Collusion Enforcement in Repeated First-Price Auctions *

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September 21, 2021

Abstract

In the context of repeated first-price auctions, we explore how a bid-rigging cartel can simultaneously overcome the difficulty of soliciting truthful private information about valuations and the difficulty of enforcing its internal mechanism. Focusing on the class of trigger-strategy collusive agreements, we explicitly characterize the optimal collusive agreement for any given discount factor. Making use of the characterization, we also explore how a long-run seller can use a reserve price to fight the cartel.

Keywords: collusion, bidding rings, first-price auctions, repeated games
JEL codes: D82

1 Introduction

Soliciting truthful private information about preferences or cost conditions from its members and internally enforcing the collusive agreement are two of the main

*I am grateful to the coeditor and two anonymous referees for their advice. This research was funded by the National Natural Science Foundation of China (Grant No. 71973124).

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obstacles that a bid-rigging cartel faces. Despite a sizable literature has investigated how cartels overcome each of these two obstacles separately, we know very little about how cartels overcome these two obstacles simultaneously, and how the joint presence of the two obstacles limits the cartel profit and constrains the cartel behaviors.

In this paper, we address these issues in the context of repeated first-price auctions. As collusive agreements to fix price cannot be written and enforced by courts, repeated interactions allow the cartel to use the threat of switching to noncooperative bidding to provide incentives for compliance. Specifically, we consider a setup where in each period of an infinite horizon, there is an item up for sale via the first-price auction. There are n long-run buyers and their valuations for the items are private information, independently and identically distributed across the buyers and time periods. The buyers form an all-inclusive cartel that allows them to exchange the private information about valuations prior to the auction, to deliberate on the allocation and the bids to be submitted at the auction, and to make side-payments after the auction. However, the cartel cannot verify the reports, control the bids, or compel the buyers to make the payments. The cartel has to overcome the adverse-selection problem and the enforcement problem simultaneously.

Focusing on the class of trigger-strategy collusive agreements, we derive the optimal truth-telling and self-enforcing collusive agreement for any given discount factor. The optimal collusive agreement and the associated collusive profit are characterized by two thresholds of the discount factor of the buyers. If the discount factor is above the higher threshold, then the discounted future collusive profits are relatively large compared to the gains from cheating on the bids or from reneging on the payments in the current period. As a result, enforcement is not an issue and the first-best collusive outcome is achieved: In each period, the buyer with the highest valuation obtains the item and pays the minimum price at the seller's auction. At the other extreme, If the discount factor is below the lower threshold, then the enforcement constraints are so tight that the cartel cannot enforce any positive payments and any bid recommendations below the second-highest valuation. As a result, the buyers have zero collusive gain and they do not have strict incentives to form the cartel in the first place. If the discount factor is in between, the enforcement constraints are binding but the buyers nevertheless manage to have positive collusive gains; as the buyers become more patient, the enforcement constraints are less stringent, larger side-payments become credible, lower bids are placed at the seller's auction, and collusive gains are higher. We derive the explicit functional forms for the optimal self-enforcing bids and side-payments for any given discount factor.

In the setting of repeated interactions, it is often difficult to obtain an explicit description of the optimal equilibrium for a given discount factor. As a result, researchers have to focus on the limiting case where all agents become infinitely patient and on the folk-theorem-type results. Unfortunately, the information obtained in the limiting case cannot be used to address issues such as comparative statics and optimal policy/intervention for the case of moderate discount factors. In this paper, the explicit characterization of the optimal collusive agreement allows us to investigate how the change in the discount factor impacts the cartel behaviors and the cartel profit.

In particular, we exploit the explicit characterization of the optimal collusive agreement by exploring how a long-run seller can use a reserve price to fight the cartel. We suppose that the seller is aware of the functioning of the optimal collusive agreement and he decides to impose a reserve price that is constant in all future periods. He anticipates that the cartel will respond optimally to the reserve price by revising the collusive agreement accordingly. We explicitly characterize the optimal reserve price that maximizes the payoff to the seller. Interestingly, we find that the optimal reserve price depends on the discount factor, and plays very different roles for different discount factors.

An important implication of this analysis for antitrust practice is that the reserve price is very effective in fighting collusion, and the optimal reserve price is non-monotone in the value of the discount factor. The seller optimally trades off between deterring effective collusion using a more aggressive but also more costly reserve price and accommodating effective collusion with a lower but less costly reserve price.

1.1 Literature

Abstracting away the issue of enforcement by assuming that cartels are able to control the bids of the buyers and enforce any side-payments, one strand of the literature on collusion in auctions has focused on the issue of soliciting private information about valuations. Graham and Marshall (1987) and Mailath and Zemsky (1991) consider the second-price auction. Graham and Marshall (1987) show that with the aid of an external risk-neutral budget-breaker, the cartel can achieve efficient collusion by having a second-price pre-auction knockout. Mailath and Zemsky (1991) show that efficient collusion can be achieved even when the buyers are *ex ante* heterogeneous. For the first-price auction, McAfee and McMillan (1992) show that with perfect enforcement there exists a cartel mechanism that achieves the first-best collusive outcome. Hendricks, Porter, and Tan (2008) consider the first-price auction with affiliated private value and common value.

Relaxing the assumption of perfect enforcement, McAfee and McMillan (1992) also consider cartels at the first-price auction that are unable to enforce any side-payments but are still able to control the bids of the cartel members (which they call weak cartels). They show that in this setup the optimal cartel mechanism is to have the buyers submit identical bids. Marshall and Marx (2007) and Lopomo, Marx, and Sun (2011) consider the mirror situation where cartels are not able to control the bids of the buyers but are able to enforce any amount of *ex ante* budget-balanced side-payments. Marshall and Marx (2007) show that in this case the first-price auction is less vulnerable to collusion than the second-price auction is. Lopomo, Marx, and Sun (2011) further show that in the case of two buyers and two possible valuations, the buyers at the first-price auction cannot do better than bidding noncooperatively. In contrast to the assumptions made in this literature, a key point of our paper is to understand how the ability of a cartel to control the bids and enforce side-payments is endogenously generated.

Abstracting away the issue of adverse selection by assuming that valuations are public information, another strand of the literature has focused on the issue of enforcement. Robinson (1985) is the first to point out that cartels at auctions have the enforcement challenge and, as a result, the first-price auction is less vulnerable to collusion than the second-price or ascending-price auction is. In a closely related paper, Chassang and Ortner (2019) consider collusion enforcement in the context of repeated government procurements, and empirically test the implications of binding enforcement constraints on the cartel behaviors using data from paving auctions in the Japanese cities of Ibaraki prefecture. In contrast to the assumption that valuations or firms' costs are public information, in our model, the cartel has to simultaneously solicit private information and enforce its internal mechanism, which is the main source of difficulty.

The setting of repeated auctions also allows one to explore what a cartel can achieve when the cartel members can not communicate with each other or can not make explicit side-payments. When communications are allowed but not side-payments, Aoyagi (2003) shows that a version of dynamic bid rotation improves upon the simple bid rotation mechanism: The buyers communicate their valuations and allocate the first period of each rotation phase to the high-valuation buyer. Aoyagi (2007) provides sufficient conditions for the cartel to extract all surplus when the buyers' valuations are finite. When neither communications nor side-payments are allowed, Skrzypacz and Hopenhayn (2004) and Blume and Heidhues (2006) show that with repeated interactions the cartel is still able to do better than bid rotation and noncooperative bidding. The intuition is that asymmetric continuation payoffs (resulting from temporary exclusions of some buyers) serve as implicit side-payments. Rachmilevitch (2013) devises a more elaborate bid rotation mechanism with two buyers that improves the cartel profit: If buyer 1 obtains the item in the current period, then, instead of buyer 2 obtaining the item for sure in the next period, buyer 2 does so only when his valuation is above a certain threshold. Rachmilevitch (2014) and Hörner and Jamison (2007) provide sufficient conditions for the cartel to extract all surplus when the buyers' valuations are finite. Athey and Bagwell (2001), Athey, Bagwell, and Sanchirico (2004), Athey and Bagwell (2008) study collusion in repeated Bertrand oligopoly game with discrete types.

Finally, several authors have also studied how to set the optimal reserve price in different contexts. In his classic work, Myerson (1981) shows that reserve price emerges as the optimal way for the seller to extract the rent in the setting of static auctions and there is no collusion. McAfee and McMillan (1992) study the optimal reserve price in the presence of strong cartels. The optimal rent-extraction reserve price in Myerson (1981) and the optimal reserve price to counter strong cartels in McAfee and McMillan (1992) correspond to, respectively, the optimal reserve price in our dynamic context when the discount factor is sufficiently small, and that when the discount factor is sufficiently close to 1. We uncover new roles of the reserve price for the discount factors in between. In the setting of repeated procurements with two buyers in separate markets and two suppliers that can serve both markets, Iossa, Loertscher, Marx, and Rey (2020) compare whether it is more difficult to sustain and initiate collusion when the two buyers hold staggered procurements or when they hold synchronized procurements. Interestingly, in that context they also find that the optimal reserve price is non-monotone in the discount factor and the buyers face the tradeoff between collusion deterrence with a more aggressive reserve price and collusion accommodation with a less costly reserve price.

The rest of this paper is organized as follows. We present the model of collusion enforcement in the next section, introduce the class of trigger-strategy collusive agreements in Section 3, and derive the optimal collusive agreement in Section 4. Making use of the explicit optimal collusive agreement, we characterize the seller's optimal reserve price in Section 5.

2 A Model of Collusion Enforcement

2.1 Collusion in repeated first-price auctions

We model a bid-rigging cartel facing two obstacles to collusion: private information and enforcement. In each period $t = 1, 2, \ldots,$ there is an item up for sale by the sealed-bid first-price auction. There are n long-run buyers, denoted by $i = 1, 2, \ldots, n$. The buyers' valuations for the items are private information. Let $v_i(t)$

denote buyer i's valuation for the item sold in period t. It is common knowledge that the valuations are drawn according to a common distribution, independently across the buyers and time periods, but each buyer only observes his own valuation for the item sold in each period at the beginning of that period. Assume that the common distribution has a cumulative distribution function F and a continuous, positive density function f with support $[0, \overline{v}]$.

To ease exposition, we assume that at the auction in each period t, each buyer i simultaneously submits a bid $b_i(t) \in [0, \infty)$ and a complementary bid $\kappa_i(t) \in \{i, i+n\}$, and all bids are announced. If $(b_i(t), \kappa_i(t)) > (b_j(t), \kappa_j(t))$, that is, either $b_i(t) > b_j(t)$, or $b_i(t) = b_j(t)$ and $\kappa_i(t) > \kappa_j(t)$, then we say that buyer i's bid is higher than that of buyer j. If buyer i's bid is the highest one, then he wins the item and pays the seller $b_i(t)$. The complementary bids allow us to express the idea that a buyer outbids the others by a "small" amount, without specifying the details caused by this small amount. The fact that the complementary bids allow no ties and the assumption that the bids are public information simplify some of the arguments about incentives.[1]

The buyers form an all-inclusive cartel that allows them to exchange the private information about valuations prior to the auction, to decide on the allocation and the bids to be submitted at the auction, and to make side-payments after the auction. Let $\widehat{v}_i(t) \in [0, \overline{v}]$ denote buyer i's report of his valuation and $\tau_{ij}(t) \in [0, \infty)$ denote his payment to buyer j in period t. For convenience, set $\tau_{ii}(t) = 0$ for all $i = 1, 2, \ldots, n$ and all $t \geq 1$. However, the cartel cannot verify the reports, control the bids, or compel the cartel members to make the payments. The central issue we address in this paper is how the cartel overcomes these difficulties simultaneously.

To summarize, the events in each period t unfold as follows. First, each buyer i learns his own valuation $v_i(t)$. Second, each buyer i simultaneously reports $\widehat{v}_i(t)$ to each other. Third, on the basis of the reports and his own valuation, each buyer i simultaneously submits a bid-pair $(b_i(t), \kappa_i(t))$ at the auction. Fourth, the seller

[1] For our purpose, it suffices to assume that the identity of the winner is announced, which is often the case by law in government procurement auctions.

announces all the bids publically. Fifth, the winner obtains the item and pays the seller the winning bid. Lastly, on the basis of the reports and the announcement, each buyer i pays each buyer j an amount $\tau_{ij}(t)$.

Buyer i's payoff in period t is, therefore, equal to

$$(v_i(t) - b_i(t))\, x_i(t) + \sum_{j=1}^{n} (\tau_{ji}(t) - \tau_{ij}(t)),$$

where $x_i(t)$ is the indicator function that equals 1 if buyer i wins the item in period t and equals 0 otherwise. The buyers discount future payoffs by a common factor $\delta < 1$. The normalized repeated-game payoff to buyer i is

$$(1 - \delta) \sum_{t=1}^{\infty} \delta^{t-1} \left[(v_i(t) - b_i(t))\, x_i(t) + \sum_{j=1}^{n} (\tau_{ji}(t) - \tau_{ij}(t)) \right].$$

Remark 1. In our model the cartel members settle their internal payments after the auctions rather than before the auctions. This modeling strategy is intended to capture the reality that cartel members often need to fund the payments from auction proceeds or through subcontracting arrangements. For example, in a conspiracy to rig the bidding on a project involving the relocation of National Weather Service facilities at Raleigh-Durham Airport, A-A-A, a North Carolina electrical construction contractor and the designated winner, submitted the lowest bid and was awarded the contract on July 5, 1979, but paid off the other conspirators in May, 1980, after it performed the contract and received the final payment for its work.[2] In a conspiracy to rig the bidding on contracts to repave portions of Interstate 35 in Oklahoma, Broce Construction Company, after winning all the four jobs, subcontracted the paving of the northernmost sections to Metropolitan Enterprises, Inc., in return for "protecting" its bids, as they agreed on in a meeting before the auction.[3]

2.2 Self-enforcing collusive agreements

The formation of a cartel merely provides the possibility of collusion. It allows the buyers to communicate and make payments, but does not guarantee that the

[2] *United States v. A-A-A Elec. Co., Inc. (788 F.2d 242; 4th Cir. 1986)*

[3] *United States v. Metropolitan Enterprises, Inc. (728 F.2d 444; 10th Cir. 1984)*

reports will be truthful, the recommendations will be followed, or the promised payments will be honored. If the interaction is only one-shot, the buyers will not have incentives to comply. With repeated interactions, the cartel can condition future collusion on what the buyers have done in the past, thereby creating incentives for compliance. Formally, a collusive agreement is self-enforcing if it forms a perfect public equilibrium of the repeated game, which we now describe.

In period t, each buyer must choose a report, a bid-pair, and a profile of side-payments. A stage-game reporting strategy for buyer i is a function $\rho_i : [0, \overline{v}] \to [0, \overline{v}]$ that maps buyer i's valuation $v_i(t)$ into a report $\widehat{v}_i(t)$. A stage-game bidding strategy for buyer i is a pair of functions $(\beta_i, \gamma_i) : [0, \overline{v}]^n \to [0, \infty) \times \{i, i+n\}$ that maps his own valuation $v_i(t)$ and the reports of the other buyers, denoted by $\widehat{v}_{-i}(t)$, into a pair of bids $(b_i(t), \kappa_i(t))$. A stage-game side-payment strategy for buyer i is a profile of functions $\varphi_i = (\varphi_{i1}, \ldots, \varphi_{in})$, where each

$$\varphi_{ij} : [0, \overline{v}]^n \times \prod_{k \neq i} ([0, \infty) \times \{k, k+n\}) \to [0, \infty)$$

maps his own valuation $v_i(t)$, the reports of the other buyers $\widehat{v}_{-i}(t)$, and the bids of the other buyers $(b_{-i}(t), \kappa_{-i}(t))$ into a side-payment $\tau_{ij}(t)$. A stage-game strategy for buyer i is a quadruple $\alpha_i \equiv (\rho_i, \beta_i, \gamma_i, \varphi_i)$. We use $\alpha \equiv (\alpha_1, \ldots, \alpha_n)$ to denote a stage-game strategy profile.

For each variable $z = v, \widehat{v}, b, \kappa$, we let $z(t) \equiv (z_1(t), \ldots, z_n(t))$ denote the value of $(z_1, \ldots, z_n)$ in period t and let $\tau(t)$ denote the value of $(\tau_{ij} : i, j = 1, \ldots, n)$ in period t. A history of the game is an infinite sequence

$$(v(1), \widehat{v}(1), b(1), \kappa(1), \tau(1), \ldots, v(t), \widehat{v}(t), b(t), \kappa(t), \tau(t), \ldots).$$

At the beginning of period t, buyer i has observed a public history h_t that consists of the reports of valuations, the bids, and the side-payments in the first $(t-1)$ periods. In addition, he has observed a private history $h_{i,t}$ that consists of his own valuations in the first $(t-1)$ periods. Buyer i's information at the beginning of period t includes both h_t and $h_{i,t}$. A repeated-game strategy for buyer i, denoted by σ_i, maps buyer i's information at the beginning of each period t into a stage-game strategy. A repeated-game strategy for a buyer is a public strategy if its stage-game strategy in any period depends only on the public history up to that period. A profile of strategies $\sigma = (\sigma_1, \ldots, \sigma_n)$ is a perfect public equilibrium if the strategy for each buyer is public and the continuation strategies after any public history form a Nash equilibrium in the continuation game.

3 Trigger-Strategy Collusive Agreements

A trigger-strategy collusive agreement is characterized by two states, a collusive one and a noncollusive one. The equilibrium starts off in the collusive state and remains in the collusive state unless some buyer deviates publically. There are two types of deviations that can be publically observed: a buyer submits a bid inconsistent with the equilibrium bidding strategy or refuses to make the required payments. If a publically observable deviation occurs, then the equilibrium switches to the noncollusive state. The noncollusive state is absorbing. In the noncollusive state, the buyers always choose the stage-game strategy profile α^{N} defined as follows. In the first-price auction with n buyers, each buyer i with valuation v_i bids

$$\beta_i^{\mathrm{N}}(v_i) \equiv v_i - \frac{\int_0^{v_i} [F(s)]^{n-1}\,\mathrm{d}s}{[F(v_i)]^{n-1}}$$

is a Bayes-Nash equilibrium.[4] Hence, the strategy profile $\alpha^{\mathrm{N}} \equiv (\rho^{\mathrm{N}}, \beta^{\mathrm{N}}, \gamma^{\mathrm{N}}, \varphi^{\mathrm{N}})$, where each buyer i always reports the constant valuation 0, bids $(\beta_i^{\mathrm{N}}(v_i), i+n)$ if his valuation is v_i, and makes no payments, is self-enforcing.

