How To Sell in a Sequential Auction Market*

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Abstract

We characterize an optimal mechanism for a seller with one unit of a good facing $N \geq 3$ buyers and a single competitor who sells another identical unit in a second-price auction. Buyers who do not get the seller's good compete in the competitor's subsequent auction. The mechanism features transfers from buyers with the two highest valuations, allocation to the buyer with the second-highest valuation, and an allocation rule that depends on the two highest valuations. It can be implemented by a modified third-price auction, and it raises significantly more revenue than would a standard second- or first-price auction with a reserve price.

1 Introduction

Sequential auctions are commonly used to sell or procure multiple products or multiple units of identical products. For example, in many states, school districts hold procurement auctions for school milk contracts sequentially during the spring of each year. Sellers on online auction platforms such as eBay sell their goods individually,

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and these auctions are sequenced by their unique arrival times. Much of the theoretical literature on sequential auctions is focused on characterizing the equilibrium behavior of bidders under the assumption that sellers are passive and non-strategic. In this paper, we analyze how a seller facing competition in the form of a subsequent auction can design a mechanism to maximize revenue. We will formulate this design problem in terms of sale auctions but it should be clear from the analysis that it also applies to procurement auctions.

We consider the design problem in a setting where losing bidders compete in another, subsequent auction or market, but where the winner has satisfied his demand and drops out of the competition. Our focus is on the allocation externality that arises in this setting and the impact it has on optimal auction design. The revenue that a seller can collect from buyers is constrained by her need to incentivize them to participate in her auction. In our setting, these incentive constraints depend on the bidder's outside option, which is his payoff from competing in the subsequent market. That payoff is endogeneous: it depends on the valuations of the other buyers and on which (if any) of them wins the auction. If the good is not allocated, or not allocated efficiently, then the losing buyers face stronger competition in the subsequent market. This creates a negative payoff externality on bidders by lowering their outside option. Our primary goal in this paper is to examine how a seller can exploit this externality to extract more surplus from the bidders.

Our model is simple. There are two sellers, each with a single unit of an identical good, who sell their units sequentially to $N \geq 3$ buyers. These buyers have unit demands. Their values are private and independently drawn from a common distribution F with density f. Any buyer who fails to obtain the good from the first seller participates in the auction of the second seller. The second auction is a second-price auction with no reserve.[1] In this case, buyers have a dominant strategy to bid their value in that auction. Thus, any information that buyers obtain about their competitors' types from the first auction has no effect on their bidding behavior in the subsequent auction. Given this situation, the first seller's problem consists of

[1]Hendricks and Wiseman [2021a] examines the case where the second seller sets a non-trivial reserve price.

designing an allocation and pricing rule to maximize revenues.

We can interpret the second seller as a non-strategic government agency or a competitive market. For instance, a private equity firm may acquire and resell spectrum licenses before a scheduled FCC auction.[2] A procurement example is the market for pharmaceutical drugs in middle-income countries such as Ecuador, studied by Brugués [2020]. For a product with a specific active ingredient, first the government runs an auction in which a small number of firms bid to supply public pharmacies. Then the same firms sell to private pharmacies and compete in prices. This second market, in which the firm with the lowest cost wins the market at a price equal to the second-lowest cost, corresponds to a second-price auction. So does a setting where, as in Carroll and Segal [2019], types are revealed after the first mechanism runs and the remaining buyers have bargaining power over the second seller and make take-it-or-leave-it offers.

Our main result is a characterization of an optimal direct mechanism. We are interested in this mechanism because it establishes how much revenue a seller can achieve and, more importantly, how she can achieve it. We show that the seller's design problem reduces to a revenue-maximization problem that can be solved using standard methods from Myerson [1981]. However, the solution is quite different from the optimal mechanism of a monopoly seller. In the latter case, a seller maximizes revenues by allocating the good to the buyer with the highest reported value as long as that report is high enough. By contrast, in our model, the seller optimally allocates her good only when the *second*-highest report is large *compared to* the third-highest. This withholding rule clearly cannot be implemented by a reserve price.

The second difference is that the seller must commit to sometimes allocate the object to the second-highest bidder rather than to the highest bidder. That policy ensures that a bidder cannot increase the probability of winning a unit, either the first or second, by underreporting his type. If the good goes to the highest bidder, then a buyer i who believes that he is unlikely to have highest value may have an incentive to underreport his value in the first mechanism in order to increase the probability

[2] https://www.npr.org/2020/11/06/932048876/hacking-the-perfect-auction.

that the good is allocated (recall that allocation is more likely when the third-highest reported value is low). The resulting reduction in future competition benefits buyer i in the second auction if he turns out to have the second-highest value. Allocating the first good to the second-highest bidder eliminates this incentive to underreport. Note however that misallocation to lower-value buyers does not occur. If the first good is not allocated, then the highest type gets the second good, and if the first good is allocated, then the highest two types get the two goods.

To understand why the optimal allocation rule in our setting depends on the second- and third-highest reported values, recall the intuition behind the optimal allocation rule when there is only one seller. The maximum surplus that the single seller can create by allocating her good is $x_{(1)}$, the highest value among the buyers. However, because the values of the buyers are private information, the seller must leave them some information rent. The most that she can extract is $\psi(x_{(1)})$, where $\psi(x) \equiv x - (1 - F(x))/f(x)$ is the virtual value of a buyer with value x. Thus, the optimal allocation rule in the single seller case is to allocate only if $\psi(x_{(1)})$ is positive, and this rule can be implemented by setting a reserve price $r = \psi^{-1}(0)$.

When there is a second seller, the maximum surplus that the first seller can create for the buyers is the difference between the surpluses generated from allocating the good and from not allocating it. If the first good is not allocated, then the second good will go to the buyer with the highest value at a price equal to the second-highest value, yielding a buyer surplus of $x_{(1)} - x_{(2)}$. If the first good is allocated to either the highest or second-highest-value buyer, then the second object will go to the other of these buyers at a price equal to the third-highest value, and total surplus for the buyers is $x_{(1)} + x_{(2)} - x_{(3)}$. The difference is $2x_{(2)} - x_{(3)}$. From this difference, the first seller can extract $\psi(x_{(2)}) + x_{(2)} - x_{(3)}$ by allocating the good to the second-highest-value buyer. This buyer gains $x_{(2)}$, but the seller must pay him an information rent to learn $x_{(2)}$, and so she can extract only $\psi(x_{(2)})$ from him. The highest-value buyer gains $x_{(2)} - x_{(3)}$, the reduction in the price in the second auction. The seller does not have to pay to learn that value, because this information belongs to the second- and third-highest-value buyers, who provide it for free because the seller's use of it does not affect them. Thus, the optimal rule is to allocate whenever the first seller's share

$\psi(x_{(2)}) + x_{(2)} - x_{(3)}$ is positive and not otherwise.

The optimal mechanism for a single seller who controlled both units would never allocate to a buyer with a value less than $\psi^{-1}(0)$, the revenue-maximizing monopoly price. In contrast, the first seller in our setting sometimes sells even when the highest report is below $\psi^{-1}(0)$. The difference is that, as described above, allocating generates an additional benefit for the buyers by reducing the price in the second auction. The first seller and the buyers split that surplus at the expense of the second seller.[3]

We show that the optimal mechanism can be implemented by a modified third-price auction that is ex post incentive compatible. An unusual feature of this auction is that both the highest and second-highest bidder make payments to the seller when the good is allocated: both benefit from the good being allocated and are willing to pay to ensure that this event occurs.

The gains from using an optimal mechanism can be substantial. An interesting benchmark is when the first seller uses a standard auction with an optimal reserve price. As in Jehiel and Moldovanu [2003], the allocation externality implies that the equilibrium of that auction involves partial pooling at the reserve price. In an example with three bidders and values distributed uniformly on the unit interval, we compute the optimal reserve price and find that the first seller could increase her expected revenue by more than 25% by using the optimal mechanism instead.

Our paper is the first to study the design problem of a seller who faces competition from subsequent sellers, and it provides a first step toward a broader study of competing mechanisms in sequential sale environments.[4] The literature on sequential auctions, following Milgrom and Weber [2000], usually assumes that the goods are sold by the same seller (or, equivalently, passive sellers).[5] An exception is Kirkegaard

[3]That effect is reminiscent of the intuition in Aghion and Bolton [1987], where a buyer and an incumbent seller contract to exclude an entrant and extract some of the entrant's potential surplus.

