Stable matching: an integer programming approach*

Chao Huang†

Abstract

This paper develops an integer programming approach to two-sided many-to-one matching by investigating stable integral matchings of a fictitious market where each worker is divisible. We show that stable matching exists in a discrete matching market when firms' preference profile satisfies a unimodularity condition that is compatible with various forms of complementarities. We provide a class of firms' preference profiles that satisfy the unimodularity condition.

Keywords: two-sided matching; stability; integer programming; many-to-one matching; complementarity; unimodularity; demand type

JEL classification: C61, C78, D47, D63

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†Institute for Social and Economic Research, Nanjing Audit University. Email: huangchao916@163.com

1 Introduction

Studies of two-sided matching originated with the seminal work of Gale and Shapley (1962) on the markets of marriage and college admission. In the past decades, the theory of two-sided matching has provided practical solutions to real-life matching problems such as hospital-doctor matching, college admission, and school choice.[1] Kelso and Crawford (1982) and Roth (1984) have recognized different versions of substitutability conditions that are sufficient for the existence of a stable matching in different settings. The substitutability condition for a discrete matching market[2] requires that any worker chosen by a firm from a set of available workers should still be chosen when the available set shrinks.[3] Most real-life matching practices rely on the substitutability condition. Complementarities in firms' preferences have been regarded as the primary source of difficulties for market design. For instance, consider the following market borrowed from Che, Kim, and Kojima (2019) (henceforth, CKK)[4] where there are two firms f_1, f_2 and two workers w_1, w_2. The agents have the following preferences.

$$\begin{aligned} &f_1 : F\{w_1, w_2\} \succ \varnothing && w_1 : \quad f_1 \succ f_2 \\ &f_2 : \{w_1\} \succ \{w_2\} \succ \varnothing && w_2 : \quad f_2 \succ f_1 \end{aligned} \tag{1.1}$$

[1]See Roth and Peranson (1999), Roth (2002), Abdulkadiroğlu and Sönmez (2003), Abdulkadiroğlu et al. (2009), Echenique and Yenmez (2015), and Kamada and Kojima (2015), among others.

[2]We use the term "discrete matching market" to distinguish it from the matching market with continuous monetary transfers and quasi-linear utilities; see Section 1.1.

[3]If a worker is chosen by a firm from a set of available workers but not chosen when the available set shrinks, we would deem that some of the worker's complements become unavailable as the set shrinks.

[4]See Section 2 of CKK. Literature on the quasi-linear market also uses examples of a similar structure to illustrate the nonexistence of equilibrium caused by complementarities; see e.g., Azevedo et al. (2013) and Baldwin and Klemperer (2019).

Firm f_1's preference violates the substitutability condition since w_1 and w_2 are complements for f_1.[5] No stable matching exists in this market.[6]

The lack of stable matching has been attributed to complementarities in firms' preferences. However, consider the following market where f_2's preference is changed.

$$\begin{aligned} & f_1 : \{w_1, w_2\} \succ \emptyset & & w_1 : \quad f_1 \succ f_2 \\ & f_2 : \{w_1, w_2\} \succ \{w_1\} \succ \{w_2\} \succ \emptyset & & w_2 : \quad f_2 \succ f_1 \end{aligned} \tag{1.2}$$

There exists a stable matching in this market where f_2 hires both w_1 and w_2. The complementarity in f_1's preference does not distort stability in this market. This paper shows that the firms' preference profile in the latter example satisfies a unimodularity condition that guarantees a stable matching for all possible preferences of workers, whereas this condition is violated in the former example. Unimodularity is a condition for a set of vectors, which roughly speaking, requires the determinant of any square matrix formed by any of the vectors to be 0 or ± 1.[7] We find that the notion of demand type proposed by Baldwin and Klemperer (2019) (henceforth, BK) is useful for analyzing firms' preferences in a discrete matching market.[8] When the available set of workers expands, f_1 would hire both workers when they are available, thus f_1 has a demand type of $\{(1, 1)\}$. When the available set of workers for f_2 expands from $\{w_2\}$ to $\{w_1, w_2\}$, f_2 would drop w_2 and hire w_1 in the former example, thus the demand type of f_2 contains $(1, -1)$, whereas the

[5]This is because w_1 would be hired by f_1 from $\{w_1, w_2\}$ but not hired by f_1 from $\{w_1\}$. Similarly, w_2 would be hired by f_1 from $\{w_1, w_2\}$ but not hired by f_1 from $\{w_2\}$.

[6]In this market, f_1 should hire both workers or neither in any stable matching. In the former case, f_2 would form a blocking coalition with w_2, who prefers f_2 to f_1. In the latter case, f_2 would be matched with w_1, leaving w_2 unmatched. Then, f_1 would form a blocking coalition with both w_1 and w_2.

[7]See Definition 5 for the formal definition of the unimodularity for a set of vectors.

[8]BK studied the exchange economy where agents have quasi-linear utilities over indivisible goods, which we call the quasi-linear market. See Section 1.1.

demand type of f_2 does not contain $(1,-1)$ in the latter example. The demand type of f_2 is $\{(1,0),(0,1)(1,-1)\}$ in the former example, and $\{(1,1),(1,0),(0,1)\}$ in the latter example.[9] Therefore, the firms' demand type is $\{(1,1),(1,0),(0,1),(1,-1)\}$ in the former example, which is not unimodular (because the determinant of $(1,1)$ and $(1,-1)$ is -2).[10] The firms' demand type is $\{(1,1),(1,0),(0,1)\}$ in the latter example, which is unimodular (because any matrix formed by two of these vectors has determinant 0 or ± 1). In this paper, we show that this unimodularity of firms' preferences guarantees the existence of a stable matching for all possible preferences of workers.

Previous studies on discrete matching markets are mostly based on the Gale-Shapley mechanism or Taski's fixed-point theorem.[11] This paper develops a new integer programming approach. We prove our results by studying stable integral matchings of a continuum market induced from the original market. We assume that each worker is divisible and construct a particular continuous preference on worker shares for each firm. This continuum market is an instance of CKK's continuum market, and CKK's existence theorem guarantees the existence of a stable matching in our continuum market. Each stable integral matching of this market corresponds to a stable matching of the original market. Finally, we can apply tools from integer programming to prove the existence of a stable integral matching in the continuum market when firms' demand type is unimodular. In a discrete matching market, unimodularity is independent of substitutability and compati-

[9] In both examples, when the available set of workers for f_2 expands from $\emptyset$ to $\{w_1\}$ or $\{w_2\}$, f_2 would hire w_1 or w_2. Thus $(1,0)$ and $(0,1)$ are in f_2's demand type in both examples. In the latter example, the demand type of f_2 also contains $(1,1)$, since when the available set of workers expands from $\emptyset$ to $\{w_1, w_2\}$, f_2 would hire both workers.

[10] See Definition 4 for the formal definition of firms' demand type, which contains the demand type of the null firm.

[11] The Gale-Shapley mechanism with its variants is a standard tool for two-sided matching. See Section 1.1 for literature on the fixed-point method.

ble with various complementarities.[12] Therefore, our result indicates that stable matching is compatible with various forms of complementarities in firms' preferences. This observation runs contrary to the common belief that complementarities distort stability.

Our existence theorem applies to a problem of matching firms with specialists, which has a practical economic meaning. We depict the structure of firms' acceptable sets of workers by a technology tree, where each vertex of the tree represents a technology that requires a set of workers to implement. Each edge of the tree is an upgrade from one technology to another that requires more workers. We say that a worker is a specialist if she engages in only one upgrade. We show that firms' demand type is unimodular when firms have preferences over the technologies in a technology tree where each worker is a specialist. See Section 4.

1.1 Related literature

There are two parallel lines in the literature of matching theory. The discrete matching market we study assumes that there are no monetary transfers (school choice and college admission) or that workers' wages are exogenously given (firm-worker matching and hospital-doctor matching). The other line assumes that there are continuous monetary transfers between firms and workers where firms have quasi-linear utilities and workers' wages are determined endogenously. This market can be viewed as an exchange economy where agents have quasi-linear utilities over indivisible goods.[13] We call both the matching market and the exchange economy of this line the quasi-linear market, where the solution concept may be stable matching or equilibrium.

[12] In a quasi-linear market, substitutability implies unimodularity (see Section 3.2 of BK). However, the two conditions are independent of each other in a discrete matching market, see Section 2.3.

[13] See Section 6.4 of BK.

Kelso and Crawford (1982) found that stable matching exists in a quasi-linear market when each firm has a gross-substitute valuation over workers. Roth (1984, 1985) found that stable matching exists in a discrete many-to-many market when all agents have substitutable preferences. In quasi-linear markets and discrete matching markets, Sun and Yang (2006, 2009) and Ostrovsky (2008) respectively generalized the restriction of substitutability to allow complementarities between two group of agents where substitutability is satisfied within each group.[14] Danilov et al. (2001) showed that a unimodularity condition is sufficient for the existence of an equilibrium in a quasi-linear market.[15] BK proved this result independently and enhanced it into the unimodularity theorem, which states that equilibrium exists for all profiles of concave valuations of a demand type if and only if the demand type is unimodular.

This paper is closely related to CKK, with BK's notion of demand type playing a critical role. In a discrete matching market, CKK found that stable matching always exists with a continuum of workers when firms have continuous preferences, which can exhibit various complementarities. Their approach enables market designers to pursue an approximately stable matching, which is guaranteed to exist in a real-life market when the market is large enough. CKK proved their existence theorem for the continuum market using the Brouwer (or Kakutani) fixed-point theorem.[16] Thus, the existence of a stable matching proved in this paper is essentially the existence of an integral Brouwer fixed point, which is quite different from previous methods in matching theory.

[14] Ostrovsky (2008) studied the problem of supply chain networks, which subsumes the two-sided matching as a special case; see also Hatfield et al. (2013).

[15] The proof was provided in Danilov and Koshevoy (2004) as a special case of Theorem 3 therein.

[16] CKK applied the Kakutani-Fan-Glicksberg theorem to prove their existence theorem in a general setting, allowing for indifferences in firms' preferences and infinite worker types. Our continuum market is the basic case with strict preferences and finite worker types, where stable matchings correspond to Brouwer fixed points.

BK's demand type considers how demand changes as prices change, while our demand type considers how demand changes as available workers change. BK proved their results using tools from tropical geometry. Tran and Yu (2019) showed that the unimodularity theorem for the quasi-linear market could also be proved via integer programming. Although we adopt the notion of demand type from BK and use a similar tool as Tran and Yu (2019), our method is quite different from theirs. The unimodularity theorem for the quasi-linear market is built on the following result: Equilibrium always exists in a quasi-linear market if and only if the aggregate valuation is concave.[17] In a discrete matching market, there is neither counterpart to the notion of the aggregate valuation nor counterpart to the concavity condition. Therefore, the methods for quasi-linear markets do not apply to discrete matching markets. Our method is to study the existence of a stable integral matching in a market with "divisible" workers, where stable matching always exists according to CKK. Moreover, the unimodularity theorem for the quasi-linear market assumes that each agent has a concave valuation. By contrast, our existence theorem does not involve any concavity condition.