We shall only consider collusive agreements in which the buyers are truth-telling in the collusive state. Let $\rho^* = (\rho_1^*, \ldots, \rho_n^*)$ denote the profile of truth-telling stage-game reporting strategies. In the standard setup of mechanism-design problems, there is a mechanism designer who can enforce the final allocation on the basis of the reports of the agents. In contrast, in our collusion setup, the final allocation is determined by the bids of the buyers at the seller's auction, the transfer payments are voluntary, and the ability of the cartel to enforce the final allocation is endogenously generated. Hence, we can not simply use the revelation

[4]See, e.g., Theorem 4.6 of Milgrom (2004).

principle to argue that it is without loss of generality to focus on the truth-telling stage-game strategy profiles.

For tractability, we focus on the following class of trigger-strategy collusive agreements. We say that a trigger-strategy collusive agreement is stationary if the buyers repeat a constant stage-game strategy profile in the collusive state. For any stage-game strategy profile α, we let $\pi_i(\alpha)$ denote buyer i's expected stage-game payoff under α and let $\pi(\alpha) \equiv \sum_{i=1}^{n} \pi_i(\alpha)$ denote the total cartel payoff. We say that a stage-game strategy profile α is payoff-symmetric if each buyer receives the same expected payoff under α, i.e.,

$$\pi_1(\alpha) = \pi_2(\alpha) = \cdots = \pi_n(\alpha).$$

Assumption 1. We consider the following trigger-strategy collusive agreements.

(a) The collusive agreement is stationary. The buyers repeats a stage-game strategy profile α in the collusive state.

(b) The stage-game strategy profile α is payoff-symmetric.

(c) Under α, only the buyer with the highest bid at the seller's auction pays the other cartel members and he pays them equally whenever he needs to pay.

Remark 2. The restriction that only the designated winner pays reflects the fact that in many cases such as the ones in the construction industry in Remark 1, the transfer payments have to be funded from the proceeds of the project or through subcontracting. The restriction of equal payment is mainly for tractability. We hope that our study provides a starting point towards a more complete investigation on cartels facing the adverse-selection problem and the enforcement problem simultaneously.

Remark 3. Assumption 1 is not needed for our results if there are only two buyers. See Appendix B.

Our aim is to characterize the trigger-strategy collusive agreement that is optimal from the cartel's point of view, i.e., the collusive agreement that maximizes the total payoff of the buyers.

In the remainder of this section, we translate the requirement that a stationary collusive agreement in which a truth-telling stage-game strategy profile α is played in the collusive state be self-enforcing into requirements on α. Specifically, there are three requirements that α needs to satisfy: It needs to (i) induce truthful revelation of valuations, (ii) enforce the bid recommendations, and (iii) enforce the payments. We consider each of these three requirements in turn.

We start with the requirement of inducing the buyers to reveal their valuations truthfully. Conditional on any draw of valuations $v = (v_1, \ldots, v_n)$, under α, buyer i's probability of winning the item, his net transfer, and his total payoff in the stage game are, respectively,

$$\begin{aligned}
q_i^\alpha(v) &\equiv \begin{cases} 1, & \text{if } (\beta_i(v), \gamma_i(v)) > (\beta_j(v), \gamma_j(v)) \text{ for all } j \neq i \\ 0, & \text{otherwise} \end{cases} \\
\chi_i^\alpha(v) &\equiv \sum_{j=1}^{n} [\varphi_{ji}(v, \beta(v), \gamma(v)) - \varphi_{ij}(v, \beta(v), \gamma(v))] \\
u_i^\alpha(v) &\equiv (v_i - \beta_i(v))\, q_i^\alpha(v) + \chi_i^\alpha(v).
\end{aligned}$$

Hence, buyer i's interim expected probability of winning and interim expected payoff are, respectively,

$$\begin{aligned}
Q_i^\alpha(v_i) &\equiv E_{v_{-i}}[q_i^\alpha(v_i, v_{-i})] \\
U_i^\alpha(v_i) &\equiv E_{v_{-i}}[u_i^\alpha(v_i, v_{-i})].
\end{aligned}$$

Under a stationary collusive agreement, the reports of valuations only affect the payoffs in the stage game. Hence, by standard argument, buyer i is willing to report his valuation truthfully if and only if

$$U_i^\alpha(v_i) = U_i^\alpha(0) + \int_0^{v_i} Q_i^\alpha(x)\, \mathrm{d}x \quad \text{for each } v_i \in [0, \overline{v}]\,, \tag{1}$$

and

$$Q_i^\alpha \text{ is nondecreasing.} \tag{2}$$

Next we turn to the issue of enforcement. Suppose that $(\beta_i(v), \gamma_i(v)) > (\beta_j(v), \gamma_j(v))$ for all $j \neq i$. That is, the cartel mechanism allocates the right to win the item at the seller's auction to buyer i. We examine the buyers' incentives to follow these bid recommendations. There are two types of deviations buyer i can choose. If he chooses a deviation $(b_i, \kappa_i) > \max_{j \neq i}(\beta_j(v), \gamma_j(v))$, then his stage-game payoff is $v_i - b_i$. The supremum of his stage-game payoff from a deviation $(b_i, \kappa_i) > \max_{j \neq i}(\beta_j(v), \gamma_j(v))$ is, therefore, equal to

$$\sup_{(b_i, \kappa_i) > \max_{j \neq i}(\beta_j(v), \gamma_j(v))} (v_i - b_i) = v_i - \max_{j \neq i} \beta_j(v).$$

If he chooses a deviation $(b_i, \kappa_i) < \max_{j \neq i}(\beta_j(v), \gamma_j(v))$, then his stage-game payoff is 0. Hence, he will follow the bid recommendation if and only if

$$(1-\delta) \max\left(v_i - \max_{j \neq i} \beta_j(v), 0\right) + \delta \pi_i(\alpha^{\mathrm{N}}) \leq (1-\delta)\left(v_i - \beta_i(v) + \chi_i^{\alpha}(v)\right) + \delta \pi_i(\alpha). \tag{3}$$

The left-hand side of (3) is the supremum of the payoffs from deviating while the right-hand side is the payoff from following the equilibrium strategy. The first and the second terms on each side are the respective stage-game payoffs and continuation payoffs. Similarly, each buyer $j \neq i$ will follow the bid recommendation if and only if

$$(1-\delta) \max\left(v_j - \beta_i(v), 0\right) + \delta \pi_j(\alpha^{\mathrm{N}}) \leq (1-\delta)\chi_j^{\alpha}(v) + \delta \pi_j(\alpha). \tag{4}$$

Finally, it is optimal for each buyer to make the payments if and only if, for all v and all $i = 1, \ldots, n$,

$$(1-\delta) \sum_{j=1}^{n} \varphi_{ji}(v, \beta(v), \gamma(v)) + \delta \pi_i(\alpha^{\mathrm{N}}) \leq (1-\delta)\chi_i^{\alpha}(v) + \delta \pi_i(\alpha). \tag{5}$$

The left-hand side of (5) is buyer i's continuation payoff from reneging on the payments $\varphi_{ij}(v, \beta(v), \gamma(v))$: he receives the payment $\varphi_{ji}(v, \beta(v), \gamma(v))$ from each buyer j, gains from not paying $\varphi_{ij}(v, \beta(v), \gamma(v))$, and receives the payoff from noncooperative bidding $\pi_i(\alpha^{\mathrm{N}})$ starting from the next period. We assume that the payments from the other buyers are always accepted. As above, the right-hand side of (5) is his continuation payoff from making the required payments.

We summarize these requirements in Lemma 1.

Lemma 1. *A stationary trigger-strategy profile in which a stage-game strategy profile $\alpha = (\rho^*, \beta, \gamma, \varphi)$ is repeated in each period in the collusive state is a perfect public equilibrium if and only if the constraints* (1)–(5) *are satisfied.*

4 Optimal Collusion

By Lemma 1, the task of designing an optimal self-enforcing collusive agreement is reduced to designing a truth-telling stage-game strategy profile α to maximize the cartel payoff $\pi(\alpha)$, subject to the revelation and enforcement constraints (1)–(5), i.e.,

$$\sup_{\alpha} \pi(\alpha) \qquad \text{(Program I)}$$
$$\text{subject to: (1)–(5).}$$

Since the choice variable for the program–the stage-game strategy profile–is a profile of functions, Program I belongs to the class of infinite-dimensional optimization problems, which are difficult to solve directly. We solve Program I in two steps. In Step 1, we derive an upper bound for the value of Program I. We first identify a set of key enforcement constraints, and then solve the relaxed program that involves only these constraints. In Step 2, we identify a candidate strategy profile for Program I and show that it attains the upper bound. It follows that the candidate must be optimal and the upper bound must be the value of the program.

4.1 Step 1: An upper bound on the cartel payoff

We define two benchmarks on the cartel payoff under a stage-game strategy profile. Let

$$\underline{\Pi} \equiv \pi(\alpha^{\mathrm{N}})$$

denote the cartel payoff under noncooperative bidding. Let

$$\overline{\Pi} \equiv E[\max(v_1, \ldots, v_n)]$$

denote the cartel payoff under the first-best collusive outcome: the buyer with the highest valuation obtains the item and pays the minimum bid of 0 at the seller's auction. Hence, $\pi(\alpha) \leq \overline{\Pi}$ for any stage-game strategy profile α.

Suppose that the cartel payoff from a trigger-strategy perfect public equilibrium is $\Pi \in [\underline{\Pi}, \overline{\Pi}]$, and the equilibrium switches to the noncollusive state. Then the loss in the cartel payoff in all future periods, evaluated from the current period, is

$$L(\Pi, \delta) \equiv \delta(1-\delta)^{-1} (\Pi - \underline{\Pi}) . \tag{6}$$

In other words, $L(\Pi, \delta)$ measures the present value of collusion.

Our derivation of the upper bound starts with the following key observation. For any profile of valuations $v = (v_1, \ldots, v_n)$, let $v_{(1)}$ and $v_{(2)}$ denote, respectively the highest and the second-highest valuations among them.

Lemma 2. *Suppose that α is a candidate for Program I that satisfies Assumptions 1b and 1c. For any profile of valuations $v = (v_1, \ldots, v_n)$, if $(\beta_i(v), \gamma_i(v)) > (\beta_j(v), \gamma_j(v))$ for all $j \neq i$, then*

$$\beta_i(v) \geq \max\left(v_{(2)} - \frac{L(\pi(\alpha), \delta)}{n-1}, 0\right). \tag{7}$$

Proof. The enforcement constraints (4) and (5) imply, respectively, for any $j \neq i$,

$$(1-\delta)\,(v_j - \beta_i(v)) + \delta\pi_j(\alpha^{\mathrm{N}}) \quad \leq \quad (1-\delta)\chi_j^{\alpha}(v) + \delta\pi_j(\alpha) \tag{8}$$

$$\delta\pi_i(\alpha^{\mathrm{N}}) \quad \leq \quad (1-\delta)\chi_i^{\alpha}(v) + \delta\pi_i(\alpha). \tag{9}$$

By Assumption 1c that the designated winner pays the co-conspirators equally whenever he needs to pay, we have

$$\chi_i^{\alpha}(v) = -(n-1)\chi_j^{\alpha}(v). \tag{10}$$

Substituting (10) into (9) for $\chi_i^{\alpha}(v)$, and dividing both sides by $(n-1)$, we have

$$\frac{1}{n-1}\delta\pi_i(\alpha^{\mathrm{N}}) \leq -(1-\delta)\chi_j^{\alpha}(v) + \frac{1}{n-1}\delta\pi_i(\alpha). \tag{11}$$

By Assumption 1b that α is payoff-symmetric, we have

$$\delta(\pi_i(\alpha) - \pi_i(\alpha^{\mathrm{N}})) = \delta(\pi_j(\alpha) - \pi_j(\alpha^{\mathrm{N}})) = (1-\delta)\frac{L(\pi(\alpha), \delta)}{n}. \tag{12}$$

Summing (8) and (11), and simplifying terms using (12), yields

$$\beta_i(v) \geq v_j - \frac{L(\pi(\alpha), \delta)}{n-1}. \tag{13}$$

This completes the proof as bids must be nonnegative. $\square$

Lemma 2 identifies a lower bound on the price paid to the seller. To apply Lemma 2, we decompose the cartel payoff $\pi(\alpha)$ into two parts,

$$\begin{aligned}\pi(\alpha) = &\int_0^{\overline{v}} \cdots \int_0^{\overline{v}} \sum_{i=1}^{n} v_i q_i^{\alpha}(v)\, \mathrm{d}F(v_1) \cdots \mathrm{d}F(v_n) \\ &- \int_0^{\overline{v}} \cdots \int_0^{\overline{v}} \max(\beta_1(v), \ldots, \beta_n(v))\, \mathrm{d}F(v_1) \cdots \mathrm{d}F(v_n).\end{aligned} \tag{14}$$

The first term on the right-hand side of (14) is the buyers' payoffs from obtaining the item. The second term is the price paid to the seller. Hence, the level of the cartel payoff depends on the extent to which the cartel can correctly allocate the item to the buyer with the highest valuation and on the extent to which it can suppress the price paid to the seller. Using the payoff from the efficient allocation as an upper bound for the first term, and the bound in Lemma 2 for the second term, we obtain an upper bound on the cartel payoff. This motivates the following definition and Lemma 3.

For any $\Pi \in [\underline{\Pi}, \overline{\Pi}]$, we define

$$g(\Pi) \equiv \overline{\Pi} - E\left[\max\left(v_{(2)} - \frac{L(\Pi, \delta)}{n-1}, 0\right)\right], \tag{15}$$

which captures the cartel payoff under a stage-game mechanism that (i) always induces the efficient allocation and (ii) pays the seller $\max(v_{(2)} - \frac{L(\Pi,\delta)}{n-1}, 0)$.

Lemma 3. *Suppose that Assumptions 1b and 1c hold. If α is a candidate for Program I, then $\pi(\alpha) \leq g(\pi(\alpha))$.*

Proof. It follows directly from applying Lemma 2 to (14) and (15). $\square$

Let

$$\Pi^* \equiv \sup\{\Pi \in [\underline{\Pi}, \overline{\Pi}] \mid \Pi \leq g(\Pi)\}. \tag{Program I$'$}$$

By Lemma 3, Π^* is an upper bound for the value of Program I. Examining the proofs of Lemmas 2 and 3 reveals that Program I$'$ relaxes Program I as it does not involve the revelation constraint and the other instances of the enforcement constraints except (8) and (9).

We characterize Π^* in Lemma 4.

Lemma 4. *(i) If $\delta \in (0, \frac{n-1}{n}]$, then $\Pi^* = \underline{\Pi}$.*

(ii) If $\delta \in (\frac{n-1}{n}, \frac{(n-1)\overline{v}}{(n-1)\overline{v}+\overline{\Pi}-\underline{\Pi}})$, then Π^* is the unique fixed point of g in $(\underline{\Pi}, \overline{\Pi})$, i.e., $\Pi^* = g(\Pi^*)$.

(iii) If $\delta \in [\frac{(n-1)\overline{v}}{(n-1)\overline{v}+\overline{\Pi}-\underline{\Pi}}, 1)$, then $\Pi^* = \overline{\Pi}$.

We shall prove a more general version of Lemma 4 in Theorem 3, in which the seller adopts a reserve price.

4.2 Step 2: An optimal stage-game mechanism

We now introduce a candidate stage-game strategy profile $\alpha^* = (\rho^*, \beta^*, \gamma^*, \varphi^*)$ that attains the upper bound Π^*. For each $M > 0$, we define an auxiliary function H_M on $[0, \overline{v}]$ by letting, for each $w \in [0, \overline{v}]$,

$$H_M(w) \equiv \begin{cases} \frac{1}{n}M, & \text{if } w > M \\ \frac{1}{n}\left(w + \frac{\int_w^M [1-F(s)]^n \, \mathrm{d}s}{[1-F(w)]^n}\right), & \text{if } w \leq M. \end{cases} \tag{16}$$

For any $\widehat{v} = (\widehat{v}_1, \ldots, \widehat{v}_n)$, let ℓ be the buyer who has the highest reported valuation. In the case of ties, let ℓ be the smallest integer i such that buyer i's reported valuation is among the highest ones. (Thus, ℓ is formally a function that maps each profile of reports into a buyer.) Define

$$\begin{aligned}
\beta_1^*(\widehat{v}) = \cdots = \beta_n^*(\widehat{v}) &\equiv \max\left(\widehat{v}_{(2)} - \frac{L(\Pi^*, \delta)}{n-1}, 0\right) \\
\gamma_\ell^*(\widehat{v}) &\equiv \ell + n \\
\gamma_j^*(\widehat{v}) &\equiv j \ \text{ for all } j \neq \ell \\
\varphi_{\ell j}^*(\widehat{v}, b, \kappa) &\equiv \begin{cases} H_{\frac{L(\Pi^*, \delta)}{n-1}}(\widehat{v}_{(2)}), & \text{if } (b, \kappa) = (\beta^*(\widehat{v}), \gamma^*(\widehat{v})) \text{ and } j \neq \ell \\ 0, & \text{otherwise} \end{cases} \\
\varphi_{jk}^*(\widehat{v}, b, \kappa) &\equiv 0 \ \text{ for all } j \neq \ell, k, \text{and } (b, \kappa).
\end{aligned}$$

Under α^*, the cartel mechanism allocates the right to win the item at the auction to the buyer who claims the highest valuation. In the case of ties, it allocates the item to the buyer who has the smallest index among them. Hence, the cartel mechanism is efficient. The designated winner is recommended to bid the difference between the second-highest reported valuation and $\frac{L(\Pi^*,\delta)}{n-1}$, whenever it is positive. All the other buyers are recommended to place a bid just below the winner's bid. There are no transfer payments if any buyer deviates from the recommendations; otherwise, the winner compensates all the other buyers according to $H_{\frac{L(\Pi^*,\delta)}{n-1}}$.

We show that the stage-game strategy profile $\alpha^* = (\rho^*, \beta^*, \gamma^*, \varphi^*)$ satisfies the constraints (1)–(5) of Program I and attains the upper bound Π^*. Combined with Lemmas 1 and 3, this implies the following characterization of the optimal trigger-strategy perfect public equilibrium.

Theorem 1. *The trigger-strategy profile in which α^* is played in every period in the collusive state is a perfect public equilibrium. It is optimal among all trigger-strategy perfect public equilibria that satisfy Assumption 1. The equilibrium payoff of the cartel is* Π^*.

Proof. See Appendix D. □

The need to simultaneously overcome the adverse-selection problem and the enforcement problem shapes the optimal collusive mechanism α^* in two ways. First, the cartel profits by suppressing the bids below the second-highest valuation, but the extent to which it can suppress the bids is at most $\frac{L(\Pi^*,\delta)}{n-1}$. Second, the transfer payments between the cartel members take the form of a specific function $H_{\frac{L(\Pi^*,\delta)}{n-1}}$. We have explained through Lemma 2 how the enforcement constraints determine the maximum extent of bid suppressions $\frac{L(\Pi^*,\delta)}{n-1}$. The reason that the losing buyers are required to place bids just below the winning bid is to prevent the designated winner from lowering his bids. Below we explain in three steps why the transfer payments take the particular functional form $H_{\frac{L(\Pi^*,\delta)}{n-1}}$.