[4]The literature on competing mechanisms is focused on static environments in which sellers with unit supply choose their mechanisms simultaneously and buyers with unit demands choose among them. Burguet and Sakovics [11] find that competition for buyers in a duopoly setting reduces the market power of sellers and forces them to lower reserve prices in second-price auctions. McAfee [23], Peters and Severinov [29], and Pai [26] show that, when the number of sellers and buyers is large, second-price auctions with zero reserve prices emerge as an equilibrium mechanism.

[5]There is a growing literature (e.g., Backus and Lewis [2020], Said [2011], and Zeithammer [2006]) that studies bidding behavior in sequential, second-price auctions in stationary environments where

and Overgaard [2008], who show that the early seller in Black and de Meza's [1992] two-period model where buyers have multi-unit demands can increase her expected revenue by offering an optimal buy-out price. Our analysis allows the early seller to consider any mechanism, in the special case of identical goods and unit demands. We find that even in this very simple environment, standard auctions are suboptimal.

This paper is closely related to the work on auctions with externalities and type-dependent outside options.[6] Jehiel and Moldovanu [2003] study the impact of interactions by buyers in a post-auction market on bidding behavior in standard auctions. Figueroa and Skreta [2009] and Jehiel et al. [1996, 1999] consider revenue-maximizing mechanisms in environments where the payoffs of the buyers can depend upon whether and to whom the object is allocated. In general, the optimal threat by the seller to minimize the continuation payoff of non-participating buyers may be complicated, and calculating the "critical type" for whom the participation constraint binds can be challenging. In our setting, the optimal threat is to maximize competition in the second auction by not allocating the first item, and the participation constraint binds only for the lowest type of buyer.

2 Model

There are $N \geq 3$ *ex ante* identical potential buyers, indexed by i, with unit demand for an indivisible good. Each buyer i's privately observed valuation for the good X_i is independently drawn from distribution F with support $[\underline{x}, \bar{x}]$, $\underline{x} \geq 0$. We will sometimes refer to a buyer's valuation as his *type*. We assume that F has a continuous density f and that the virtual valuation $\psi(x) \equiv x - (1 - F(x))/f(x)$ is increasing in x. Order the valuations from highest to lowest $X_{(1)}, X_{(2)}, \ldots, X_{(N)}$.

There are two sellers who sell identical units of the good. Each seller sells one unit. They sell their units sequentially over two periods and we refer to them in the

new buyers and sellers enter the market each period. These papers make behavioral assumptions that effectively rule out the allocation externality.

[6] This paper is also related to the recent literature on optimal design of auctions (and disclosure rules) in which the externalities are due to resale (e.g., Bergemann et al. [2020], Calzolari and Pavan [2006], Carroll and Segal [2019], Dworczak [2020], and Virág [2016]).

order that they sell. The second seller uses a second-price auction with no reserve price. Both sellers' valuations of the good are normalized to zero. This structure is common knowledge. We will characterize the revenue-maximizing mechanism for the seller in the first period, given that any buyer who does not obtain the first object will participate in the auction for the second object. In what follows, we typically refer to the first seller as just "the seller."

In our model, it is a weakly dominant strategy for any buyer who did not obtain the first object to submit a bid equal to his valuation in the second auction. Thus, in designing her mechanism, the seller does not have to be concerned about the leakage problem. Any information buyers acquire in the first period about the types of competitors does not influence their bidding behavior in the second period. As a result, buyers have no incentive to bid untruthfully in period one to affect behavior in period two. However, in period one, a buyer's bid may still influence the allocation of the first object, which does affect outcomes in the second period. The design of the revenue-maximizing mechanism for the seller must take that incentive into account.

Without loss of generality, we restrict attention to direct mechanisms in which buyers report their types. Let $\mathbf{x} \in [\underline{x}, \bar{x}]^N$ denote the vector of *reported* types. A direct mechanism in our context specifies, for any given $\mathbf{x}$, the probability that each bidder i gets the good $P_i(\mathbf{x}) \geq \mathbf{0}$ with $\Sigma_{i=1}^N P_i(\mathbf{x}) \leq \mathbf{1}$ and the payment $t_i(\mathbf{x})$ that he must make.

3 The Optimal Mechanism

Fix a buyer i with valuation X_i, and denote the valuations of the other $N-1$ buyers, ordered from highest to lowest, by $Y_{(1)}, Y_{(2)}, \ldots, Y_{(N-1)}$.The payoff to buyer i in the second period, provided that he did not obtain the first object, depends on whether or not the first object was allocated to his competitor with the highest type $Y_{(1)}$. If so, then buyer i's payoff, $\max\left\{X_i - Y_{(2)}, 0\right\}$, is a function of the highest remaining competitor's type $Y_{(2)}$. If not, then buyer i's payoff is $\max\left\{X_i - Y_{(1)}, 0\right\}$. All else equal, buyer i prefers that the first object go to his strongest competitor so that competition in the subsequent auction is reduced. Thus, the expected payoff to a buyer depends on the two highest valuations among his competitors. We denote the highest-type competitor of bidder i by j (so that $X_j = Y_{(1)}$). Then, the expected payoff to a bidder i with type x_i given vector of reports $\mathbf{x}$, excluding any payment to the first seller, is

$$P_i(\mathbf{x}) \cdot x_i + P_j(\mathbf{x}) \cdot \max\left\{x_i - y_{(2)}, 0\right\} + (1 - P_i(\mathbf{x}) - P_j(\mathbf{x})) \cdot \max\left\{x_i - y_{(1)}, 0\right\}. \quad (1)$$

The standard approach defines the payoffs of the bidders and the allocation rule in terms of the vector of reported types. However, in our case, a bidder's payoff depends not only upon reported types but also upon the highest actual types among his competitors who do not get the first object. This dependence creates problems summing Expression 1 across bidders because the set of competitors varies with the identity of the bidder. To deal with this issue, we exploit the symmetry of the bidders and re-define payoffs and allocations in terms of the vector of reported realizations of order statistics. The notation is as follows. For any vector of reported types $\mathbf{x}$, define $\hat{\mathbf{x}}$ as the vector of reported types ordered from highest to lowest (with ties broken arbitrarily). Thus, the k-th element of $\hat{\mathbf{x}}$ is the k-th highest reported type in $\mathbf{x}$ (i.e., $\widehat{x}_k = x_{(k)}$). Let $\hat{\mathbf{f}}$ denote the joint density of $\widehat{\mathbf{x}}$.

Similarly, let $\widehat{\mathbf{y}}$ denote the ordered vector of competitors' reported types facing a single buyer. Given a bidder's type x and competitors' types $\widehat{\mathbf{y}}$, let $(x; \widehat{\mathbf{y}})$ denote the ordered vector of all N types. Finally, for each $k \in \{1, \ldots, N\}$, let $\hat{p}^k(\hat{\mathbf{x}})$ denote the probability that the mechanism allocates the object to the bidder with the k-th highest report, given $\hat{\mathbf{x}}$.

Using this notation, the interim payoff of a buyer of type x who reports truthfully when other buyers are also reporting truthfully is given by

$$\Pi(x|x) = E_{\widehat{\mathbf{Y}}}\left[\begin{array}{ll} & \mathbf{1}_{x > \widehat{Y}_1} \cdot \left(\hat{p}^1 \cdot x + \hat{p}^2 \cdot \left[x - \widehat{Y}_2\right] + (1 - \hat{p}^1 - \hat{p}^2) \cdot \left[x - \widehat{Y}_1\right]\right) \\ + & \mathbf{1}_{\widehat{Y}_1 \geq x > \widehat{Y}_2} \cdot \left(\hat{p}^1 \cdot \left[x - \widehat{Y}_2\right] + \hat{p}^2 \cdot x\right) \\ + & \sum_{k=2}^{N-1} \mathbf{1}_{\widehat{Y}_k \geq x > \widehat{Y}_{k+1}} \cdot \left(\hat{p}^{k+1} \cdot x\right) \end{array}\right],$$

where for the sake of readability we omit the dependence of $\hat{p}^k$ on $(x; \widehat{\mathbf{y}})$.[7] More generally, in Appendix A.1 we derive the payoff $\Pi(q|x)$ to a buyer of type x who reports his type as q.