Our method also relates to the linear programming (henceforth, LP) method for selecting stable matchings of specific properties. Vande Vate (1989) showed the polytope defined by the stable matchings in the marriage market and solved the optimal marriage problem as a linear program. This method was then developed by Rothblum (1992), Roth et al. (1993), and Baïou and Balinski (2000). Teo and Sethuraman (1998) and Sethuraman et al. (2006) used this method to find a median stable matching, which is a compromise between agents of the two sides among stable matchings. We discuss the relation between the LP method and our method in Section 3.4. Biró et al. (2014) and Ágoston et al. (2016) used integer programming methods to study the complexities of matching with couples and special college admission problems, respectively.

[17] See e.g., Lemma 2 of Tran and Yu (2019).

Stable matchings in a discrete matching market have been characterized as Taski's fixed points; see Adachi (2000), Fleiner (2003), Echenique and Oviedo (2004), Hatfield and Milgrom (2005), and Echenique and Oviedo (2006), among others. A more straightforward method for our purpose would be studying the system defined by the fixed-point characterization under the unimodularity condition. The fixed-point method is appealing for our problem because the fixed point characterization provides a necessary and sufficient condition that can be used for studying further generalizations or variations of the unimodularity condition.[18] Although it is so far unknown how to obtain our result using the fixed-point method, proving our result using the fixed-point method would be useful for future research.

Complementarities were also considered in the problem of matching with couples (e.g., Klaus and Klijn, 2005 and Nguyen and Vohra, 2018) and matching with peer effects (see Echenique and Yenmez, 2007 and Pycia, 2012). CKK is a part of the literature that treated the problem of complementarity by pursuing approximate stable outcomes; see also Azevedo et al. (2013), Kojima et al. (2013), and Azevedo and Hatfield (2018), among others. The setting of discrete matching has been generalized to allow discrete contract terms between firms and workers (see Roth, 1984 and Hatfield and Milgrom, 2005), where contracts may specify wages, insurances, retirement plans, etc. Under the generalized setting, Hatfield and Kojima (2010) proposed weakened substitutability conditions that are not implied by the Kelso-Crawford gross substitutes condition (see Echenique, 2012); see also Hatfield and Kominers (2019). We can also extend our model to the framework of matching with contracts by studying stable integral matchings of the market in Section S.9 of CKK.[19] The author's recent works, Huang (2021a) and Huang (2021b), provide

[18]For instance, we can not tell whether stable matching exists under some generalized unimodularity condition if this condition does not guarantee an integral solution to the constructed linear equations in Section 3. By contrast, we can confirm that stable matching does not exist when there is no fixed point for the related operator in the fixed point method.

[19]CKK extended their results to the framework of matching with contract terms in their Section

sufficient conditions for stable matching where firms' preferences may violate both the substitutability condition and the unimodularity condition.

The remainder of this paper is organized as follows. Section 2 presents the model of a discrete matching market and the existence theorem. Section 3 elaborates on our method for the proof of the existence theorem. Section 4 provides an application of matching firms with specialists. Proofs are relegated to the Appendix.

2 Model

2.1 Preliminaries

There is a set $F = \{f_1, \ldots, f_m\}$ of m firms, and a set $W = \{w_1, \ldots, w_n\}$ of n workers. Let ø be the null firm, representing not being matched with any firm. Each worker $w \in W$ has a strict, transitive, and complete preference $\succ_w$ over $\widetilde{F} := F \cup \{ø\}$. For any $f, f' \in \widetilde{F}$, we write $f \succ_w f'$ when w prefers f to f' according to $\succ_w$. We write $f \succeq_w f'$ if either $f \succ_w f'$ or $f = f'$. Let $\succ_W$ denote the preference profile of all workers. Each firm $f \in F$ has a strict, transitive and complete preference $\succ_f$ over 2^W. For any $S, S' \subseteq W$, we write $S \succ_f S'$ when f prefers S to S' according to $\succ_f$. We write $S \succeq_f S'$ if either $S \succ_f S'$ or $S = S'$. Let $\succ_F$ be the preference profile of all firms. A matching market can be summarized as a tuple $\Gamma = (W, F, \succ_W, \succ_F)$.

Let Ch_f be the choice function of f such that for any $S \subseteq W$, $Ch_f(S) \subseteq S$ and $Ch_f(S) \succeq_f S'$ for any $S' \subseteq S$. By convention, let $Ch_ø(S) = S$ for all $S \subseteq W$. For any $f \in F$, $w \in W$ and $S \subseteq W$, we say that f is acceptable to w if $f \succ_w ø$; we say that S is acceptable to f if $S \succ_f \emptyset$.[20]

Definition 1. A **matching** μ is a function from the set $\widetilde{F} \cup W$ into $\widetilde{F} \cup 2^W$ such that

S.9.

[20]Note that ø is the null firm whereas $\emptyset$ is the empty set of workers.

for all $f \in \widetilde{F}$ and $w \in W$,

(i) $\mu(w) \in \widetilde{F}$;

(ii) $\mu(f) \in 2^W$;

(iii) $\mu(w) = f$ if and only if $w \in \mu(f)$.

We say that a matching μ is **individually rational** if $\mu(w) \succeq_w ø$ for all $w \in W$ and $\mu(f) = Ch_f(\mu(f))$ for all $f \in F$. We say that a firm f and a subset of workers $S \subseteq W$ form a **blocking coalition** that blocks μ if $f \succeq_w \mu(w)$ for all $w \in S$, and $S \succ_f \mu(f)$. In words, individual rationality requires that each matched worker prefer her current employer to being unmatched and that no firm wish to unilaterally drop any of its employees. When f and S block μ, S may contain workers that are matched with f in μ. Thus, we require each worker $w \in S$ to weakly prefer f to $\mu(w)$. f should strictly prefer S to $\mu(f)$ since we require $S \neq \mu(f)$.

Definition 2. A matching μ is **stable** if it is individually rational and there is no blocking coalition that blocks μ.[21]

Stable matching is guaranteed to exist in a matching market when the preference of each firm satisfies the following substitutability condition; see Chapter 6 of Roth and Sotomayor (1990).[22]

Definition 3. Firm f has a **substitutable** preference if for any $S \subseteq W$ and any $\{w, w'\} \subseteq S$, $w \in Ch_f(S)$ implies $w \in Ch_f(S \setminus \{w'\})$.

If, on the contrary, there exist $S \subseteq W$ and $\{w, w'\} \subseteq S$ such that $w \in Ch_f(S)$ but $w \notin Ch_f(S \setminus \{w'\})$, then we would consider w and w' to be complements in f's preference because w becomes undesired when w' is not available.

[21] The no blocking coalition condition implies the individual rationalities of firms: $\mu(f) = Ch_f(\mu(f))$ for all $f \in F$. This is also the case for Definition 6 (see footnote 28 of CKK). The exposition of these two definitions follows the convention in the literature.

[22] The stability of Definition 2 is called the core defined by weak domination in Roth and Sotomayor (1990).

2.2 Demand type

Now we introduce BK's concept of demand type to discrete matching markets. For any subset $S \subseteq W$ of workers, we let $ind(S) \in \{0,1\}^W$ denote the indicator vector of S. For any $S, S' \subseteq W$ such that $ind(S) = \mathbf{x}$ and $ind(S') = \mathbf{x}'$, we abuse the notation to write $\mathbf{x} \succeq_f \mathbf{x}'$ when $S \succeq_f S'$, $\mathbf{x} \succ_f \mathbf{x}'$ when $S \succ_f S'$, and $Ch_f(\mathbf{x}) = \mathbf{x}'$ when $Ch_f(S) = S'$.

Definition 4. For each $f \in F$, let $\mathcal{D}_f = \{\mathbf{d} \in \{0,1\}^W \mid \mathbf{d} \neq \mathbf{0}$ and $\mathbf{d} = ind(Ch_f(S)) - ind(Ch_f(S'))$ for some S, S' such that $S' \subset S \subseteq W\}$ be f's **demand type**. Let the demand type of the null firm $\varnothing$ be the set of n-dimensional unit vectors $\mathcal{D}_\varnothing = \{\mathbf{e}^1, \mathbf{e}^2, \ldots, \mathbf{e}^n\}$.[23] The demand type for the firms' preference profile is $\mathcal{D} = \cup_{f \in \widetilde{F}} \mathcal{D}_f$.

Recall firms' preferences in (1.1): we have $\mathcal{D}_{f_1} = \{(1,1)\}$, $\mathcal{D}_{f_2} = \{(1,0),(0,1),(1,-1)\}$, and $\mathcal{D}_\varnothing = \{(1,0),(0,1)\}$. The firms' demand type is $\mathcal{D} = \{(1,1),(1,0),(0,1),(1,-1)\}$. Recall firms' preferences in (1.2): we have $\mathcal{D}_{f_1} = \{(1,1)\}$, $\mathcal{D}_{f_2} = \{(1,1),(1,0),(0,1)\}$, and $\mathcal{D}_\varnothing = \{(1,0),(0,1)\}$. The firms' demand type is $\mathcal{D} = \{(1,1),(1,0),(0,1)\}$.

We say that $k < n$ linearly independent vectors $\mathbf{a}^1, \ldots, \mathbf{a}^k$ can be extended to a basis for $\mathbb{R}^n$ if there exist vectors $\mathbf{a}^{k+1}, \ldots, \mathbf{a}^n$ such that $\mathbf{a}^1, \ldots, \mathbf{a}^n$ is a basis for $\mathbb{R}^n$. By the "determinant" of n vectors in $\mathbb{Z}^n$ we mean the determinant of the $n \times n$ matrix that has them as its columns.

Definition 5. A set of vectors in $\mathbb{Z}^n$ is **unimodular** if every linearly independent subset can be extended to a basis for $\mathbb{R}^n$, of integer vectors, with determinant ± 1.

Given a set of vectors in $\mathbb{Z}^n$, if its maximal subset of linearly independent elements contains n vectors, then unimodularity simply requires the determinant of every subset with n elements to be 0 or ± 1. This is true for the firms' demand type $\mathcal{D}$ of (1.2). The following characterization is useful for testing unimodularity.

[23] The definition of the demand type of the null firm is different from that of the firms in F. If we define the demand type of the null firm as that of the firms in F, $\mathcal{D}_\varnothing$ would contain all nonnegative 0-1 vectors.

Fact 1. A set of $k \leq n$ linearly independent vectors in $\mathbb{Z}^n$ is unimodular if and only if among the determinants of all the $k \times k$ matrices consisting of k rows of the $n \times k$ matrix whose columns are these k vectors, the greatest common factor is 1.[24]

See Truemper (1978) for other characterizations for unimodularity. The unimodularity for a given set of vectors can be tested in polynomial time, see, for example, Walter and Truemper (2013). In a practical problem, suppose a firm f reports a preference that contains N acceptable sets. If S and S' are two sets such that $Ch_f(S') = S'$ and $S \succ_f S'$, we should check whether $S = Ch_f(S \cup S')$ holds to determine whether $ind(S) - ind(S')$ belongs to f's demand type. Hence, it takes $\mathcal{O}(N^3)$ time to obtain f's demand type, which contains at most $N(N+1)/2$ vectors.

2.3 Existence theorem

We find that stable matching exists in a discrete matching market for all possible preferences of workers, provided the firms' demand type is unimodular.

Theorem 1. There exists a stable matching if the firms' demand type $\mathcal{D}$ is unimodular.

We explain our method for the proof of this theorem in Section 3. Unimodular demand types are prevalent and compatible with various complementarities. We provide a class of firms' preference profiles in Section 4 that exhibits both unimodularity and complementarities. In a quasi-linear market, both conditions of gross substitutes (Kelso and Crawford, 1982) and gross substitutes and complements (Sun and Yang, 2006) imply unimodularity.[25] However, unimodularity is independent of substitutability in a discrete matching market. In a quasi-linear market, the

[24]This characterization is used in integer programming to define the unimodular matrix, which is a matrix where the set of its columns is unimodular (see, e.g., Section 21.4 of Schrijver, 1986).

