Bargaining with Evolving Private Information*

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July 6, 2022

Abstract

I study how the arrival of new private information affects bargaining outcomes. A seller makes offers to a buyer. The buyer is privately informed about her valuation, and the seller privately observes her stochastically changing cost of delivering the good. Prices fall gradually at the early stages of negotiations, and trade is inefficiently delayed. The first-best is implementable via a mechanism, whereas all equilibrium outcomes of the bargaining game are inefficient.

KEYWORDS: bargaining, inefficient delay, Coase conjecture, evolving private information, two-sided private information.

*I am indebted to Nageeb Ali, Brendan Daley, Marina Halac, Johannes Horner, Bart Lipman, Qingmin Liu, Chiara Margaria, Dilip Mookherjee and Jawwad Noor, and to seminar participants at Arizona State, Bonn, BU, Chicago, EUI, Michigan, Penn State, Seoul National University, the 2020 Stony Brook Conference and Yale for helpful comments and suggestions. Pablo Cuellar Tapia, Duoxi Li and Beixi Zhou provided excellent research assistance.

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1 Introduction

In many bargaining settings, new private information may arrive as negotiations proceed. Consider, for instance, a producer of a new intermediate good negotiating a sale with a potential industrial buyer. Since the good for sale is new, production costs are likely to be initially high. Over time, costs may fall as the seller privately becomes more efficient. Markets for new durable goods also typically feature declining production costs, driven by efficiency gains and falling input prices. The goal of this paper is to study how the arrival of new private information affects bargaining outcomes.

I study a bargaining game in which a seller makes offers to a privately informed buyer.[1] The seller's cost of producing the good changes stochastically over time, and is privately observed by the seller. The seller's cost can take two values, high or low, and it evolves over time as a Markov chain. For most of the analysis I focus on separating Perfect Bayesian Equilibria (PBE), under which the seller's price each period reveals her cost. These equilibria are intuitive, tractable, and provide a natural point of comparison with prior papers in the literature (e.g. Fudenberg et al., 1985, Gul et al., 1985, Cho, 1990, Ortner, 2017).

The analysis delivers three main results. First, I provide a characterization of the set of separating PBE. In any separating equilibrium, buyer and seller trade at a slow rate when the seller's cost is high, and prices fall gradually. When the seller's cost falls, equilibrium becomes Coasian: buyer and seller trade fast at a low price. Market dynamics under separating PBE are broadly consistent with dynamics typically observed in markets for new durable goods, where prices fall gradually during the early stages, and market penetration raises slowly (Conlon, 2012). Moreover, without loss, separating PBE can be taken to be weakly stationary.

The key drivers of these equilibrium dynamics are the information revelation con-

[1] As usual, this bargaining model is mathematically equivalent to a model in which a durable good monopolist sells to a population of heterogenous buyers.

straints that arise as a result of the seller's evolving private information. In any separating equilibrium, a seller whose cost just fell must not gain by mimicking a high cost seller and posting a high price. The slow rate at which buyer and seller trade when costs are high makes this deviation unprofitable, since a low cost seller has a stronger incentive to trade fast. An implication is that information revelation constraints lead to inefficiencies relative to the first-best outcome.

The second main result studies the frequent-offers limit of (most efficient) separating equilibria. I show that this limit is characterized by a system of differential equations, which specifies how prices and probability of trade change over time while the seller's cost is high. This tractable characterization allows me to derive several comparative statics. An increase in the seller's high cost increases equilibrium prices, and lowers the speed with which buyer and seller trade. An increase in the distribution of buyer values (in terms of the reverse hazard rate), or an increase in the rate at which costs fall, have similar effects on bargaining dynamics. Lastly, seller's profits become negligible as the buyer's lowest valuation converges to zero, as in classic Coasian bargaining games (Fudenberg et al., 1985, Gul et al., 1985). The difference, however, is that this fall in seller's profits comes together with a drop in social welfare.

My third main result shows that, under some conditions, the environment that I study admits an efficient mechanism satisfying individually rationality, incentive compatibility and budget balance. An implication is that equilibrium dynamics lead to greater inefficiencies than those implied by feasibility. This relates my work with Deneckere and Liang (2006), who study settings with interdependent values and show that bargaining outcomes are not second-best whenever the first-best outcome is not implementable.

Related literature. This paper fits into the literature on dynamic bargaining with private information. Early contributions in this literature illustrate how, in settings with one-sided private information, the uninformed party's inability to commit to future offers limits the rents she can extract (Bulow (1982), Fudenberg et al. (1985), Gul et al. (1985), Gul and Sonnenschein (1988)). Stationary equilibria satisfy the Coase conjecture when offers are frequent (Coase, 1972): the seller posts a low initial price, and buyer and seller reach an immediate agreement.