Next, we use the first-order incentive compatibility constraints to express the transfer payments from buyers in terms of their payoffs and the allocation rule, and then choose the allocation rule to maximize the sum of payments. Let $t(q)$ denote the expected transfer to a seller from a buyer who reports q. From the envelope theorem, the equilibrium payoff to a buyer of type x is

$$U(x) = U(\underline{x}) + \int_{\underline{x}}^{x} \Pi_2(x'|x')dx', \tag{2}$$

where $\Pi_2(x|x)$ is the partial derivative of $\Pi(q|x)$ with respect to the second argument (the buyer's true type) evaluated at the truthful report. It is given by

$$\Pi_2(x|x) = E_{\widehat{\mathbf{Y}}}\left[\mathbf{1}_{x>\widehat{Y}_1} + \mathbf{1}_{\widehat{Y}_1 \geq x > \widehat{Y}_2} \cdot \left(\hat{p}^1 + \hat{p}^2\right) + \sum_{k=2}^{N-1} \mathbf{1}_{\widehat{Y}_k \geq x > \widehat{Y}_{k+1}} \cdot \hat{p}^{k+1}\right].$$

That is, $\Pi_2(x|x)$ equals the equilibrium probability that a type-x buyer gets an object, either the first or the second. The *ex ante* expected buyer payoff is thus

$$E\left[U(X)\right] = \int_{\underline{x}}^{\bar{x}} \int_{\underline{x}}^{x} \Pi_2(x'|x')dx' f(x) = \int_{\underline{x}}^{\bar{x}} \tfrac{1-F(x)}{f(x)} \Pi_2(x|x) = E\left[\tfrac{1-F(X)}{f(X)} \Pi_2(X|X)\right],$$

so the expected transfer is

$$Et(X) = E\left[\Pi(X|X) - U(X)\right] = t(\underline{x}) - \Pi(\underline{x}|\underline{x}) + E\left[\Pi(X|X) - \tfrac{1-F(X)}{f(X)} \Pi_2(X|X)\right].$$

[7] For completeness, set $\widehat{y}_{k+1} = \underline{x}$ when $k = N-1$.

Plugging in the expressions for $\Pi(x|x)$ and $\Pi_2(x|x)$, we get

$$Et(X) = t(\underline{x}) - \Pi(\underline{x}|\underline{x}) + E\left[\begin{array}{l} \quad \mathbf{1}_{\widehat{X}_1 = X} \cdot \left(\psi(X) - \widehat{X}_2 + \hat{p}^1 \cdot \widehat{X}_2 + \hat{p}^2 \cdot \left[\widehat{X}_2 - \widehat{X}_3\right]\right) \\ + \ \mathbf{1}_{\widehat{X}_2 = X} \cdot \left(\hat{p}^1 \cdot \left[\psi(X) - \widehat{X}_3\right] + \hat{p}^2 \cdot \psi(X)\right) \\ + \ \sum_{k=3}^{N} \mathbf{1}_{\widehat{X}_k = X} \cdot \left(\hat{p}^k \cdot \psi(X)\right) \end{array}\right].$$

Because the probability that a given bidder has the k-th highest value is $1/N$ for each $k \in \{1, \dots, N\}$, we can rewrite the expected transfer as

$$\begin{aligned} Et(X) &= t(\underline{x}) - \Pi(\underline{x}|\underline{x}) \\ &+ \tfrac{1}{N} E\left(\psi(\widehat{X}_1) - \widehat{X}_2 + \hat{p}^1 \cdot \widehat{X}_2 + \hat{p}^2 \cdot \left[\widehat{X}_2 - \widehat{X}_3\right]\right) \\ &+ \tfrac{1}{N} E\left(\hat{p}^1 \cdot \left[\psi(\widehat{X}_2) - \widehat{X}_3\right] + \hat{p}^2 \cdot \psi(\widehat{X}_2)\right) \\ &+ \sum_{k=3}^{N} \tfrac{1}{N} E\left(\hat{p}^k \cdot \psi(\widehat{X}_k)\right). \end{aligned} \tag{3}$$

Next, we use that way of writing the expected transfer in solving the seller's problem. The seller maximizes expected revenue $ER(\widehat{\mathbf{X}}) = N \cdot Et(X)$ subject to incentive compatibility and individual rationality. To find the optimal allocation rule, we ignore the constraints and maximize the expectation pointwise. Given any vector of ordered types $\hat{\mathbf{x}}$, taking the derivative of the seller's expected revenue with respect to $\hat{p}^k(\hat{\mathbf{x}})$ yields

$$\frac{\partial ER(\widehat{\mathbf{X}})}{\partial \hat{p}^1(\hat{\mathbf{x}})} = \frac{\partial ER(\widehat{\mathbf{X}})}{\partial \hat{p}^2(\hat{\mathbf{x}})} = \left[\psi(\widehat{x}_2) + \widehat{x}_2 - \widehat{x}_3\right] \hat{\mathbf{f}}(\widehat{\mathbf{x}}),$$

and for all $k > 2$,

$$\frac{\partial ER(\widehat{\mathbf{X}})}{\partial \hat{p}^k(\hat{\mathbf{x}})} = \psi(\widehat{x}_k)\, \hat{\mathbf{f}}(\widehat{\mathbf{x}}).$$

There are two things to note about the derivatives. First, the marginal revenue from increasing the probability $\hat{p}^1$ of allocating to the highest bidder is exactly the same as from increasing the probability $\hat{p}^2$ of allocating to the second-highest bidder. Both are $\psi(x_{(2)}) + x_{(2)} - x_{(3)}$. Intuitively, allocating the unit to either bidder means that both will obtain a good since the other bidder gets the second good at the third-highest valuation, $x_{(3)}$. Not allocating the good means that only the highest bidder will get a good (the second one), and he will pay the second-highest valuation, $x_{(2)}$. The difference in surplus between the first case $(x_{(1)} + x_{(2)} - x_{(3)})$ and the second case $(x_{(1)} - x_{(2)})$ is $2x_{(2)} - x_{(3)}$. Leaving some surplus for the buyers to incentivize truth-telling results in replacing one of the $x_{(2)}$ terms with the corresponding virtual valuation $\psi(x_{(2)})$, and so the seller's marginal revenue is $\psi(x_{(2)}) + x_{(2)} - x_{(3)}$.

The second thing to note is that the marginal benefit from increasing $\hat{p}^1$ or $\hat{p}^2$ exceeds the marginal benefit from increasing the probability $\hat{p}^k$ of allocating to any lower-ranked bidder $k > 2$. Because $x_{(2)} \geq x_{(3)}$ and the virtual valuation $\psi(\cdot)$ is increasing, we have

$$\psi(x_{(2)}) + x_{(2)} - x_{(3)} \geq \psi(x_{(k)}),$$

with strict inequality if $x_{(2)} > x_{(k)}$. The solution to the seller's maximization problem, then, is to allocate to one of the top two bidders $(\hat{p}^1 + \hat{p}^2 = 1)$ as long as

$$\psi(x_{(2)}) + x_{(2)} - x_{(3)} \geq 0, \tag{4}$$

and not to allocate ($\hat{p}^k = 0$ for all k) otherwise.[8]

That reserve rule is a function of the second- and third-highest valuations. The unit is allocated for certain if $\psi(x_{(2)}) \geq 0$ because in that case the inequality holds. If $x_{(2)} + \psi(x_{(2)}) < 0$, then the unit is certain not to be allocated. For intermediate values of $x_{(2)}$, it may or may not be allocated, depending on the realization of the third order statistic: for low values of $x_{(3)}$, the unit may be allocated even when $\psi(x_{(2)}) < 0$ so that $x_{(2)}$ is below the usual monopoly price.

Finally, in order to verify that we have found the optimal mechanism we need to check that the solution to the relaxed problem satisfies the incentive compatibility and individual rationality constraints. Because allocating the first object to any buyer

[8] If $\psi(\underline{x})$, the virtual valuation of the lowest possible type, is positive, then the object is always allocated, because $\psi(x_{(2)}) + x_{(2)} \geq x_{(2)} \geq x_{(3)}$.

weakly increases the total payoff to every buyer, withholding the first object minimizes the buyers' payoffs. Thus, the individual rationality condition is that $U(x)$ exceeds the expected payoff that a buyer of type x could get from the second auction given that the first seller allocates her unit to no one. As usual, incentive compatibility implies that the mechanism is individually rational for all types if it is individually rational for a buyer of the lowest type $\underline{x}$. The above argument implies that $t(\underline{x}) = \Pi(\underline{x}|\underline{x}) = 0$, so individual rationality for a type-$\underline{x}$ buyer is satisfied.