First, the transfer payments arise from the need to induce truthful revelation of valuations. The revelation constraints (1) and (2) show that the transfer payments are determined by both the allocation of the item and the price paid to the seller. Since the equilibrium allocation of the item is efficient, the price paid to the seller $\beta_\ell(\widehat{v})$ alone determines the transfer payments through the constraints (1) and (2).

Second, by the payment-enforcement constraint, the total transfer payment that the designated winner can credibly promised is $\frac{L(\Pi^*,\delta)}{n}$. Transfer payments in the form of $H_{\frac{L(\Pi^*,\delta)}{n-1}}$ is the most "economical" way to arrange the payments in terms of minimizing the tightness of the payment-enforcement constraint. This is best illustrated by the case that the payment-enforcement constraint is *just* binding, which happens if $\delta = \frac{(n-1)\overline{v}}{(n-1)\overline{v}+\overline{\Pi}-\underline{\Pi}}$. Note that in this case $\frac{L(\Pi^*,\delta)}{n-1} = \overline{v}$ and the price paid to the seller is always 0. The payment from the designated winner to each co-conspirator becomes

$$H_{\overline{v}}(\widehat{v}_{(2)}) = \frac{1}{n}\left(\widehat{v}_{(2)} + \frac{\int_{\widehat{v}_{(2)}}^{\overline{v}}[1-F(s)]^n\,\mathrm{d}s}{[1-F(\widehat{v}_{(2)})]^n}\right) \quad \text{for all } \widehat{v}_{(2)}. \tag{17}$$

The total transfer payment that the designated winner has to make attains the maximum at $\widehat{v}_{(2)} = \overline{v}$:

$$(n-1)H_{\overline{v}}(\overline{v}) = (n-1)\frac{\overline{v}}{n} = \frac{L(\Pi^*,\delta)}{n}.$$

Hence, the enforcement-constraint holds with equality at $\overline{v}$ and only at $\overline{v}$. The defining feature of $H_{\overline{v}}$ is that, among the arrangements of transfer payments that are able to induce truthful revelation of valuations (i.e., satisfying (1) and (2)), it has the smallest maximum total payment, which is $\frac{n-1}{n}\overline{v}$ in this case.

Third, in the general case of the payment function $H_{\frac{L(\Pi^*,\delta)}{n-1}}$, there is a cap $\frac{L(\Pi^*,\delta)}{n-1}$ on the first term $\widehat{v}_{(2)}$ inside the bracket of (17), and the upper bound of the integral is replaced by $\frac{L(\Pi^*,\delta)}{n-1}$. These two changes arise because in general the cartel needs to pay a nonzero price the seller, which takes the specific functional form $\beta_\ell(\widehat{v})$.

4.3 Discounting, cartel behaviors and collusive profit

To better understand how the cartel behaviors and collusive gains are constrained by the need to simultaneously resolve the adverse-selection problem and the enforcement problem, we specify the characterization of Π^* in Lemma 4 and the optimal collusive mechanism α^* to the two special cases of $\delta \in (0, \frac{n-1}{n}]$ and of $\delta \in [\frac{(n-1)\overline{v}}{(n-1)\overline{v}+\overline{\Pi}-\underline{\Pi}}, 1)$, and discuss the comparative statics in the remaining case of $\delta \in (\frac{n-1}{n}, \frac{(n-1)\overline{v}}{(n-1)\overline{v}+\overline{\Pi}-\underline{\Pi}})$.

Case 1: $\delta \in (0, \frac{n-1}{n}]$.

As a corollary to Theorem 1, we obtain the following simplification of the optimal collusive mechanism in this case.

Corollary 1. *Suppose that $\delta \in (0, \frac{n-1}{n}]$. Then the equilibrium cartel payoff is $\underline{\Pi}$ and the optimal collusive mechanism α^* is simplified as follows: For all $\widehat{v}$,*

$$\begin{aligned}
\beta_1^*(\widehat{v}) = \cdots = \beta_n^*(\widehat{v}) &= \widehat{v}_{(2)} \\
\gamma_\ell^*(\widehat{v}) &= \ell + n \\
\gamma_j^*(\widehat{v}) &= j \quad \textit{for all } j \neq \ell \\
\varphi_{jk}^*(\widehat{v}, b, \kappa) &= 0 \quad \textit{for all } j, k, \textit{and } (b, \kappa).
\end{aligned}$$

That is, the cartel's equilibrium payoff Π^* equals the buyers' total payoffs from noncooperative bidding $\pi(\alpha^{\mathrm{N}})$. The buyer who has the highest valuation wins the item and pays the seller the second-highest valuation. There are no transfer payments between the buyers. Effectively, through its internal collusive mechanism, the cartel switches the first-price auction of the seller into the second-price auction.

There are no collusive gains from forming the cartel. Hence, we say that the collusion in this case is *ineffective*. This might seem surprising: Since the actions are continuous, if the collusive gains are not big enough to enforce very profitable collusive mechanisms that require substantial bid suppressions or side-payments, the cartel should be able to enforce collusive mechanisms that are marginally profitable and require only moderate bid suppressions or side-payments. This intuition is not correct because, when the cartel enforces marginally profitable collusive mechanisms, the resulting collusive gains are also marginal and, hence, the cartel has only marginal resources to enforce.

This case demonstrates that enforcement can be a decisive obstacle to profitable collusion. For all discount factors below $(n-1)/n$, the enforcement constraints are so tight that forming a cartel that makes communication and exchanging side-payments feasible does not bring any collusive gains. This is in sharp contrast with what we have learned from the earlier investigations on cartel operations that abstract away the issue of enforcement, e.g., McAfee and McMillan (1992), where it is shown that, in the absence of the enforcement constraints, the first-best collusive outcome is achievable. In Section 5, we show how a long-run seller can exploit this insight in combating the cartel.

Case 2: $\delta \in [\frac{(n-1)\overline{v}}{(n-1)\overline{v}+\overline{\Pi}-\underline{\Pi}}, 1)$.

Corollary 2 simplifies Theorem 1 for this case.

Corollary 2. *Suppose that* $\delta \in [\frac{(n-1)\overline{v}}{(n-1)\overline{v}+\overline{\Pi}-\underline{\Pi}}, 1)$. *Then the equilibrium cartel payoff is* $\overline{\Pi}$ *and the optimal collusive mechanism* α^* *is simplified as follows: For all* $\widehat{v}$*,*

$$\begin{aligned}
\beta_1^*(\widehat{v}) = \cdots = \beta_n^*(\widehat{v}) &= 0 \\
\gamma_\ell^*(\widehat{v}) &= \ell + n \\
\gamma_j^*(\widehat{v}) &= j \quad \textit{for all } j \neq \ell \\
\varphi_{\ell j}^*(\widehat{v}, b, \kappa) &= \begin{cases} \frac{1}{n}\left(\widehat{v}_{(2)} + \frac{\int_{\widehat{v}_{(2)}}^{\overline{v}} [1-F(s)]^n \, \mathrm{d}s}{[1-F(\widehat{v}_{(2)})]^n}\right), & \textit{if } (b,\kappa) = (\beta^*(\widehat{v}), \gamma^*(\widehat{v})) \textit{ and } j \neq \ell \\ 0, & \textit{otherwise} \end{cases} \\
\varphi_{jk}^*(\widehat{v}, b, \kappa) &= 0 \quad \textit{for all } j \neq \ell, k, \textit{ and } (b,\kappa).
\end{aligned}$$

The first-best collusive outcome is achieved in this case: The cartel mechanism always allocates the right to win the item to a buyer with the highest valuation, who then wins the item with the minimum bid of 0 at the seller's auction. The cartel extracts all the surplus. We say that *full collusion* is attained in this case.

By the insights from the folk theorems in the repeated-game literature (e.g., Fudenberg, Levine, and Maskin, 1994), when the discount factor is close enough to 1, each cartel member's share of future collusive gains outweighs the temporary gains from any one-shot deviation. As a result, the first-best outcomes can be achieved. Our contribution in this case is the explicit identification of the lower bound for the discount factors (i.e., $\frac{(n-1)\overline{v}}{(n-1)\overline{v}+\overline{\Pi}-\underline{\Pi}}$) such that the enforcement constraints are no longer an obstacle to collusion and the identification of the optimal mechanism for which this lower bound is the lowest one (among all the lower bounds that are associated with a stage-game mechanism that attains the first-best collusive outcome).

In this case, $\frac{L(\Pi^*,\delta)}{n-1} \geq \overline{v}$. Hence, the cap in the function $H_{\frac{L(\Pi^*,\delta)}{n-1}}$ is not binding and

$$H_{\frac{L(\Pi^*,\delta)}{n-1}}(w) = H_{\overline{v}}(w) = \frac{1}{n}\left(w + \frac{\int_w^{\overline{v}}[1-F(s)]^n \,\mathrm{d}s}{[1-F(w)]^n}\right).$$

The function $H_{\overline{v}}$ is related to a bidding game called $(k+1)$-price auction in the literature of partnership dissolution (Cramton, Gibbons, and Klemperer, 1987). In that bidding game (with $k = 1$), each buyer i submits a bid $b_i \in [0,\infty)$; the buyer with the highest bid wins the item and ties are broken randomly; all losing buyers receive an equal share of the second-highest bid from the winner. Cramton, Gibbons, and Klemperer (1987) show that the symmetric equilibrium bidding strategy in this bidding game is $nH_{\overline{v}}$. Hence, the equilibrium payment from the winner to each losing buyer is given by $H_{\overline{v}}$.

Case 3: $\delta \in (\frac{n-1}{n}, \frac{(n-1)\overline{v}}{(n-1)\overline{v}+\overline{\Pi}-\underline{\Pi}})$**.**

In this case, the enforcement constraints are binding. Chassang and Ortner (2019) show that certain patterns in the bidding data from paving auctions in the Japanese cities of Ibaraki prefecture are likely due to the binding enforcement constraints. Hence, at least for the construction industry, the case of binding enforcement constraints seems to be the more relevant one. Since collusion brings strictly positive collusive gains, we say that the collusion in this case is *effective.*

The explicit characterization of the optimal collusive mechanism and the associated cartel profit for each discount factor allows us to study the comparative statics with respect to the change in the discount factor.

Theorem 2. *Suppose that* $\delta \in (\frac{n-1}{n}, \frac{(n-1)\overline{v}}{(n-1)\overline{v}+\overline{\Pi}-\underline{\Pi}})$. *Then*

$$\frac{\mathrm{d}\Pi^*}{\mathrm{d}\delta} = \frac{g'(\Pi^*)}{1-g'(\Pi^*)}\frac{\Pi^*-\underline{\Pi}}{\delta(1-\delta)} > 0 \tag{18}$$

$$\frac{\mathrm{d}L(\Pi^*,\delta)}{\mathrm{d}\delta} = \frac{1}{1-g'(\Pi^*)}\frac{\Pi^*-\underline{\Pi}}{(1-\delta)^2} > 0, \tag{19}$$

where

$$g'(\Pi^*) = \frac{\delta}{(1-\delta)(n-1)}\left(1-n\left[F\left(\frac{L(\Pi^*,\delta)}{n-1}\right)\right]^{n-1}+(n-1)\left[F\left(\frac{L(\Pi^*,\delta)}{n-1}\right)\right]^{n}\right).$$

In addition, for all i *and all* $\widehat{v}$,

$$\frac{\mathrm{d}\beta_i^*(\widehat{v})}{\mathrm{d}\delta} = \begin{cases} -\frac{1}{n-1}\frac{\mathrm{d}L(\Pi^*,\delta)}{\mathrm{d}\delta}, & \text{if } \widehat{v}_{(2)} > \frac{L(\Pi^*,\delta)}{n-1} \\ 0, & \text{if } \widehat{v}_{(2)} \leq \frac{L(\Pi^*,\delta)}{n-1}; \end{cases} \tag{20}$$

$$\frac{\mathrm{d}H_{\frac{L(\Pi^*,\delta)}{n-1}}(\widehat{v}_{(2)})}{\mathrm{d}\delta} = \begin{cases} \frac{1}{n(n-1)}\frac{\mathrm{d}L(\Pi^*,\delta)}{\mathrm{d}\delta}, & \text{if } \widehat{v}_{(2)} > \frac{L(\Pi^*,\delta)}{n-1} \\ \frac{1}{n(n-1)}\frac{\left[1-F\left(\frac{L(\Pi^*,\delta)}{n-1}\right)\right]^n}{[1-F(\widehat{v}_{(2)})]^n}\frac{\mathrm{d}L(\Pi^*,\delta)}{\mathrm{d}\delta}, & \text{if } \widehat{v}_{(2)} \leq \frac{L(\Pi^*,\delta)}{n-1}. \end{cases} \tag{21}$$

Proof. See Appendix E. □

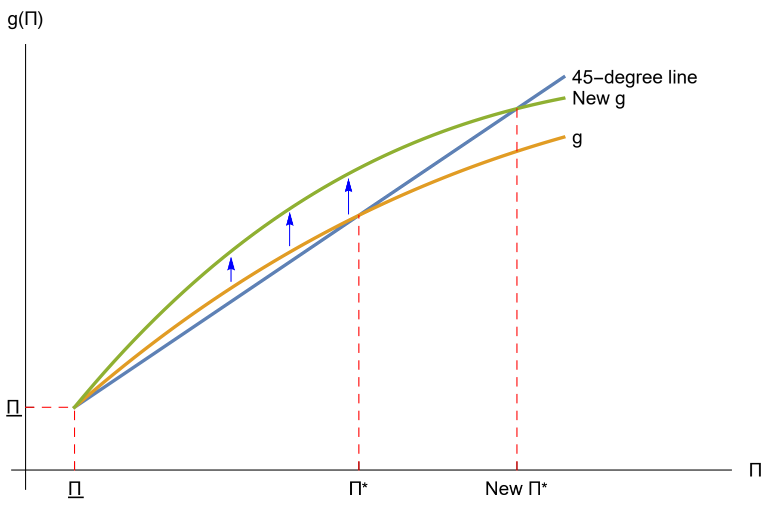


Figure 1: Illustration of the change in the equilibrium cartel profit as δ increases.

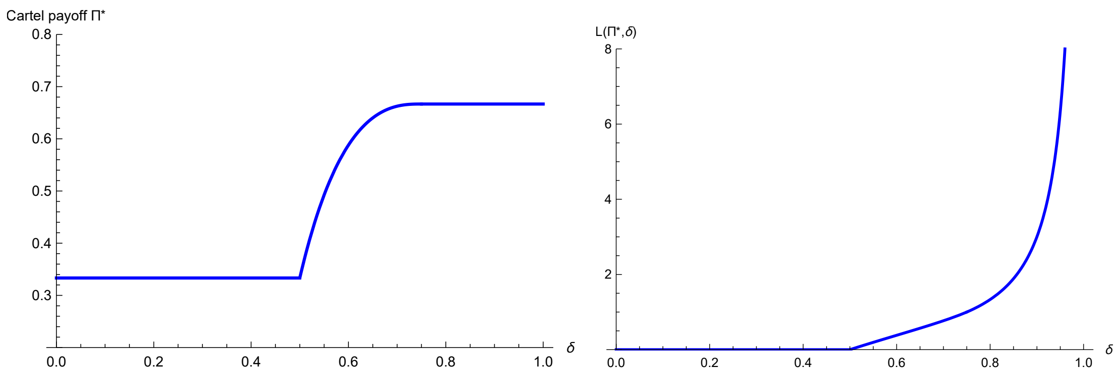
By Case (ii) of Lemma 4, the value of Π^* is characterized by the fixed-point relation

$$\Pi^* = g(\Pi^*). \tag{22}$$

Intuitively, as δ increases, the value of future collusion increases. This increases the cartel's ability to enforce the transfer payments and suppress the bids at the seller's auction. Lowering the payment to the seller shifts the graph of g upward, resulting in a larger fixed point. See Figure 1.

As δ increases, the *equilibrium* ability of enforcement $L(\Pi^*, \delta)$ increases through two channels. First, $L(\Pi^*, \delta)$ itself is a function of δ as it defines the value of *future* collusion. Second, $L(\Pi^*, \delta)$ is also a function of Π^*; hence, it increases as the value of future *collusion* increases.

To visualize these two comparative-statics results, we plot Π^* and $L(\Pi^*, \delta)$ in Figure 2, for the example of two buyers and uniform distribution of valuations. If $\delta \in \left(0, \frac{1}{2}\right]$, we have $\Pi^* = \underline{\Pi} = \frac{1}{3}$ and $L(\Pi^*, \delta) = 0$. This corresponds to Case 1 where the cartel has no collusive gain and is unable to enforce any transfer payment or bid suppression. If $\delta \in \left(\frac{1}{2}, \frac{3}{4}\right)$, both Π^* and $L(\Pi^*, \delta)$ increase continuously as δ increases. The cartel manages to have positive collusive gains and the enforcement constraints become less stringent. If $\delta \in [\frac{3}{4}, 1)$, then $\Pi^* = \overline{\Pi} = \frac{2}{3}$ and $L(\Pi^*, \delta) \geq 1$. This corresponds to Case 2 where a mechanism that achieves the first-best outcome is enforceable and the cartel extracts all the surplus.



(a) The cartel payoff Π^*.

(b) The cartel's ability of enforcement $L(\Pi^*, \delta)$.

Figure 2: Illustrations of Π^* and $L(\Pi^*, \delta)$ as the discount factor varies. Assume that there are two buyers and F is uniform on $[0, 1]$.

Theorem 2 also shows how the equilibrium bids and transfer payments change as the buyers become more patient. The binding enforcement constraints limit the cartel behaviors in two ways: they limit the degree of bid suppressions and bound the maximum amount of transfer payment that can be credibly promised. As both the collusive mechanism and competitive bidding are efficient, the collusive gains come solely from the bid suppression. In equilibrium, the cartel suppresses the designated winning bids below the second-highest valuation $\widehat{v}_{(2)}$ by $\frac{L(\Pi^*,\delta)}{n-1}$, whenever it is possible. Otherwise, the bid is suppressed down to 0. Since $L(\Pi^*, \delta)$ is increasing in δ, as the buyers become more patient, the variable part of the bidding functions β_i^* shifts downward. See Figure 3. As a result, for all $\widehat{v}$, the

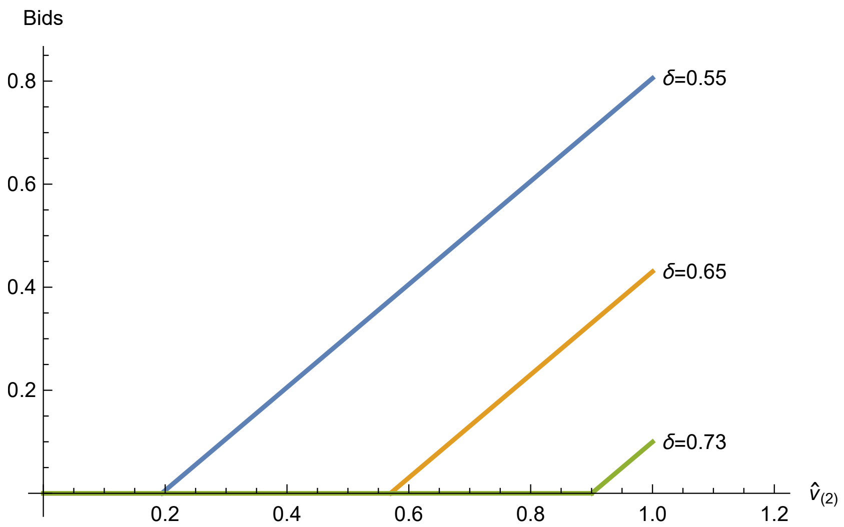


Figure 3: Illustration of the equilibrium bids β_i^* as a function of the second-highest reported valuation, for the discount factors $\delta = 0.55$, 0.65, and 0.73. Assume that there are two buyers and F is uniform on $[0, 1]$.

designated winning bid is weakly smaller if the buyers are more patient.

