot \exp\left(-\frac{\sigma^2}{2}t + \sigma W_t\right)\right\}\right) \mathrm{d}t \,\middle|\, \theta_0\right] \\ &= \mathbb{E}\left[\int_0^\infty e^{-(r+\gamma)t} x_t \exp\left(-\frac{\sigma^2}{2}t + \sigma W_t\right) \mathrm{d}t \,\middle|\, \theta_0\right] \\ &= q(\theta_0),\end{aligned}$$

where the last line follows from the definition of the aggregate quantity $q(\theta_0)$ given earlier in (5). Similarly, we can express the revenue of the seller in terms of the dynamic virtual value as given earlier in (9),

$$J_t(\theta_t) \triangleq \theta_t - \frac{1-F(\theta_0)}{f(\theta_0)}\frac{\mathrm{d}\theta_t}{\mathrm{d}\theta_0},$$

and we observe that using (34), we can express the derivative equivalently as

$$\frac{\mathrm{d}\theta_t}{\mathrm{d}\theta_0} = \frac{\theta_t}{\theta_0}.$$

The expected revenue of the seller can, therefore, be expressed as

$$\begin{aligned}&\mathbb{E}\left[\int_0^\infty e^{-(r+\gamma)t} p_t \,\mathrm{d}t\right] \\ &\quad= \mathbb{E}\left[\int_0^\infty e^{-(r+\gamma)t} x_t \left(\theta_t - \frac{1-F(\theta_0)}{f(\theta_0)}\frac{\mathrm{d}\theta_t}{\mathrm{d}\theta_0}\right) \mathrm{d}t\right] - V(0) \\ &\quad= \mathbb{E}\left[\int_0^\infty e^{-(r+\gamma)t} x_t \theta_t \left(1 - \frac{1-F(\theta_0)}{f(\theta_0)\theta_0}\right) \mathrm{d}t\right] - V(0) \\ &\quad= \mathbb{E}\left[\int_0^\infty e^{-(r+\gamma)t} x_t \frac{\theta_t}{\theta_0} \left(\theta_0 - \frac{1-F(\theta_0)}{f(\theta_0)}\right) \mathrm{d}t\right] - V(0) \\ &\quad= \int_0^{\overline{\theta}} \mathbb{E}\left[\int_0^\infty e^{-(r+\gamma)t} x_t \frac{\theta_t}{\theta_0} \left(\theta_0 - \frac{1-F(\theta_0)}{f(\theta_0)}\right) \mathrm{d}t \,\middle|\, \theta_0\right] f(\theta_0)\,\mathrm{d}\theta_0 - V(0) \\ &\quad= \int_0^{\overline{\theta}} \left(\theta_0 - \frac{1-F(\theta_0)}{f(\theta_0)}\right) \mathbb{E}\left[\int_0^\infty e^{-rt} x_t \frac{\theta_t}{\theta_0} \,\mathrm{d}t \,\middle|\, \theta_0\right] f(\theta_0)\,\mathrm{d}\theta_0 - V(0).\end{aligned}$$

Plugging in the explicit representation of θ_t given by (34) yields that the expected revenue satisfies

$$\begin{aligned}&\mathbb{E}\left[\int_0^\infty e^{-(r+\gamma)t} p_t \,\mathrm{d}t\right] \\ &\quad= \int_0^{\overline{\theta}} J(\theta_0) \mathbb{E}\left[\int_0^\infty e^{-(r+\gamma)t} x_t \exp\left(-\frac{\sigma^2}{2}t + \sigma W_t\right) \mathrm{d}t \,\middle|\, \theta_0\right] f(\theta_0)\,\mathrm{d}\theta_0 - V(0),\end{aligned}$$

where the expectation term inside the square bracket equals the aggregate quantity $q(\theta_0)$, i.e.,

$$q(\theta_0) = \mathbb{E}\left[\int_0^\infty e^{-(r+\gamma)t} x_t \exp\left(-\frac{\sigma^2}{2}t + \sigma W_t\right) \mathrm{d}t \,\middle|\, \theta_0\right].$$