[25]See Section 3.2 of BK.

demand type for a gross-substitute valuation only involves vectors that each has at most one coordinate of $+1$ and at most one coordinate of -1.[26] But this is not the case for the demand type of a firm's substitutable preference in a discrete matching market. Next is an example of a substitutable preference profile with demand type that fails unimodularity.

Example 1. There are two firms f_1, f_2, and two workers w_1, w_2. The firms have the following preferences.

$$f_1 : \{w_1, w_2\} \succ \{w_1\} \succ \{w_2\} \succ \emptyset \qquad\qquad f_2 : \{w_1\} \succ \{w_2\} \succ \emptyset$$

Both firms have substitutable preferences. However, we have $\mathcal{D}_{f_1} = \{(1,1),(1,0),(0,1)\}$, $\mathcal{D}_{f_2} = \{(1,0),(0,1),(1,-1)\}$, and $\mathcal{D}_{ø} = \{(1,0),(0,1)\}$. The firms' demand type is $\mathcal{D} = \{(1,1),(1,0),(0,1),(1,-1)\}$, which is not unimodular.

The necessity part of BK's unimodularity theorem says that given a demand type that is not unimodular, there must be some profile of concave valuations of this demand type for which equilibrium does not exist.[27] It is unknown whether the following counterpart holds in our context: In a discrete matching market Γ with firms' demand type $\mathcal{D}$ not being unimodular (such as the market of Example 1), there must be some market Γ' with the same demand type for firms, for which stable matching does not exist (such as the market of (1.1), the firms' demand type in (1.1) is the same as that in Example 1).

3 Method

We elaborate on our method for the proof of Theorem 1 in this section. We present the market with "divisible" workers in Section 3.1, the stability-preserving turnover-

[26]See Definition 3.5 and Proposition 3.6 of BK.

[27]See Corollary 4.4 of BK.

s in this market in Section 3.2, and an illustrative example in Section 3.3. We discuss the relation to the LP method for stable matching in Section 3.4.

3.1 A continuum market

We construct an instance of the continuum market in CKK from a matching market Γ by assuming that each worker in W is divisible. In this setting, W is called the set of worker types. There is a divisible mass of quantity 1 of each worker type $w \in W$. A vector $\mathbf{x} \in [0,1]^W$ is called a **subpopulation** where $x(w)$ is the quantity of the type-w workers for each $w \in W$. We say that a subset of worker types $S \subseteq W$ (or $\mathbf{z} \in \{0,1\}^W$) is available at subpopulation $\mathbf{x} \in [0,1]^W$ if $w \in S$ implies $x(w) > 0$ (resp. $z(w) = 1$ implies $x(w) > 0$). Each worker type $w \in W$ has a strict preference $\succ_w$ over $\widetilde{F}$, where $\succ_w$ is the preference of worker w in the original market Γ. Each firm $f \in \widetilde{F}$ has a choice function $\widehat{Ch}_f : [0,1]^W \to [0,1]^W$, where $\widehat{Ch}_f(\mathbf{x}) \leq \mathbf{x}$. By convention, let $\widehat{Ch}_\varnothing(\mathbf{x}) = \mathbf{x}$ for all $\mathbf{x} \in [0,1]^W$. We now construct a choice function $\widehat{Ch}_f$ for each firm $f \in F$ based on its preference $\succ_f$ in Γ.

Let $\mathbf{u}^1 \succ_f \mathbf{u}^2 \succ_f \cdots \succ_f \mathbf{u}^L \succ_f \mathbf{0}$ be the preference order of firm f over its acceptable sets of workers in Γ, where $\mathbf{u}^j \in \{0,1\}^W$ for all $j \in \{1, \ldots, L\}$ and $\mathbf{0}$ is the empty set of workers. Before proceeding to the formal definition of $\widehat{Ch}_f$, we want to give an intuitive description of this choice function, which resembles that of the Probabilistic Serial assignment of Bogomolnaia and Moulin (2001). Consider the workers of each worker type of $\mathbf{x}$ as a "divisible commodity". The firm first consumes the workers of each type of $\mathbf{u}^j$ from $\mathbf{x}$ simultaneously at speed 1, where j is the smallest index such that $\mathbf{u}^j$ is available at $\mathbf{x}$. When the workers of one worker type of $\mathbf{u}^j$ is exhausted, the firm switches to consume the workers of $\mathbf{u}^{j'}$ simultaneously at speed 1, where j' is the smallest index such that $\mathbf{u}^{j'}$ is available at the remaining subpopulation. This procedure goes on until the time reaches 1, or until there is no acceptable set of worker types available at the remaining subpopulation.

Example 2. If f has the preference $\{w_1, w_2\} \succ \{w_2, w_3\} \succ \{w_3\} \succ \emptyset$ in Γ, then $\widehat{Ch}_f((0.6, 0.6, 0.5)) = (0.6, 0.6, 0.4)$: It first consumes 0.6 of each of the type-w_1 and type-w_2 workers. Then, since $\{w_2, w_3\}$ is not available at the remaining subpopulation (0,0,0.5), it switches to consume 0.4 of the type-w_3 workers when the time reaches 1. In another case, $\widehat{Ch}_f((0.1, 0.4, 0.1)) = (0.1, 0.2, 0.1)$: The firm first consumes 0.1 of each of the type-w_1 and type-w_2 workers, and then, it consumes 0.1 of each of the type-w_2 and type-w_3 workers from the remaining subpopulation (0,0.3,0.1). The procedure then terminates since there is no acceptable set of worker types available at the remaining subpopulation (0,0.2,0).

Formally, for any $\mathbf{x} \in [0,1]^W$, $\widehat{Ch}_f(\mathbf{x})$ is defined by the following recursive procedure. Let $t_0 = 0$, $\mathbf{z}^0 = \mathbf{x}$. Suppose t_0, $\mathbf{z}^0$, ..., t_{k-1}, $\mathbf{z}^{k-1}$ ($k \in \{1, 2, \ldots, L\}$) are already defined, then define

$$\begin{aligned} t_k &= \min\{1 - \sum_{j=0}^{k-1} t_j, z_i^{k-1} \mid i \in \{1, 2, \ldots, n\} \quad \text{and} \quad u_i^k \neq 0\} \\ \mathbf{z}^k &= \mathbf{z}^{k-1} - t_k \mathbf{u}^k \end{aligned} \tag{3.1}$$

After the sequences $(t_k)_{k=0}^L$ and $(\mathbf{z}^k)_{k=0}^L$ have been defined, we define

$$\widehat{Ch}_f(\mathbf{x}) = \sum_{k=1}^{L} t_k \mathbf{u}^k \tag{3.2}$$

When we consider this procedure as the consumption process described above, for each $k \in \{1, 2, \ldots, L\}$, t_k is the time the firm spends in the consumption of $\mathbf{u}^k$; $\mathbf{z}^k$ is the remaining subpopulation after $\mathbf{u}^k$ has been consumed. If $\sum_{j=1}^{k'} t_j = 1$ for some $k' \in \{1, 2, \ldots, L-1\}$, then $t_{k''} = 0$ for all $k'' \in \{k'+1, \ldots, L\}$. It turns out that the choice function $\widehat{Ch}_f$ constructed above is continuous and satisfies the revealed preference property.

Lemma 1. $\widehat{Ch}_f$ is continuous and satisfies the **revealed preference property**: For any $\mathbf{x}, \mathbf{x}' \in [0,1]^W$ with $\mathbf{x}' \leq \mathbf{x}$, $\widehat{Ch}_f(\mathbf{x}) \leq \mathbf{x}'$ implies $\widehat{Ch}_f(\mathbf{x}') = \widehat{Ch}_f(\mathbf{x})$.

The revealed preference property stated above is known as Sen's property α, which is a premise of CKK's existence theorem. Let $\widehat{Ch}_F$ be the set of choice functions of all firms constructed in the above way. A continuum market induced from a matching market Γ is also summarized as a tuple $\widehat{\Gamma} = (W, F, \succ_W, \widehat{Ch}_F)$, where W, F and $\succ_W$ are the same as those of Γ. We can interpret the induced market $\widehat{\Gamma}$ as a schedule matching market of Alkan and Gale (2003), where each worker schedules her time among different firms, and each firm schedules its time among different groups of workers. Different groups of workers bring different outputs per unit time for each firm, which decrease along with the firm's preference order in the original market Γ. For instance, suppose f has the preference $\{w_1, w_2\} \succ \{w_2, w_3\} \succ \{w_3\} \succ \emptyset$ in Γ. In the induced market, $\{w_1, w_2\}$ brings more output per unit time than $\{w_2, w_3\}$, which brings more output per unit time than $\{w_3\}$. When a firm faces an available supply of working time from workers, the firm solves a linear program to find the optimal schedule, which is the same as the choice function $\widehat{Ch}_f$.[28]

A **matching** M in $\widehat{\Gamma}$ assigns each firm a subpopulation of workers: $M = (M_f)_{f\in\widetilde{F}}$ such that $M_f \in [0,1]^W$ for all $f \in \widetilde{F}$ and $\sum_{f\in\widetilde{F}} M_f(w) = 1$ for all $w \in W$. For any subpopulations $\mathbf{x}$,$\mathbf{x}' \in [0,1]^W$, we let $\mathbf{x} \vee \mathbf{x}'$ denote the subpopulation whose quantity of type-w workers is $max\{x(w), x'(w)\}$. We use firms' choice functions to define firms' preferences over matchings. Given two matchings M and M', we say that firm f prefers M'_f to M_f, denoted as $M'_f \succeq_f M_f$, if $M'_f = \widehat{Ch}_f(M'_f \vee M_f)$. This relation is known as Blair's partial order in the literature (Blair, 1984). We write $M'_f \succ_f M_f$ to indicate that $M'_f \succeq_f M_f$ and $M'_f \neq M_f$. For any matching M and firm f, the subpopulation of workers assigned to firm f or some firm worse than f (according to their preferences) in M is denoted as

[28] I thank coeditor Federico Echenique for this insightful interpretation.

$A^{\preceq f}(M) \in [0,1]^W$, where

$$A^{\preceq f}(M)(w) = \sum_{f' \in \widetilde{F}: f \succeq_w f'} M_{f'}(w) \tag{3.3}$$

for each $w \in W$. $A^{\preceq f}(M)$ refers to the available subpopulation for f in M since it measures the amount of workers of each type that would rather match with f in M. The concept of stability in the continuum market $\widetilde{\Gamma}$ is then defined as follows:

Definition 6. A matching $M = (M_f)_{f \in \widetilde{F}}$ in $\widehat{\Gamma}$ is **stable** if

(i) (Individual Rationality) For each $f \in F$, $M_f = \widehat{Ch}_f(M_f)$; and for all $w \in W$, $M_f(w) = 0$ for any f that satisfies $\varnothing \succ_w f$.

(ii) (No Blocking Coalition) There are no $f \in F$ and $M'_f \in [0,1]^W$ such that $M'_f \succ_f M_f$ and $M'_f \leq A^{\preceq f}(M)$.

CKK proved that stable matching exists when each firm has a continuous choice function.[29] Then by Lemma 1, we know that there exists a stable matching in the continuum market $\widehat{\Gamma}$.