Similarly, the binding enforcement constraints bound the equilibrium payment by the cap $\frac{L(\Pi^*,\delta)}{n-1}$: the payment function is increasing in $\widehat{v}_{(2)}$ if $\widehat{v}_{(2)}$ is less than $\frac{L(\Pi^*,\delta)}{n-1}$, and becomes constant when $\widehat{v}_{(2)}$ reaches $\frac{L(\Pi^*,\delta)}{n-1}$. Since $L(\Pi^*, \delta)$ is increasing in δ, Theorem 2 implies that both the variable part and the constant part of the payment function are increasing in δ. As a result, as the buyers become more patient, the payment function $H_{\frac{L(\Pi^*,\delta)}{n-1}}$ shifts upward and, for all $\widehat{v}_{(2)}$, the payment

$H_{\frac{L(\Pi^*,\delta)}{n-1}}(\widehat{v}_{(2)})$ becomes larger. See Figure 4.

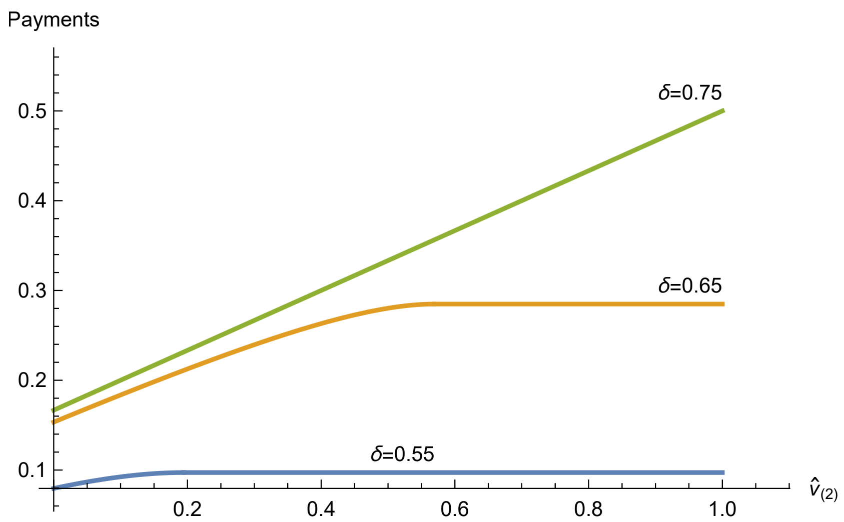


Figure 4: Illustration of the equilibrium payment $H_{\frac{L(\Pi^*,\delta)}{n-1}}$ as a function of the second-highest reported valuation, for the discount factors $\delta = 0.55$, 0.65, and 0.75. Assume that there are two buyers and F is uniform on $[0,1]$.

We have seen in Case 2 that, if the discount factor is above $\frac{(n-1)\overline{v}}{(n-1)\overline{v}+\overline{\Pi}-\underline{\Pi}}$, the payment function $H_{\frac{L(\Pi^*,\delta)}{n-1}}$ is equivalent to the equilibrium bidding strategy in the $(k+1)$-price auction with $k = 1$. This payment function corresponds to the case of $\delta = 0.75$ in Figure 4. This equivalence does not hold if $\delta \in (\frac{n-1}{n}, \frac{(n-1)\overline{v}}{(n-1)\overline{v}+\overline{\Pi}-\underline{\Pi}})$, because of the cap on the maximum payment. Hence, the equilibrium payment in our collusion setup and that in the $(k+1)$-price auction are different. The difference arises from the additional enforcement constraints in our collusion setup.

5 Fighting Collusion by Reserve Price

Suppose that the seller is long-run and he is aware of the functioning of the optimal collusive agreement. Suppose that he decides to fight the cartel by imposing a reserve price that is constant over all future periods. He anticipates that the cartel will respond optimally to the reserve price by revising the collusive agreement accordingly. In this section, we show how the seller should choose the reserve price optimally, and how effective it is.

We proceed in two steps. In Section 5.1, we show, for every possible reserve price chosen by the seller, how the cartel will respond optimally by revising the collusive agreement. In Section 5.2, we show how the seller should choose the reserve price optimally, anticipating the cartel's optimal response derived in Section 5.1.

5.1 Collusion enforcement in auctions with a reserve price

The seller sets a reserve price r for all auctions starting from period 1. In any period, if all bids are below the reserve price, then he retains the item. His valuations for all the items are normalized to zero.

We follow the same steps to derive the optimal revised agreement as in the case of no reserve price, focusing on the need to accommodate the presence of the reserve price and its impact on the optimal agreement.

5.1.1 Trigger-strategy with a reserve price

The stage-game strategy, repeated-game strategy, perfect public equilibrium, etc., are defined in the same way as before. A difference in the definition of trigger-strategy perfect public equilibrium is the strategy in the noncollusive state. When the equilibrium switches to the noncollusive state, the buyers now compete in the first-price auction with a reserve price r. With a reserve price, every buyer i with valuation v_i bidding

$$\beta^{\mathrm{N}}_{r,i}(v_i) \equiv \begin{cases} v_i - \frac{\int_r^{v_i}[F(s)]^{n-1}\,\mathrm{d}s}{[F(v_i)]^{n-1}}, & \text{if } v_i > r; \\ 0, & \text{if } v_i \leq r. \end{cases}$$

is a Bayes-Nash equilibrium.[5] In the noncollusive state, the buyers now choose the stage-game strategy profile $\alpha^{\mathrm{N}}_r \equiv (\rho^{\mathrm{N}}_r, \beta^{\mathrm{N}}_r, \gamma^{\mathrm{N}}_r, \varphi^{\mathrm{N}}_r)$, where each buyer i always reports the constant valuation 0, bids $(\beta^{\mathrm{N}}_{r,i}(v_i), i+n)$ if his valuation is v_i, and makes no payments. Clearly, α^{N}_r is self-enforcing.

[5]See, e.g., Theorem 4.6 of Milgrom (2004).

5.1.2 Optimal revised agreement

Let $\underline{\Pi}_r \equiv \pi(\alpha_r^{\mathrm{N}})$ denote the cartel payoff under the strategy profile α_r^{N}; let

$$\overline{\Pi}_r \equiv E[v_{(1)} - r | v_{(1)} \geq r]$$

denote the cartel payoff under the first-best collusive outcome. For any $\Pi \in [\underline{\Pi}_r, \overline{\Pi}_r]$, define

$$L_r(\Pi, \delta) \equiv \delta(1-\delta)^{-1}\left(\Pi - \underline{\Pi}_r\right)$$

and

$$g_r(\Pi) \equiv E\left[v_{(1)} - \max\left(v_{(2)} - \frac{L_r(\Pi,\delta)}{n-1}, r\right) \middle| v_{(1)} \geq r\right]. \tag{23}$$

Let

$$\Pi_r^* \equiv \sup\left\{\Pi \in [\underline{\Pi}_r, \overline{\Pi}_r] \mid \Pi \leq g_r(\Pi)\right\}.$$

The variables $\underline{\Pi}_r$, $\overline{\Pi}_r$, Π_r^* and functions L_r, g_r are the generalizations of $\underline{\Pi}$, $\overline{\Pi}$, Π^*, L, and g to accommodate the reserve price.

We can now introduce the stage-game strategy profile $\alpha_r^* = (\rho^*, \beta_r^*, \gamma_r^*, \varphi_r^*)$ that is repeated in the collusive state of the optimal trigger-strategy equilibrium. For any $M, r > 0$, we define a function $H_{M,r}$ on $[0, \overline{v}]$ by letting, for each $w \in [0, \overline{v}]$,

$$H_{M,r}(w) = \begin{cases} \frac{1}{n}M, & \text{if } w > M + r \\ \frac{1}{n}\left(w - r + \frac{\int_w^{M+r}[1-F(s)]^n\,\mathrm{d}s}{[1-F(w)]^n}\right), & \text{if } r < w \leq M + r \\ \frac{1}{n}\frac{\int_r^{M+r}[1-F(s)]^n\,\mathrm{d}s}{[1-F(w)]^n}, & \text{if } w \leq r \end{cases}.$$

For any $\widehat{v} = (\widehat{v}_1, \ldots, \widehat{v}_n)$, recall that ℓ is the buyer who has the highest reported valuation and, in the case of ties, is the buyer who, in addition, has the smallest index. Define, for all $\widehat{v}$,

$$\begin{aligned}
\beta^*_{r,1}(\widehat{v}) = \cdots = \beta^*_{r,n}(\widehat{v}) &\equiv \begin{cases} \max\left(\widehat{v}_{(2)} - \frac{L_r(\Pi^*_r,\delta)}{n-1}, r\right), & \text{if } \widehat{v}_{(1)} \geq r \\ 0, & \text{otherwise} \end{cases} \\
\gamma^*_{r,\ell}(\widehat{v}) &\equiv \ell + n \\
\gamma^*_{r,j}(\widehat{v}) &\equiv j \ \text{ for all } j \neq \ell \\
\varphi^*_{r,\ell j}(\widehat{v}, b, \kappa) &\equiv \begin{cases} H_{\frac{L_r(\Pi^*_r,\delta)}{n-1},r}(\widehat{v}_{(2)}), & \text{if } (b,\kappa) = (\beta^*_r(\widehat{v}), \gamma^*_r(\widehat{v})) \text{ and } j \neq \ell \\ 0, & \text{otherwise} \end{cases} \\
\varphi^*_{r,jk}(\widehat{v}, b, \kappa) &\equiv 0 \ \text{ for all } j \neq \ell, k, \text{and } (b,\kappa).
\end{aligned}$$

Compared to α^*, with a reserve price there are three differences in the stage-game mechanism α^*_r. First, the buyers submit positive bids only when the valuation of at least one buyer is above the reserve price. Second, because in the noncollusive state the buyers compete noncooperatively in the auction with a reserve price, in the definition of $L_r(\Pi^*_r, \delta)$ the cartel payoff $\underline{\Pi}$ is now replaced by $\underline{\Pi}_r$. Third, the payment function $H_{\frac{L_r(\Pi^*_r,\delta)}{n-1},r}$ is also modified to accommodate the reserve price.

Following the same steps that lead to Lemma 4 and Theorem 1, we obtain the following characterization of the optimal trigger-strategy perfect public equilibrium for auctions with a reserve price.

Theorem 3. *Suppose that the seller sets a reserve price r in every period. Then the trigger-strategy profile in which the buyers repeat α^*_r in every period of the collusive state is a perfect public equilibrium. It is optimal among all trigger-strategy perfect public equilibria that satisfy Assumption 1. The equilibrium payoff of the cartel is Π^*_r and its value is characterized as follows.*

(i) If $\delta \in (0, \frac{n-1}{n-n[F(r)]^{n-1}+(n-1)[F(r)]^n}]$, then $\Pi^*_r = \underline{\Pi}_r$.

(ii) If $\delta \in (\frac{n-1}{n-n[F(r)]^{n-1}+(n-1)[F(r)]^n}, \frac{(n-1)(\overline{v}-r)}{(n-1)(\overline{v}-r)+\overline{\Pi}_r-\underline{\Pi}_r})$, then Π^*_r is the unique fixed point of g_r in $(\underline{\Pi}_r, \overline{\Pi}_r)$, i.e., $\Pi^*_r = g_r(\Pi^*_r)$.

(iii) If $\delta \in [\frac{(n-1)(\overline{v}-r)}{(n-1)(\overline{v}-r)+\overline{\Pi}_r-\underline{\Pi}_r}, 1)$, then $\Pi^*_r = \overline{\Pi}_r$.

See Appendix F for the proof of Theorem 3.

5.2 Optimal reserve price

We now proceed to the seller's problem of choosing an optimal reserve price to counter the cartel activity. For expositional clarity and tractability, we assume for the remaining section that there are two buyers and the distribution of valuations F is the uniform distribution on $[0, 1]$. This specification allows us to derive an analytical expression for the seller's equilibrium payoff as a function of the reserve price. Let $S(r)$ denote the seller's equilibrium payoff when he sets a reserve price r in every period.

Lemma 5. *Suppose that there are two buyers, the valuations are uniformly distributed on $[0, 1]$, and the seller sets a reserve price r in every period. Then the equilibrium payoffs to the cartel and the seller are, respectively,*

$$\Pi_r^* = \begin{cases} \underline{\Pi}_r, & \text{if } r \geq 1 - \sqrt{\frac{1-\delta}{\delta}}; \\ \underline{\Pi}_r + \frac{1-\delta}{2\delta}\left(3(1-r) - \sqrt{\frac{12(1-\delta)}{\delta} - 3(1-r)^2}\right), & \text{if } 1 - \sqrt{\frac{3(1-\delta)}{\delta}} < r < 1 - \sqrt{\frac{1-\delta}{\delta}}; \\ \overline{\Pi}_r, & \text{if } r \leq 1 - \sqrt{\frac{3(1-\delta)}{\delta}}. \end{cases}, \tag{24}$$

and

$$S(r) = \overline{\Pi}_r + r(1 - r^2) - \Pi_r^*. \tag{25}$$

Proof. See Appendix G. □

The analytical expression for the cartel's equilibrium payoff Π_r^* follows from applying Theorem 3 to the special case of two buyers and uniform distribution of valuations. The three cases in the expression for Π_r^* correspond to the three cases of the discount factor that we discussed in Section 4.3. In the case of (r, δ) that satisfies $1 - \sqrt{\frac{3(1-\delta)}{\delta}} < r < 1 - \sqrt{\frac{1-\delta}{\delta}}$, by Theorem 3, the value of Π_r^* is characterized by the following fixed-point relation,

$$\Pi_r^* = g_r(\Pi_r^*). \tag{26}$$

The explicit expression for Π_r^* in the lemma is obtained by solving (26), which is made possible by the assumption that there are two buyers and F is the uniform distribution on $[0, 1]$.

With a reserve price r, the total surplus in each period can be decomposed as follows,

$$\begin{aligned} E[v_{(1)}|v_{(1)} \geq r] =& E[v_{(1)} - r|v_{(1)} \geq r] + E[r|v_{(1)} \geq r] \\ =& \overline{\Pi}_r + r(1 - r^2). \end{aligned} \tag{27}$$

The rent accrued to the seller is therefore equal to

$$S(r) = \overline{\Pi}_r + r(1 - r^2) - \Pi_r^*. \tag{28}$$

Equation (28) shows that imposing a reserve price affects the seller's payoff through two channels: (i) It affects the size of the total surplus $E[v_{(1)}|v_{(1)} \geq r]$. (ii) It affects the share of the total surplus obtained by the cartel. Interestingly, note that the effect of the first channel is independent of the discount factor, while, as can be seen from (24), the cartel payoff is a function of the discount factor.

Using this analytical expression for $S(r)$, we can now characterize the optimal reserve price that maximizes $S(r)$.

Theorem 4. *Suppose that there are two buyers and the valuations are uniformly distributed on* $[0, 1]$. *Then, depending on the value of the discount factor, the optimal reserve price* r^* *that maximizes the seller's payoff* $S(r)$ *is given as follows.*

(i) If $\delta \leq \frac{4}{5}$, *then* $r^* = \frac{1}{2}$.

(ii) If $\frac{4}{5} < \delta \leq \frac{3+\sqrt{5}}{6} (\approx 0.87)$, *then* $r^* = 1 - \sqrt{\frac{1-\delta}{\delta}}$.

(iii) If $\frac{3+\sqrt{5}}{6} < \delta \leq \frac{9(13+2\sqrt{3})}{157} (\approx 0.94)$, *then* r^* *is the unique solution to the following first-order condition*

$$S'(r) = 2r - 4r^2 + \frac{3(1-\delta)}{2\delta} \left(1 + \frac{1-r}{\sqrt{\frac{12(1-\delta)}{\delta} - 3(1-r)^2}} \right) = 0 \tag{29}$$

in the interval $(1 - \sqrt{\frac{3(1-\delta)}{\delta}}, 1 - \sqrt{\frac{1-\delta}{\delta}})$, *and is thus decreasing in* δ.

(iv) If $\delta > \frac{9(13+2\sqrt{3})}{157}$, *then* $r^* = \frac{\sqrt{3}}{3}$.

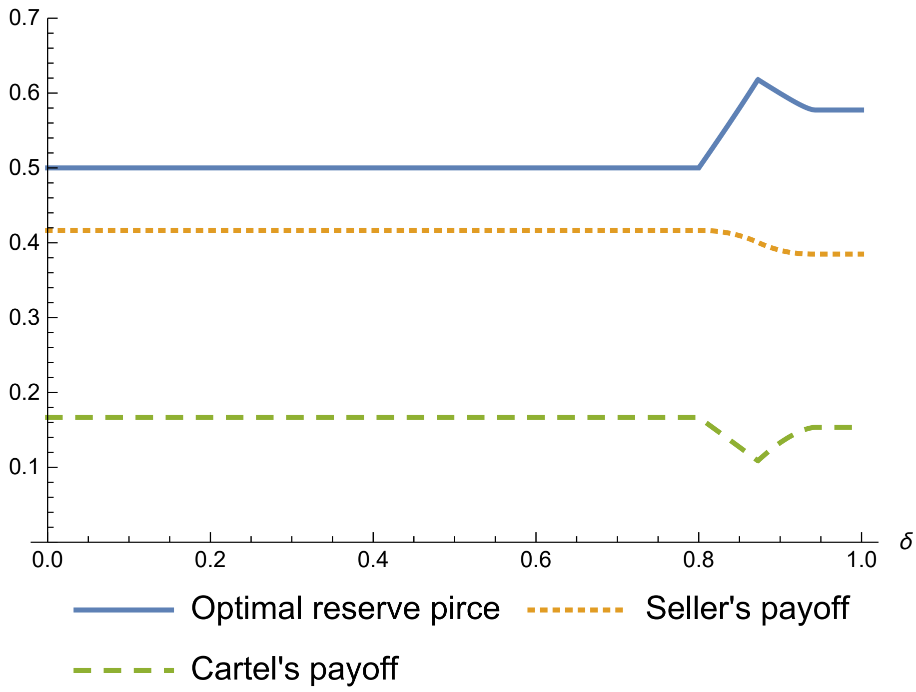


Figure 5: Illustration of the optimal reserve price, the seller's payoff and the cartel payoff as the discount factor varies.