Several papers have identified economic forces that push towards inefficient bargaining outcomes within the one-sided private information framework. Bargaining inefficiencies can arise when bargainers strategically delay trade to signal their types (Admati and Perry, 1987), when bargainers use non-stationary strategies (Ausubel and Deneckere, 1989), when the seller faces capacity constraints (Kahn, 1986, McAfee and Wiseman, 2008), or when values are interdependent (Evans, 1989, Vincent, 1989, Deneckere and Liang, 2006, Gerardi et al., 2021). Costly delays can also arise in the presence of deadlines (Güth and Ritzberger, 1998, Hörner and Samuelson, 2011, Fuchs and Skrzypacz, 2013), when bargainers have outside options (Board and Pycia, 2014), or when bargainers seek to build a reputation for being obstinate (Myerson, 2013, Abreu and Gul, 2000).[2]

A smaller literature studies how inefficiencies arise when there is two-sided private information (Cramton, 1984, 1992, Chatterjee and Samuelson, 1987, 1988, Cho, 1990, Ausubel and Deneckere, 1992). The closest work within this literature is Cho (1990), who studies separating stationary equilibria of a two-sided private information game. Cho's main result establishes a version of the Coase conjecture. In particular, when buyer and seller trade, they do it without delay, and at a price equal to the buyer's lowest value. Moreover, bargaining outcomes are efficient if and only if gains from trade are common knowledge. The current paper adds to this literature by analyzing a model in which the seller's cost privately evolves over time. Separating equilibria in this model generate non-trivial price dynamics. In addition, bargaining outcomes are inefficient even with common-knowledge gains from trade, and even when the efficient outcome is implementable.

[2] See also Abreu and Pearce (2007), Fanning (2016, 2018), Sanktjohanser (2017).

The current paper also relates to Ortner (2017), who studies a continuous-time durable goods monopoly model in which the seller's cost is publicly observed, and changes stochastically over time.[3] Ortner (2017) shows that time-varying costs allow the seller to extract rents when buyer values are discrete. With a continuum of buyer types (as in the current paper), the seller is unable to extract rents, and the market outcome is efficient.[4]

Fuchs and Skrzypacz (2010) and Daley and Green (2020) study bargaining games with one-sided private information in which players may receive public news while negotiating. Their results shed light into how the arrival of public information affects bargaining outcomes and can lead to costly delays and inefficiencies. In contrast, the current paper highlights the inefficiencies generated by the arrival of new private information.

Hwang (2018) studies how the arrival of new private information affects trading dynamics between a long-run seller and a sequence of short-term buyers. I instead study how new private information affects bargaining dynamics between two long-run agents. Kennan (2001) studies a repeated bargaining game with imperfectly persistent one-sided private information, and shows that this may give rise to path-dependent bargaining outcomes.

Lastly, several papers construct models to rationalize sales in durable goods markets. Conlisk et al. (1984) and Sobel (1984, 1991) propose theories of sales driven by entry of new consumers. Board (2008), Board and Skrzypacz (2016) and Dilmé and Li (2019) show that sales can be part of an optimal selling scheme when demand is time-varying. Dilmé and Garrett (2017) show that sellers might extract additional

[3]See also Acharya and Ortner (2017), who study how public shocks affect equilibrium dynamics in environments with perfectly persistent private information.

[4]A previous version of this paper (Ortner, 2021) compares equilibrium outcomes in the current model with a model in which the seller's evolving cost is publicly observed, as in Ortner (2017). Stationary equilibria of the game with public costs retain two key features of the Coasian model: equilibrium outcomes are efficient in the frequent-offers limit, and the seller is unable to extract rents. Hence, privately observed costs lead to lower social welfare, higher seller revenues and lower buyer surplus relative to a setting with public costs.

rents by offering random price discounts. The current paper adds to this literature by providing a theory of sales driven by changes in the seller's cost of production.

The paper proceeds as follows. Section 2 introduces the model. Section 3 characterizes the set of separating PBE. Section 4 studies the frequent-offers limit of welfare maximizing separating PBE and derives several comparative statics. Section 5 shows that, under certain conditions, the game admits an efficient mechanism satisfying IC, IR and budget balance. Section 6 discusses other (non-separating) equilibria. Proofs are collected in the Appendix.