The mechanism is incentive compatible if for any type x and any reports q, q' such that $q > x > q'$, we have $\Pi_2(q|x) \geq \Pi_2(x|x) \geq \Pi_2(q'|x)$. It turns out that there is a subtlety relative to the standard mechanism design environment. Those conditions correspond to the requirement that that a bidder cannot increase the probability of the event that he wins a unit, either the first or the second, by underreporting his type, or decrease the probability of this event by overreporting his type. Allocating to the second-highest bidder (conditional on the good being allocated at all) satisfies that requirement, but allocating to the top bidder may not.

The reason that assigning the object to the bidder with the highest valuation may violate incentive compatibility comes from the fact that the condition for allocating the good, Expression 4, is decreasing in the third-highest report $x_{(3)}$. To understand why this can be a problem, suppose x_i is the second-highest value and $x_i + \psi(x_i) < x_{(3)}$. Conditional on the realized types of other bidders, reporting truthfully means that bidder i does not get a unit, because the first unit will not be allocated and the highest bidder will get the second. However, if bidder i reports a type $q < x_{(3)}$ such that $x_{(3)} + \psi(x_{(3)}) > q$, then the first unit will be allocated. In that case, bidder i gets the second unit if the first unit is allocated to the highest bidder but not if it is allocated to the second-highest bidder.

That intuition shows how a downward misreport can increase bidder i's probability of getting an object conditional on other bidders' types.It is straightforward to construct examples for the interim stage, when the bidder knows only his own value, in order to show that allocating to the highest bidder is not incentive compatible. Specifically, we show that this is the case when values are distributed on $[0, 1]$ with cdf $F(x) = x^2$. We then derive a more general sufficient condition for incentive compatibility to be violated.

Suppose that the first seller in our example allocates only to the highest type, and for simplicity assume that $N = 3$. Consider a buyer i with value x_i just above $\hat{x} = \sqrt{1/5}$, where $\hat{x}$ solves $\hat{x} + \psi(\hat{x}) = 0$. By deviating to a report $q < x_i$, buyer i can affect his probability of getting an object only if he has the second-highest value: if he has the highest type but not the highest reported type, then he would win the second auction, and if there are two higher types, then he would not get the first unit and would lose the second auction. Conditional on having the second-highest value, if buyer i reports his type truthfully, $q = x_i$, then he gets an object (the second) if $x_{(3)} \approx 0$, because then the first unit is allocated to the highest bidder. If instead he reports the lowest possible value, $q = 0$, then that allocation occurs if $x_{(3)} \in \left[\sqrt{1/5}, x_i\right]$. Because the probability density $f(x) = 2x$ is higher at $x = \sqrt{1/5}$ than at $x = 0$, this deviation increases buyer i's chance of getting an object.

Next, we show how to generalize that example. We introduce the following notation.

Definition. For $x \in [\underline{x}, \bar{x}]$, define $a(x) \equiv \min\left\{a \in [\underline{x}, \bar{x}] : a + \psi(a) \geq x\right\}$.

Using this notation, the optimality condition for allocating the good is that $x_{(2)}$ exceeds $\max\left\{a\left(x_{(3)}\right), x_{(3)}\right\}$. Note that $a(x) > x$ when $\psi(x) < 0$ and $a(x) \leq x$ when $\psi(x) \geq 0$. Therefore, the condition implies that the first seller always allocates when $\psi\left(x_{(2)}\right) \geq 0$, never allocates when $x_{(2)} < a(\underline{x})$, and sometimes allocates when $x_{(2)} \geq a(\underline{x})$ and $\psi\left(x_{(2)}\right) < 0$, depending upon the value of $x_{(3)}$.

We can duplicate the reasoning from the example. Suppose that $\psi(\underline{x}) < 0$, so that $a(\underline{x}) > \underline{x}$, and suppose as before that the first seller allocates only to the highest type. For a buyer i with value x_i just above $a(\underline{x})$ (so that $\psi(x_i) + x_i$ is just above $\underline{x}$), reporting the lowest possible value, $q = \underline{x}$, increases his chance of getting an object when $x_{(3)}$ is more likely, conditional on being below x_i, to fall above $a(\underline{x})$ than below $a^{-1}(x_i)$.

That difference in likelihood holds, roughly, when the distribution F has more density around $a(\underline{x})$ than around $\underline{x}$. A sufficient condition for that difference to hold is that $f(a(\underline{x})) > f(\underline{x}) \cdot [1 + \psi'(a(\underline{x}))]$. When that sufficient condition is satisfied, then for any number of buyers N there is a neighborhood of values above $a(\underline{x})$ such that buyers with values in that neighborhood can increase the probability of allocation by reporting $q = \underline{x}$. As a result, incentive compatibility requires that the mechanism allocate to the second-highest bidder and not to the highest bidder whenever the second-highest bid $x_{(2)}$ falls in that neighborhood and $x_{(2)} \geq a\left(x_{(3)}\right)$. The sufficient condition is satisfied if values are distributed on $[0, 1]$ with cdf $F(x) = x^{\alpha}$ for any $\alpha > 1$.[9]

Thus, allocating to the second-highest bidder when Expression 4 is satisfied ensures that the mechanism is incentive compatible. (Details are in Appendix A.3.) We summarize that optimal mechanism in Theorem 1 below. Other, more complicated allocation rules that sometimes assign the object to the highest bidder also work. For example, allocating to the highest bidder (or randomizing between the top two bidders) when $\psi\left(x_{(2)}\right) \geq 0$ and to the second-highest bidder in other cases satisfies incentive compatibility.[10]

Theorem 1. The following is an optimal (direct) mechanism for the first seller. (Ties are broken randomly.)

a. ***Allocation rule:*** *The seller allocates the good to the bidder with the second-highest valuation if* $\psi(x_{(2)}) + x_{(2)} - x_{(3)} \geq 0$*, and does not allocate otherwise.*

b. ***Transfers:***

   i. *If the good is allocated, then the bidder with the highest valuation pays* $\max\left\{a\left(x_{(3)}\right) - x_{(3)}, 0\right\}$*, the bidder with the second-highest valuation pays* $\max\left\{a\left(x_{(3)}\right), x_{(3)}\right\}$*, and the other bidders pay nothing.*

   ii. *If the good is not allocated, then there are no payments.*

[9] If buyers discount between periods, then they prefer getting the first unit to getting the second unit. When this is the case, the first seller is no longer indifferent between allocating her unit to the highest or second-highest bidder: the marginal revenue is strictly higher from allocating to the highest bidder than to the second-highest bidder. As a result, the solution to the relaxed problem fails to be incentive compatible whenever $f\left(a\left(\underline{x}\right)\right) > f\left(\underline{x}\right) \cdot \left[1 + \psi'\left(a\left(\underline{x}\right)\right)\right]$.

[10] The downward deviation described above cannot increase the allocation probability for a bidder with value x_i such that $\psi\left(x_i\right) \geq 0$, because conditional on having the second-highest value, truthful reporting always leads to allocation.

c. ***Revenue:*** *The expected revenue is* $E \max\left\{\psi(X_{(2)}) + X_{(2)} - X_{(3)}, 0\right\}$.

Note that $a(x) > x$ exactly when $\psi(x) < 0$. The highest bidder, then, makes a payment only when $\psi\left(x_{(3)}\right) < 0$ and the good is allocated ($x_{(2)} \geq a\left(x_{(3)}\right)$). For the second-highest bidder, $a\left(x_{(3)}\right)$ acts as a reserve price that depends on the value of $x_{(3)}$. Paralleling the standard mechanism design setting, a bidder who gets an object (either the first or the second) pays a transfer equal to his gross payoff minus the gap between his valuation and the smallest valuation at which he would still get an item, holding fixed the types of the other bidders. For example, suppose that $\psi\left(x_{(3)}\right) < 0$ and $x_{(2)} \geq a\left(x_{(3)}\right)$, so that the first object is allocated. Then the bidder with the highest type wins the second auction and gets a gross payoff of $x_{(1)} - x_{(3)}$. The lowest valuation at which he would get an object is $a\left(x_{(3)}\right)$: if his valuation were below $a\left(x_{(3)}\right)$, then the first object would not be allocated and he would lose the second auction. Thus, his payment is $x_{(1)} - x_{(3)} - \left[x_{(1)} - a\left(x_{(3)}\right)\right] = a\left(x_{(3)}\right) - x_{(3)}$.

The expected revenue expression is obtained by substituting the optimal allocation rule into Expression 3, integrating, and recognizing that $E[\psi(X_{(1)}] = E[X_{(2)}]$. For comparison, we note that the maximum expected revenue that the seller can obtain when he must sell the unit with probability 1 is $E[X_{(3)}]$.[11] The optimal "must sell" mechanism is revenue equivalent to a first or second-price auction with no reserve price (Milgrom and Weber [2000]).