Figure 5 shows the optimal reserve price, and the payoffs to the seller and the cartel under the optimal reserve price, for each level of discounting.

Depending on the value of the discount factor, the optimal reserve price plays very different roles. Below we explain the intuition for the optimal reserve price derived in each case of Theorem 4.

Case (i): $\delta \in (0, \frac{4}{5}]$. If δ is in the subinterval $(0, \frac{1}{2}]$, then this is Case 1 in Section 4.3 where the buyers effectively convert the first-price auction of the seller into the second-price auction and there is no effective collusion. The optimal reserve price $\frac{1}{2}$ serves purely the role of optimal rent extraction, as in Myerson (1981), where it is shown that, in the static setting with no collusion, setting the reserve price $\frac{1}{2}$ is the best way for the seller to extract the surplus.

If δ is in the subinterval $(\frac{1}{2}, \frac{4}{5}]$, the reserve price $\frac{1}{2}$ continues to play the role of optimal rent extraction, but it also has a collusion-reduction effect. Comparing

Lemmas 4 and 5 shows that, when $r = \frac{1}{2}$, the cartel has effective collusion if and only if $\delta > \frac{4}{5}$. By contrast, when $r = 0$, this constraint is $\delta > \frac{1}{2}$. Hence, the reserve price $\frac{1}{2}$ also prevents effective collusion if $\delta \in (\frac{1}{2}, \frac{4}{5}]$. The intuition is as follows. The constraint on δ such that the cartel has effective collusion is related to the benefit from a marginal increase in $L_r(\Pi_r^*, \delta)$ from 0. With a reserve price r, the cartel cannot suppress the bids below r. As a result, the benefit from a marginal increase in $L_r(\Pi_r^*, \delta)$ from 0 is smaller and a larger δ is needed to ensure that collusion is profitable.

Case (ii): $\delta \in (\frac{4}{5}, \frac{3+\sqrt{5}}{6} \approx 0.87)$. The optimal rent-extracting reserve price $\frac{1}{2}$ is not able to deter effective collusion now. The optimal reserve price $r^* = 1 - \sqrt{\frac{1-\delta}{\delta}}$ is the knife-edge reserve price separating the case of ineffective collusion and the case of effective collusion. Thus, the seller increases the reserve price to the level that is just sufficient to deter effective collusion. The role of the reserve price in this case is, therefore, to deter effective collusion. Unlike the previous case, deterring effective collusion is costly to the seller now. Figure 5 shows that, for $\delta \in (\frac{4}{5}, \frac{3+\sqrt{5}}{6})$, the payoff to the seller falls as δ and hence the reserve price increases.

Case (iii): $\delta \in (\frac{3+\sqrt{5}}{6} \approx 0.87, \frac{9(13+2\sqrt{3})}{157} \approx 0.94)$. In this case, it is too costly to deter effective collusion. Instead, the seller's optimal strategy is to accommodate the operation of the cartel mechanism and try to reduce the bid suppressions only to the extent that is optimal. As δ increases, r^* is actually decreasing.

Under effective collusion, the reserve price serves to fight the cartel in two ways. First, it reduces the bid suppressions *below the reserve price* r by imposing the lowest possible bid. This occurs when $\big(\widehat{v}_{(2)} - L_r(\Pi_r^*, \delta)\big)$ is below r. Second, it reduces the bid suppressions *above the reserve price* r by decreasing the cartel's ability of enforcement $L_r(\Pi_r^*, \delta)$. As both Π_r^* and $\underline{\Pi}_r$ depend on the reserve price, increasing the reserve price lowers $L_r(\Pi_r^*, \delta)$, and thus reduces the bid suppressions when $\big(\widehat{v}_{(2)} - L_r(\Pi_r^*, \delta)\big)$ is above r.[6]

Case (iv): $\delta \in (\frac{9(13+2\sqrt{3})}{157} \approx 0.94, 1)$. As δ becomes close to 1, the cartel is able to enforce any bid suppressions and transfer payments. It is no longer optimal to fight the cartel by trying to reduce the bid suppressions *above the reserve price.*

[6]Using Lemma 5, it is straightforward to show that $\frac{\mathrm{d}L_r(\Pi_r^*, \delta)}{\mathrm{d}r} < 0$.

Instead, the seller tolerates the fully-collusive mechanism, and aims to obtain a share of the surplus as large as possible by reducing the bid suppressions *below the reserve price*. The main tradeoff here is that increasing the reserve price will increase the chance that the seller has to retain the object, and hence, decrease the size of the total surplus. Under full collusion where all bids are suppressed down to the reserve price, the seller's revenue from setting a reserve price r is equal to

$$r(1-r^2).$$

The reserve price $r^* = \frac{\sqrt{3}}{3}$ is the one that maximizes this revenue. Since the seller tolerates the fully-collusive mechanism, this reserve price is identical to what McAfee and McMillan (1992) derive for strong cartels in the static setting, which can control the bids and enforce any amount of payments.

We summarize the discussion of Theorem 4 with a few takeaways.

First, the optimal reserve prices play different roles for different values of the discount factor.

Second, the reserve prices are very effective in fighting collusion for the intermediate values of the discount factor. The cartel does not have effective collusion for all $\delta \leq 0.87$. For all $\delta > 0.87$, the cartel payoff under the reserve price is actually lower than that from noncooperative bidding. Thus, the buyers would be better off if they could commit not to collude and the seller does not increase the reserve price above the optimal rent-extraction level.

Third, the optimal reserve price is non-monotone in the value of the discount factor. The seller optimally trades off between deterring effective collusion using a more aggressive but also more costly reserve price and accommodating effective collusion with a lower but less costly reserve price at different values of the discount factor. The intuition for this dependence on δ can be explained using the decomposition (28). As δ varies, the cost of imposing a reserve price – the decrease in the size of the total surplus $E[v_{(1)}|v_{(1)} \geq r]$ – remains constant. However, the benefit depends on δ. Using Lemma 5, it is straightforward to verify that $\frac{\mathrm{d}L_r(\Pi_r^*,\delta)}{\mathrm{d}\delta} > 0$. That is, as δ increases, the reserve price is less effective in reducing the bid suppressions above reserve price.

A Stationary Collusive Agreements

In this appendix, we show that it is without loss of generality to focus on the stationary collusive agreements if we do not impose Assumptions 1b and 1c. Although only the case of two buyers is needed for Theorem 3A in Appendix B, we state and prove Lemma 6 for general n.

Lemma 6. *Suppose that an optimal trigger-strategy perfect public equilibrium exists. Then there is a* stationary *one that delivers the same equilibrium payoffs.*

Fix an optimal trigger-strategy perfect public equilibrium σ. Let $V_i(\sigma)$ denote the equilibrium payoff to buyer i. Let $\alpha = (\rho, \beta, \gamma, \varphi)$ be the first-period stage-game strategy profile. Let

$$\eta \equiv (\widehat{v}, \beta(\widehat{v}), \gamma(\widehat{v}), \varphi(\widehat{v}, \beta(\widehat{v}), \gamma(\widehat{v})))$$

denote the first-period public information in the collusive state. Let $w_i(\eta)$ denote buyer i's continuation payoff following η. We have the decomposition

$$V_i(\sigma) = (1-\delta)\pi_i(\alpha) + \delta E[w_i(\eta)|\alpha] \tag{30}$$

for each $i = 1, \ldots, n$. Since σ is optimal, we have, for all η in the collusive state,

$$\sum_{i=1}^{n} V_i(\sigma) = \sum_{i=1}^{n} \pi_i(\alpha) = \sum_{i=1}^{n} w_i(\eta). \tag{31}$$

The idea of the proof is to replace the variations in the continuation payoffs $w_i(\eta)$ by direct side-payments between the buyers. We proceed in three steps.

In Step 1, we define a profile of new transfer strategies $\widetilde{\varphi} = (\widetilde{\varphi}_1, \ldots, \widetilde{\varphi}_n)$ such that for each $i = 1, \ldots, n$, and each history $\eta_0 \equiv (\widehat{v}, \beta(\widehat{v}), \gamma(\widehat{v}))$,

$$\text{if } \widetilde{\varphi}_{ij}(\eta_0) > 0, \text{ then } \widetilde{\varphi}_{ji}(\eta_0) = 0, \tag{32}$$

and

$$(1-\delta)\sum_{j=1}^{n}(\widetilde{\varphi}_{ji}(\eta_0) - \widetilde{\varphi}_{ij}(\eta_0)) + \delta V_i(\sigma) = (1-\delta)\sum_{j=1}^{n}(\varphi_{ji}(\eta_0) - \varphi_{ij}(\eta_0)) + \delta w_i(\eta_0, \varphi(\eta_0)). \tag{33}$$

Let $\widetilde{\alpha} \equiv (\rho, \beta, \gamma, \widetilde{\varphi})$ be the stage-game strategy profile that is obtained from α by replacing φ by $\widetilde{\varphi}$. Let $\widetilde{\sigma}$ be the stationary trigger-strategy profile in which the buyers repeat $\widetilde{\alpha}$ in the collusive state.

In Step 2, we show that

$$\pi_i(\widetilde{\alpha}) = V_i(\sigma) \quad \text{for each } i = 1, \dots, n. \tag{34}$$

In Step 3, we show that $\widetilde{\sigma}$ is self-enforcing.

Since $\widetilde{\sigma}$ is stationary, by (34), buyer i's payoff under $\widetilde{\sigma}$ is $V_i(\sigma)$ for each $i = 1$, $\dots$, n. Hence, it is also an optimal equilibrium.

A.1 Step 1: Defining $\widetilde{\varphi}$.

Condition (32) ensures that only one buyer makes a positive payment between any pair of buyers. Condition (33) ensures that (i) the continuation payoff starting from the next period is $V_i(\sigma)$ for each buyer i, and (ii) all variations in $w_i(\eta_0, \varphi(\eta_0))$ are absorbed into the new side-payments $\widetilde{\varphi}$.

For any buyer i, let

$$\xi_i(\eta_0) \equiv (1-\delta) \sum_{j=1}^{n} (\varphi_{ji}(\eta_0) - \varphi_{ij}(\eta_0)) + \delta w_i(\eta_0, \varphi(\eta_0)) - \delta V_i(\sigma) \tag{35}$$

denote the difference between buyer i's continuation following η_0 under σ and the "target" continuation payoff starting from the next period, $V_i(\sigma)$. If $\xi_i(\eta_0)$ is strictly positive, then buyer i needs to be paid by $\xi_i(\eta_0)$ to make sure that his continuation payoff at the beginning of the next period is $V_i(\sigma)$ and his incentives remain intact.

Let N_1 and N_2 be the sets of buyer i such that $\xi_i(\eta_0)$ is strictly positive and strictly negative, respectively. By (31), we must have

$$\sum_{j \in N_1} \xi_j(\eta_0) = -\sum_{i \in N_2} \xi_i(\eta_0). \tag{36}$$

That is, the total shortfall of all buyers in N_1 must be equal to the total surplus of all buyers in N_2. We define the new transfers $\widetilde{\varphi}_{ij}(\eta_0)$ by making each buyer i in

N_2 pay each buyer j in N_1 a proportional share of buyer j's shortfall. Formally, for each $i \in N_2$ and $j \in N_1$, let

$$\widetilde{\varphi}_{ij}(\eta_0) \equiv \xi_j(\eta_0) \frac{\xi_i(\eta_0)}{\sum_{k \in N_2} \xi_k(\eta_0)}. \tag{37}$$

Otherwise, let $\widetilde{\varphi}_{ij}(\eta_0) = 0$.

It is straightforward to verify that the new transfers $\widetilde{\varphi}_{ij}(\eta_0)$ satisfy Conditions (32) and (33).

A.2 Step 2: Verifying $\pi_i(\widetilde{\alpha}) = V_i(\sigma)$ for each i.

Since the allocation rule under $\widetilde{\alpha}$ and that under α are the same, we have

$$q_i^{\widetilde{\alpha}}(v) = q_i^{\alpha}(v) \quad \text{for all } v. \tag{38}$$

Buyer i's payoff under $\widetilde{\alpha}$ is, therefore, equal to

$$\begin{aligned}
\pi_i(\widetilde{\alpha}) =& E\left[(v_i - \beta_i(v))\, q_i^{\widetilde{\alpha}}(v) + \sum_{j=1}^{n} (\widetilde{\varphi}_{ji}(\eta_0) - \widetilde{\varphi}_{ij}(\eta_0))\right] \\
=& E\left[(v_i - \beta_i(v))\, q_i^{\alpha}(v) + \sum_{j=1}^{n} (\varphi_{ji}(\eta_0) - \varphi_{ij}(\eta_0)) + (1-\delta)^{-1}\delta[w_i(\eta_0, \varphi(\eta_0)) - V_i(\sigma)]\right] \\
=& (1-\delta)^{-1} V_i(\sigma) - (1-\delta)^{-1}\delta V_i(\sigma) \\
=& V_i(\sigma),
\end{aligned}$$

where the second line follows from (38) and substituting for $\sum_{j=1}^{n}(\widetilde{\varphi}_{ji}(\eta_0) - \widetilde{\varphi}_{ij}(\eta_0))$ using (33), and the third from (30).

A.3 Step 3: Verifying that $\widetilde{\sigma}$ is self-enforcing.

We now show that $\widetilde{\sigma}$ is self-enforcing. By stationarity, it suffices to focus on the incentives in the first period. The buyers have incentives to follow the reporting-strategy profile ρ and bidding-strategy profile (β, γ) because, by (33), the continuation payoffs following the reports and the bids are the same as those under the original equilibrium σ.

It remains to verify the incentives to make the payments. Suppose that buyer i is to make strictly positive payments, i.e., $\xi_i(\eta_0) < 0$. By (33),

$$\begin{aligned}(1-\delta)\sum_{j=1}^{n}(\widetilde{\varphi}_{ji}(\eta_0)-\widetilde{\varphi}_{ij}(\eta_0))+\delta V_i(\sigma) =&(1-\delta)\sum_{j=1}^{n}(\varphi_{ji}(\eta_0)-\varphi_{ij}(\eta_0))+\delta w_i(\eta_0,\varphi(\eta_0))\\ \geq&(1-\delta)\sum_{j=1}^{n}\varphi_{ji}(\eta_0)+\delta\pi_i(\alpha^{\mathrm{N}})\\ \geq&(1-\delta)\sum_{j=1}^{n}\widetilde{\varphi}_{ji}(\eta_0)+\delta\pi_i(\alpha^{\mathrm{N}}).\end{aligned}$$

The first inequality follows from the fact that buyer i has incentives to make the payment $\sum_{j=1}^{n}\varphi_{ij}(\eta_0)$ under σ. The second inequality follows since, by construction, if buyer i is to make positive payments, he will not receive payments from the others, i.e., $\sum_{j=1}^{n}\widetilde{\varphi}_{ji}(\eta_0) = 0$.

B Collusion Enforcement with Two Buyers

In this appendix, we show that Assumption 1 can be removed from Theorem 3 if there are two buyers.

Theorem 3A. *If there are two buyers, then the trigger-strategy perfect public equilibrium in Theorem 3 is optimal among all trigger-strategy perfect public equilibria.*

Assumption 1 is used in two places in the proof of Theorem 1. First, Assumptions 1b and 1c are needed to prove Lemma 3, which is based on Lemma 2. Second, Assumptions 1b and 1c restrict the ways side-payments are used. As a result, we cannot use Lemma 6 in Appendix A to argue that it is without loss of generality to focus on the stationary collusive agreements. Instead, we impose Assumption 1a.

In the following, we show that Assumptions 1b and 1c are not needed in Lemmas 2 and 3 if there are only two buyers. Hence, Assumption 1a can also be removed by using Lemma 6 in Appendix A.

Lemma 2A. *If there are only two buyers, then Lemma 2 holds without Assumptions 1b and 1c.*

Proof. If there are only two buyers, Equation (10) holds without Assumption 1c. Equation (13) follows from summing (8) and (11) directly, without using (12). Hence, Assumption 1b is not needed either. □

Lemma 3A. *If there are only two buyers, then Lemma 3 holds without Assumptions 1b and 1c.*

Proof. Instead of Lemma 2, we apply Lemma 2A to (14) and (15). □

C Properties of the Function g_r

In this section, we prove several properties of the function g_r.

Lemma 7. *Suppose that* $\delta < \frac{(n-1)(\overline{v}-r)}{(n-1)(\overline{v}-r)+\overline{\Pi}_r-\underline{\Pi}_r}$.

(i) The function g_r *is strictly concave on the interval* $(\underline{\Pi}_r, \overline{\Pi}_r)$.

(ii) We have

$$\begin{aligned} g_r(\underline{\Pi}_r) &= \underline{\Pi}_r && (39)\\ g_r(\overline{\Pi}_r) &< \overline{\Pi}_r && (40)\\ g'_r(\underline{\Pi}_r) &= \frac{\delta}{(1-\delta)(n-1)}(1-n[F(r)]^{n-1}+(n-1)[F(r)]^n). && (41) \end{aligned}$$

(iii) We have

$$g'_r(\underline{\Pi}_r) \geq 1 \iff \delta \geq \frac{n-1}{n-n[F(r)]^{n-1}+(n-1)[F(r)]^n}. \qquad (42)$$

Suppose that $\delta \geq \frac{(n-1)(\overline{v}-r)}{(n-1)(\overline{v}-r)+\overline{\Pi}_r-\underline{\Pi}_r}$.

(iv) We have

$$g_r(\overline{\Pi}_r) = \overline{\Pi}_r. \qquad (43)$$

Proof. Part (i). By rearranging terms, $\delta < \frac{(n-1)(\overline{v}-r)}{(n-1)(\overline{v}-r)+\overline{\Pi}_r-\underline{\Pi}_r}$ implies that

$$\begin{aligned}\frac{L_r(\overline{\Pi}_r,\delta)}{n-1}+r =&\frac{\delta(\overline{\Pi}_r-\underline{\Pi}_r)}{(n-1)(1-\delta)}+r \\ <&\overline{v}.\end{aligned} \tag{44}$$

Hence, $\frac{L_r(\Pi,\delta)}{n-1}+r<\overline{v}$ for all $\Pi \in (\underline{\Pi}_r, \overline{\Pi}_r)$.