2 Model

A seller with the technology to deliver a good faces a buyer. The buyer's valuation for the seller's good, v, is her private information, and is drawn from distribution F with support $[\underline{v}, \overline{v}]$ and continuous density $F' = f$ satisfying $f(v) > 0$ for all $v \in [\underline{v}, \overline{v}]$. I assume that $\underline{v} > 0$. Time is discrete, with $t \in T(\Delta) \equiv \{0, \Delta, 2\Delta, ..., \infty\}$.

The seller's cost of delivering the good (or, equivalently, her opportunity cost of selling it) changes over time. The seller's cost can take two values: $c_H > 0$ or $c_L = 0$. At $t = 0$, the seller's cost c_0 takes value c_H with probability $q \in (0, 1)$ and c_L with probability $1 - q$. For all times $t \in T(\Delta)$, $\text{prob}(c_{t+\Delta} = c_H | c_t = c_H) = e^{-\lambda\Delta}$ and $\text{prob}(c_{t+\Delta} = c_L | c_t = c_L) = 1$, where $\lambda > 0$ is a strictly positive constant. The assumption that low cost c_L is absorbing simplifies the exposition, but is not necessary.[5] The seller is privately informed about her production cost: she privately observes her current cost realization at the start of each period $t \in T(\Delta)$.

The timing within each period t is as follows. At $t = 0$, the buyer privately learns her valuation and the seller privately learns her initial cost. Then, the seller offers price $p_0 \in \mathbb{R}_+$, and the buyer chooses to accept or reject this price. At any time $t > 0$,

[5] A previous version of this paper, Ortner (2021), shows that the paper's main results extend when c_L is not absorbing. Intuitively, the seller has an incentive to trade fast when her cost is c_L, regardless of whether or not c_L is absorbing.

if the buyer hasn't yet accepted a price, the seller first privately observes current cost c_t. After observing c_t, the seller offers price $p_t \in \mathbb{R}_+$, and the buyer chooses to accept or reject this price. If the buyer accepts the seller's offer at time t, trade happens and the game ends, with the buyer obtaining payoff $e^{-rt}(v - p_t)$ and the seller obtaining payoff $e^{-rt}(p_t - c_t)$, where $r > 0$ is the common discount rate.

Histories and strategies. At any period t before agreement is reached, the seller's history $h_t^S = \{c_s, p_s\}_{s<t}$ records all previous cost realizations and all previous prices, and the buyer's history $h_t^B = \{v, \{p_s\}_{s<t}\}$ records her valuation and all previous prices. A (pure) strategy for the seller $\sigma_S : h_t^S \times c_t \mapsto p_t$ maps seller's histories h_t^S and current cost c_t into a price. A (pure) strategy for the buyer $\sigma_B : h_t^B \times p_t \mapsto d_t \in \{accept, reject\}$ maps buyer's histories h_t^B and the seller's current price p_t into a decision of whether or not to accept price p_t. For any buyer history $h_t^B = \{v, \{p_s\}_{s<t}\}$ and current price p_t, I use $h_t^B \sqcup p_t$ to denote the buyer history $h_{t+\Delta}^B = \{v, \{p_s\}_{s<t+\Delta}\}$.

Solution concept. For most of the paper, I focus on *separating* Perfect Bayesian Equilibrium (PBE) under which, at every seller history, the seller's price reveals her current cost (Section 6 discusses other equilibria). As the analysis below shows, these equilibria are intuitive and tractable. Moreover, they provide a natural point of comparison with prior papers in the literature (e.g. Cho, 1990, Ortner, 2017).

Let (σ, μ) be a PBE, where $\sigma = (\sigma_S, \sigma_B)$ are players' strategies and $\mu = (\mu_S, \mu_B)$ are players' beliefs: $\mu_S(h_t^S)$ is the seller's beliefs over the buyers' type after history h_t^S, and $\mu_B(h_t^B)$ the buyer's beliefs over the seller's realized costs $\{c_s\}_{s<t}$ after history h_t^B. I look for PBE (σ, μ) with the property that, for every seller history h_t^S, $\mathsf{supp}\, \sigma^B(h_t^S)(c_H) \cap \mathsf{supp}\, \sigma^B(h_t^S)(c_L) = \emptyset$. That is, for every history h_t^S, the seller charges a different price if her cost at time t is c_H than if it is c_L. As a result, for every on-path buyer history $h_t^B \sqcup p_t$, $\mu_B(h_t^B \sqcup p_t)$ assigns probability 1 to the seller's true realized costs $\{c_s\}_{s \le t}$.