3.1 Example: Three bidders, uniform valuations

To illustrate the working of the optimal mechanism, suppose that there are three buyers whose valuations are distributed uniformly between zero and one. That is, $N = 3$ and $F(x) = x$. In that case, virtual valuations are given by $\psi(x) = 2x - 1$. The reserve rule is to allocate when $3x_{(2)} - 1 > x_{(3)}$. Then the good is always allocated when $x_{(2)} \geq \psi^{-1}(0) = \frac{1}{2}$ and never allocated when $x_{(2)} < a(0) = \frac{1}{3}$. Figure 1 illustrates the combinations of values of $x_{(2)}$ and $x_{(3)}$ that lead to allocation. The allocation region

[11] It follows from the above analysis that the expected revenue from the optimal "must sell" mechanism is $E\left[\psi(X_{(2)}) + X_{(2)} - X_{(3)}\right]$, and we know from Loertscher and Marx's [2020] Lemma 1 that $E[\psi\left(X_{(2)}\right)] = 2E[X_{(3)}] - E[X_{(2)}]$.

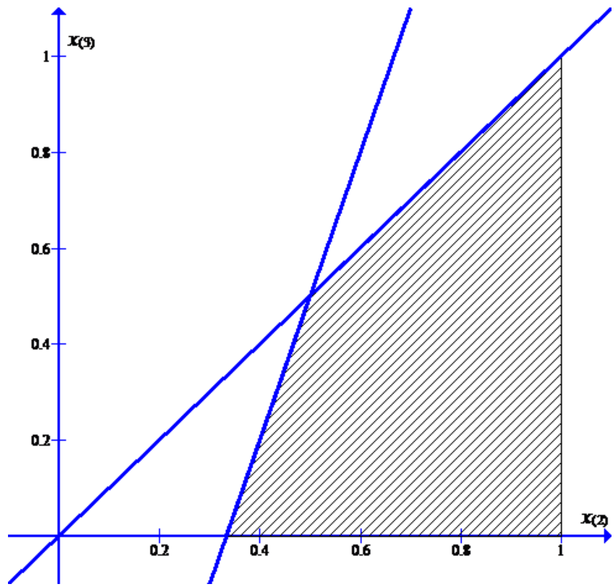

Figure 1: Allocation region.

is given by the striped area below the diagonal.

The seller's expected revenue is $\frac{55}{144} \approx 0.382$. We can also compute the expected revenue to the second seller: $\frac{125}{432} \approx 0.289$. By contrast, in the absence of any reserve rule, both sellers earn $E[X_{(3)}] = 0.25$. Thus, the second seller also benefits when the first seller uses the optimal reserve rule.

4 Implementing the Optimal Mechanism

It is straightforward to implement the payments and allocation rule from Theorem 1 in a version of a third-price auction. Define the *modified third-price auction* as follows: each buyer submits a bid in $[\underline{x}, \bar{x}]$. As a function of the vector of bids $\mathbf{b}$, the good is allocated to the second-highest bidder if and only if $b_{(2)} \geq a(b_{(3)})$. If the unit is not allocated, then no one makes any payments. If the unit is allocated, then the payments are based on the third-highest bid, $b_{(3)}$. When $\psi(b_{(3)}) > 0$, the highest bidder pays nothing and the second-highest bidder pays $b_{(3)}$; when $\psi(b_{(3)}) < 0$, then the highest bidder pays $a(b_{(3)}) - b_{(3)} > 0$ and the second-highest bidder pays $a(b_{(3)})$.

Theorem 2. Truthful bidding is an ex post equilibrium of the modified third-price auction, and that equilibrium yields the optimal expected revenue for the first seller.

In order to see that truthful reporting is ex post incentive compatible, consider, for example, the highest-valuation buyer in the case where $\psi(x_{(3)}) < 0$ and the item is allocated ($x_{(2)} \geq a(x_{(3)})$). Truthfully bidding $b = x_{(1)}$ yields a payoff of

$$x_{(1)} - x_{(3)} - \left[a(x_{(3)}) - x_{(3)}\right] = x_{(1)} - a(x_{(3)});$$

the bidder transfers $a(x_{(3)}) - x_{(3)})$ to the first seller and then wins the second auction at price $x_{(3)}$. Any bid above $x_{(2)}$ yields that same payoff. A bid between $a(x_{(3)})$ and $x_{(2)}$ also results in payoff $x_{(1)} - a(x_{(3)})$: the bidder gets the first item and transfers $a(x_{(3)})$ to the first seller. Any bid below $a(x_{(3)})$ gives a lower payoff, $x_{(1)} - x_{(2)}$, because the first item will not be allocated, no transfers will be made to the first seller, and the bidder will win the second item at price $x_{(2)}$. The other cases are similar.

It is also possible to implement the optimal mechanism using a modified first-price or "pay your bid" auction where buyers submit a bid according to a strictly increasing bid function. If the item is allocated, then both the highest and second-highest bidders pay their bids. If the item is not allocated, then only the highest bidder pays his bid. In either case, the highest bidder then gets a rebate equal to the sale price of the second item ($x_{(3)}$ if the first item is allocated, $x_{(2)}$ if it is not) assuming that he wins the second auction. The highest bidder does not get a rebate if he does not win the second auction. Details of the construction are available in the working paper version (Hendricks and Wiseman [2021b]).

5 Revenue Comparisons

In this section, we compare the expected revenue of the optimal mechanism in our uniform example to the expected revenue of a standard auction with an optimal reserve price. Relative to an optimal standard auction, how much better does the optimal mechanism with its more complicated reserve rule do? We show that the gains can be substantial. The first step is to derive the symmetric equilibrium in the first auction with a positive reserve price, r_1. This derivation turns out to be a significant challenge. Allocating the good in the first auction generates a positive externality for the losing buyers, and Jehiel and Moldovanu [2003] show that a second-price auction with positive externalities does not have a pure-strategy symmetric separating equilibrium. Instead, a symmetric equilibrium with partial pooling at the reserve price can exist. In that equilibrium, an interval of types $[\hat{x}, \hat{\hat{x}}]$ all bid r_1, types above $\hat{\hat{x}}$ bid according to a strictly increasing $\beta(x)$, and types below $\hat{x}$ do not bid. We construct such an equilibrium for our example and then calculate the optimal reserve price and revenues for the first seller.

We calculate that the optimal reserve price is $r_1^* \approx 0.379$. The corresponding values of $\hat{x}$ and $\hat{\hat{x}}$ are $\hat{x} \approx 0.60$ and $\hat{\hat{x}} \approx 0.82$, and the resulting maximal revenue is $R_1(r_1^*) \approx 0.303$. (Details are in Appendix B.)

The revenue of the second seller is the second-highest valuation $x_{(2)}$ if the first seller does not allocate and the third-highest valuation $x_{(3)}$ otherwise – except if all three valuations are between $\hat{x}$ and $\hat{\hat{x}}$ and the first seller randomly allocates to the buyer with valuation $x_{(3)}$, in which case the second seller gets $x_{(2)}$ instead of $x_{(3)}$. Overall, the expected revenue for the second seller when the first seller sets the optimal reserve price is $R_2(r_1^*) \approx 0.282$.

A striking feature of this equilibrium is that the threshold for bidding, $\hat{x}$, is significantly higher than the optimal reserve price, r_1^*. The outside option of winning the second auction at a price below r_1^* causes types between r_1^* and $\hat{x}$ not to bid in the first auction. Their lack of participation gives the high types an incentive to participate because they are more likely to win the first auction at price equal to r_1^*. As a result, only 40% of the buyers bid in the first auction and roughly half of them bid the reserve price.

Table 1 summarizes the revenue results for our uniform example.

Table 1: Revenue Comparisons		
	First Seller Revenues	Second Seller Revenues
Optimal Mechanism	0.382	0.289
Must-Sell Mechanism	0.250	0.250
Optimal Second Price Auction	0.303	0.282

In comparison to the must-sell mechanism, the optimal mechanism increases the expected revenues of both sellers, by 53% for the first seller and by 16% for the second seller. A standard auction with an optimal reserve price gives the second seller essentially the same increase but gives the first seller only a 21% increase in revenues. This example suggests that reserve prices in standard auctions are not a very effective way for the first seller to increase revenues in a sequential auction setting.