By (27), we have

$$\begin{aligned}g_r(\Pi) \equiv& E\left[v_{(1)} - \max\left(v_{(2)} - \frac{L_r(\Pi,\delta)}{n-1}, r\right)\middle| v_{(1)} \geq r\right] \\ =&E\left[v_{(1)}-r\middle| v_{(1)}\geq r\right] - E\left[\max\left(v_{(2)}-\frac{L_r(\Pi,\delta)}{n-1}-r,0\right)\middle| v_{(1)}\geq r\right] \\ =&\overline{\Pi}_r - \int_0^{\overline{v}} \max\left(w-\frac{L_r(\Pi,\delta)}{n-1}-r,0\right)\,\mathrm{d}\left(n[F(w)]^{n-1}[1-F(w)]+[F(w)]^n\right) \\ =&\overline{\Pi}_r - \int_{\frac{L_r(\Pi,\delta)}{n-1}+r}^{\overline{v}} \left(w-\frac{L_r(\Pi,\delta)}{n-1}-r\right)\,\mathrm{d}\left(n[F(w)]^{n-1}[1-F(w)]+[F(w)]^n\right) \\ =&\overline{\Pi}_r - \int_{\frac{L_r(\Pi,\delta)}{n-1}+r}^{\overline{v}} \left(1-n[F(w)]^{n-1}[1-F(w)]-[F(w)]^n\right)\,\mathrm{d}w,\end{aligned} \tag{45}$$

where the third line follows from the fact that the c.d.f. for the second-highest order statistic is $(n[F(w)]^{n-1}[1-F(w)]+[F(w)]^n)$ and the last from integration by parts.

Differentiating g_r with respect to Π yields

$$\begin{aligned}g_r'(\Pi) = \frac{\delta}{(1-\delta)(n-1)}\Bigg(1-n\left[F\left(\frac{L_r(\Pi,\delta)}{n-1}+r\right)\right]^{n-1} \\ +(n-1)\left[F\left(\frac{L_r(\Pi,\delta)}{n-1}+r\right)\right]^n\Bigg).\end{aligned} \tag{46}$$

The derivative of the polynomial $[1-nx^{n-1}+(n-1)x^n]$ with respect to x is

$$-n(n-1)x^{n-2}(1-x),$$

which is negative for $x\in[0,1]$. Hence, $[1-nx^{n-1}+(n-1)x^n]$ is decreasing. It follows that

$$[1-nx^{n-1}+(n-1)x^n] > [1-n\cdot 1^{n-1}+(n-1)\cdot 1^n] = 0 \tag{47}$$

for $x \in [0,1)$. Hence,

$$\begin{aligned} g_r'(\Pi) =& \frac{\delta}{(1-\delta)(n-1)} \left(1 - n \left[F\left(\frac{L_r(\Pi,\delta)}{n-1} + r \right) \right]^{n-1} \right. \\ & \left. + (n-1) \left[F\left(\frac{L_r(\Pi,\delta)}{n-1} + r \right) \right]^{n} \right). \\ >& \frac{\delta}{(1-\delta)(n-1)} \left(1 - n \cdot 1^{n-1} + (n-1) \cdot 1^n \right) \\ =& 0. \end{aligned} \tag{48}$$

Differentiating g_r' in (46) again, we have

$$\begin{aligned} g_r''(\Pi) =& - \frac{\delta^2 n}{(1-\delta)^2(n-1)} \left[1 - F\left(\frac{L_r(\Pi,\delta)}{n-1} + r \right) \right] \left[F\left(\frac{L_r(\Pi,\delta)}{n-1} + r \right) \right]^{n-2} \\ & \cdot f\left(\frac{L_r(\Pi,\delta)}{n-1} + r \right) \\ <& 0. \end{aligned} \tag{49}$$

Hence, g_r is strictly concave.

Part (ii). Since $L_r(\underline{\Pi}_r, \delta) = 0$, by (45),

$$\begin{aligned} g_r(\underline{\Pi}_r) =& \overline{\Pi}_r - \int_r^{\overline{v}} \left(1 - n[F(w)]^{n-1}[1-F(w)] - [F(w)]^n \right) \mathrm{d}w \\ =& \int_r^{\overline{v}} w - r \, \mathrm{d}[F(w)]^n - \int_r^{\overline{v}} \left(1 - n[F(w)]^{n-1}[1-F(w)] - [F(w)]^n \right) \mathrm{d}w \\ =& \int_r^{\overline{v}} n[F(w)]^{n-1}[1-F(w)] \, \mathrm{d}w \\ =& \underline{\Pi}_r, \end{aligned}$$

where the last line follows since, by the definition of the bidding strategy in the noncollusive state,

$$\begin{aligned} \underline{\Pi}_r =& \sum_{i=1}^n \pi_i(\alpha_r^{\mathrm{N}}) \\ =& \sum_{i=1}^n \int_r^{\overline{v}} \left[v_i - \beta_{r,i}^{\mathrm{N}}(v_i) \right] [F(v_i)]^{n-1} \, \mathrm{d}F(v_i) \\ =& n \int_r^{\overline{v}} [F(w)]^{n-1}[1-F(w)] \, \mathrm{d}w. \end{aligned}$$

Similarly, by (45),

$$
\begin{aligned}
g_r(\overline{\Pi}_r) =& \overline{\Pi}_r - \int_{\frac{L_r(\overline{\Pi}_r,\delta)}{n-1}+r}^{\overline{v}} \left(1 - n[F(w)]^{n-1}[1-F(w)] - [F(w)]^n\right) \mathrm{d}w \\
<& \overline{\Pi}_r,
\end{aligned}
$$

where the last line follows from (47) and $\frac{L_r(\overline{\Pi}_r,\delta)}{n-1} + r < \overline{v}$.

By (46),

$$
g_r'(\underline{\Pi}_r) = \frac{\delta}{(1-\delta)(n-1)}(1 - n[F(r)]^{n-1} + (n-1)[F(r)]^n).
$$

Part (iii). Part (iii) follows from rearranging terms.

Part (iv). By (44), $\frac{L(\overline{\Pi}_r,\delta)}{n-1} + r \geq \overline{v}$. Hence, by (45),

$$
\begin{aligned}
g_r(\overline{\Pi}_r) =& \overline{\Pi}_r - \int_{\frac{L_r(\overline{\Pi}_r,\delta)}{n-1}+r}^{\overline{v}} \left(1 - n[F(w)]^{n-1}[1-F(w)] - [F(w)]^n\right) \mathrm{d}w \\
=& \overline{\Pi}_r.
\end{aligned}
$$

□

D Proof of Theorem 1

It remains to show that the stage-game strategy profile $\alpha^* = (\rho^*, \beta^*, \gamma^*, \varphi^*)$ satisfies the constraints (1)–(5) of Program I and attains the upper bound Π^*. We divide the proof into three parts. In Part I, we show that α^* attains the upper bound Π^*. In Part II, we show that α^* satisfies the incentive constraints (1) and (2). In Part III, we show that α^* satisfies the enforcement constraints (3), (4), and (5).

D.1 Part I: α^* attains the upper bound Π^*.

Under α^*, the allocation is efficient and the winner pays the seller the amount

$$
\max\left(v_{(2)} - \frac{L(\Pi^*,\delta)}{n-1}, 0\right).
$$

Hence,

$$\begin{aligned}\pi(\alpha^*) =& \overline{\Pi} - E\left[\max\left(v_{(2)} - \frac{L(\Pi^*, \delta)}{n-1}, 0\right)\right] \\ =& g(\Pi^*).\end{aligned} \tag{50}$$

Using Lemma 4, we proceed in three cases to show that $g(\Pi^*) = \Pi^*$. Combined with (50), this implies that

$$\pi(\alpha^*) = g(\Pi^*) = \Pi^*,$$

which proves that α^* attains the upper bound Π^*.

Suppose that $\delta \in \left(0, \frac{n-1}{n}\right]$. By Lemma 4, $\Pi^* = \underline{\Pi}$. By (39), $g(\underline{\Pi}) = \underline{\Pi}$. Hence, $g(\Pi^*) = \Pi^*$.

Suppose that $\delta \in \left(\frac{n-1}{n}, \frac{(n-1)\overline{v}}{(n-1)\overline{v}+\overline{\Pi}-\underline{\Pi}}\right)$. By Lemma 4, $\Pi^* = g(\Pi^*)$.

Suppose that $\delta \in \left[\frac{(n-1)\overline{v}}{(n-1)\overline{v}+\overline{\Pi}-\underline{\Pi}}, 1\right)$. By Lemma 4, $\Pi^* = \overline{\Pi}$. By (43), $g(\overline{\Pi}) = \overline{\Pi}$. Hence, $g(\Pi^*) = \Pi^*$.

This completes the proof that α^* attains the upper bound Π^*.

D.2 Part II: Verifying the incentive constraints (1) and (2).

Since the cartel mechanism is efficient, we have, for all v_i and all i,

$$Q_i^{\alpha^*}(v_i) = [F(v_i)]^{n-1},$$

which is increasing in v_i. To prove (1), it suffices to show that

$$\frac{\mathrm{d}}{\mathrm{d}v_i} U_i^{\alpha^*}(v_i) = [F(v_i)]^{n-1}. \tag{51}$$

Under α^*, each buyer i with valuation v_i pays the seller a positive amount

$$v_{(2)} - \frac{L(\Pi^*, \delta)}{n-1}$$

only when $v_i \geq v_{(2)}$ and $v_{(2)} \geq \frac{L(\Pi^*,\delta)}{n-1}$; hence, his expected payment to the seller is

$$P_i(v_i) \equiv \begin{cases} \int_{\frac{L(\Pi^*,\delta)}{n-1}}^{v_i} \left(w - \frac{L(\Pi^*,\delta)}{n-1}\right) \cdot (n-1)[F(w)]^{n-2} f(w)\,\mathrm{d}w, & \text{if } v_i > \frac{L(\Pi^*,\delta)}{n-1}; \\ 0, & \text{if } v_i \leq \frac{L(\Pi^*,\delta)}{n-1}. \end{cases}, \tag{52}$$

where the second term $(n-1)[F(w)]^{n-2}f(w)$ inside the integrand is the probability density of the highest order statistic among $(n-1)$ i.i.d. random variables. His expected net payment to the other cartel members is

$$\begin{aligned}T_i(v_i) \equiv &-(n-1)\int_0^{v_i} H_{\frac{L(\Pi^*,\delta)}{n-1}}(w)\cdot(n-1)[F(w)]^{n-2}f(w)\,\mathrm{d}w\\ &+H_{\frac{L(\Pi^*,\delta)}{n-1}}(v_i)\cdot(n-1)[1-F(v_i)][F(v_i)]^{n-2}\\ &+\int_{v_i}^{\overline{v}} H_{\frac{L(\Pi^*,\delta)}{n-1}}(w)\cdot(n-1)(n-2)[1-F(w)][F(w)]^{n-3}f(w)\,\mathrm{d}w.\end{aligned} \tag{53}$$

The first term of (53) is his payment to all the other $(n-1)$ buyers when buyer i's valuation is the highest one, and the integrand is the payment times the probability density of the highest order statistic among $(n-1)$ i.i.d. random variables. The second term is what buyer i will receive if his valuation is the second-highest one times the probability that this case occurs. The last term is the expected payment to him when his valuation is below the second-highest one, and the integrand is the payment times the probability density of the second-highest order statistic among $(n-1)$ i.i.d. random variables. Hence, the interim payoff to buyer i with valuation v_i can be written as

$$U_i^{\alpha^*}(v_i) = v_i[F(v_i)]^{n-1} - P_i(v_i) + T_i(v_i),$$

where the first term on the right-hand side is buyer i's expected payoff from winning the item.

We now differentiate $U_i^{\alpha^*}$. As the expression for T_i in (53) depends on whether $n=2$ or $n\geq 3$, we proceed in two cases.

Case 1. Suppose that $n=2$. Differentiating $U_i^{\alpha^*}$ and rearranging terms yields

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}v_i}U_i^{\alpha^*}(v_i) =&(F(v_i)+v_if(v_i)) - \frac{\mathrm{d}}{\mathrm{d}v_i}P_i(v_i) + \Big(-H_{\frac{L(\Pi^*,\delta)}{n-1}}(v_i)f(v_i)\\ &+[1-F(v_i)]\frac{\mathrm{d}}{\mathrm{d}v_i}H_{\frac{L(\Pi^*,\delta)}{n-1}}(v_i) - H_{\frac{L(\Pi^*,\delta)}{n-1}}(v_i)f(v_i)\Big)\\ =&F(v_i) - \frac{\mathrm{d}}{\mathrm{d}v_i}P_i(v_i) + f(v_i)\left[v_i - 2H_{\frac{L(\Pi^*,\delta)}{n-1}}(v_i)\right]\\ &+[1-F(v_i)]\frac{\mathrm{d}}{\mathrm{d}v_i}H_{\frac{L(\Pi^*,\delta)}{n-1}}(v_i).\end{aligned}$$

Case 2. Suppose that $n \geq 3$. Differentiating $U_i^{\alpha^*}$ and rearranging terms yields

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}v_i}U_i^{\alpha^*}(v_i) = &\big([F(v_i)]^{n-1} + (n-1)v_i[F(v_i)]^{n-2}f(v_i)\big) - \frac{\mathrm{d}}{\mathrm{d}v_i}P_i\left(v_i\right) \\
&+ \Bigg(-(n-1)H_{\frac{L(\Pi^*,\delta)}{n-1}}(v_i) \cdot (n-1)[F(v_i)]^{n-2}f(v_i) \\
&\quad + \frac{\mathrm{d}}{\mathrm{d}v_i}H_{\frac{L(\Pi^*,\delta)}{n-1}}(v_i) \cdot (n-1)[1-F(v_i)][F(v_i)]^{n-2} \\
&\quad + H_{\frac{L(\Pi^*,\delta)}{n-1}}(v_i) \cdot (n-1)[-f(v_i)][F(v_i)]^{n-2} \\
&\quad + H_{\frac{L(\Pi^*,\delta)}{n-1}}(v_i) \cdot (n-1)[1-F(v_i)](n-2)[F(v_i)]^{n-3}f(v_i) \\
&\quad - H_{\frac{L(\Pi^*,\delta)}{n-1}}(v_i) \cdot (n-1)(n-2)[1-F(v_i)][F(v_i)]^{n-3}f(v_i)\Bigg) \\
=&[F(v_i)]^{n-1} - \frac{\mathrm{d}}{\mathrm{d}v_i}P_i(v_i) + (n-1)[F(v_i)]^{n-2}f(v_i)\left[v_i - nH_{\frac{L(\Pi^*,\delta)}{n-1}}(v_i)\right] \\
&+ (n-1)[1-F(v_i)][F(v_i)]^{n-2}\frac{\mathrm{d}}{\mathrm{d}v_i}H_{\frac{L(\Pi^*,\delta)}{n-1}}(v_i).
\end{aligned}$$

Note that the final expressions for $\frac{\mathrm{d}}{\mathrm{d}v_i}U_i^{\alpha^*}(v_i)$ in both cases can be written as

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}v_i}U_i^{\alpha^*}(v_i) =&[F(v_i)]^{n-1} - \frac{\mathrm{d}}{\mathrm{d}v_i}P_i(v_i) + (n-1)[F(v_i)]^{n-2}f(v_i)\left[v_i - nH_{\frac{L(\Pi^*,\delta)}{n-1}}(v_i)\right] \\
&+ (n-1)[1-F(v_i)][F(v_i)]^{n-2}\frac{\mathrm{d}}{\mathrm{d}v_i}H_{\frac{L(\Pi^*,\delta)}{n-1}}(v_i).
\end{aligned} \tag{54}$$

We proceed in two cases to evaluate the right-hand side of (54), depending on whether v_i is below or above $\frac{L(\Pi^*,\delta)}{n-1}$.

Case 1. Suppose that $v_i \leq \frac{L(\Pi^*,\delta)}{n-1}$. By (52), P_i is a constant. Hence,

$$\frac{\mathrm{d}}{\mathrm{d}v_i}P_i\left(v_i\right) = 0. \tag{55}$$

We differentiate $H_{\frac{L(\Pi^*,\delta)}{n-1}}$, following its definition in (16), to get

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}v_i} H_{\frac{L(\Pi^*,\delta)}{n-1}}(v_i) \\
=&\frac{1}{n}\frac{\mathrm{d}}{\mathrm{d}v_i}\left(v_i + \frac{\int_{v_i}^{\frac{L(\Pi^*,\delta)}{n-1}} [1-F(s)]^n \,\mathrm{d}s}{[1-F(v_i)]^n}\right) \\
=&\frac{1}{n}\left(1 + \frac{-[1-F(v_i)]^n \cdot [1-F(v_i)]^n + \int_{v_i}^{\frac{L(\Pi^*,\delta)}{n-1}} [1-F(s)]^n \,\mathrm{d}s \cdot n[1-F(v_i)]^{n-1} f(v_i)}{[1-F(v_i)]^{2n}}\right) \\
=&\frac{f(v_i)\int_{v_i}^{\frac{L(\Pi^*,\delta)}{n-1}} [1-F(s)]^n \,\mathrm{d}s}{[1-F(v_i)]^{n+1}}.
\end{aligned}
\tag{56}
$$

Substituting (55), the definition of $H_{\frac{L(\Pi^*,\delta)}{n-1}}$ in (16), and (56) into (54), we have

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}v_i} U_i^{\alpha^*}(v_i) \\
=&[F(v_i)]^{n-1} - \frac{\mathrm{d}}{\mathrm{d}v_i} P_i(v_i) + (n-1)[F(v_i)]^{n-2} f(v_i)\left[v_i - nH_{\frac{L(\Pi^*,\delta)}{n-1}}(v_i)\right] \\
&+ (n-1)[1-F(v_i)][F(v_i)]^{n-2}\frac{\mathrm{d}}{\mathrm{d}v_i} H_{\frac{L(\Pi^*,\delta)}{n-1}}(v_i) \\
=&[F(v_i)]^{n-1} - 0 + (n-1)[F(v_i)]^{n-2} f(v_i)\left[v_i - n\cdot\frac{1}{n}\left(v_i + \frac{\int_{v_i}^{\frac{L(\Pi^*,\delta)}{n-1}} [1-F(s)]^n \,\mathrm{d}s}{[1-F(v_i)]^n}\right)\right] \\
&+ (n-1)[1-F(v_i)][F(v_i)]^{n-2}\frac{f(v_i)\int_{v_i}^{\frac{L(\Pi^*,\delta)}{n-1}} [1-F(s)]^n \,\mathrm{d}s}{[1-F(v_i)]^{n+1}} \\
=&[F(v_i)]^{n-1}.
\end{aligned}
$$

Case 2. Suppose that $v_i > \frac{L(\Pi^*,\delta)}{n-1}$. By the definition of $P_i(v_i)$ in (52),

$$
\frac{\mathrm{d}}{\mathrm{d}v_i} P_i(v_i) = \left[v_i - \frac{L(\Pi^*,\delta)}{n-1}\right] \cdot (n-1)[F(v_i)]^{n-2} f(v_i). \tag{57}
$$

By (16), $H_{\frac{L(\Pi^*,\delta)}{n-1}}(v_i) = \frac{1}{n(n-1)} L(\Pi^*,\delta)$ is a constant. Hence,

$$
\frac{\mathrm{d}}{\mathrm{d}v_i} H_{\frac{L(\Pi^*,\delta)}{n-1}}(v_i) = 0. \tag{58}
$$

Substituting (57), the definition of $H_{\frac{L(\Pi^*,\delta)}{n-1}}$ in (16), and (58) into (54), we have

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}v_i}U_i^{\alpha^*}(v_i)\\
=&[F(v_i)]^{n-1} - \frac{\mathrm{d}}{\mathrm{d}v_i}P_i(v_i) + (n-1)[F(v_i)]^{n-2}f(v_i)\left[v_i - nH_{\frac{L(\Pi^*,\delta)}{n-1}}(v_i)\right]\\
&+(n-1)[1-F(v_i)][F(v_i)]^{n-2}\frac{\mathrm{d}}{\mathrm{d}v_i}H_{\frac{L(\Pi^*,\delta)}{n-1}}(v_i)\\
=&[F(v_i)]^{n-1} - \left[v_i - \frac{L(\Pi^*,\delta)}{n-1}\right]\cdot(n-1)[F(v_i)]^{n-2}f(v_i)\\
&+(n-1)[F(v_i)]^{n-2}f(v_i)\left[v_i - n\cdot\frac{1}{n(n-1)}L(\Pi^*,\delta)\right]+0\\
=&[F(v_i)]^{n-1}.
\end{aligned}$$

This proves (51) and completes the verification of the revelation constraints (1) and (2).