In addition, I impose the following restriction on the buyer's beliefs: if at any history h_t^B the buyer assigns probability 1 to the seller's current cost being c_L, then I require that for all histories that follow h_t^B, the buyer continues to assign probability 1 to the seller's cost being c_L. This restriction is natural, since cost c_L is absorbing.[6] Let $\Sigma^{\mathsf{S}}(\Delta)$ denote the set of PBE satisfying these conditions, under which the seller uses a pure action while her costs are c_H.[7]

Successive skimming. Any PBE must satisfy the *skimming property*: if at time t a buyer with valuation $v \in [\underline{v}, \overline{v})$ finds it optimal to accept the current price p_t, then a buyer with valuation $v' > v$ finds it strictly optimal to accept p_t. The reason for this is that it is more costly for high-value buyers to delay trade.[8] The skimming property implies that, after any buyer history $h_t^B \sqcup p_t$, there exists a cutoff $\kappa_{t+\Delta}$ such that a buyer with valuation $v > \kappa_{t+\Delta}$ accepts the current offer p_t, and a buyer with valuation $v < \kappa_{t+\Delta}$ rejects the offer. Hence, if the buyer rejects all of the seller's offers $\{p_s\}_{s \leq t}$ up to time t, the seller believes that the buyer's valuation is distributed according to $\text{prob}(v \leq \tilde{v}) = \frac{F(\tilde{v})}{F(\kappa_{t+\Delta})}$ for all $\tilde{v} \in [\underline{v}, \kappa_{t+\Delta}]$.

First-best. Define $\rho(\Delta) \equiv \frac{e^{-r\Delta}(1-e^{-\lambda\Delta})}{1-e^{-(r+\lambda)\Delta}}$ to be the expected discounted time until costs fall to c_L, when current cost is c_H. Let $v^*(\Delta)$ be the solution to $v^*(\Delta) - c_H = \rho(\Delta)v^*(\Delta)$. Under the first-best outcome, the seller sells to a buyer with valuation $v \geq v^*(\Delta)$ at $t = 0$, regardless of the initial cost, and sells to a buyer with valuation $v < v^*(\Delta)$ the first time costs fall to c_L. Define $\tau_L \equiv \min\{t \in T(\Delta) : c_t = c_L\}$ to be the random time at which costs fall to c_L. The following proposition summarizes the first-best outcome.

Proposition 1 (First best)**.** *Under the first best, a buyer with valuation $v \geq v^*(\Delta)$ buys at time $t = 0$, and a buyer with valuation $v < v^*(\Delta)$ buys at time τ_L.*

[6]This condition is similar to the "Never Dissuaded Once Convinced" condition often used in bargaining models with private information (e.g., Osborne and Rubinstein, 1990).

[7]The restriction to equilibria under which the seller uses a pure strategy when her costs are c_H greatly simplifies the exposition. In Appendix A, I briefly discuss equilibria under which the seller mixes while her costs are c_H.

[8]See Lemma 1 in Fudenberg et al. (1985) for a formal proof.

Throughout the paper, I maintain the following assumption.

Assumption 1. $v^*(\Delta) \in (\underline{v}, \overline{v})$.

Since $v^*(\Delta) > c_H$, Assumption 1 is consistent both with gains from trade being common knowledge (i.e., $\underline{v} \geq c_H$) and with settings in which some buyer types only trade when costs are low (i.e, $\underline{v} < c_H$). Note however that $\lim_{\lambda \to 0} v^*(\Delta) = c_H$. Hence, the counterpart of Assumption 1 in a model with time-invariant costs is $c_H \in (\underline{v}, \overline{v})$.

3 Separating Equilibria

This section studies equilibrium set $\Sigma^{\mathsf{S}}(\Delta)$. I start with a few preliminary observations. Note that in any PBE in $\Sigma^{\mathsf{S}}(\Delta)$, when costs fall to c_L the buyer's beliefs about the seller's cost remain concentrated at c_L at all future periods. Hence, the continuation game is strategically equivalent to the one-sided incomplete information game in Fudenberg et al. (1985) and Gul et al. (1985). This game has a unique equilibrium (since $\underline{v} > c_L = 0$), which is weakly stationary: the buyer's acceptance rule at histories at which the current price is the lowest among all past prices depends solely on her valuation (see Fudenberg et al. (1985) and Gul et al. (1985)). For any $\kappa \in [\underline{v}, \overline{v}]$, let $p^L(\kappa)$ denote the price that a seller posts in the one-sided incomplete information game when her belief cutoff is κ, and let $U^L(\kappa)$ denote the seller's equilibrium continuation profits given belief cutoff κ.