6 Concluding Remarks

We view our analysis as a first step in studying competition in mechanisms between sellers in a sequential setting. The design of the first seller's optimal mechanism can be straightforwardly extended to environments in which either or both sellers have multiple units, as long as buyers have unit demand. In Hendricks and Wiseman [2021a], we analyze the case where the second seller's auction has a non-trivial reserve price in order to study a simplified form of competition, where the second seller chooses her reserve price knowing that the first seller will respond with an optimal mechanism.[12]

More broadly, we hope that our analysis can be used as a framework for studying sequential competition in mechanisms between sellers, in a setting where the second

[12] A positive reserve price for the second seller introduces a significant complication: the solution to the first seller's optimization problem given first-order incentive constraints turns out to violate global incentive compatibility. Nevertheless, we show that the basic features of the optimal mechanism for the no-reserve case are preserved.

seller also acts strategically in choosing her mechanism. There are many ways of modeling such competition with regard to the timing of moves and information revelation. One interesting possibility is to assume that types are revealed after the first seller runs her auction (as is done in Bergemann et al. [2020] and in Carroll and Segal [2019]) and to model the second stage as a Nash bargaining game between the second seller and the remaining buyer with the highest value. In that case, the surplus from allocating the first object to the highest bidder strictly exceeds the surplus from allocating to the second-highest bidder. As a result, the pointwise revenue maximizing solution specifies allocation to the highest type only, and so it may fail to be incentive compatible.

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Appendix

A Proving Theorem 1

A.1 Payoff from false report

We derive the payoff to a type-x buyer who reports his type as q. If $q \geq x$, then

$$\Pi(q|x) = E_{\widehat{\mathbf{Y}}} \left[\begin{array}{cl} & \mathbf{1}_{x>\widehat{Y}_1} \cdot \left(x - \widehat{Y}_1 + \hat{p}^1 \left(\left(q; \widehat{\mathbf{Y}} \right) \right) \cdot \widehat{Y}_1 + \hat{p}^2 \left(\left(q; \widehat{\mathbf{Y}} \right) \right) \cdot \left[\widehat{Y}_1 - \widehat{Y}_2 \right] \right) \\ + & \mathbf{1}_{q>\widehat{Y}_1 \geq x > \widehat{Y}_2} \cdot \left(\hat{p}^1 \left(\left(q; \widehat{\mathbf{Y}} \right) \right) \cdot x + \hat{p}^2 \left(\left(q; \widehat{\mathbf{Y}} \right) \right) \cdot \left[x - \widehat{Y}_2 \right] \right) \\ + & \mathbf{1}_{q>\widehat{Y}_1 > \widehat{Y}_2 \geq x} \cdot \left(\hat{p}^1 \left(\left(q; \widehat{\mathbf{Y}} \right) \right) \cdot x \right) \\ + & \mathbf{1}_{\widehat{Y}_1 \geq q > \geq x > \widehat{Y}_2} \cdot \left(\hat{p}^1 \left(\left(q; \widehat{\mathbf{Y}} \right) \right) \cdot \left[x - \widehat{Y}_2 \right] + \hat{p}^2 \left(\left(q; \widehat{\mathbf{Y}} \right) \right) \cdot x \right) \\ + & \displaystyle\sum_{k=1}^{N-1} \mathbf{1}_{\widehat{Y}_k \geq q > \widehat{Y}_{k+1}, \widehat{Y}_2 \geq x} \cdot \left(\hat{p}^{k+1} \left(\left(q; \widehat{\mathbf{Y}} \right) \right) \cdot x \right) \end{array} \right].$$

If $q < x$, then

$$\Pi(q|x) = E_{\widehat{\mathbf{Y}}} \left[\begin{array}{cl} & \mathbf{1}_{q>\widehat{Y}_1} \cdot \left(x - \widehat{Y}_1 + \hat{p}^1 \left(\left(q; \widehat{\mathbf{Y}} \right) \right) \cdot \widehat{Y}_1 + \hat{p}^2 \left(\left(q; \widehat{\mathbf{Y}} \right) \right) \cdot \left[\widehat{Y}_1 - \widehat{Y}_2 \right] \right) \\ + & \displaystyle\sum_{k=1}^{N-1} \mathbf{1}_{\widehat{Y}_k \geq q > \widehat{Y}_{k+1}, x > \widehat{Y}_1} \cdot \left(\begin{array}{c} x - \widehat{Y}_1 \\ + \hat{p}^1 \left(\left(q; \widehat{\mathbf{Y}} \right) \right) \cdot \left[\widehat{Y}_1 - \widehat{Y}_2 \right] \\ + \hat{p}^{k+1} \left(\left(q; \widehat{\mathbf{Y}} \right) \right) \cdot \widehat{Y}_1 \end{array} \right) \\ + & \displaystyle\sum_{k=1}^{N-1} \mathbf{1}_{\widehat{Y}_k \geq q > \widehat{Y}_{k+1}, \widehat{Y}_1 \geq x > \widehat{Y}_2} \cdot \left(\begin{array}{c} \hat{p}^1 \left(\left(q; \widehat{\mathbf{Y}} \right) \right) \cdot \left[x - \widehat{Y}_2 \right] \\ + \hat{p}^{k+1} \left(\left(q; \widehat{\mathbf{Y}} \right) \right) \cdot x \end{array} \right) \\ + & \displaystyle\sum_{k=2}^{N-1} \mathbf{1}_{\widehat{Y}_k \geq q > \widehat{Y}_{k+1}, \widehat{Y}_2 \geq x} \cdot \left(\hat{p}^{k+1} \left(\left(q; \widehat{\mathbf{Y}} \right) \right) \cdot x \right) \end{array} \right].$$

A.1.1 $\Pi_2(q|x)$

The derivative of the payoff with respect to its second argument (the buyer's true type), $\Pi_2(q|x)$, will be used below. If $q \geq x$, then we calculate that derivative as

$$\Pi_2(q|x) = E_{\widehat{\mathbf{Y}}}\left[\begin{array}{ll} & \mathbf{1}_{x>\widehat{Y}_1} + \mathbf{1}_{\widehat{Y}_1 \geq x > \widehat{Y}_2} \cdot \left(\hat{p}^1\left(\left(q;\widehat{\mathbf{Y}}\right)\right) + \hat{p}^2\left(\left(q;\widehat{\mathbf{Y}}\right)\right)\right) \\ + & \mathbf{1}_{q>\widehat{Y}_1>\widehat{Y}_2 \geq x} \cdot \left(\hat{p}^1\left(\left(q;\widehat{\mathbf{Y}}\right)\right)\right) \\ + & \sum_{k=1}^{N-1} \mathbf{1}_{\widehat{Y}_k \geq q > \widehat{Y}_{k+1}, \widehat{Y}_2 \geq x} \cdot \left(\hat{p}^{k+1}\left(\left(q;\widehat{\mathbf{Y}}\right)\right)\right) \end{array}\right].$$

If $q < x$, then

$$\Pi_2(q|x) = E_{\widehat{\mathbf{Y}}}\left[\begin{array}{ll} & \mathbf{1}_{x>\widehat{Y}_1} + \sum_{k=1}^{N-1} \mathbf{1}_{\widehat{Y}_k \geq q > \widehat{Y}_{k+1}, \widehat{Y}_1 \geq x > \widehat{Y}_2} \cdot \left(\begin{array}{c} \hat{p}^1\left(\left(q;\widehat{\mathbf{Y}}\right)\right) \\ +\hat{p}^{k+1}\left(\left(q;\widehat{\mathbf{Y}}\right)\right) \end{array}\right) \\ + & \sum_{k=2}^{N-1} \mathbf{1}_{\widehat{Y}_k \geq q > \widehat{Y}_{k+1}, \widehat{Y}_2 \geq x} \cdot \left(\hat{p}^{k+1}\left(\left(q;\widehat{\mathbf{Y}}\right)\right)\right) \end{array}\right].$$

A.2 Convexity

We show that the payoff $\Pi(q|x)$ is convex in its second argument (the buyer's true type). That convexity implies that $U(x)$, as the maximum of convex functions, is also convex. It is therefore absolutely continuous and so differentiable almost everywhere, and thus Expression 2 is valid.