Example 3. Consider a matching market Γ where there are two firms $F = \{f_1, f_2\}$ and three workers $W = \{w_1, w_2, w_3\}$. The preferences of firms and workers are as follows.

$$f_1 : \{w_1, w_2\} \succ \{w_3\} \succ \emptyset \qquad w_1 : \ f_1 \succ f_2 \succ \varnothing$$

$$f_2 : \{w_1, w_2\} \succ \emptyset \qquad w_2 : \ f_2 \succ f_1 \succ \varnothing$$

$$w_3 : \ f_1 \succ \varnothing$$

[29] See Theorem 2 of CKK. CKK's framework accommodates indifferences in firms' preferences. The requirements of continuity and convex-valuedness on firms' choice correspondences in their setting reduce to the continuity of firms' choice functions in our setting.

The above matching market induces a continuum market $\widehat{\Gamma}$, where the choice functions $\widehat{Ch}_{f_1}$ and $\widehat{Ch}_{f_2}$ are generated from the above firms' preference profile via (3.1) and (3.2). By Lemma 1 and CKK's existence theorem, there is at least one stable matching in this continuum market. For instance, the matching M in $\widehat{\Gamma}$, where f_1 is matched with $(0.5, 0.5, 0.5)$, f_2 with $(0.5, 0.5, 0)$, and ø with $(0, 0, 0.5)$, is stable. Under our constructed firms' preferences, $\widehat{Ch}_{f_1}((0.5, 0.5, 0.5)) = (0.5, 0.5, 0.5)$ and $\widehat{Ch}_{f_2}((0.5, 0.5, 0)) = (0.5, 0.5, 0)$ indicate the individual rationality of M. Hiring more type-w_3 workers does not benefit f_1. f_1 and f_2 would both like to hire more workers of type-w_1 and type-w_2 at the ratio 1:1, but neither can draw workers of this ratio from the other. For example, f_1 would be better off when matched with $(0.6, 0.6, 0.4)$. This is because $\widehat{Ch}_{f_1}((0.6, 0.6, 0.4) \vee (0.5, 0.5, 0.5)) = (0.6, 0.6, 0.4)$ indicates $(0.6, 0.6, 0.4) \succ_{f_1} (0.5, 0.5, 0.5)$. However, f_1 cannot draw any type-w_2 workers from f_2 since the type-w_2 workers prefer f_2 to f_1. In the language of Definition 6, f_1 and $(0.6, 0.6, 0.4)$ do not form a blocking coalition because although $(0.6, 0.6, 0.4) \succ_{f_1} (0.5, 0.5, 0.5)$ holds, $(0.6, 0.6, 0.4) \leq A^{\preceq f_1}(M)$ does not hold where $A^{\preceq f_1}(M) = (1, 0.5, 1)$.

We say that a matching M in $\widehat{\Gamma}$ is integral if $M_f(w) \in \{0, 1\}$ for all $f \in \widetilde{F}$ and $w \in W$, otherwise we say that M is fractional. Another observation is that each stable integral matching in the continuum market $\widehat{\Gamma}$ is also a stable matching in the original market Γ. For instance, the stable integral matching in $\widehat{\Gamma}$ where f_1 matches with (0,0,0), f_2 with (1,1,0), and ø with $(0, 0, 1)$ is also a stable matching in Γ.

Lemma 2. If M is a stable integral matching in the continuum market $\widehat{\Gamma}$, then μ is a stable matching in the original matching market Γ, where $\mu(w) = f$ and $w \in \mu(f)$ if $M_f(w) = 1$ for each $w \in W$ and $f \in \widetilde{F}$.

3.2 Stability-preserving turnovers

Every matching market Γ induces a continuum market $\widehat{\Gamma}$. Motivated by Lemma 2, we turn to investigate when there exists a stable integral matching in $\widehat{\Gamma}$. According to CKK's existence theorem, there always exists a stable matching M in $\widehat{\Gamma}$, which may be fractional or integral. Consider the stable fractional matching in Example 3, where f_1 is matched with $(0.5, 0.5, 0.5)$, f_2 with $(0.5, 0.5, 0)$, and ø with $(0, 0, 0.5)$. We obtain a stable integral matching when all the workers matched with f_2 switch to f_1 and the type-w_3 workers previously hired by f_1 become unemployed. Such "turnover" of workers in the continuum market preserves stability and produces a stable integral matching. However, consider the continuum market induced by (1.1) and the stable fractional matching where f_1 is matched with $(0.5, 0.5)$, and f_2 with $(0.5, 0.5)$. We cannot obtain a stable integral matching by any stability-preserving turnover of workers.[30]

Therefore, stable matching always exists in the original market when stability-preserving turnover toward a stable integral matching exists in the continuum market. To formalize some of the stability-preserving turnovers, we generalize the concept of matching to "pseudo-matching" in the continuum market and introduce "stable transformations" that operate on pseudo-matchings in a succinct manner. We find that some special stability-preserving turnovers toward an integral matching can be expressed as a series of our stable transformations.

In the continuum market $\widehat{\Gamma}$, we call $M = (M_f)_{f \in \widetilde{F}}$ a **pseudo-matching** if $M_f \in [0,1]^W$ for all $f \in \widetilde{F}$. A pseudo-matching may assign more or less than quantity 1 of some worker type to firms. A pseudo-matching is a matching if $\sum_{f \in \widetilde{F}} M_f(w) = 1$ for each $w \in W$. We say that a pseudo-matching M is **stable** if it satisfies condition (i) and (ii) in Definition 6. In other words, a pseudo-matching M is stable if M is a stable matching when the quantity of each worker type w in the market is adjusted

[30] It happens that there is no stability-preserving turnovers of workers in this continuum market, because there is only one stable matching; see Example 3 of CKK.

to its current quantity $\sum_{f \in \widetilde{F}} M_f(w)$ in M.

Example 4. Consider the continuum market $\widehat{\Gamma}$ in Example 3. Let M be the pseudo-matching where $M_{f_1} = (0.6, 0.6, 0.3)$, $M_{f_2} = (0.3, 0.3, 0)$, and $M_ø = (0, 0, 0)$, then M is a stable pseudo-matching. The reason is similar to that in Example 3.

Let M' be the pseudo-matching where $M'_{f_1} = (0, 0, 0)$, $M'_{f_2} = (0.3, 0.3, 0)$ and $M'_ø = (0, 0, 0)$. Readers can check that M' is also a stable pseudo-matching.

Given a stable pseudo-matching M in $\widehat{\Gamma}$, each of the following transformations on M produces a stable pseudo-matching M'.

Type-1 stable transformation: Choose a firm f' from F such that $\sum_{j=1}^{L} t_j < 1$ holds in the procedure (3.1) that computes $\widehat{Ch}_{f'}(M_{f'})$. Let $M'_{f'} = \mathbf{0}$ and let $M'_f = M_f$ for all $f \neq f'$.

Type-2 stable transformation: Choose a firm f' from F. Consider the procedure (3.1) that computes $\widehat{Ch}_{f'}(M_{f'}) = \sum_{j=1}^{L} t_j \mathbf{u}^j$. Choose an index $k \in \{1, 2, \ldots, L\}$ that satisfies $t_k > 0$, let $M'_{f'} = \mathbf{u}^k$. Let $M'_f = M_f$ for all $f \neq f'$.

Type-3 stable transformation: Choose a worker type $w' \in W$ that satisfies $M_ø(w') \in (0, 1)$; let $M'_ø(w') = 0$ or 1. Let $M'_ø(w) = M_ø(w)$ for all $w \neq w'$, and $M'_f = M_f$ for all $f \in F$.

Example 5. Consider the continuum market $\widehat{\Gamma}$ in Example 3 and the stable pseudo-matchings M and M' in Example 4. $M \to M'$ is a type-1 stable transformation. Note that in the procedure (3.1) that computes $\widehat{Ch}_{f_1}(M_{f_1})$, we have $\sum_{j=1}^{L} t_j = 0.9 < 1$. Since f_1 can not draw any subpopulation that includes both types from f_2 in M, f_1 can do no such thing in M' either, and thus, this transformation maintains stability.

Let M'' be the pseudo-matching where $M''_{f_1} = (1, 1, 0)$, $M''_{f_2} = (0.3, 0.3, 0)$, and $M''_ø = (0, 0, 0)$; then, $M \to M''$ is a type-2 stable transformation. This transformation maintains stability because f_2 can not draw any subpopulation that includes both types from f_1 no matter when f_1 is matched with $(0.6, 0.6, 0.3)$ or $(1, 1, 0)$.

Let $\widetilde{M}$ be the pseudo-matching where $\widetilde{M}_{f_1} = (0,0,0)$, $\widetilde{M}_{f_2} = (0.3, 0.3, 0)$, and $\widetilde{M}_{ø} = (0, 0.3, 0)$. Let $\widetilde{M}'$ be the pseudo-matching where $\widetilde{M}'_{f_1} = (0,0,0)$, $\widetilde{M}'_{f_2} = (0.3, 0.3, 0)$ and $\widetilde{M}'_{ø} = (0,1,0)$. Then, $\widetilde{M} \to M'$ and $\widetilde{M} \to \widetilde{M}'$ are both type-3 stable transformations. Unmatched workers in a stable matching can be viewed as redundant for firms; thus, stability is not affected when the quantity of unmatched workers varies.

Lemma 3. Each of the type-1, type-2, and type-3 stable transformations on a stable pseudo-matching produces a stable pseudo-matching.

When we implement several stable transformations on a stable fractional matching, the matching becomes "less fractional" and ultimately becomes integral, but may it assign more or less than quantity 1 of some worker types. Thus, we also want a series of transformations to be "balanced", that is, to assign 1 quantity of each worker type in the output. Because the stable transformations transform stable fractional matchings in a succinct manner, we can reduce "balanced" stable transformations to integral solutions to a system of linear equations, where the unimodularity condition applies.

3.3 Illustrative example

Example 6. Consider the market of Example 3 and the stable matching M in the continuum market where $M_{f_1} = (0.5, 0.5, 0.5)$, $M_{f_2} = (0.5, 0.5, 0)$, and $M_{ø} = (0, 0, 0.5)$. In the following, we represent a pseudo-matching in $\widehat{\Gamma}$ with a matrix, the rows of which represent the subpopulations matched with the firms. M is then represented by the following 3×3 matrix.

	w_1	w_2	w_3
f_1	0.5	0.5	0.5
f_2	0.5	0.5	0
ø	0	0	0.5

Consider the following transformations on M.

$$\begin{pmatrix} 0.5 & 0.5 & 0.5 \\ 0.5 & 0.5 & 0 \\ 0 & 0 & 0.5 \end{pmatrix} \xrightarrow{(1)} \begin{pmatrix} 0.5 & 0.5 & 0.5 \\ 0 & 0 & 0 \\ 0 & 0 & 0.5 \end{pmatrix} \xrightarrow{(2)} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0.5 \end{pmatrix}$$
$$\xrightarrow{(3)} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Transformations (1), (2), and (3) are type-1, type-2, and type-3 stable transformations, respectively. It turns out that we finally reach a stable integral matching M' in $\widehat{\Gamma}$, where $M'_{f_1} = (1,1,0)$, $M'_{f_2} = (0,0,0)$, and $M'_{ø} = (0,0,1)$. The outcome is not only a stable integral pseudo-matching but also a matching that assigns precisely quantity 1 of each worker type.