Consider next equilibrium behavior at periods at which costs are high. Note that for any $(\sigma, \mu) \in \Sigma^{\mathsf{S}}(\Delta)$, on-path behavior at times t with $c_t = c_H$ is characterized by a sequence $\{p_t^H, \kappa_t^H\}_{t \in T(\Delta)}$ such that: p_t^H is the price that the seller charges at time t if $c_t = c_H$, and κ_t^H is the seller's belief cutoff at the start of time t if her cost last period was c_H. Hence, on the equilibrium path, at any time $t \in T(\Delta)$ with $c_t = c_H$, the buyer accepts the seller's price if her valuation lies in $[\kappa_{t+\Delta}^H, \kappa_t^H)$; and the conditional probability with which buyer and seller trade is $\frac{F(\kappa_t^H) - F(\kappa_{t+\Delta}^H)}{F(\kappa_t^H)}$.

At any time $t \in T(\Delta)$ with $c_t = c_L$, continuation play is given by the continuation equilibrium of the one-sided private information game.

For any sequence $\{p_\tau^H, \kappa_\tau^H\}$, and for all times t, let $U_t^H(\{p_\tau^H, \kappa_\tau^H\})$ be the seller's on-path continuation payoff if $c_t = c_H$, when play is given by $\{p_\tau^H, \kappa_\tau^H\}$:

$$U_t^H(\{p_\tau^H, \kappa_\tau^H\}) = (p_t^H - c_H)\frac{F(\kappa_t^H) - F(\kappa_{t+\Delta}^H)}{F(\kappa_t^H)} + e^{-(r+\lambda)\Delta}\frac{F(\kappa_{t+\Delta}^H)}{F(\kappa_t^H)}U_{t+\Delta}^H(\{p_\tau^H, \kappa_\tau^H\})$$
$$+ e^{-r\Delta}(1 - e^{-\lambda\Delta})\frac{F(\kappa_{t+\Delta}^H)}{F(\kappa_t^H)}U^L(\kappa_{t+\Delta}^H).$$

Theorem 1. *There exists $\overline{\Delta} > 0$ such that for all $\Delta \leq \overline{\Delta}$: (i) $\Sigma^{\mathsf{S}}(\Delta)$ is non-empty; and (ii) for every equilibrium $(\sigma, \mu) \in \Sigma^{\mathsf{S}}(\Delta)$, there exists a weakly stationary equilibrium $(\sigma^{\mathsf{ws}}, \mu^{\mathsf{ws}}) \in \Sigma^{\mathsf{S}}(\Delta)$ that induces the same outcome as (σ, μ).*

To establish Theorem 1, I show that in any equilibrium in $\Sigma^{\mathsf{S}}(\Delta)$, for all $t \in T(\Delta)$, prices and belief cutoffs $\{p_\tau^H, \kappa_\tau^H\}$ satisfy the following three conditions:[9]

$$\kappa_{t+\Delta}^H - p_t^H = e^{-(r+\lambda)\Delta}(\kappa_{t+\Delta}^H - p_{t+\Delta}^H) + e^{-r\Delta}(1 - e^{-\lambda\Delta})(\kappa_{t+\Delta}^H - p^L(\kappa_{t+\Delta}^H)), \quad (1)$$

$$\frac{F(\kappa_t^H) - F(\kappa_{t+\Delta}^H)}{F(\kappa_t^H)}p_t^H \leq U^L(\kappa_t^H) - e^{-r\Delta}\frac{F(\kappa_{t+\Delta}^H)}{F(\kappa_t^H)}U^L(\kappa_{t+\Delta}^H), \quad (2)$$

$$U_t^H(\{p_\tau^H, \kappa_\tau^H\}) \geq \rho(\Delta)U^L(\kappa_t^H). \quad (3)$$

I further show that, for all $\Delta \leq \overline{\Delta}$ (where $\overline{\Delta}$ is the cutoff in Theorem 1), and for any sequence $\{p_\tau^H, \kappa_\tau^H\}$ satisfying (1)-(3) with $\{\kappa_\tau^H\}$ decreasing, there exists an equilibrium $(\sigma, \mu) \in \Sigma^{\mathsf{S}}(\Delta)$ that induces $\{p_\tau^H, \kappa_\tau^H\}$. Hence, for Δ small, these three conditions fully characterize $\Sigma^{\mathsf{S}}(\Delta)$.

Equation (1) is the standard indifference condition of the marginal buyer: for all periods t with $c_t = c_H$, the marginal buyer $\kappa_{t+\Delta}^H$ is indifferent between trading at the current price p_t^H, or waiting and trading at time $t + \Delta$.