To establish monotonicity, consider the deviation where the agent of type θ_0 reports to be of type $\hat{\theta}_0$ at time 0 and at every later point in time t reports a value of $\hat{\theta}_t = \theta_t \frac{\hat{\theta}_0}{\theta_0}$. Note that under this deviation, the agent's value evolves as if the agent would have been of initial type $\hat{\theta}_0$. Thus, this deviation generates a value of

$$\mathbb{E}\left[\int_0^\infty e^{-(r+\gamma)t}\left(x_t\theta_0 \exp\left(-\frac{\sigma^2}{2}t + \sigma W_t\right) - p_t\right) \mathrm{d}t \,\middle|\, \hat{\theta}_0\right] = \theta_0 q(\hat{\theta}_0) - T(\hat{\theta}_0),$$

where $T(\hat{\theta}_0) = \mathbb{E}[\int_0^\infty e^{-(r+\gamma)t} p_t \,\mathrm{d}t \mid \hat{\theta}_0]$ is the total expected transfer made by an agent of initial type $\hat{\theta}_0$. It follows from the monotone selection theorem that q is nondecreasing whenever truthful reporting of the initial type is incentive-compatible. □

Proof of Lemma 2. For $\theta_0 \geq x$, the buyer stops immediately and, thus, the statement is true. For $\theta_0 < x$, we have that

$$\begin{aligned}
\mathbb{E}\left[e^{-(r+\gamma)\tau_x} \mid \theta_0\right] &= \mathbb{E}\left[e^{-(r+\gamma)\tau_x}\left(\frac{\theta_{\tau_x}}{\theta_{\tau_x}}\right)^\beta \,\middle|\, \theta_0\right] = \mathbb{E}\left[e^{-(r+\gamma)\tau_x}\left(\frac{\theta_0 e^{-\frac{\sigma^2}{2}\tau_x + \sigma W_{\tau_x}}}{x}\right)^\beta \,\middle|\, \theta_0\right] \\
&= \mathbb{E}\left[e^{-[(r+\gamma)-\frac{\sigma^2}{2}\beta]\tau_x + \beta\sigma W_{\tau_x}}\left(\frac{\theta_0}{x}\right)^\beta \,\middle|\, \theta_0\right] \\
&= \mathbb{E}\left[e^{-[(r+\gamma)+\frac{\sigma^2}{2}\beta(1-\beta)]\tau_x} e^{-\frac{\sigma^2\beta^2}{2}\tau_x + \beta\sigma W_{\tau_x}}\left(\frac{\theta_0}{x}\right)^\beta \,\middle|\, \theta_0\right].
\end{aligned}$$

As $(r+\gamma) + \frac{\sigma^2}{2}\beta - \frac{\sigma^2\beta^2}{2} = 0$ and $t \mapsto e^{-\frac{\sigma^2\beta^2}{2}t + \beta\sigma W_t}$ is a martingale whose value at time τ_x is uniformly integrable, it follows from Doob's optional sampling theorem that

$$\mathbb{E}\left[e^{-r\tau_x} \mid \theta_0\right] = \mathbb{E}\left[\left(\frac{\theta_0}{x}\right)^\beta \,\middle|\, \theta_0\right].$$

Define $\Delta \equiv g - h$. Suppose that there exists a point θ' such that $\Delta(\theta') > 0$. As $\Delta(0) \leq 0$ and as Δ is absolute continuous, there exists a point θ'' such that $\Delta(\theta'') = 0$, and as $\Delta' \geq 0$, we have that $\Delta(\theta) \geq 0$ for all $\theta \in [\theta'', \theta']$. Combined with the uniform Lipschitz continuity of Φ, this implies that there exists a constant $L > 0$ such that for all $\theta \in [\theta'', \theta']$,

$$\begin{aligned}
\Delta'(\theta) = g'(\theta) - h'(\theta) &\leq \Phi\big(g(\theta), \theta\big) - \Phi\big(h(\theta), \theta\big) \leq \big|\Phi\big(g(\theta), \theta\big) - \Phi\big(h(\theta), \theta\big)\big| \\
&\leq L\big|g(\theta) - h(\theta)\big| = L\big|\Delta(\theta)\big| = L\Delta(\theta).
\end{aligned}$$

By Gronwall's inequality, we thus have that $\Delta(\theta') \leq \Delta(\theta'')e^{L(\theta' - \theta'')} = 0$, which contradicts the assumption that $\Delta(\theta') > 0$. □

Lemma 4 (Generalized Comparison Principle). *Let $g, h : [0, \overline{\theta}] \to \mathbb{R}$ be absolutely continuous and satisfy $g'(\theta) \leq \Phi(g(\theta), \theta)$ and $h'(\theta) \geq \Phi(h(\theta), \theta)$, where $\Phi : \mathbb{R} \times [0, \overline{\theta}] \to \mathbb{R}$ is uniformly Lipschitz continuous in the first variable. If $g(\hat{\theta}) = h(\hat{\theta})$, we have that $g(\theta) \leq h(\theta)$ for all $\theta \in [\hat{\theta}, \overline{\theta}]$ and $g(\theta) \geq h(\theta)$ for all $\theta \in [0, \hat{\theta}]$.*