A critical observation is that we can reach a stable integral matching through stable transformations on M when there is a nonnegative integral solution to the following system of linear equations.

$$\begin{array}{cc} & \\ & \\ B^* & \left\{ \begin{array}{c} \\ \\ \\ \end{array} \right. \end{array} \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

Let B denote the 5×5 matrix on the left. The first and second rows of B are constraints for preserving stability; the third to fifth rows of B are constraints for assigning quantity 1 of each worker type. Let B^* denote the 3×5 submatrix that includes the third to fifth rows of B. Let B_i and B^*_i be the vectors of the i-th column of B and B^*, respectively. B^*_1 and B^*_2 correspond to $\{w_1, w_2\}$ and $\{w_3\}$ in f_1's preference list, respectively. B^*_3 and B^*_4 correspond to $\{w_1, w_2\}$ and $\emptyset$ in f_2's preference list, respectively. B^*_5 corresponds to the type-w_3 workers matched with firm ø. $\mathbf{z} = (0.5, 0.5, 0.5, 0.5, 0.5)$ is a solution to this system, which refers to matching

M as follows.

$$M = \begin{pmatrix} f_1 & f_2 & \varnothing \\ z_1(1,1,0) + z_2(0,0,1) & z_3(1,1,0) + z_4(0,0,0) & z_5(0,0,1) \end{pmatrix} \tag{3.4}$$

$(z_1, z_2) = (0.5, 0.5)$ corresponds to (t_1, t_2) in the procedure (3.1) that computes $\widehat{Ch}_{f_1}(0.5, 0.5, 0.5)$. $(z_3, z_4) = (0.5, 0.5)$, corresponds to $(t_1, 1 - t_1)$ in the procedure (3.1) that computes $\widehat{Ch}_{f_2}(0.5, 0.5, 0)$. $z_5 = 0.5$ corresponds to $M_\varnothing = (0, 0, 0.5)$. Note that (3.4) with any integral $\mathbf{z} \in \{0, 1\}^5$ that satisfies $z_1 + z_2 = 1$ and $z_3 + z_4 = 1$ (guaranteed by the first and second rows of B) corresponds to a stable integral pseudo-matching obtained from M via stable transformations. Then, any $\mathbf{z} \in \{0, 1\}^5$ that satisfies $B\mathbf{z} = \mathbf{1}$ corresponds to a stable integral matching since $B^*\mathbf{z} = \mathbf{1}$ requires that the workers of each type assigned to firms is of quantity 1. For instance, $\mathbf{z}' = (1, 0, 0, 1, 1)$ is a solution to the system, and we obtain the stable integral matching M' by plugging $\mathbf{z}'$ into (3.4).

Therefore, given a stable matching M in the continuum market $\widehat{\Gamma}$, we can construct a system of linear equations $B\mathbf{z} = \mathbf{1}$. Our construction guarantees that the polytope $\{\mathbf{z} \mid B\mathbf{z} = \mathbf{1}, \mathbf{z} \geq 0\}$ is nonempty because M corresponds to a nonnegative solution to this system of equations. Our construction also guarantees that every integral point of this polytope corresponds to a stable matching in the original market. Now, we can apply a standard result from integer programming to this problem. Under the condition that the set of columns of B is unimodular, all vertices of the polytope $\{\mathbf{z} \mid B\mathbf{z} = \mathbf{1}, \mathbf{z} \geq 0\}$ are integral (see, e.g., Theorem 21.5 of Schrijver, 1986). Since the polytope is nonempty, we further know that there is at least an integral vertex on this polytope. Finally, it is not difficult to find out that the set of the columns of B is unimodular when the firms' demand type $\mathcal{D}$ is unimodular. In this example, we have $\mathcal{D}_{f_1} = \{(1, 1, 0), (1, 1, -1), (0, 0, 1)\}$, $\mathcal{D}_{f_2} = \{(1, 1, 0)\}$, and $\mathcal{D}_\varnothing = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$. The firms' demand type is $\mathcal{D} = \{(1,1,0),(1,1,-1),(1,0,0),(0,1,0),(0,0,1)\}$, which is unimodular. Now, we illustrate why the unimodularity of $\mathcal{D}$ implies the unimodularity of the set of columns from matrix B. For instance, B_1, B_2, B_3, and B_5 are linearly independent. The unimodularity of $\mathcal{D}$ guarantees that the set of $\{B_1^* - B_2^*, B_5^*\}$ can be extended to a basis for $\mathbb{R}^3$, of integer vectors, with determinant ± 1 (e.g., by extending with $(1,0,0)$). Then, we know that $\{B_1, B_3\} \cup \{B_1 - B_2, B_5\}$ can be extended to a basis for $\mathbb{R}^5$, of integer vectors, with determinant ± 1 (e.g., by extending with $(0,0,1,0,0)$). Therefore, such extension also exists for the set $\{B_1, B_2, B_3, B_5\}$.[31]

3.4 Relation to the LP method

Linear programming has been used to select stable matchings of specific properties, such as optimal marriage and median stable matching. Both the LP method and our method represent matchings by their indicator vectors and study polytopes related to these vectors. We discuss the differences between the two methods below.

First, the LP method is used to find stable matchings of specific properties in a discrete matching market when firms have so-called *responsive* preferences, a special case of substitutable preferences. By contrast, we investigated under what conditions stable matching exist in the market with complementarities. Second, the LP method uses the definition of stable matching to define a polytope whose vertices are the stable matchings of the market; then, the problems of optimal marriage and median stable matching reduce to linear programs. By contrast, we address our problem by constructing a fictitious market where each worker is divisible and investigating the stable integral matchings of this market. We incorporate the results of CKK and tools from integer programming to obtain our results.

[31]Since subtracting one column from another column leaves the determinant unchanged, we know that $\{B_1, B_2, B_3, B_5\} \cup \{(0,0,1,0,0)\}$ is also an integral basis for $\mathbb{R}^5$, with determinant ± 1.

4 Application

This section presents an application of matching firms with specialists, which has a practical economic meaning. We describe the structure of firms' acceptable sets of workers as a directed rooted tree, which we call the "technology tree". Each vertex of the technology tree represents a technology that requires a set of workers to implement. Each edge of the tree is an upgrade from one technology to another that requires more workers. A worker is called a specialist if she engages in only one upgrade. We show that firms' demand type is unimodular when firms have unit-demand preferences over the technologies of a technology tree where each worker is a specialist. We first provide an illustrative example as follows.

Example 7. Consider a market with two firms f_1, f_2, and four workers w_1, w_2, w_3, w_4. A technology tree is depicted as follows.

$$\begin{tikzcd}
 & v_0 : \emptyset \arrow[dl] \arrow[dr] & & \\
v_1 : \{w_1, w_2\} & & v_2 : \{w_3\} \arrow[dr] & \\
 & & & v_3 : \{w_3, w_4\}
\end{tikzcd}$$

Each vertex from $\{v_0, v_1, v_2, v_3\}$ represents a technology that requires the set of workers on the right to implement. The root v_0 represents no technology and requires no worker. Each directed edge is an upgrade from one technology to another, where more workers should be employed to implement the upgrade. If $e = vv'$ is an edge from vertex v to vertex v', where w is not demanded by v but demanded by v', we say that w engages in the upgrade e or vv'. For example, w_1 and w_2 both engage in the upgrade v_0v_1. Each firm possesses some of the technologies and has a preference order over the technologies it possesses, which induces its preference over the sets of workers required for the technologies. For instance, f_1

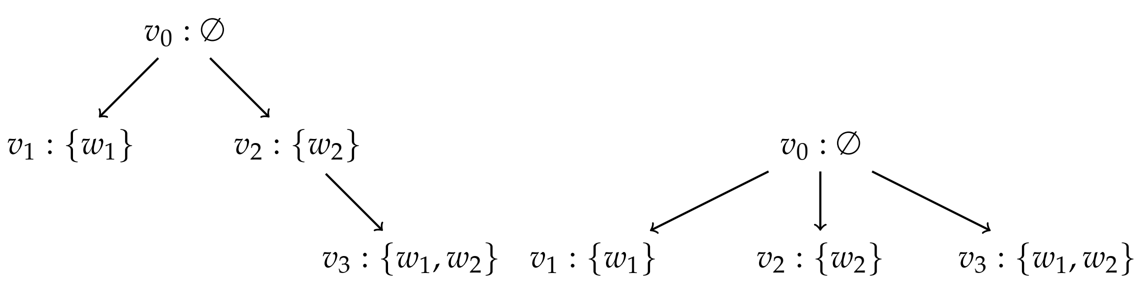
may possess v_1 and v_2; f_2 may possess v_1 and v_3; and v_0 is trivially possessed by both firms. f_1 and f_2 may have the following preferences.

$$
\begin{aligned}
&f_1 : \{w_1, w_2\} \succ \{w_3\} \succ \emptyset \\
&f_2 : \{w_3, w_4\} \succ \{w_1, w_2\} \succ \emptyset
\end{aligned}
\tag{4.1}
$$

With firms' preferences induced in this way, we find that firms' demand type is unimodular when the technology tree satisfies the following condition.

$$\textit{Each worker is a specialist that engages in only one upgrade.} \tag{4.2}$$

For example, given that w_1 engages in the upgrade v_0v_1, this condition requires that w_1 cannot engage in other upgrades. When firms' acceptable sets of workers are from a technology tree that satisfies this condition, the firms' demand type is unimodular since its elements form a matrix that can be obtained by taking operations on a *network matrix* (Tutte, 1965, see also Chapter 19.3 of Schrijver, 1986) while preserving the total unimodularity of the network matrix.[32] See an illustration in Section 5.3. To see the role of (4.2), consider the firms' preference profile in (1.1). There are two possible structures for the technology tree that induces the firms' preferences.



Both technology trees violate condition (4.2): On the left, w_1 engages in both upgrades v_0v_1 and v_2v_3. On the right, w_1 engages in both upgrades v_0v_1 and v_0v_3.

[32] A totally unimodular matrix is a matrix for which every square submatrix has determinant $0, +1$, or -1. Each subset of the column vectors of a totally unimodular matrix is unimodular.

Formally, a **technology tree** $T = (V, E, W)$ is a directed rooted tree (V, E) defined on a set of workers W. $V = \{v_0, v_1, \ldots, v_l\}$ is a set of vertices with v_0 as the root. Each vertex $v \in V$ represents a technology and requires a subset of workers $W^v \subseteq W$ to implement the technology. The root v_0 represents no technology and requires no worker: $W^{v_0} = \emptyset$. E is a set of directed edges, all of which point away from the root. For each edge $e \in E$ from vertex v to vertex v', $W^v \subset W^{v'}$, and we let $W^e = W^{v'} \setminus W^v$. We say that worker w is a **specialist** in T if she engages in only one upgrade: $| \{e \in E \mid w \in W^e\} |= 1$. We study firms' preferences where each firm wants to hire one set of workers from a common technology tree.

Definition 7. Firms have unit-demand preferences over a technology tree $T = (V, E, W)$ if for each $f \in F$ and each $S \subseteq W$, $S \succ_f \emptyset$ if $S = W^v$ for some $v \in V$.

In words, firms have unit-demand preferences over a technology tree if those sets of workers not on the tree are not acceptable for all firms. Each firm wants to employ one set from the technology tree and can have arbitrary preference order over the sets of workers on the tree.[33] We then have the following theorem.

Theorem 2. Firms' demand type is unimodular if firms have unit-demand preferences over a technology tree where each worker is a specialist.