[9] As I explain in Appendix A, while condition (1) need not hold at periods in which buyer and seller trade with zero probability, it is without loss to focus on equilibria in which the condition does hold for all t.

Inequality (2) shows that the probability $(F(\kappa_t^H) - F(\kappa_{t+\Delta}^H))/F(\kappa_t^H)$ with which buyer and seller trade at a period t with $c_t = c_H$ cannot be too large. As a result, equilibrium trade is slow relative to the first-best outcome. To see why (2) holds, suppose that the seller's belief cutoff at t is κ_t^H, and that her cost falls from c_H to $c_L = 0$ at this period. The seller's profit from posting price $p^L(\kappa_t^H)$ and revealing that her cost is c_L is $U^L(\kappa_t^H)$. The seller's profit from mimicking a high cost seller for one period, and revealing her cost at $t + \Delta$, is

$$\frac{F(\kappa_t^H) - F(\kappa_{t+\Delta}^H)}{F(\kappa_t^H)} p_t^H + e^{-r\Delta} \frac{F(\kappa_{t+\Delta}^H)}{F(\kappa_t^H)} U^L(\kappa_{t+\Delta}^H).$$

Inequality (2) guarantees that this deviation is not profitable.

Lastly, equation (3) shows that the seller's equilibrium payoff when her cost is c_H must be at least as large as what she would get by delaying trade until her cost falls to c_L, and playing the continuation equilibrium from that point onwards.

Proposition 2. *In any equilibrium in $\Sigma^{\mathsf{S}}(\Delta)$, a buyer with value $v < v^*(\Delta)$ only trades when the seller's cost is low: if $\{p_\tau^H, \kappa_\tau^H\}$ is induced by $(\sigma, \mu) \in \Sigma^{\mathsf{S}}(\Delta)$, then for all τ, $\kappa_\tau^H \geq v^*(\Delta)$.*

Proposition 2 shows that any inefficiency takes the form of too much delay: trade cannot happen earlier than under the first best. To see why the result holds, fix $\{p_\tau^H, \kappa_\tau^H\}$ induced by a separating equilibrium (σ, μ), and suppose by contradiction that there exists t with $\kappa_{t+\Delta}^H < \kappa_t^H$ and $\kappa_{t+\Delta}^H < v^*(\Delta)$. Note that, at time t, a buyer of type $\kappa_{t+\Delta}^H$ can delay trade until the seller's cost falls to c_L and get the good at a price weakly lower than $p^L(\kappa_{t+\Delta}^H)$. Hence, we have that

$$\begin{aligned}
\kappa_{t+\Delta}^H - p_t^H &\geq \rho(\Delta)(\kappa_{t+\Delta}^H - p^L(\kappa_{t+\Delta}^H)) \\
\iff p_t^H &\leq \kappa_{t+\Delta}^H(1 - \rho(\Delta)) + \rho(\Delta) p^L(\kappa_{t+\Delta}^H)) \\
&< c_H + \rho(\Delta) p^L(\kappa_{t+\Delta}^H)
\end{aligned} \tag{4}$$

where the strict inequality uses $\kappa_{t+\Delta}^{H} < v^{*}(\Delta) = \frac{c_H}{1-\rho(\Delta)}$. In words, inequality (4) states that the seller's profits from serving buyer of type $\kappa_{t+\Delta}^{H}$ when her cost is c_H are strictly lower than her expected discounted profits from serving this buyer when her cost falls to $c_L = 0$. As the proof of Proposition 2 shows (see Lemma A.2), this means that the seller has a profitable deviation at time t: her payoff from delaying all trade until costs fall to c_L is strictly larger than what she gets under (σ, μ). Hence, (σ, μ) can't be an equilibrium.

I end this section by noting that, in any equilibrium in $\Sigma^{\mathsf{S}}(\Delta)$, the probability with which buyer and seller trade while seller's cost is high is bounded by inequality (2). This delayed trade is socially costly. Therefore, under the equilibrium in $\Sigma^{\mathsf{S}}(\Delta)$ that maximizes the sum of players' payoffs, constraint (2) binds at all periods t with $\kappa_t^H > \kappa_{t+\Delta}^H$, except possibly the last period at which a cost c_H seller makes a sale.