D.3 Part III: Verifying the enforcement constraints (3), (4), and (5).

Fix any profile of valuations v. Recall that buyer ℓ denotes the designated winner. Hence, under truthful reporting, $v_\ell = v_{(1)} \geq v_j$ for all $j \neq \ell$.

First we consider the incentives of buyer ℓ. Since

$$\begin{aligned}
\beta_1^*(v) = \cdots = \beta_n^*(v) &= \max\left(v_{(2)} - \frac{L(\Pi^*,\delta)}{n-1}, 0\right)\\
&\leq v_{(2)} \leq v_\ell\\
\varphi_{\ell j}^*(v, \beta^*(v), \gamma^*(v)) &= H_{\frac{L(\Pi^*,\delta)}{n-1}}(v_{(2)}) \quad \text{for all } j \neq \ell\\
\varphi_{jk}^*(v, \beta^*(v), \gamma^*(v)) &= 0 \quad \text{for all } j \neq \ell \text{ and all } k,
\end{aligned}$$

both the enforcement constraints (3) and (5) reduce to the same constraint

$$\delta\pi_i(\alpha^{\mathrm{N}}) \leq -(1-\delta)(n-1)H_{\frac{L(\Pi^*,\delta)}{n-1}}(v_{(2)}) + \delta\pi_i(\alpha^*),$$

which, by rearranging terms, is

$$(1-\delta)(n-1)H_{\frac{L(\Pi^*,\delta)}{n-1}}(v_{(2)}) \leq \delta(\pi_i(\alpha^*) - \pi_i(\alpha^{\mathrm{N}})). \tag{59}$$

To verify (59), note that by (56), $H_{\frac{L(\Pi^*,\delta)}{n-1}}$ is increasing if $v_{(2)} \leq \frac{L(\Pi^*,\delta)}{n-1}$ and equals the cap $\frac{L(\Pi^*,\delta)}{n(n-1)}$ if $v_{(2)} \geq \frac{L(\Pi^*,\delta)}{n-1}$. Hence,

$$\begin{aligned}(1-\delta)(n-1)H_{\frac{L(\Pi^*,\delta)}{n-1}}(v_{(2)}) \leq &(1-\delta)\frac{n-1}{n(n-1)}L(\Pi^*,\delta)\\ =&\frac{\delta}{n}(\Pi^* - \underline{\Pi})\\ =&\delta(\pi_i(\alpha^*) - \pi_i(\alpha^{\mathrm{N}})),\end{aligned}$$

where the second line follows from the definition of $L(\Pi^*,\delta)$, and the last from the fact that the buyers share the total profit equally.

Next we consider the incentives of buyer $j \neq \ell$. Similarly, the enforcement constraints (4) and (5) reduce to

$$(1-\delta)\max(v_j - \beta^*_\ell(v), 0) + \delta\pi_j(\alpha^{\mathrm{N}}) \leq (1-\delta)H_{\frac{L(\Pi^*,\delta)}{n-1}}(v_{(2)}) + \delta\pi_j(\alpha^*) \qquad (60)$$

and

$$\delta\pi_j(\alpha^{\mathrm{N}}) \leq \delta\pi_j(\alpha^*), \qquad (61)$$

respectively.

Constraint (60) is equivalent to

$$\max(v_j - \beta^*_\ell(v), 0) \leq H_{\frac{L(\Pi^*,\delta)}{n-1}}(v_j) + \frac{1}{n}L(\Pi^*,\delta). \qquad (62)$$

We proceed in two cases to verify (62), depending on whether $v_{(2)}$ is below or above $\frac{L(\Pi^*,\delta)}{n-1}$.

Case 1. Suppose that $v_{(2)} < \frac{L(\Pi^*,\delta)}{n-1}$. Then

$$\begin{aligned}\max(v_j - \beta^*_\ell(v), 0) \leq &\max\left(v_{(2)} - \beta^*_\ell(v), 0\right)\\ =&v_{(2)} - \max\left(v_{(2)} - \frac{L(\Pi^*,\delta)}{n-1}, 0\right)\\ =&v_{(2)}\\ \leq&\frac{1}{n}\cdot v_{(2)} + \frac{n-1}{n}\cdot\frac{1}{n-1}L(\Pi^*,\delta)\\ \leq&H_{\frac{L(\Pi^*,\delta)}{n-1}}(v_{(2)}) + \frac{1}{n}L(\Pi^*,\delta),\end{aligned}$$

where the last inequality follows from the fact that by the definition of $H_{\frac{L(\Pi^*,\delta)}{n-1}}$, we have $\frac{v_{(2)}}{n} \leq H_{\frac{L(\Pi^*,\delta)}{n-1}}(v_{(2)})$ if $v_{(2)} < \frac{L(\Pi^*,\delta)}{n-1}$.

Case 2. Suppose that $v_{(2)} \geq \frac{L(\Pi^*,\delta)}{n-1}$. Then

$$H_{\frac{L(\Pi^*,\delta)}{n-1}}(v_{(2)}) = \frac{L(\Pi^*,\delta)}{n(n-1)},$$

and

$$\begin{aligned}\max\left(v_j - \beta_\ell^*(v), 0\right) \leq & \max\left(v_{(2)} - \beta_\ell^*(v), 0\right) \\ \leq & v_{(2)} - \beta_\ell^*(v) \\ = & v_{(2)} - \max\left(v_{(2)} - \frac{L(\Pi^*,\delta)}{n-1}, 0\right) \\ = & \frac{L(\Pi^*,\delta)}{n-1} \\ = & \frac{1}{n}\cdot\frac{L(\Pi^*,\delta)}{n-1} + \frac{n-1}{n}\cdot\frac{L(\Pi^*,\delta)}{n-1} \\ = & H_{\frac{L(\Pi^*,\delta)}{n-1}}(v_{(2)}) + \frac{1}{n}L(\Pi^*,\delta).\end{aligned}$$

This proves (62).

By definition, $\Pi^* \geq \underline{\Pi}$. Since the buyers share the total profit equally, we have $\pi_j(\alpha^*) \geq \pi_j(\alpha^{\mathrm{N}})$. This proves (61) and completes the verification of the enforcement constraints (3), (4), and (5).

E Proof of Theorem 2

By Case (ii) of Lemma 4, the value of Π^* is characterized by the following fixed-point relation,

$$\Pi^* = g(\Pi^*). \tag{63}$$

Note that by (45) and (46) of Appendix C, we have

$$g(\Pi) = \overline{\Pi} - \int_{\frac{L(\Pi,\delta)}{n-1}}^{\overline{v}} \left(1 - n[F(w)]^{n-1}[1-F(w)] - [F(w)]^n\right) \mathrm{d}w, \tag{64}$$

and

$$g'(\Pi) = \frac{\delta}{(1-\delta)(n-1)} \left(1 - n \left[F\left(\frac{L(\Pi,\delta)}{n-1} \right) \right]^{n-1} + (n-1) \left[F\left(\frac{L(\Pi,\delta)}{n-1} \right) \right]^{n} \right).$$

Using (64) and differentiating both sides of (63) with respect to δ, we have

$$\begin{aligned}
\frac{\mathrm{d}\Pi^*}{\mathrm{d}\delta} =& g'(\Pi^*)\frac{\mathrm{d}\Pi^*}{\mathrm{d}\delta} \\
&+ \left(1 - n \left[F\left(\frac{L(\Pi^*,\delta)}{n-1} \right) \right]^{n-1} + (n-1) \left[F\left(\frac{L(\Pi^*,\delta)}{n-1} \right) \right]^{n} \right) \frac{\Pi^* - \underline{\Pi}}{n-1} \frac{1}{(1-\delta)^2} \\
=& g'(\Pi^*)\frac{\mathrm{d}\Pi^*}{\mathrm{d}\delta} + g'(\Pi^*) \left[\frac{\delta}{(1-\delta)(n-1)} \right]^{-1} \frac{\Pi^* - \underline{\Pi}}{n-1} \frac{1}{(1-\delta)^2} \\
=& g'(\Pi^*)\frac{\mathrm{d}\Pi^*}{\mathrm{d}\delta} + g'(\Pi^*) \frac{\Pi^* - \underline{\Pi}}{\delta(1-\delta)}.
\end{aligned}$$

Rearranging terms yields

$$\frac{\mathrm{d}\Pi^*}{\mathrm{d}\delta} = \frac{g'(\Pi^*)}{1-g'(\Pi^*)} \frac{\Pi^* - \underline{\Pi}}{\delta(1-\delta)}. \tag{65}$$

To prove (19), differentiating $L(\Pi^*, \delta)$ using its definition in (6), we have

$$\begin{aligned}
\frac{\mathrm{d}L(\Pi^*,\delta)}{\mathrm{d}\delta} =& \frac{\delta}{1-\delta}\frac{\mathrm{d}\Pi^*}{\mathrm{d}\delta} + \frac{\Pi^* - \underline{\Pi}}{(1-\delta)^2} \\
=& \frac{1}{1-g'(\Pi^*)} \frac{\Pi^* - \underline{\Pi}}{(1-\delta)^2},
\end{aligned} \tag{66}$$

where the second line follows from (65) and rearranging terms.

By (48) of Appendix C and (72) of Appendix F, we have $g'(\Pi^*) \in (0, 1)$. Hence, the right-hand sides of (65) and (66) are both strictly positive.

For all i and $\widehat{v}$, differentiating $\beta_i^*(\widehat{v})$ with respect to δ, we have,

$$\begin{aligned}
\frac{\mathrm{d}\beta_i^*(\widehat{v})}{\mathrm{d}\delta} =& \frac{\mathrm{d}\beta_1^*(\widehat{v})}{\mathrm{d}L(\Pi^*,\delta)} \cdot \frac{\mathrm{d}L(\Pi^*,\delta)}{\mathrm{d}\delta} \\
=& \begin{cases} -\frac{1}{n-1}\frac{\mathrm{d}L(\Pi^*,\delta)}{\mathrm{d}\delta}, & \text{if } \widehat{v}_{(2)} > \frac{L(\Pi^*,\delta)}{n-1} \\ 0, & \text{if } \widehat{v}_{(2)} \le \frac{L(\Pi^*,\delta)}{n-1}. \end{cases}.
\end{aligned} \tag{67}$$

Since

$$\frac{\mathrm{d}H_M(w)}{\mathrm{d}M} = \begin{cases} \frac{1}{n}, & \text{if } w > M \\ \frac{1}{n}\frac{[1-F(M)]^n}{[1-F(w)]^n}, & \text{if } w \le M. \end{cases}, \tag{68}$$

we have, for all $\widehat{v}_{(2)}$,

$$\frac{\mathrm{d}H_{\frac{L(\Pi^*,\delta)}{n-1}}(\widehat{v}_{(2)})}{\mathrm{d}\delta} = \frac{\mathrm{d}H_{\frac{L(\Pi^*,\delta)}{n-1}}(\widehat{v}_{(2)})}{\mathrm{d}\frac{L(\Pi^*,\delta)}{n-1}} \cdot \frac{1}{n-1}\frac{\mathrm{d}L(\Pi^*,\delta)}{\mathrm{d}\delta}$$
$$= \begin{cases} \frac{1}{n(n-1)}\frac{\mathrm{d}L(\Pi^*,\delta)}{\mathrm{d}\delta}, & \text{if } \widehat{v}_{(2)} > \frac{L(\Pi^*,\delta)}{n-1} \\ \frac{1}{n(n-1)}\frac{\left[1-F\left(\frac{L(\Pi^*,\delta)}{n-1}\right)\right]^n}{[1-F(\widehat{v}_{(2)})]^n}\frac{\mathrm{d}L(\Pi^*,\delta)}{\mathrm{d}\delta}, & \text{if } \widehat{v}_{(2)} \leq \frac{L(\Pi^*,\delta)}{n-1}. \end{cases} \tag{69}$$

F Proof of Theorem 3

We proceed in three steps to prove Theorem 3. In Step 1, we consider the impact of the reserve price on the revelation and enforcement constraints that α needs to satisfy. This allows us to focus on an optimization problem similar to Program I in the main text. In Step 2, following the same steps in Section 4, we can show that Π_r^* is an upper bound of the program with a reserve price and the stage-game strategy profile α_r^* described in the main text is a candidate for the problem and attains the upper bound Π_r^*. This shows simultaneously that α_r^* is an optimal stage-game mechanism and Π_r^* is the value of the program. Hence, Π_r^* is the equilibrium cartel payoff. In Step 3, we characterize the value of Π_r^*.

F.1 Step 1. Trigger-strategy collusive agreement with a reserve price

With a reserve price, to win the item buyer i needs a bid greater than or equal to r; hence, conditional on v, buyer i's probability of winning the item under α is

$$q_i^\alpha(v) \equiv \begin{cases} 1, & \text{if } (\beta_i(v),\gamma_i(v)) > (\beta_j(v),\gamma_j(v)) \text{ for all } j \neq i \text{ and } \beta_i(v) \geq r; \\ 0, & \text{otherwise.} \end{cases}$$

Since the revelation constraints (1) and (2) and the payment-enforcement constraint (5) do not directly involve the bids, the introduction of a reserve price does not affect the statements of these constraints. It remains to consider the bid-enforcement constraints.

Suppose that $(\beta_i(v),\gamma_i(v)) > (\beta_j(v),\gamma_j(v))$ for all $j \neq i$. There are two cases to consider depending on whether $\beta_i(v)$ is greater than or less than r.

Case 1. Suppose that $\beta_i(v) \geq r$. Now if buyer i wants to deviate to a bid (b_i, κ_i) but still wins the item, his bid b_i must be greater than r as well. Hence, buyer i's bid-enforcement constraint (3) is now replaced by

$$(1-\delta)\max\left(v_i - \max\left(\max_{j\neq i}\beta_j(v), r\right), 0\right) + \delta\pi_i(\alpha_r^{\mathrm{N}}) \\ \leq (1-\delta)\left(v_i - \beta_i(v) + \chi_i^{\alpha}(v)\right) + \delta\pi_i(\alpha). \quad (70)$$

If buyer j wants to deviate and win the item, his bid must be greater than buyer i's bid $\beta_i(v)$, which by assumption is greater than r. Hence, buyer j's bid-enforcement constraint (4) remains the same.

Case 2. Suppose that $\beta_i(v) < r$. That is, the cartel decides not to win the item from the seller. In this case, each buyer $k = 1, \ldots, n$ will follow the bid recommendation instead of submitting a deviating bid to win at the auction if and only if

$$(1-\delta)(v_k - r) + \delta\pi_k(\alpha_r^{\mathrm{N}}) \leq (1-\delta)\chi_k^{\alpha}(v) + \delta\pi_k(\alpha). \quad (71)$$

To conclude, with a reserve price r, the cartel's optimal collusion problem becomes one of maximizing the cartel payoff $\pi(\alpha)$ subject to the same revelation constraints (1) and (2), the same payment-enforcement constraint (5), but a different set of bid-enforcement constraints (70), (71), and (4), i.e.,

$$\sup_{\alpha} \pi(\alpha) \quad \text{(Program II)}$$

$$\text{subject to: (1), (2), (70), (71), (4), and (5).}$$

F.2 Step 2. Showing Π_r^* is an upper bound of the value of Program II and α_r^* is a candidate for Program II

The proof of Step 2 follows exactly the same steps in Section 4 and, hence, is omitted.

F.3 Step 3. Characterizing the cartel payoff with a reserve price

In Step 3, we prove the characterization of Π_r^* in the theorem.

Case (i). Suppose that $\delta \in \left(0, \frac{n-1}{n-n[F(r)]^{n-1}+(n-1)[F(r)]^n}\right]$. By (39) and (41) and Part (iii) of Lemma 7, we have

$$
\begin{aligned}
g_r(\underline{\Pi}_r) &= \underline{\Pi}_r \\
g_r'(\underline{\Pi}_r) &= \frac{\delta}{(1-\delta)(n-1)}(1-n[F(r)]^{n-1}+(n-1)[F(r)]^n) \leq 1.
\end{aligned}
$$

That is, the graph of g_r starts from the point $(\underline{\Pi}_r, \underline{\Pi}_r)$ and is initially below the 45-degree line. By Part (i) of Lemma 7, g_r is concave. Hence, the graph of g_r lies below the 45-degree line throughout the interval $(\underline{\Pi}_r, \overline{\Pi}_r)$. It follows that $\Pi \leq g_r(\Pi)$ if and only if $\Pi = \underline{\Pi}_r$. See the left panel of Figure 6.