Therefore, we know that stable matching always exists when firms have unit-demand preferences over a technology tree where each worker is a specialist. S-tudying this problem motivated the author's subsequent work, Huang (2021b), which generalizes condition (4.2) and the unit demand of technologies. Both generalizations allow for firms' demand types that are not unimodular.

[33]Note that if a firm has the preference $\{w_3\} \succ \{w_3, w_4\} \succ \emptyset$ in Example 7, this preference is essentially equivalent to $\{w_3\} \succ \emptyset$.

5 Appendix

5.1 Proofs of the lemmata

Proof of Lemma 1

(1) Revealed preference property. For any $\mathbf{x}, \widetilde{\mathbf{x}} \in [0,1]^W$ with $\widetilde{\mathbf{x}} \leq \mathbf{x}$ and $\widehat{Ch}_f(\mathbf{x}) \leq \widetilde{\mathbf{x}}$, consider the procedures (3.1) that compute $\widehat{Ch}_f(\mathbf{x})$ and $\widehat{Ch}_f(\widetilde{\mathbf{x}})$. Let $(t_k)_{k=1}^L$ and $(\mathbf{z}^k)_{k=1}^L$ be the parameters in computing $\widehat{Ch}_f(\mathbf{x})$. Let $(\widetilde{t}_k)_{k=1}^L$, and $(\widetilde{\mathbf{z}}^k)_{k=1}^L$ be the parameters in computing $\widehat{Ch}_f(\widetilde{\mathbf{x}})$.

Since $\widetilde{\mathbf{x}} \leq \mathbf{x}$, we have $\widetilde{t}_1 \leq t_1$. Suppose $\widetilde{t}_1 < t_1$, we have $\widehat{Ch}_f(\mathbf{x}) \geq t_1\mathbf{u}^1$ but $\widetilde{\mathbf{x}} \geq t_1\mathbf{u}^1$ does not hold. This contradicts $\widehat{Ch}_f(\mathbf{x}) \leq \widetilde{\mathbf{x}}$. Thus, we have $t_1 = \widetilde{t}_1$, and then $\widetilde{\mathbf{z}}^1 \leq \mathbf{z}^1$.

Suppose for all $j \in \{1, 2, \ldots, k-1\}$, we have $t_j = \widetilde{t}_j$ and $\mathbf{z}^j \geq \widetilde{\mathbf{z}}^j$. Then, since $\mathbf{z}^{k-1} \geq \widetilde{\mathbf{z}}^{k-1}$, we have $\widetilde{t}_k \leq t_k$. Suppose $\widetilde{t}_k < t_k$, we have $\widehat{Ch}_f(\mathbf{x}) \geq \sum_{j=1}^k t_j\mathbf{u}^j$ but $\widetilde{\mathbf{x}} \geq \sum_{j=1}^k t_j\mathbf{u}^j$ does not hold. This contradicts $\widehat{Ch}_f(\mathbf{x}) \leq \widetilde{\mathbf{x}}$. Thus, we have $t_k = \widetilde{t}_k$ and $\widetilde{\mathbf{z}}^k \leq \mathbf{z}^k$.

According to the above inductive arguments, the procedures that compute $\widehat{Ch}_f(\mathbf{x})$ and $\widehat{Ch}_f(\widetilde{\mathbf{x}})$ coincide, and we have $\widehat{Ch}_f(\mathbf{x}) = \widehat{Ch}_f(\widetilde{\mathbf{x}})$.

(2) Continuity. For any $\mathbf{x}, \widetilde{\mathbf{x}} \in [0,1]^W$, let $r(\mathbf{x}, \widetilde{\mathbf{x}}) = \max_{w \in W} \mid x(w) - \widetilde{x}(w) \mid$ be the maximum metric. Consider the procedures (3.1) that compute $\widehat{Ch}_f(\mathbf{x})$ and $\widehat{Ch}_f(\widetilde{\mathbf{x}})$, where $r(\mathbf{x}, \widetilde{\mathbf{x}}) < v$. Let $(t_k)_{k=1}^L$ and $(\mathbf{z}^k)_{k=1}^L$ be the parameters in computing $\widehat{Ch}_f(\mathbf{x})$. Let $(\widetilde{t}_k)_{k=1}^L$, and $(\widetilde{\mathbf{z}}^k)_{k=1}^L$ be the parameters in computing $\widehat{Ch}_f(\widetilde{\mathbf{x}})$.

Since $r(\mathbf{x}, \widetilde{\mathbf{x}}) < v$, we have $\mid t_1 - \widetilde{t}_1 \mid < v$ and $r(t_1\mathbf{u}^1, \widetilde{t}_1\mathbf{u}^1) < v$. Then, $r(\mathbf{z}^1, \widetilde{\mathbf{z}}^1) = \max_{i \in \{1,2,\ldots,n\}} \mid x_i - t_1u_i^1 - \widetilde{x}_i + \widetilde{t}_1u_i^1 \mid < 2v$.

We then establish the following inductive arguments. Suppose we have $\mid t_j - \widetilde{t}_j \mid < 2^{j-1}v$ and $r(\mathbf{z}^j, \widetilde{\mathbf{z}}^j) < 2^jv$ for all $j \in \{1, 2, \ldots, k-1\}$. Then, we have $\mid \sum_{j=1}^{k-1} t_j - \sum_{j=1}^{k-1} \widetilde{t}_j \mid < (2^{k-1} - 1)v$. We then consider four cases:

(i) $\sum_{j=1}^k t_j < 1$ and $\sum_{j=1}^k \widetilde{t}_j < 1$. Since $r(\mathbf{z}^{k-1}, \widetilde{\mathbf{z}}^{k-1}) < 2^{k-1}v$, we have $\mid t_k - \widetilde{t}_k \mid <$

$2^{k-1}v$.

(ii) $\sum_{j=1}^{k} t_j = \sum_{j=1}^{k} \widetilde{t}_j = 1$. We have $\mid t_k - \widetilde{t}_k \mid < (2^{k-1} - 1)v$;

(iii) $\sum_{j=1}^{k} t_j = 1$ and $\sum_{j=1}^{k} \widetilde{t}_j < 1$. We have $\sum_{j=1}^{k} \widetilde{t}_j = \sum_{j=1}^{k-1} \widetilde{t}_j + \widetilde{t}_k < 1 = \sum_{j=1}^{k-1} t_j + t_k$, and thus $t_k - \widetilde{t}_k > \sum_{j=1}^{k-1} \widetilde{t}_j - \sum_{j=1}^{k-1} t_j > -(2^{k-1} - 1)v$. Since $r(\mathbf{z}^{k-1}, \widetilde{\mathbf{z}}^{k-1}) < 2^{k-1}v$ and $\sum_{j=1}^{k} \widetilde{t}_j < 1$ also imply $t_k < \widetilde{t}_k + 2^{k-1}v$, we have $\mid t_k - \widetilde{t}_k \mid < 2^{k-1}v$.

(iv) $\sum_{j=1}^{k} t_j < 1$ and $\sum_{j=1}^{k} \widetilde{t}_j = 1$, this is symmetric to (iii) and we also have $\mid t_k - \widetilde{t}_k \mid < 2^{k-1}v$.

According to the above inductive arguments, for each $j \in \{1, 2, \ldots, L\}$, we have $\mid t_j - \widetilde{t}_j \mid < 2^{j-1}v$ and then $r(t_j \mathbf{u}^j, \widetilde{t}_j \mathbf{u}^j) < 2^{j-1}v$. We have $r(\widehat{Ch}_f(\mathbf{x}), \widehat{Ch}_f(\widetilde{\mathbf{x}})) < (2^L - 1)v$.

Therefore, for any $\epsilon > 0$, there exists $\delta = \epsilon / [(2^L - 1)\sqrt{n}]$ such that $\mid \mathbf{x} - \widetilde{\mathbf{x}} \mid < \delta$ implies $r(\mathbf{x}, \widetilde{\mathbf{x}}) < \delta$, and then $r(\widehat{Ch}_f(\mathbf{x}), \widehat{Ch}_f(\widetilde{\mathbf{x}})) < \epsilon / \sqrt{n}$, which further implies $\mid \widehat{Ch}_f(\mathbf{x}) - \widehat{Ch}_f(\widetilde{\mathbf{x}}) \mid < \epsilon$.[34]

Proof of Lemma 2

Since M is an integral matching in $\widehat{\Gamma}$, there exists $f \in \widetilde{F}$ such that $M_f(w) = 1$ for each $w \in W$, and thus μ is a matching in Γ.

We then prove that for any integral $\mathbf{x} \in \{0, 1\}^W$, $\widehat{Ch}_f(\mathbf{x}) = Ch_f(\mathbf{x})$. Let $j \in \{1, 2, \ldots, L\}$ be the index such that $\mathbf{u}^j \leq \mathbf{x}$, and $\mathbf{u}^k \leq \mathbf{x}$ does not hold for all $k < j$. (If such j does not exist, we have $\widehat{Ch}_f(\mathbf{x}) = Ch_f(\mathbf{x}) = \mathbf{0}$.) Then, we have $t_k = 0$ for all $k < j$, $t_j = 1$, and $t_k = 0$ for all $j < k \leq L$. Therefore, we have $\widehat{Ch}_f(\mathbf{x}) = Ch_f(\mathbf{x}) = \mathbf{u}^j$.

Let $M = (M_f)_{f \in \widetilde{F}}$ be a stable integral matching in a continuum market $\widehat{\Gamma}$. Let μ be the matching in Γ such that $\mu(w) = f$ and $w \in \mu(f)$ if $M_f(w) = 1$ for each $w \in W$ and $f \in \widetilde{F}$.

(1) Individual rationality. For any $f \in F$, since M_f is integral, we have $\widehat{Ch}_f(M_f) =$

[34] $\mid \cdot \mid$ is the Euclidean distance.

$Ch_f(M_f)$. The individual rationality of firms in M in $\widehat{\Gamma}$ implies that for each $f \in F$, $M_f = \widehat{Ch}_f(M_f) = Ch_f(M_f)$. Because the set of workers available at $workers(M_f)$ is precisely $\mu(f)$, we have $\mu(f) = Ch_f(\mu(f))$. The individual rationality of workers in M in $\widehat{\Gamma}$ also implies the individual rationality of workers in μ.

(2) No blocking coalition. Suppose f and a subset of workers $S \subseteq W$ block μ in Γ, where $ind(S) = M'$. Since $S \succ_f \mu(f)$ in Γ, we have $\widehat{Ch}_f(M' \vee M) = Ch_f(M' \vee M) \neq M$. Then by the revealed preference property, we have $\widehat{Ch}_f(M' \vee M) \succ_f M$ in $\widehat{\Gamma}$. Since $f \succeq_w \mu(w)$ for all $w \in S$ in Γ, then $\widehat{Ch}_f(M' \vee M) \leq M' \vee M \leq A^{\preceq f}(M)$. Thus f and $\widehat{Ch}_f(M' \vee M)$ block M in $\widehat{\Gamma}$. A contradiction.

Proof of Lemma 3

We first prove the following two lemmata. Given a subpopulation $\mathbf{x} \in [0,1]^W$, let $integral(\mathbf{x}) = \mathbf{y} \in \{0,1\}^W$ where $y(w) = 0$ if $x(w) = 0$ and $y(w) = 1$ if $x(w) > 0$ for each $w \in W$. Given a pseudo-matching M, for each $f \in F$ we still define $B^{\preceq f}(M)$ by (3.3). Let $A^{\prec f}(M) \in [0,1]^W$ be the subpopulation such that $A^{\prec f}(M)(w) = \sum_{f' \in \widetilde{F}: f \succ_w f'} M_{f'}(w)$ for each $w \in W$. $A^{\prec f}(M)$ is the subpopulation of workers assigned to some firm worse than f in M according to their preferences. Note that $A^{\prec f}(M) + M_f = A^{\preceq f}(M)$ for each $f \in F$.