The intuition is as follows: the derivative of $\Pi(q|x)$ with respect to the buyer's type corresponds to the probability that the buyer gets an item (either the first or the second). Conditional on the report, that probability is increasing in the buyer's type because a buyer with a higher valuation is more likely to win the second auction if he does not win the first item. Formally, the second derivative of the payoff $\Pi(q|x)$

with respect to the buyer's true type, $\Pi_{22}(q|x)$, when $q \geq x$ is given by

$$\Pi_{22}(q|x) = E_{\widehat{\mathbf{Y}}} \left[\begin{array}{ll} & \mathbf{1}_{\widehat{Y}_1 = x} \cdot \left(1 - \hat{p}^1\left(\left(q; \widehat{\mathbf{Y}}\right)\right) - \hat{p}^2\left(\left(q; \widehat{\mathbf{Y}}\right)\right)\right) \\ + & \mathbf{1}_{q > \widehat{Y}_1, \widehat{Y}_2 = x} \cdot \left(\hat{p}^2\left(\left(q; \widehat{\mathbf{Y}}\right)\right)\right) \\ + & \mathbf{1}_{\widehat{Y}_1 \geq q, \widehat{Y}_2 = x} \cdot \left(\hat{p}^1\left(\left(q; \widehat{\mathbf{Y}}\right)\right)\right) \end{array} \right].$$

The first line represents the increase in the chance of getting an item when the buyer's type moves from just below the highest competitor's type $\widehat{Y}_1$ to just above it: if $x > \widehat{Y}_1$, then the buyer gets an item for sure because he would win the second auction. If $x < \widehat{Y}_1$, then he gets an item only if he or the highest competitor gets the first item. Similarly, the second and third lines represent the increase in the chance of getting an item when the buyer's type moves from just below the second highest competitor's type $\widehat{Y}_2$ to just above it. Each of the three terms is weakly positive, so $\Pi_{22}(q|x) \geq 0$.

Analogously, when $q < x$, $\Pi_{22}(q|x)$ is given by

$$\Pi_{22}(q|x) = E_{\widehat{\mathbf{Y}}} \left[\begin{array}{ll} & \sum_{k=1}^{N-1} \mathbf{1}_{\widehat{Y}_k \geq q > \widehat{Y}_{k+1}, \widehat{Y}_1 = x} \cdot \left(1 - \hat{p}^1\left(\left(q; \widehat{\mathbf{Y}}\right)\right) - \hat{p}^{k+1}\left(\left(q; \widehat{\mathbf{Y}}\right)\right)\right) \\ + & \sum_{k=2}^{N-1} \mathbf{1}_{\widehat{Y}_k \geq q > \widehat{Y}_{k+1}, \widehat{Y}_2 = x} \cdot \left(\hat{p}^1\left(\left(q; \widehat{\mathbf{Y}}\right)\right)\right) \end{array} \right].$$

Thus, $\Pi(q|x)$ is convex in the buyer's valuation, as desired.

A.3 Incentive compatibility

Truthful reporting is a best response if and only if for all $x, q \in [\underline{x}, \bar{x}]$,

$$U(x) = \Pi(x|x) - t(x) \geq \Pi(q|x) - t(q) = U(q) + \Pi(q|x) - \Pi(q|q) = U(q) + \int_q^x \Pi_2(q|x')dx'. \tag{5}$$

By substituting Expression 2 into Expression 5, we can rewrite the incentive compatibility condition as

$$\int_q^x \Pi_2(x'|x')dx' \geq \int_q^x \Pi_2(q|x')dx'.$$

That condition holds if for any type x and any reports q, q' such that $q > x > q'$, we have $\Pi_2(q|x) \geq \Pi_2(x|x) \geq \Pi_2(q'|x)$. The allocation rule in Theorem 1 has the property that $\hat{p}^k(\widehat{\mathbf{x}}) = 0$ when $k > 2$ for all $\widehat{\mathbf{x}}$, so the expressions for $\Pi_2(q|x) - \Pi_2(x|x)$ simplify. If $q > x$, then

$$\Pi_2(q|x) - \Pi_2(x|x) = E_{\widehat{\mathbf{Y}}}\left[\begin{array}{rl} & \mathbf{1}_{\widehat{Y}_1 \geq x > \widehat{Y}_2} \cdot \left(\begin{array}{rc} & \hat{p}^1\left(\left(q; \widehat{\mathbf{Y}}\right)\right) + \hat{p}^2\left(\left(q; \widehat{\mathbf{Y}}\right)\right) \\ - & \left(\hat{p}^1\left(\left(x; \widehat{\mathbf{Y}}\right)\right) + \hat{p}^2\left(\left(x; \widehat{\mathbf{Y}}\right)\right)\right) \end{array}\right) \\ + & \mathbf{1}_{q > \widehat{Y}_1 > \widehat{Y}_2 \geq x} \cdot \left(\hat{p}^1\left(\left(q; \widehat{\mathbf{Y}}\right)\right)\right) \\ + & \mathbf{1}_{\widehat{Y}_1 \geq q > \widehat{Y}_2 \geq x} \cdot \left(\hat{p}^2\left(\left(q; \widehat{\mathbf{Y}}\right)\right)\right) \end{array}\right]. \tag{6}$$

If $q < x$, then

$$\Pi_2(x|x) - \Pi_2(q|x) = E_{\widehat{\mathbf{Y}}}\left[\begin{array}{rl} & \mathbf{1}_{\widehat{Y}_1 \geq x > q > \widehat{Y}_2} \cdot \left(\begin{array}{rc} & \hat{p}^1\left(\left(x; \widehat{\mathbf{Y}}\right)\right) + \hat{p}^2\left(\left(x; \widehat{\mathbf{Y}}\right)\right) \\ - & \left(\hat{p}^1\left(\left(q; \widehat{\mathbf{Y}}\right)\right) + \hat{p}^2\left(\left(q; \widehat{\mathbf{Y}}\right)\right)\right) \end{array}\right) \\ + & \mathbf{1}_{\widehat{Y}_1 \geq x > \widehat{Y}_2 \geq q} \cdot \left(\begin{array}{rc} & \hat{p}^1\left(\left(x; \widehat{\mathbf{Y}}\right)\right) + \hat{p}^2\left(\left(x; \widehat{\mathbf{Y}}\right)\right) \\ - & \hat{p}^1\left(\left(q; \widehat{\mathbf{Y}}\right)\right) \end{array}\right) \end{array}\right]. \tag{7}$$

Because the allocation rule in Theorem 1 has the property that $\hat{p}^1(\widehat{\mathbf{x}}) + \hat{p}^2(\widehat{\mathbf{x}})$ is weakly increasing in $\widehat{x}_1$ and $\widehat{x}_2$, Expressions 6 and 7 are positive, as desired. We conclude that the mechanism in Theorem 1 is incentive compatible.

B Second-Price Auction with Reserve Price

To characterize the equilibrium, we introduce some notation. For $k \in \{0, 1, 2\}$, define $p_k(\hat{x}, \hat{\hat{x}})$ as the probability that a buyer has exactly k rivals with types between $\hat{x}$ and $\hat{\hat{x}}$ and no rivals with a higher type:

$$p_0 = [F(\hat{x})]^{N-1} = (\hat{x})^2,$$

$$p_1 = (N-1)\left[F(\hat{\hat{x}}) - F(\hat{x})\right][F(\hat{x})]^{N-2} = 2(\hat{\hat{x}} - \hat{x})\hat{x},$$

$$p_2 = \frac{(N-1)(N-2)}{2}\left[F(\hat{\hat{x}}) - F(\hat{x}\right]^2 [F(\hat{x})]^{N-3} = (\hat{\hat{x}} - \hat{x})^2.$$

Define D_k as the expected value of the highest rival type in the second auction, conditional on k and conditional on one of the k rivals winning the first auction if $k > 0$:

$$D_0 = E\left[Y_1 | Y_1 < \hat{x}\right] = \frac{2}{3}\hat{x},$$

$$D_1 = E\left[Y_2 | Y_1 \in \left[\hat{x}, \hat{\hat{x}}\right], Y_2 < \hat{x}\right] = \frac{1}{2}\hat{x},$$

$$D_2 = \frac{1}{2}E\left[Y_1 | Y_1, Y_2 \in \left[\hat{x}, \hat{\hat{x}}\right]\right] + \frac{1}{2}E\left[Y_2 | Y_1, Y_2 \in \left[\hat{x}, \hat{\hat{x}}\right]\right] = \frac{1}{2}(\hat{\hat{x}} + \hat{x}).$$

Finally, for $x \in [\hat{x}, \hat{\hat{x}}]$, let

$$L(x) \equiv E\left[\mathbf{1}_{\widehat{Y}_1 \geq \hat{\hat{x}}, \widehat{Y}_2 < x} \cdot [x - \widehat{Y}_2]\right]$$

denote the expected payoff in the second auction conditional on the winner of the first auction having a type above $\hat{\hat{x}}$, times the probability of that event. The dependence of p_k, D_k, and $L(x)$ on $\hat{x}$ and $\hat{\hat{x}}$ is suppressed for readability.