Proof. The first part of the result follows by considering the functions $\tilde{g}(s) = g(\hat{\theta} + s)$ and $h(s) = \bar{y}(\hat{\theta} + s)$, and applying Lemma 3. The second part follows by considering the functions $\tilde{g}(s) = -g(\hat{\theta} - s)$ and $\tilde{h}(s) = -h(\hat{\theta} - s)$ for $s \in [0, \hat{\theta}]$, and applying Lemma 3, which implies that for all $s \in [0, \hat{\theta}]$,

$$\tilde{g}(s) \leq \tilde{h}(s) \quad \Leftrightarrow \quad -g(\hat{\theta} - s) \leq -h(\hat{\theta} - s) \quad \Leftrightarrow \quad g(\hat{\theta} - s) \geq h(\hat{\theta} - s). \qquad \square$$

Lemma 5. *Suppose that $\mathcal{J} : [0, \overline{\theta}]$ is a nondecreasing function with $\mathcal{J}(\overline{\theta}) \leq 0$ and that $g, h : [0, \overline{\theta}] \to \mathbb{R}$ are absolutely continuous with $g \geqq h$. Then*

$$\int_0^{\overline{\theta}} \mathcal{J}(\theta) g'(\theta) \, \mathrm{d}t + \mathcal{J}(0) g(0) \leq \int_0^{\overline{\theta}} \mathcal{J}(\theta) h'(\theta) \, \mathrm{d}\theta + \mathcal{J}(0) h(0).$$

Proof. The result follows from integration by parts and the assumption that $\mathcal{J}(\overline{\theta}) \leq 0$:

$$\begin{aligned}
\int_0^{\overline{\theta}} \mathcal{J}(\theta) g'(\theta) \, \mathrm{d}t + \mathcal{J}(0) g(0) &= \big[\mathcal{J}(\theta) g(\theta)\big]_{\theta=0}^{\theta=\overline{\theta}} - \int_0^{\overline{\theta}} g(\theta) \, \mathrm{d}\mathcal{J}(\theta) + \mathcal{J}(0) g(0) \\
&= \mathcal{J}(\overline{\theta}) g(\overline{\theta}) - \mathcal{J}(0) g(0) - \int_0^{\overline{\theta}} g(\theta) \, \mathrm{d}\mathcal{J}(\theta) + \mathcal{J}(0) g(0) \\
&\leq \mathcal{J}(\overline{\theta}) h(\overline{\theta}) - \mathcal{J}(0) h(0) - \int_0^{\overline{\theta}} h(\theta) \, \mathrm{d}\mathcal{J}(\theta) + \mathcal{J}(0) h(0) \\
&= \big[\mathcal{J}(\theta) h(\theta)\big]_{\theta=0}^{\theta=\overline{\theta}} - \int_0^{\overline{\theta}} h(\theta) \, \mathrm{d}\mathcal{J}(\theta) + \mathcal{J}(0) h(0) \\
&= \int_0^{\overline{\theta}} \mathcal{J}(\theta) h'(\theta) \, \mathrm{d}\theta + \mathcal{J}(0) h(0). \qquad \square
\end{aligned}$$

Proof of Theorem 1. Let g be an arbitrary feasible policy in the optimization problem (25). Define $\theta^\star = \inf\{\theta \colon \mathcal{J}(\theta) \geq 0\}$. As $\mathcal{J}$ is continuous, $\mathcal{J}(\theta^\star) = 0$. Let $h : [0, \overline{\theta}] \to \mathbb{R}$ be the solution to

$$\begin{aligned}
h'(\theta) &= \Phi\big(h(\theta), \theta\big) \\
h\big(\theta^\star\big) &= g\big(\theta^\star\big).
\end{aligned}$$

The proof proceeds in two steps: first we establish that h leads to a higher value of the integral (25) above $\theta^\star$; in the second step, we establish the analogous result below $\theta^\star$.