4 Frequent-offers Limit

This section studies the frequent-offers limit of the most efficient separating equilibrium. For each $\Delta > 0$, let $(\sigma^{\Delta}, \mu^{\Delta})$ be an equilibrium in $\Sigma^{\mathsf{S}}(\Delta)$ achieving the largest social welfare (among equilibria in $\Sigma^{\mathsf{S}}(\Delta)$). Let $\{p_t^H(\Delta), \kappa_t^H(\Delta)\}$ denote the prices and belief cutoffs induced by $(\sigma^{\Delta}, \mu^{\Delta})$ at periods at which the seller's costs are c_H. Note that $p_t^H(\Delta)$ and $\kappa_t^H(\Delta)$ are defined for all $t \in T(\Delta)$. I extend both of these functions to all $t \geq 0$ so that $p_t^H(\Delta)$ and $\kappa_t^H(\Delta)$ are piece-wise constant in t.[10]

Recall that, when the seller's costs fall to c_L, continuation play under any equilibrium in $\Sigma^{\mathsf{S}}(\Delta)$ is equivalent to the continuation equilibrium in a game with one-sided private information. By Fudenberg et al. (1985) and Gul et al. (1985), as $\Delta \to 0$, for all $\kappa \in [\underline{v}, \overline{v}]$ price $p^L(\kappa)$ converges to $\underline{v}$, and trade happens with essentially no delay. Hence, under any equilibrium in $\Sigma^{\mathsf{S}}(\Delta)$, the seller's continuation profits when her cost is c_L converge to $\underline{v}$ as $\Delta \to 0$. Define $\hat{v} \equiv \lim_{\Delta \to 0} v^{*}(\Delta) = \frac{r+\lambda}{r} c_H$ to be the

[10] That is, for all $s \in T(\Delta)$, and all $t \in [s, s+\Delta)$, $p_t^H(\Delta) = p_s^H(\Delta)$ and $\kappa_t^H(\Delta) = \kappa_s^H(\Delta)$.

efficient cutoff as $\Delta \to 0$.

Theorem 2. *There exists functions $p^H : \mathbb{R}_+ \to \mathbb{R}_+$ and $\kappa^H : \mathbb{R}_+ \to [\underline{v}, \overline{v}]$ such that, for all $t \geq 0$, $\lim_{\Delta \to 0} p_t^H(\Delta) = p^H(t)$ and $\lim_{\Delta \to 0} \kappa_t^H(\Delta) = \kappa^H(t)$.*

Functions $p^H(t)$ and $\kappa^H(t)$ satisfy

$$-\frac{dp^H(t)}{dt} = r(\kappa^H(t) - p^H(t)) + \lambda(\underline{v} - p^H(t)), \tag{5}$$

$$-\frac{d\kappa^H(t)}{dt} = \frac{F(\kappa^H(t))}{f(\kappa^H(t))} \frac{r\underline{v}}{(p^H(t) - \underline{v})}, \tag{6}$$

for all $t \leq \hat{t} \equiv \inf\{t \geq 0 : \kappa^H(t) = \hat{v}\}$, with boundary conditions $\kappa^H(0) = \overline{v}$ and $p^H(\hat{t}) = c_H + \frac{\lambda}{r+\lambda}\underline{v}$. For all $t > \hat{t}$, $\frac{dp^H(t)}{dt} = \frac{d\kappa^H(t)}{dt} = 0$.

Theorem 2 shows that the frequent-offers limit of welfare maximizing equilibrium is characterized by a system of differential equations. The intuition behind equation (5) is as follows. The buyer's benefit from delaying her purchase for an instant at time t while $c_t = c_H$ is

$$-\frac{dp^H(t)}{dt} + \lambda(p^H(t) - \underline{v}).$$

Indeed, the seller's price falls at rate $\frac{dp^H(t)}{dt}$ if costs remain high, and drops from $p^H(t)$ to $\underline{v}$ if costs fall to c_L. By equation (5), this benefit must equal the cost $r(\kappa^H(t) - p^H(t))$ that the marginal buyer type $\kappa^H(t)$ incurs from delaying trade for an instant.

To see the intuition for (6), note that the equation can be written as

$$-\frac{d\kappa^H(t)}{dt} \frac{f(\kappa^H(t))}{F(\kappa^H(t))}(p^H(t) - \underline{v}) = r\underline{v}. \tag{7}$$

The left-hand side of equation (7) is the net benefit that a seller whose cost fell to $c_L = 0$ at time t obtains from pretending that her cost is c_H for an instant longer. Indeed, the seller makes a sale with instantaneous probability $-\frac{d\kappa^H(t)}{dt} \frac{f(\kappa^H(t))}{F(\kappa^H(t))}$ if she pretends to have cost c_H, and sells at price $p^H(t)$ instead of $\underline{v}$. The right-hand side of (7) is the cost in terms of delayed trade that the seller incurs by following such a mimicking strategy. The speed of trade $-\frac{d\kappa^H(t)}{dt} \frac{f(\kappa^H(t))}{F(\kappa^H(t))}$ under a welfare maximizing equilibrium is such that the net gain from pretending to have a high cost is equal to the cost of delayed trade.