Lemma 4. Let M be a pseudo-matching and there is a firm $f' \in F$ such that $\widehat{Ch}_{f'}(M_{f'}) = M_{f'}$. Consider the procedure (3.1) that computes $\widehat{Ch}_{f'}(M_{f'}) = \sum_{j=1}^{L} t_j \mathbf{u}^j$. There exists a blocking coalition that involves f' if and only if there exists $k \in \{1, 2, \ldots, L\}$ such that $\sum_{j=1}^{k} t_j < 1$ and $\mathbf{u}^k \leq integral(A^{\prec f'}(M))$.

Proof. "$\Leftarrow$". $\mathbf{u}^k \leq integral(A^{\prec f'}(M))$ implies that there exists $\epsilon > 0$ such that $\epsilon \mathbf{u}^k \leq A^{\prec f'}(M)$. Then, the revealed preference property and $\sum_{j=1}^{k} t_j < 1$ imply $\widehat{Ch}_{f'}(\epsilon \mathbf{u}^k + M_{f'}) = \widehat{Ch}_{f'}[\widehat{Ch}_{f'}(\epsilon \mathbf{u}^k + M_{f'}) \vee M_{f'}] \neq M_{f'}$. We also have $\widehat{Ch}_{f'}(\epsilon \mathbf{u}^k + M_{f'}) \leq \epsilon \mathbf{u}^k + M_{f'} \leq A^{\prec f'}(M) + M_{f'} = A^{\preceq f'}(M)$. Hence, f' and $\widehat{Ch}_{f'}(\epsilon \mathbf{u}^k + M_{f'})$ form a blocking coalition.

"$\Rightarrow$". If $t_1 = 1$, then $M_{f'} = \mathbf{u}^1$ and there is no blocking coalition that involves f'. Thus there exists $k \in \{1, 2, \ldots, L\}$ such that $\sum_{j=1}^{k} t_j < 1$. Suppose for all such k, $\mathbf{u}^k \leq \mathit{integral}(A^{\prec f'}(M))$ does not hold. Then, consider the procedures (3.1) that compute $\widehat{Ch}_{f'}(M_{f'})$ and $\widehat{Ch}_{f'}(A^{\preceq f'}(M)) = \sum_{j=1}^{L} \widehat{t}_j \mathbf{u}^j$. Since $A^{\preceq f'}(M) = M_{f'} + A^{\prec f'}(M)$, and $\mathbf{u}^k \leq \mathit{integral}(A^{\prec f'}(M))$ does not hold for all k when $\sum_{j=1}^{k} t_j < 1$, we know that $t_j = \widehat{t}_j$ for all $j \in \{1, 2, \ldots, L\}$. Thus we have $\widehat{Ch}_{f'}(M_{f'}) = \widehat{Ch}_{f'}(A^{\preceq f'}(M))$, and by revealed preference property there is no blocking coalition that involves f'. $\square$

Lemma 5. Let M be a stable pseudo-matching. Choose $f' \in F$ and let M' be a pseudo-matching such that $M'_{f'}(w) = 0$ if $M_{f'}(w) = 0$ for each $w \in W$, and $M'_f = M_f$ for all $f \neq f'$. Then, there exists no blocking coalition that involves $f \neq f'$ in M'.

Proof. For any $f \neq f'$ consider the procedure (3.1) that computes $Ch_f(M_f) = Ch_f(M'_f) = \sum_{j=1}^{L} t_j \mathbf{u}^j$. Since M is stable, by Lemma 4 for all $k \in \{1, 2, \ldots, L\}$ such that $\sum_{j=1}^{k} t_j < 1$, $\mathbf{u}^k \leq \mathit{integral}(A^{\prec f}(M))$ does not hold. Since $M'_{f'}(w) = 0$ if $M_{f'}(w) = 0$, we have $\mathit{integral}(A^{\prec f}(M')) \leq \mathit{integral}(A^{\prec f}(M))$. Therefore, for all $k \in \{1, 2, \ldots, L\}$ such that $\sum_{j=1}^{k} t_j < 1$, $\mathbf{u}^k \leq \mathit{integral}(A^{\prec f}(M'))$ does not hold for all $f \neq f'$. By Lemma 4, there exists no blocking coalition that involves $f \neq f'$ in M'. $\square$

In each of the three transformations, since we set $M'_f(w) = 1$ only when $M_f(w) > 0$ for each $f \in \widetilde{F}$ and each $w \in W$, the workers' individual rationality holds for M' from the individual rationality of M. It is also straightforward to see that the firms' individual rationality also holds. By Lemma 5, in each of the type-1 and type-2 transformations, there exists no blocking coalition that involves $f \neq f'$ in M'. Let $\mathbf{u}^1 \succ \mathbf{u}^2 \succ \cdots \succ \mathbf{u}^L \succ \mathbf{0}$ be the preference order of firm f' over its acceptable sets of workers in Γ.

(1) Type-1. Suppose in M', there exists a blocking coalition that involves f'.

Consider the procedure (3.1) that computes $\widehat{Ch}_{f'}(M'_{f'}) = \sum_{j=1}^{L} t'_j \mathbf{u}^j$. Since $M'_{f'} = \mathbf{0}$, then $t'_j = 0$ for all $j \in \{1, 2, \ldots, L\}$. By Lemma 4, there exists $k \in \{1, 2, \ldots, L\}$ such that $\mathbf{u}^k \leq integral(A^{\prec f'}(M')) = integral(A^{\prec f'}(M))$. Consider the procedure (3.1) that computes $\widehat{Ch}_{f'}(M_{f'}) = \sum_{j=1}^{L} t_j \mathbf{u}^j$. Since $\sum_{j=1}^{L} t_j < 1$ and there exists $k \in \{1, 2, \ldots, L\}$ such that $\mathbf{u}^k \leq integral(A^{\prec f'}(M))$, by Lemma 4, M is not stable. A contradiction.

(2) Type-2. Suppose in M', there exists a blocking coalition that involves f'. Consider the procedure (3.1) that computes $\widehat{Ch}_{f'}(M'_{f'}) = \sum_{j=1}^{L} t'_j \mathbf{u}^j$. Since $M'_{f'} = \mathbf{u}^k$, by Lemma 4, there exists $k' < k$ such that $\mathbf{u}^{k'} \leq integral(A^{\prec f'}(M'))$. Then, Consider the procedure (3.1) that computes $\widehat{Ch}_{f'}(M_{f'}) = \sum_{j=1}^{L} t_j \mathbf{u}^j$. Since $t_k > 0$, we have $\sum_{j=1}^{k'} t_j < 1$, and $\mathbf{u}^{k'} \leq integral(A^{\prec f'}(M')) = integral(A^{\prec f'}(M))$. By Lemma 4, M is not stable. A contradiction.

(3) Type-3. If we set $M'_{\o}(w') = 0$ for some $w' \in W$ such that $M_{\o}(w') \in (0, 1)$, then for each $f \in F$, $B^{\preceq f}(M') \leq A^{\preceq f}(M)$. Any $f \in F$ that can form a blocking coalition with some M''_f in M', can also form a blocking coalition with M''_f in M.

If we set $M'_{\o}(w') = 1$ for some $w' \in W$ such that $M_{\o}(w') \in (0, 1)$, then for any $f \in F$, $integral(A^{\prec f}(M)) = integral(A^{\prec f}(M'))$. If there exists a blocking coalition that involves f in M', then by Lemma 4, there also exists a blocking coalition that involves f in M.

5.2 Proof of Theorem 1

The proof of Theorem 1 is illustrated by Example 6. Every matching market Γ induces a continuum market $\widehat{\Gamma}$, where Lemma 1 and the existence theorem of CKK indicate that a stable matching M is guaranteed to exist in $\widehat{\Gamma}$. For each $f \in F$, consider the procedure (3.1) that computes $\widehat{Ch}_f(M_f) = \sum_{j=1}^{L} t_j \mathbf{u}^j = M_f$, where the last equality is implied by the individual rationality of firms in M. We then construct a matrix B^{*f} as follows.

$$B^{*f} = \begin{cases} [\mathbf{u}^{k_1}, \mathbf{u}^{k_2}, \ldots, \mathbf{u}^{k_s}], & \text{if} \quad \sum_{j=1}^{L} t_j = 1. \\ [\mathbf{u}^{k_1}, \mathbf{u}^{k_2}, \ldots, \mathbf{u}^{k_s}, \mathbf{0}], & \text{if} \quad \sum_{j=1}^{L} t_j < 1. \end{cases} \tag{5.1}$$

where $t_l > 0$ for each $l \in \{k_1, k_2, \ldots, k_s\}$ and $k_1 < k_2 < \cdots < k_s$.[35]

We then construct matrix $B^{ø}$ and B^f for each $f \in F$. Each column of $B^{ø}$ is an $(m+n)$- dimensional unit vector, where the $(m+j)$-th unit vector is in $B^{ø}$ if $M_{ø}(w_j) > 0$.[36] We construct B^f from B^{*f} for each $f \in F$. Each B^f has $m+n$ rows, where the first m components of each column are the components of the i-th unit vector of dimension m when $f = f_i$. The last n components of each column of B^f are those of each column of B^{*f}, thus the number of columns of each B^f is s or $s+1$ in its corresponding formula (5.1). Then, let matrix $B = [B^{f_1}, B^{f_2}, \cdots, B^{f_m}, B^{ø}]$. Let $cl(f)$ be the number of columns of B^f for each $f \in \widetilde{F}$. Thus, B is an $(m+n) \times \sum_{f \in \widetilde{F}} cl(f)$ matrix. Theorem 1 is then implied by the following lemma and Lemma 2.

Lemma 6. Consider the system of linear equations $B\mathbf{z} = \mathbf{1}$, where $\mathbf{z}$ is a $\sum_{f \in \widetilde{F}} cl(f)$-dimensional vector, and $\mathbf{1}$ is the $(m+n)$-dimensional vector with all its coordinates being 1. If the firms' demand type $\mathcal{D}$ is unimodular, then there exists a solution $\mathbf{z} = (\mathbf{z}^{f_1}, \mathbf{z}^{f_2}, \ldots, \mathbf{z}^{f_m}, \mathbf{z}^{ø}) \in \{0,1\}^{\sum_{f \in \widetilde{F}} cl(f)}$ to $B\mathbf{z} = \mathbf{1}$, where $\mathbf{z}^f$ is a $cl(f)$-dimensional vector for each $f \in \widetilde{F}$.[37] Moreover, M' is a stable integral matching in $\widehat{\Gamma}$, where

[35] In (5.1), for each $l \in \{k_1, k_2, \ldots, k_s\}$, $\mathbf{u}^l$ is a column of matrix B^{*f}. B^{*f} is an $n \times s$ or $n \times (s+1)$ matrix. The numbers $\{k_1, k_2, \ldots, k_s\}$ and s are different for different firms.

[36] The $(m+j)$-th unit vector is the $(m+n)$-dimensional unit vector with its $(m+j)$-th coordinate being 1. In Example 6, $B^{ø}$ is the matrix with a single column B_5.