Step 1. As $g'(\theta) \leq \Psi(g(\theta), \theta)$, it follows from Lemma 4 that $g(\theta) \leq h(\theta)$ for $\theta \in [\theta^\star, \overline{\theta}]$ and $g(\theta) \geq h(\theta)$ for $\theta \in [a, \theta^\star]$ for every $a > 0$. As g and h are continuous, it follows that $g(0) \geq h(0)$. The monotonicity of Φ in the first variable implies that for $\theta \geq \theta^\star$,

$$g'(\theta) \leq \Phi\big(g(\theta), \theta\big) \leq \Phi\big(h(\theta), \theta\big) = h'(\theta).$$

As $\mathcal{J}(\theta^\star) = 0$ and $\theta \mapsto \min\{\mathcal{J}(\theta), 0\}$ is nondecreasing, we have that $\mathcal{J}(\theta) \geq 0$; for $\theta \geq \theta^\star$, we have that

$$\int_{\theta^\star}^{\overline{\theta}} \mathcal{J}(\theta) g'(\theta) \,\mathrm{d}\theta \leq \int_{\theta^\star}^{\overline{\theta}} \mathcal{J}(\theta) h'(\theta) \,\mathrm{d}\theta. \tag{35}$$

Step 2. Note that by Lemma 4, $g(\theta) \geq h(\theta)$ for $\theta \leq \theta^\star$. Furthermore, by definition of $\theta^\star$, we have that $\mathcal{J}(\theta) = \min\{\mathcal{J}(\theta), 0\}$ for $\theta \leq \theta^\star$. As $\theta \mapsto \min\{\mathcal{J}(\theta), 0\}$ is nondecreasing, $\mathcal{J}(\theta)$ is nondecreasing for $\theta \leq \theta^\star$. Lemma 5 implies that

$$\int_0^{\theta^\star} \mathcal{J}(\theta) g'(\theta) \,\mathrm{d}\theta + \mathcal{J}(0) g(0) \leq \int_0^{\theta^\star} \mathcal{J}(\theta) h'(\theta) \,\mathrm{d}\theta + \mathcal{J}(0) h(0). \tag{36}$$

Combining the inequalities (35) and (36) with the assumption that $\mathcal{J}(0) = -1$ yields that

$$\int_0^{\overline{\theta}} \mathcal{J}(\theta) g'(\theta) \,\mathrm{d}t - g(0) \leq \int_0^{\overline{\theta}} \mathcal{J}(\theta) h'(\theta) \,\mathrm{d}\theta - h(0).$$

As Φ is continuous in both variables, it follows that h is continuously differentiable, and, thus, feasible and an optimal policy. $\square$

Proof of Theorem 2. Define $\mathcal{J}(\theta) = J(\theta) f(\theta)$ and recall that $\theta^\circ = \min\{\theta : \mathcal{J}(\theta) = 0\}$. We first note that $\mathcal{J}(\theta)$ is negative for $\theta < \theta^\circ$ and $\mathcal{J}(0) = -1$. Consider the problem of solving

$$\begin{aligned} &\max_V \int_0^{\overline{\theta}} V'(z) \mathcal{J}(z) \,\mathrm{d}z - V(0) \\ &\text{subject to } V'(\theta) \leq \Psi\big(V(\theta), \theta\big) \quad \text{for all } \theta \in [\underline{\theta}_k, \overline{\theta}], \end{aligned}$$

where $\Psi(v, \theta) = \min\{\beta \frac{v}{\theta}, \frac{1}{r+\gamma}\}$. By Proposition 3, there exists an optimal policy that solves

$$V'(\theta) = \Psi(v, \theta). \tag{37}$$

We have that all solutions to the ODE (37) are of the form

$$V(\theta) = \begin{cases} \left(\dfrac{\theta}{\theta'}\right)^\beta V(\theta^\star) & \text{for } \theta \leq \theta^\star \\ V(\theta^\star) + \dfrac{\theta - \theta^\star}{r+\gamma} & \text{for } \theta \leq \theta^\star, \end{cases}$$

where $\frac{1}{r+\gamma} = V'(\theta^\star) = \frac{\beta}{\theta'} V(\theta^\star)$. Thus, plugging in $V(\theta^\star)$ yields that

$$V(\theta) = \begin{cases} \left(\dfrac{\theta}{\theta^\star}\right)^\beta \dfrac{\theta^\star/\beta}{r+\gamma} & \text{for } \theta \leq \theta^\star \\ \dfrac{\theta^\star/\beta}{r+\gamma} + \dfrac{\theta - \theta^\star}{r+\gamma} & \text{for } \theta \leq \theta^\star. \end{cases}$$

We note that $V \geqq 0$ and V' is increasing. It is thus feasible in the control problem (21)–(24) and we have hence found an optimal policy. $\square$

References

Ashlagi, Itai, Costis Daskalakis, and Nima Haghpanah (2016), "Sequential mechanisms with ex-post participation guarantees." In *ACM 2016 Conference on Economics and Computation*. [1010]

Baron, David and David Besanko (1984), "Regulation and information in a continuing relationship." *Information Economics and Policy*, 1, 267–302. [1009]

Battaglini, Marco (2005), "Long-term contracting with Markovian consumers." *American Economic Review*, 95, 637–658. [1009]