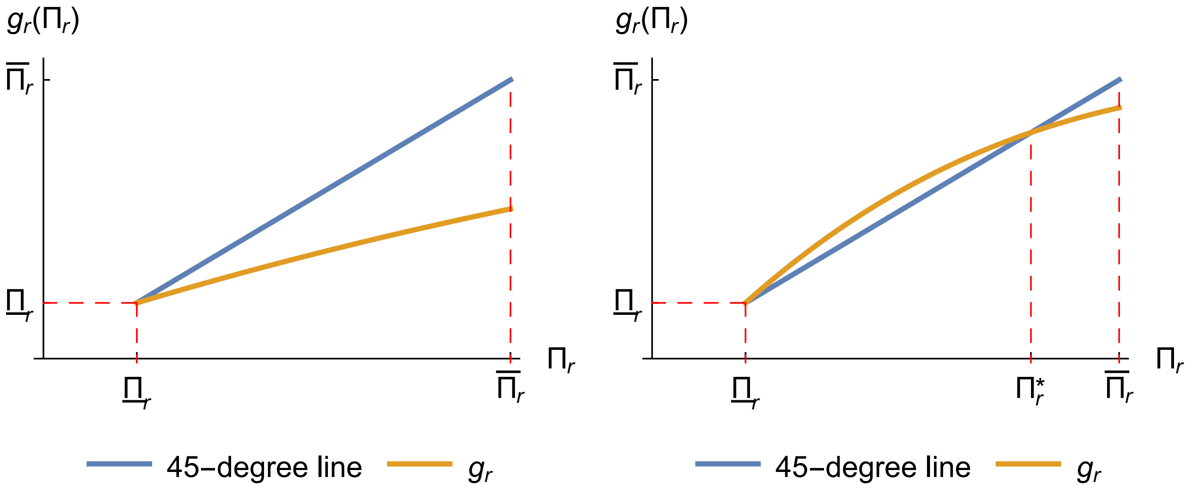


Figure 6: Illustrations of Case (i) and Case (ii).

Case (ii). Suppose that $\delta \in \left(\frac{n-1}{n-n[F(r)]^{n-1}+(n-1)[F(r)]^n}, \frac{(n-1)(\overline{v}-r)}{(n-1)(\overline{v}-r)+\overline{\Pi}_r-\underline{\Pi}_r}\right)$. By (39) and (41) and Part (iii) of Lemma 7, we have

$$
\begin{aligned}
g_r(\underline{\Pi}_r) &= \underline{\Pi}_r \\
g_r'(\underline{\Pi}_r) &= \frac{\delta}{(1-\delta)(n-1)}(1-n[F(r)]^{n-1}+(n-1)[F(r)]^n) > 1.
\end{aligned}
$$

Now the graph of g_r starts from the point $(\underline{\Pi}_r, \underline{\Pi}_r)$ but is initially above the 45-degree line. By (40), the graph must cross the 45-degree line at some point in the interval $(\underline{\Pi}_r, \overline{\Pi}_r)$. Since g_r is strictly concave, it crosses the 45-degree line exactly once. Hence, Π_r^* is the unique fixed point of g_r in $(\underline{\Pi}_r, \overline{\Pi}_r)$. Moreover, the slope of g_r at the point of intersection Π_r^* must be less than 1, i.e.,

$$g_r'(\Pi_r^*) < 1. \tag{72}$$

See the right panel of Figure 6.

Case (iii). Suppose that $\delta \in \left[\frac{(n-1)(\overline{v}-r)}{(n-1)(\overline{v}-r)+\overline{\Pi}_r-\underline{\Pi}_r}, 1\right)$. By (43), $g_r(\overline{\Pi}_r) = \overline{\Pi}_r$. It follows that $\Pi_r^* = \overline{\Pi}_r$.

G Proof of Lemma 5

In order to apply the characterization of Π_r^* in Theorem 3, we need to first compute $\overline{\Pi}_r$, $\underline{\Pi}_r$, and the difference $(\underline{\Pi}_r - \overline{\Pi}_r)$. By definition, with two buyers and uniform distribution of valuations, we have

$$\begin{aligned}
\overline{\Pi}_r =& E[\max(v_1, v_2) - r | \max(v_1, v_2) \geq r] \\
=& 2\int_r^1 \int_0^{v_1} v_1 \, \mathrm{d}v_2 \, \mathrm{d}v_1 - r(1-r^2) \\
=& 2\int_r^1 v_1^2 \, \mathrm{d}v_1 - r(1-r^2) \\
=& \frac{2}{3}(1-r^3) - r(1-r^2),
\end{aligned}$$

and

$$\begin{aligned}
\underline{\Pi}_r =& \pi(\alpha_r^{\mathrm{N}}) \\
=& 2\int_r^1 \left[v_i - \beta_{r,i}^{\mathrm{N}}(v_i)\right] \cdot v_i \, \mathrm{d}v_i \\
=& 2\int_r^1 \left[v_i - \left(v_i - \frac{\int_r^{v_i} s \, \mathrm{d}s}{v_i}\right)\right] \cdot v_i \, \mathrm{d}v_i \\
=& 2\int_r^1 \int_r^{v_i} s \, \mathrm{d}s \, \mathrm{d}v_i \\
=& \frac{1}{3} - r^2 + \frac{2}{3}r^3.
\end{aligned}$$

Hence,

$$\begin{aligned}\overline{\Pi}_r - \underline{\Pi}_r &= \left(\frac{2}{3}(1-r^3) - r(1-r^2)\right) - \left(\frac{1}{3} - r^2 + \frac{2}{3}r^3\right) \\ &= \frac{1}{3} - r + r^2 - \frac{1}{3}r^3 \\ &= \frac{1}{3}(1-r)^3.\end{aligned} \tag{73}$$

Using (73), by rearranging terms, we have

$$\delta \geq \frac{1-r}{1-r+\overline{\Pi}_r - \underline{\Pi}_r} \iff r \leq 1 - \sqrt{\frac{3(1-\delta)}{\delta}}. \tag{74}$$

Similarly, by rearranging terms, we have

$$\delta \leq \frac{1}{1+(1-r)^2} \iff r \geq 1 - \sqrt{\frac{1-\delta}{\delta}}. \tag{75}$$

Combining (74) and (75), we also have

$$\frac{1}{1+(1-r)^2} < \delta < \frac{1-r}{1-r+\overline{\Pi}_r - \underline{\Pi}_r} \iff 1 - \sqrt{\frac{3(1-\delta)}{\delta}} < r < 1 - \sqrt{\frac{1-\delta}{\delta}}. \tag{76}$$

We can now compute Π_r^* using Theorem 3.

Case (i). Suppose that $r \geq 1 - \sqrt{\frac{1-\delta}{\delta}}$. By (75), we have $\delta \leq \frac{1}{1+(1-r)^2}$. Hence, by Case (i) of Theorem 3, we have $\Pi_r^* = \underline{\Pi}_r$.

Case (ii). Suppose that $1 - \sqrt{\frac{3(1-\delta)}{\delta}} < r < 1 - \sqrt{\frac{1-\delta}{\delta}}$. By (76), we have

$$\frac{1}{1+(1-r)^2} < \delta < \frac{1-r}{1-r+\overline{\Pi}_r - \underline{\Pi}_r}.$$

Hence, by Case (ii) of Theorem 3, Π_r^* is the unique fixed point of g_r in $(\underline{\Pi}_r, \overline{\Pi}_r)$. The conclusion then follows from the following claim.

Claim 1. *If $\delta \in (\frac{1}{1+(1-r)^2}, \frac{1-r}{1-r+\overline{\Pi}_r - \underline{\Pi}_r})$, then the unique fixed point of g_r in the interval $(\underline{\Pi}_r, \overline{\Pi}_r)$ equals*

$$\Pi_r^* = \underline{\Pi}_r + \frac{1-\delta}{2\delta}\left(3(1-r) - \sqrt{\frac{12(1-\delta)}{\delta} - 3(1-r)^2}\right). \tag{77}$$

Case (iii). Suppose that $r \leq 1 - \sqrt{\frac{3(1-\delta)}{\delta}}$. By (74), we have $\delta \geq \frac{1-r}{1-r+\overline{\Pi}_r - \underline{\Pi}_r}$. Hence, by Case (iii) of Theorem 3, we have $\Pi_r^* = \overline{\Pi}_r$.

To complete the proof of Lemma 5, it remains to prove Claim 1.

G.1 Proof of Claim 1

By rearranging terms, $\delta < \frac{1-r}{1-r+\overline{\Pi}_r-\underline{\Pi}_r}$ implies that

$$\begin{aligned} L_r(\Pi_r^*, \delta) + r =& \frac{\delta}{1-\delta}(\overline{\Pi}_r - \underline{\Pi}_r) + r \\ <&1. \end{aligned} \tag{78}$$

Hence, by (45),

$$\begin{aligned} g_r(\Pi_r^*) =& \overline{\Pi}_r - \int_{L_r(\Pi_r^*,\delta)+r}^{1} \left(1 - 2w(1-w) - w^2\right) \mathrm{d}w \\ =& \overline{\Pi}_r - \frac{1}{3}[1 - L_r(\Pi_r^*, \delta) - r]^3. \end{aligned} \tag{79}$$

Using (79), the fixed-point relation can be simplified as follows,

$$\begin{aligned} \Pi_r^* =& g_r(\Pi_r^*) \\ =& \overline{\Pi}_r - \frac{1}{3}[1 - L_r(\Pi_r^*, \delta) - r]^3 \\ =& \underline{\Pi}_r + \frac{1}{3}(1-r)^3 - \frac{1}{3}[1 - L_r(\Pi_r^*, \delta) - r]^3, \end{aligned} \tag{80}$$

where the last line follows from substituting for $\overline{\Pi}_r$ using (73). Rearranging the terms of (80), we have

$$\begin{aligned} \Pi_r^* - \underline{\Pi}_r =& \frac{1}{3}(1-r)^3 - \frac{1}{3}[1 - L_r(\Pi_r^*, \delta) - r]^3 \\ =& \frac{1}{3}(1-r)^3 - \frac{1}{3}[(1-r) - L_r(\Pi_r^*, \delta)]^3 \\ =& \frac{1}{3}(1-r)^3 - \frac{1}{3}\left\{(1-r)^3 - 3(1-r)^2 L_r(\Pi_r^*, \delta) \right. \\ & \left. + 3(1-r)[L_r(\Pi_r^*, \delta)]^2 - [L_r(\Pi_r^*, \delta)]^3\right\} \\ =& -\frac{1}{3}\left\{-3(1-r)^2 L_r(\Pi_r^*, \delta) + 3(1-r)[L_r(\Pi_r^*, \delta)]^2 - [L_r(\Pi_r^*, \delta)]^3\right\} \\ =& \frac{1}{3} L_r(\Pi_r^*, \delta)\left\{3(1-r)^2 - 3(1-r)L_r(\Pi_r^*, \delta) + [L_r(\Pi_r^*, \delta)]^2\right\}. \end{aligned} \tag{81}$$

Since Π_r^* is an interior point in the interval $\left(\underline{\Pi}_r, \overline{\Pi}_r\right)$, we have $\Pi_r^* - \underline{\Pi}_r > 0$. Substituting the definition

$$L_r(\Pi_r^*, \delta) = \delta(1-\delta)^{-1}\left(\Pi_r^* - \underline{\Pi}_r\right)$$

into (81), and cancelling the term $(\Pi_r^* - \underline{\Pi}_r)$ from both sides, we have

$$1 = \frac{1}{3}\delta(1-\delta)^{-1}\Big\{3(1-r)^2 - 3(1-r)\delta(1-\delta)^{-1}\left(\Pi_r^* - \underline{\Pi}_r\right) + \left[\delta(1-\delta)^{-1}\left(\Pi_r^* - \underline{\Pi}_r\right)\right]^2\Big\}. \quad (82)$$

Rearranging the terms of (82) yields

$$\left[\delta(1-\delta)^{-1}\left(\Pi_r^* - \underline{\Pi}_r\right)\right]^2 - 3(1-r)\left[\delta(1-\delta)^{-1}\left(\Pi_r^* - \underline{\Pi}_r\right)\right] + 3(1-r)^2 - 3(1-\delta)\delta^{-1} = 0. \quad (83)$$

Treating (83) as a quadratic equation in the variable

$$\delta(1-\delta)^{-1}\left(\Pi_r^* - \underline{\Pi}_r\right),$$

and solving the equation yields

$$\begin{aligned}\delta(1-\delta)^{-1}\left(\Pi_r^* - \underline{\Pi}_r\right) &= \frac{3(1-r) - \sqrt{[3(1-r)]^2 - 4\left(3(1-r)^2 - 3(1-\delta)\delta^{-1}\right)}}{2} \\ &= \frac{3(1-r) - \sqrt{12(1-\delta)\delta^{-1} - 3(1-r)^2}}{2}.\end{aligned} \quad (84)$$

The other root of the equation,

$$\frac{3(1-r) + \sqrt{12(1-\delta)\delta^{-1} - 3(1-r)^2}}{2},$$

is discarded because it is greater than $(1-r)$, which violates the constraint (78). By rearranging terms, we have

$$\Pi_r^* = \underline{\Pi}_r + \frac{1-\delta}{2\delta}\left(3(1-r) - \sqrt{\frac{12(1-\delta)}{\delta} - 3(1-r)^2}\right).$$

H Proof of Theorem 4

For simplicity of notation, denote $d \equiv \sqrt{(1-\delta)/\delta}$. Combining (24), (25), (73), and (77), we have

$$S(r) = \begin{cases} \frac{1}{3}(1-r)^3 + r(1-r^2), & \text{if } r \geq 1-d \\ \frac{1}{3}(1-r)^3 - \frac{3d^2(1-r) - d^2\sqrt{12d^2 - 3(1-r)^2}}{2} + r(1-r^2), & \text{if } 1-\sqrt{3}d < r < 1-d \\ r(1-r^2), & \text{if } r \leq 1-\sqrt{3}d \end{cases}. \tag{85}$$

In Claim 2 below we solve the problem of maximizing S in the case of $d \geq \frac{1}{2}$.

Claim 2. *Suppose that $d \geq \frac{1}{2}$. Then S is maximized at $r = \frac{1}{2}$.*

It remains to consider $d < \frac{1}{2}$. By (85), we have an explicit expression for S on each of the three intervals $\left[0, 1-\sqrt{3}d\right]$, $\left(1-\sqrt{3}d, 1-d\right)$, and $[1-d, 1]$. In Claim 3 below we first solve the problem of maximizing S on each of the three intervals.

Claim 3. *Suppose that $d < \frac{1}{2}$.*

(i) The unique reserve price in $\left[0, 1-\sqrt{3}d\right]$ that maximizes

$$S(r) = r(1-r^2)$$

is

$$r = \begin{cases} \frac{\sqrt{3}}{3}, & \text{if } d \leq \frac{\sqrt{3}-1}{3} \approx 0.24 \\ 1-\sqrt{3}d, & \text{if } \frac{\sqrt{3}-1}{3} \leq d \leq \frac{1}{2} \end{cases}.$$

(ii) The unique reserve price in $\left[1-\sqrt{3}d, 1-d\right]$ that maximizes

$$S(r) = \frac{1}{3}(1-r)^3 - \frac{3d^2(1-r) - d^2\sqrt{12d^2 - 3(1-r)^2}}{2} + r(1-r^2)$$

is

$$r = \begin{cases} 1-\sqrt{3}d, & \text{if } d \leq \frac{\sqrt{3}-1}{3} \\ r_0, & \text{if } \frac{\sqrt{3}-1}{3} \leq d \leq \frac{3-\sqrt{5}}{2} \approx 0.38 \\ 1-d, & \text{if } \frac{3-\sqrt{5}}{2} \leq d \leq \frac{1}{2} \end{cases},$$

where r_0 is the unique interior point that solves $S'(r) = 0$.

(iii) The unique reserve price in $[1-d, 1]$ *that maximizes*

$$S(r) = \frac{1}{3}(1-r)^3 + r(1-r^2)$$

is $r = 1 - d$.

We now use Claim 3 to solve for the optimal reserve price in the case of $d < \frac{1}{2}$. By Case (iii) of Claim 3, all reserve prices strictly greater than $1-d$ are strictly dominated by $1-d$. Hence, the optimal solution is in the interval $[0, 1-d]$.

If $d \leq \frac{\sqrt{3}-1}{3}$, by the conclusion in Case (ii), all reserve prices in the interval $(1-\sqrt{3}d, 1-d]$ are strictly dominated by $1-\sqrt{3}d$. Hence, the optimal solution is in the interval $[0, 1-\sqrt{3}d]$. Now, by Case (i), the optimal reserve price is $\frac{\sqrt{3}}{3}$.

If $d \geq \frac{\sqrt{3}-1}{3}$, by Case (i), all reserve prices in the interval $[0, 1-\sqrt{3}d)$ are strictly dominated by $1-\sqrt{3}d$. Hence, the optimal solution is in the interval $[1-\sqrt{3}d, 1-d]$. Now, by Case (ii), the optimal reserve price is either r_0 or $1-d$ depending on whether $d \leq \frac{3-\sqrt{5}}{2}$ or $d \geq \frac{3-\sqrt{5}}{2}$.

To summarize, when $d < \frac{1}{2}$, S is maximized at

$$r = \begin{cases} \frac{\sqrt{3}}{3}, & \text{if } d \leq \frac{\sqrt{3}-1}{3} \\ r_0, & \text{if } \frac{\sqrt{3}-1}{3} \leq d \leq \frac{3-\sqrt{5}}{2} \\ 1-d, & \text{if } \frac{3-\sqrt{5}}{2} < d < \frac{1}{2} \end{cases}.$$

Combining this with the conclusion in Claim 2 for the case of $d \geq \frac{1}{2}$, we conclude that

$$r^* = \begin{cases} \frac{\sqrt{3}}{3}, & \text{if } d \leq \frac{\sqrt{3}-1}{3} \\ r_0, & \text{if } \frac{\sqrt{3}-1}{3} \leq d \leq \frac{3-\sqrt{5}}{2} \\ 1-d, & \text{if } \frac{3-\sqrt{5}}{2} < d < \frac{1}{2} \\ \frac{1}{2}, & \text{if } d \geq \frac{1}{2} \end{cases}.$$

Using the relation $d = \sqrt{(1-\delta)/\delta}$, we express r^* in terms of δ,

$$r^* = \begin{cases} \frac{\sqrt{3}}{3}, & \text{if } \delta > \frac{9(13+2\sqrt{3})}{157} \approx 0.94 \\ r_0, & \text{if } \frac{3+\sqrt{5}}{6} \approx 0.87 < \delta \leq \frac{9(13+2\sqrt{3})}{157} \\ 1-\sqrt{\frac{1-\delta}{\delta}}, & \text{if } \frac{4}{5} < \delta \leq \frac{3+\sqrt{5}}{6} \\ \frac{1}{2}, & \text{if } \delta \leq \frac{4}{5} \end{cases}.$$

Finally, Claim 4 shows that in the case of $\delta \in (\frac{3+\sqrt{5}}{6}, \frac{9(13+2\sqrt{3})}{157})$, the optimal reserve price r^* is decreasing in δ.

Claim 4. *Suppose that* $\frac{3+\sqrt{5}}{6} < \delta \leq \frac{9(13+2\sqrt{3})}{157}$. *Then the optimal reserve price* r^* *is decreasing in* δ.

To complete the proof of Theorem 4, it remains to prove Claims 2, 3, and 4, which is provided in an online appendix.

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