[37] By $\mathbf{z} = (\mathbf{z}^{f_1}, \mathbf{z}^{f_2}, \ldots, \mathbf{z}^{f_m}, \mathbf{z}^{ø})$ we mean the first $cl(f_1)$ components of $\mathbf{z}$ are the components of $\mathbf{z}^{f_1}$, the $(cl(f_1)+1)$-th to $(cl(f_1)+cl(f_2))$-th components are the components of $\mathbf{z}^{f_2}$, and so on. In Example 6, $\mathbf{z} = (1,0,0,1,1)$ is such a solution where $\mathbf{z}^{f_1} = (1,0)$, $\mathbf{z}^{f_2} = (0,1)$, and $\mathbf{z}^{ø} = 1$.

$M'_f = \sum_{j=1}^{cl(f)} z_j^f B_j^{*f}$ for each $f \in \widetilde{F}$.[38]

Proof. We first show that the polytope $\{\mathbf{z} \mid B\mathbf{z} = \mathbf{1}, \mathbf{z} \geq 0\}$ is nonempty. For each $f \in F$, consider the procedure (3.1) that computes $\widehat{Ch}_f(M_f) = \sum_{j=1}^{L} t_j \mathbf{u}^j = M_f$ and let

$$\widehat{\mathbf{z}}^f = \begin{cases} (t_{k_1}, t_{k_2}, \dots, t_{k_s}), & \text{if } \sum_{j=1}^{L} t_j = 1. \\ (t_{k_1}, t_{k_2}, \dots, t_{k_s}, 1 - \sum_{i=1}^{s} t_{k_i}), & \text{if } \sum_{j=1}^{L} t_j < 1. \end{cases}$$

where $k_1, k_2, \dots, k_s$ are those in (5.1). Let $\widehat{\mathbf{z}}^{ø}$ be the $cl(ø)$-dimensional vector where $\widehat{z}_l^{ø} = M_ø(w_j)$ if $B_l^{ø}$ is the $(m+j)$-th unit vector for each $l \in \{1, \dots, cl(ø)\}$. Let B^{**} be the matrix constituted of the first m rows of B, and B^* the matrix constituted of the last n rows of B. Let $\widehat{\mathbf{z}} = (\widehat{\mathbf{z}}^{f_1}, \widehat{\mathbf{z}}^{f_2}, \dots, \widehat{\mathbf{z}}^{f_m}, \widehat{\mathbf{z}}^{ø})$. $\sum_{i=1}^{cl(f)} \widehat{z}_i^f = 1$ for each $f \in F$ implies $B^{**}\widehat{\mathbf{z}} = \mathbf{1}$, and $\sum_{f \in \widetilde{F}} M_f(w) = 1$ for each $w \in W$ implies $B^*\widehat{\mathbf{z}} = \mathbf{1}$. Hence, we know $\widehat{\mathbf{z}}$ is in $\{\mathbf{z} \mid B\mathbf{z} = \mathbf{1}, \mathbf{z} \geq 0\}$.

If the set of columns of B is unimodular, all vertices of the polytope $\{\mathbf{z} \mid B\mathbf{z} = \mathbf{1}, \mathbf{z} \geq 0\}$ are integral (see, e.g., Theorem 21.5 of Schrijver, 1986). Then, since the polytope is nonempty, we know that there is at least an integral vertex on this polytope. We now show that the set of columns of B is unimodular if the firms' demand type $\mathcal{D}$ is unimodular. For any linearly independent subset $\widehat{B}$ of columns from B, we partition $\widehat{B}$ into $\widehat{B} = \cup_{f \in \widetilde{F}} \widehat{B}^f$, where $\widehat{B}^f$ is the collection of vectors of $\widehat{B}$ from B^f for each $f \in \widetilde{F}$. $\widehat{B}^f$ is possibly empty for some $f \in \widetilde{F}$.[39] Let $\mathbf{b}_1^f, \mathbf{b}_2^f, \cdots$ denote the elements of $\widehat{B}^f$ for each $f \in \widetilde{F}$ if $\widehat{B}^f$ is not empty. For each $i \in \{1, \dots, m\}$, if $\widehat{B}^{f_i}$ is empty, let $\mathbf{b}_1^{f_i}$ be the i-th unit vector of $m+n$ dimensions. For each $f \in F$, let $B^f = \{\mathbf{b}_2^f - \mathbf{b}_1^f, \mathbf{b}_3^f - \mathbf{b}_1^f, \cdots\}$ if there are at least 2 elements in

[38] B_j^{*f} is the vector of the j-th column of B^{*f}.

[39] For instance, if we consider $\widehat{B} = \{B_1, B_2, B_5\}$ in Example 6, then $\widehat{B}^{f_2}$ is empty. In the following extension to a basis for $\mathbb{R}^{m+n}$ from $\widehat{B}$, we put the i-th unit vector of $m+n$ dimensions into $\widehat{B}$ when $\widehat{B}^{f_i}$ is empty for each $i \in \{1, \dots, m\}$.

$\widehat{B}^f$, and $B^f = \emptyset$ otherwise. Let $\widetilde{B} = (\cup_{f\in F} B^f) \cup \widehat{B}^\varnothing$. Note that the first m coordinates of each vector from $\widetilde{B}$ are 0. Let $\widetilde{B}^*$ be the set of n-dimensional vectors by removing the first m components of each vector from $\widetilde{B}$. If $\mathcal{D}$ is unimodular, $\widetilde{B}^*$ can be extended to a basis for $\mathbb{R}^n$, of integer vectors, with determinant ± 1. Then, the set $\{\mathbf{b}_1^{f_1}, \mathbf{b}_1^{f_2}, \cdots, \mathbf{b}_1^{f_m}\} \cup \widetilde{B}$ can be extended to a basis for $\mathbb{R}^{m+n}$, of integer vectors, with determinant ± 1. Since adding one column to another column leaves the determinant unchanged, such extension also exists for $\{\mathbf{b}_1^{f_1}, \mathbf{b}_1^{f_2}, \cdots, \mathbf{b}_1^{f_m}\} \cup \widehat{B}$, which can be extended from $\widehat{B}$. Therefore, the set of columns of B is unimodular.

Now we know there is at least an integral vertex on the polytope $\{\mathbf{z} \mid B\mathbf{z} = \mathbf{1}, \mathbf{z} \geq 0\}$ when the firms' demand type $\mathcal{D}$ is unimodular. According to the structure of matrix B, any nonnegative integral solution to $B\mathbf{z} = \mathbf{1}$ must be a 0-1 vector. Let $\mathbf{z} = (\mathbf{z}^{f_1}, \mathbf{z}^{f_2}, \ldots, \mathbf{z}^{f_m}, \mathbf{z}^\varnothing)$ be an integral point of $\{\mathbf{z} \mid B\mathbf{z} = \mathbf{1}, \mathbf{z} \geq 0\}$, where $\mathbf{z}^f$ is a $cl(f)$-dimensional vector for each $f \in \widetilde{F}$. Let M' be the integral pseudo-matching in $\widehat{\Phi}$, where $M'_f = \sum_{j=1}^{cl(f)} z_j^f B_j^{*f}$ for each $f \in \widetilde{F}$. Because $B^{**}\mathbf{z} = \mathbf{1}$ (i.e., $\sum_{j=1}^{cl(f)} z_j^f = 1$ for each $f \in F$), we know that M' can be obtained via stable transformations on M. By Lemma 3, M' is a stable integral pseudo-matching. Since $B^*\mathbf{z} = \mathbf{1}$ implies $\sum_{f\in\widetilde{F}} M'_f(w) = 1$ for each $w \in W$, we know that M' is a stable integral matching. □

5.3 Proof of Theorem 2

We present the proof first and then an example to illustrate the proof.

Suppose the firms have unit-demand preferences over a technology tree $T = (V, E, W)$. Let $G = (V, E')$ be a complete directed graph where the set of vertices V is the same as that of T, and the direction of each edge $e \in E'$ is arbitrary. We use G and the directed tree (V, E) to define matrix H as follows. For each $e \in E$ and

$e' = (v, v') \in E'$,

$$
\begin{aligned}
H_{e,e'} = +1 & \quad \text{if the unique } v - v' \text{ path in } (V, E) \text{ passes through } e \text{ forwardly;} \\
-1 & \quad \text{if the unique } v - v' \text{ path in } (V, E) \text{ passes through } e \text{ backwardly;} \\
0 & \quad \text{if the unique } v - v' \text{ path in } (V, E) \text{ does not pass through } e.
\end{aligned}
$$

H is a network matrix, which is totally unimodular (see e.g., Chapter 19.3 of Schrijver, 1986).

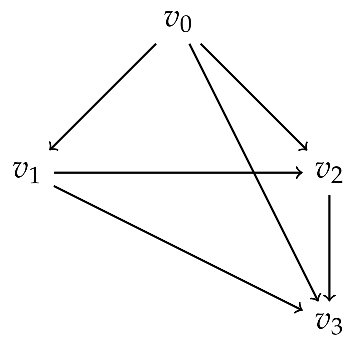
Because each worker is a specialist in T, we can define e^w to be the unique edge that w engages for each $w \in W$. We use G and the technology tree $T = (V, E, W)$ to define matrx H' as follows. For each $w \in W$ and $e' = (v, v') \in E'$,

$$
\begin{aligned}
H'_{w,e'} = +1 & \quad \text{if the unique } v - v' \text{ path in } (V, E) \text{ passes through } e^w \text{ forwardly;} \\
-1 & \quad \text{if the unique } v - v' \text{ path in } (V, E) \text{ passes through } e^w \text{ backwardly;} \\
0 & \quad \text{if the unique } v - v' \text{ path in } (V, E) \text{ does not pass through } e^w.
\end{aligned}
$$

H' can be obtained from H by repeating some rows. In particular, if $| W^e |= k$ for edge e in the technology tree, then the row of edge e of H is repeated k times in H'. Thus we know that H' is also totally unimodular.

If $\mathbf{d}$ belongs to the demand type of a firm $f \in F$, then either $\mathbf{d}$ is a column of H', or $-\mathbf{d}$ is a column of H' (see an illustration below). Since totally unimodularity is preserved by adding a column of a unit vector, we know that the vectors from firms' demand type $\mathcal{D}$ also form a column submatrix of a totally unimodular matrix. Therefore, $\mathcal{D}$ is unimodular.

For instance, consider the technology tree in Example 7, and let $G = (V, E')$ be the following complete directed graph.



The directed graph G and the directed tree (V, E) generate matrix H as follows.

	v_0v_1	v_0v_2	v_0v_3	v_1v_2	v_1v_3	v_2v_3
v_0v_1	1	0	0	-1	-1	0
v_0v_2	0	1	1	1	1	0
v_2v_3	0	0	1	0	1	1

The directed graph G and the technology tree $T = (V, E, W)$ generate matrix H' as follows.

	v_0v_1	v_0v_2	v_0v_3	v_1v_2	v_1v_3	v_2v_3
w_1	1	0	0	-1	-1	0
w_2	1	0	0	-1	-1	0
w_3	0	1	1	1	1	0
w_4	0	0	1	0	1	1

The matrix H' is exactly the one by repeating the first row of the matrix H. Now we consider the firms' preferences of (4.1). Let H'_j be the j-th column of matrix H'. We find that f_1's demand type is $\{H'_1, H'_2, -H'_4\}$, and f_2's demand type is $\{H'_1, H'_3, H'_5\}$.

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