The cutoff values $\hat{x}$ and $\hat{\hat{x}}$ are characterized by two indifference conditions. A buyer of type $\hat{\hat{x}}$ is indifferent between bidding r (and tying with other types in $[\hat{x}, \hat{\hat{x}}]$) and bidding just above r; a buyer of type $\hat{x}$ is indifferent between bidding r and not bidding. That is,

$$\begin{aligned}
&p_0(\hat{\hat{x}} - r) + p_1\left[\frac{1}{2}(\hat{\hat{x}} - r) + \frac{1}{2}(\hat{\hat{x}} - D_1\right] + p_2\left[\frac{1}{3}(\hat{\hat{x}} - r) + \frac{2}{3}(\hat{\hat{x}} - D_2)\right] + L(\hat{\hat{x}}) \\
&= p_0(\hat{\hat{x}} - r) + p_1(\hat{\hat{x}} - r) + p_2(\hat{\hat{x}} - r) + L(\hat{\hat{x}})
\end{aligned}$$

and

$$\begin{aligned}
&p_0(\hat{x} - r) + p_1\left[\frac{1}{2}(\hat{x} - r) + \frac{1}{2}(\hat{x} - D_1\right] + p_2\frac{1}{3}(\hat{x} - r) + L(\hat{x}) \\
&= p_0(\hat{x} - D_0) + p_1(\hat{x} - D_1) + p_2 \cdot 0 + L(\hat{x}).
\end{aligned}$$

Taking differences, $\hat{\hat{x}}$ solves

$$p_1 \frac{1}{2}(D_1 - r) + p_2 \frac{2}{3}(D_2 - r) = 0$$

and $\hat{x}$ solves

$$p_0(D_0 - r) + p_1 \frac{1}{2}(D_1 - r) + p_2 \frac{1}{3}(\hat{x} - r) = 0.$$

Plugging in the values of p_k and D_k gives

$$\hat{x}\left(\frac{1}{2}\hat{x} - r\right) + (\hat{\hat{x}} - \hat{x})\frac{2}{3}\left(\frac{1}{2}(\hat{\hat{x}} + \hat{x}) - r\right) = 0 \tag{8}$$

and

$$(\hat{x})^2\left(\frac{2}{3}\hat{x} - r\right) + (\hat{\hat{x}} - \hat{x})\hat{x}\left(\frac{1}{2}\hat{x} - r\right) + \frac{1}{3}(\hat{\hat{x}} - \hat{x})^2(\hat{x} - r) = 0. \tag{9}$$

The solutions to Expressions 8 and 9 are $\hat{x} = (1 + 1/\sqrt{3})r$ and $\hat{\hat{x}} = (1 + 2/\sqrt{3})r$.

Let F_1 denote the distribution of the first order statistic $X_{(1)}$, with density f_1. Let $F_{2|x_{(1)}}$ and $f_{2|x_{(1)}}$ denote the distribution and density, respectively, of the second order statistic $X_{(1)}$ conditional on the value of the first order statistic $X_{(1)} = x_{(1)}$. Then we can write the revenue of the first seller as

$$\begin{aligned}
R_1(r_1) &= \left[F_1(\hat{\hat{x}}) - F_1(\hat{x})\right] r_1 + \int_{\hat{\hat{x}}}^{1} \left[F_{2|x_{(1)}}(\hat{\hat{x}}) r_1 + \int_{\hat{\hat{x}}}^{x_{(1)}} \beta(x_{(2)}) f_{2|x_{(1)}}(x_{(2)})\right] f_1(x_{(1)}) \\
&= \left[(\hat{\hat{x}})^3 - (\hat{x})^3\right] r_1 + \int_{\hat{\hat{x}}}^{1} \left[\frac{(\hat{\hat{x}})^2}{(x_{(1)})^2} r_1 + \int_{\hat{\hat{x}}}^{x_{(1)}} \frac{x_{(2)}}{2} \frac{2x_{(2)}}{(x_{(1)})^2}\right] 3(x_{(1)})^2 \\
&= \frac{1}{4} - (\hat{\hat{x}})^3 + \frac{3}{4}(\hat{\hat{x}})^4 + r_1\left[3(\hat{\hat{x}})^2 - 2(\hat{\hat{x}})^3 - (\hat{x})^3\right],
\end{aligned}$$

where the last line follows from a lot of tedious calculations. Plugging in $\hat{x} = (1 + 1/\sqrt{3})r$ and $\hat{\hat{x}} = (1 + 2/\sqrt{3})r$ and performing more tedious calculation gives

$$R_1(r_1) = \frac{1}{4} + r_1^3\left[\frac{6\sqrt{3} + 10}{3\sqrt{3}}\right] - r_1^4\left[\frac{47\sqrt{3} + 80}{12\sqrt{3}}\right].$$

We can now determine the optimal reserve price. Differentiating with respect to r_1 yields the first-order condition

$$3r_1^2\left[\frac{6\sqrt{3}+10}{3\sqrt{3}}\right] - 4r_1^3\left[\frac{47\sqrt{3}+80}{12\sqrt{3}}\right] = 0.$$

Solving for the optimal reserve price r_1^* yields

$$r_1^* = \frac{3\left[6\sqrt{3}+10\right]}{\left[47\sqrt{3}+80\right]} \approx 0.379.$$

The corresponding values of $\hat{x}$ and $\hat{\hat{x}}$ are $\hat{x} = (1 + 1/\sqrt{3})r_1^* \approx 0.60$ and $\hat{\hat{x}} = (1 + 2/\sqrt{3})r_1^* \approx 0.82$.

Substituting the optimal reserve into the revenue function yields the maximal revenue, which is

$$\begin{aligned} R_1(r_1^*) &= \frac{1}{4} + (r_1^*)^3\left[\frac{6\sqrt{3}+10}{3\sqrt{3}}\right] - (r_1^*)^4\left[\frac{47\sqrt{3}+80}{12\sqrt{3}}\right] \\ &= \frac{1}{4} + \frac{27}{256}\frac{\left(\frac{6\sqrt{3}+10}{3\sqrt{3}}\right)^4}{\left(\frac{47\sqrt{3}+80}{12\sqrt{3}}\right)^3} \approx 0.303. \end{aligned}$$

We can also compute the expected revenue for the second seller when the first seller sets the optimal reserve price. She gets the second-highest valuation $x_{(2)}$ if the first seller does not allocate and the third-highest valuation $x_{(3)}$ otherwise – except if all three valuations are between $\hat{x}$ and $\hat{\hat{x}}$ and the first seller randomly allocates to the buyer with valuation $x_{(3)}$, in which case the second seller gets $x_{(2)}$ instead of $x_{(3)}$:

$$
\begin{aligned}
R_2(r_1^*) &= \int_0^{\hat{x}} E\left[X_{(2)}|X_{(1)}=x_{(1)}\right] f_1(x_{(1)}) + \int_{\hat{x}}^{\overline{x}} E\left[X_{(3)}|X_{(1)}=x_{(1)}\right] f_1(x_{(1)}) \\
&+ \frac{1}{3}E\left[X_{(2)}-X_{(3)}|X_{(1)},X_{(2)},X_{(3)} \in \left[\hat{x},\hat{\hat{x}}\right]\right] \Pr\left[X_{(1)},X_{(2)},X_{(3)} \in \left[\hat{x},\hat{\hat{x}}\right]\right] \\
&= \int_0^{\hat{x}} \frac{2}{3}x_{(1)}3(x_{(1)})^2 + \int_{\hat{x}}^{\overline{x}} \frac{1}{3}x_{(1)}3(x_{(1)})^2 \\
&+ \frac{1}{3}\left[\left(\frac{1}{2}\hat{x}+\frac{1}{2}\hat{\hat{x}}\right) - \left(\frac{3}{4}\hat{x}+\frac{1}{4}\hat{\hat{x}}\right)\right]\left(\hat{\hat{x}}-\hat{x}\right)^3 \\
&= \frac{1}{4}+\frac{1}{4}(\hat{x})^4 - \frac{1}{12}\left(\hat{\hat{x}}-\hat{x}\right)^4 \approx 0.282.
\end{aligned}
$$