Bergemann, Dirk, Francisco Castro, and Gabriel Weintraub (2020), "The scope of sequential screening with ex-post participation constraints." *Journal of Economic Theory*, 188, 1–62. [1010]

Bergemann, Dirk and Philipp Strack (2015), "Dynamic revenue maximization: A continuous time approach." *Journal of Economic Theory*, 159, 819–853. [1008, 1009, 1015, 1016, 1017, 1030, 1031]

Bergemann, Dirk and Juuso Välimäki (2019), "Dynamic mechanism design: An introduction." *Journal of Economic Literature*, 57, 235–274. [1009]

Besanko, David (1985), "Multi-period contracts between principal and agent with adverse selection." *Economics Letters*, 17, 33–37. [1009]

Board, Simon (2008), "Durable-goods monopoly with varying demand." *Review of Economic Studies*, 75, 391–413. [1010, 1032]

Conlisk, John, Eitan Gerstner, and Joel Sobel (1984), "Cyclic pricing by a durable goods monopolist." *Quarterly Journal of Economics*, 99, 489–505. [1010]

Courty, Pascal and Hao Li (2000), "Sequential screening." *Review of Economic Studies*, 67, 697–717. [1009, 1010]

Deb, Rahul (2014), *Intertemporal Price Discrimination With Stochastic Values.* Technical report, University of Toronto. [1010]

Deb, Rahul and Maher Said (2015), "Dynamic screening with limited commitment." *Journal of Economic Theory*, 159, 891–928. [1010]

Dixit, Avinash and Robert Pindyck (1994), *Investment Under Uncertainty*. Princeton University Press, Princeton. [1018, 1019]

El Karoui, Nicole, Shige Peng, and Marie-Claire Quenez (1997), "Backward stochastic differential equations in finance." *Mathematical Finance*, 7, 1–71. [1024, 1025]

Eső, Peter and Balasz Szentes (2007), "Optimal information disclosure in auctions." *Review of Economic Studies*, 74, 705–731. [1009]

Evans, Lawrence (1983), *An Introduction to Mathematical Optimal Control Theory*. University of California. [1024]

Garrett, Daniel (2016), "Intertemporal price discrimination: Dynamic arrivals and changing values." *American Economic Review*, 106, 3275–3299. [1008, 1009, 1010]

Garrett, Daniel (2017), "Dynamic mechanism design: Dynamic arrivals and changing values." *Games and Economic Behavior*, 104, 595–612. [1010]

Gershkov, Alex, Benny Moldovanu, and Philipp Strack (2015), *Efficient Dynamic Allocation With Strategic Arrivals.* Technical report, Hebrew University. [1023]

Gershkov, Alex, Benny Moldovanu, and Philipp Strack (2018), "Revenue maximizing mechanisms with strategic customers and unknown Markovian demand." *Management Science*, 64, 2031–2046. [1023]

Judd, Kenneth (1985), "The law of large numbers with a continuum of i.i.d. random variables." *Journal of Economic Theory*, 35, 19–25. [1011]

Krähmer, Daniel and Roland Strausz (2015), "Ex post information rents in sequential screening." *Games and Economic Behavior*, 90, 257–273. [1010]

Kunita, Hiroshi (1997), *Stochastic Flows and Stochastic Differential Equations*. Cambridge University Press, Cambridge. [1015]

Lobel, Ilan and Renato Paes Leme (2019), "Dynamic mechanism design under positive commitment." In *Proceedings of Conference on Artificial Intelligence (AAAI)*, 2101–2108. [1010]

Mussa, Michael and Sherwin Rosen (1978), "Monopoly and product quality." *Journal of Economic Theory*, 18, 301–317. [1008]

Pavan, Alessandro, Ilya Segal, and Juuso Toikka (2014), "Dynamic mechanism design: A Myersonian approach." *Econometrica*, 82, 601–653. [1009, 1015]

Peng, Shige (1992), "Stochastic Hamilton–Jacobi–Bellman equations." *SIAM Journal on Control and Optimization*, 30, 284–304. [1024]

Peskir, Goran and Albert Shiryaev (2006), *Optimal Stopping and Free Boundary Problems*. Birkhauser Verlag, Basel. [1032]

Skreta, Vasiliki (2006), "Sequentially optimal auctions." *Review of Economic Studies*, 73, 1085–1111. [1010]

Skreta, Vasiliki (2015), "Optimal auction design under non-commitment." *Journal of Economic Theory*, 159, 854–890. [1010]

Sobel, Joel (1991), "Durable goods monopoly with entry of new consumers." *Econometrica*, 59, 1455–1485. [1010]

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