Functions $p^H(t)$ and $\kappa^H(t)$ satisfy two boundary conditions: $\kappa^H(0) = \overline{v}$ and $p^H(\hat{t}) = c_H + \frac{\lambda}{r+\lambda}\underline{v}$, where $\hat{t} = \inf\{t \geq 0 : \kappa^H(t) = \hat{v}\}$. The first boundary condition holds since at $t = 0$ the seller believes that $v \sim F$ (recall that $\mathsf{supp}\, F = [\underline{v}, \overline{v}]$).

To understand the second boundary condition, note that in the frequent-offers limit, while costs are c_H the seller trades with the buyer until her belief cutoff reaches the efficient cutoff $\hat{v} = \frac{r+\lambda}{r} c_H$; i.e., until time $\hat{t} = \inf\{t \geq 0 : \kappa^H(t) = \hat{v}\}$. Price $p^H(\hat{t})$ at which a buyer with type $\hat{v}$ trades leaves this buyer indifferent between buying at $\hat{t}$, or waiting and buying at price $\underline{v}$ when costs fall to c_L:

$$\hat{v} - p^H(\hat{t}) = \frac{\lambda}{r+\lambda}(\hat{v} - \underline{v}) \Longleftrightarrow p^H(\hat{t}) = c_H + \frac{\lambda}{r+\lambda}\underline{v},$$

where the second equality uses $\hat{v} = \frac{r+\lambda}{r} c_H$.

It is worth noting that equation (6) implies that the equilibrium outcome with frequent offers is inefficient: $\hat{t} = \inf\{t \geq 0 : \kappa^H(t) = \hat{v}\}$ is bounded away from 0, so high value buyers trade slow relative to the first best. To see why, note that for all $t < \hat{t}$ we have $p^H(t) \geq p^H(\hat{t}) > \underline{v}$.[11] Hence, $-\frac{d\kappa^H(t)}{dt}$ is bounded, and so $\hat{t} > 0$.[12] At the same time, buyers with a value below cutoff $\hat{v}$ trade at the efficient time $\tau_L = \inf\{t : c_t = c_L\}$.

Lastly, it is also worth noting that knowledge of $p^H(t)$ and $\kappa^H(t)$ allows us to compute the seller's limiting equilibrium payoffs U_t^H at time t conditional on $c_t = c_H$ (seller's limiting payoffs conditional on $c_t = c_L$ are $\underline{v}$). Indeed, for all $t < \hat{t}$ we have

$$U_t^H = \int_{s=t}^{\hat{t}} e^{-(r+\lambda)(s-t)}(p^H(s) - c_H)\frac{f(\kappa^H(s))}{F(\kappa^H(t))}(-\dot{\kappa}^H(s))ds + \int_{s=t}^{\infty} \lambda e^{-(r+\lambda)(s-t)}\frac{F(\kappa^H(s))}{F(\kappa^H(t))}\underline{v}ds,$$

[11] Indeed, since $p^H(t)$ is decreasing in t, we have $p^H(t) > p^H(\hat{t})$ for all $t < \hat{t}$. Moreover, note that $p^H(\hat{t}) - \underline{v} = c_H - \underline{v}\frac{r}{r+\lambda} > 0$, where the last inequality follows since, by Assumption 1, $\hat{v} = \lim_{\Delta\to 0} v^*(\Delta) = \frac{r+\lambda}{r} c_H > \underline{v}$.

[12] These inefficiencies persist in the limit as $\lambda \to 0$. Indeed, cutoff $\hat{v}$ converges to c_H as λ goes to zero. If the equilibrium outcome were efficient, a seller with initial cost c_H would sell immediately to all buyers with value $v \geq c_H$, at a price weakly larger than c_H. But this would violate (6).

where $\dot{\kappa}^H(s) = d\kappa^H(s)/ds$. The first term corresponds to the seller's expected discounted profits while her costs are c_H, and the second term are the seller's expected discounted profits at the time costs reach $c_L = 0$.

De-coupling equation (5). The system of differential equations (5)-(6) is coupled. I now show how to transform (5)-(6) to obtain a decoupled ODE for prices.

For each $\kappa \in [\hat{v}, \overline{v}]$, let $P^H(\kappa)$ denote the price at which a buyer with value κ trades when costs are c_H; that is, for all $t \in [0, \hat{t}]$, $P^H(\kappa^H(t)) = p^H(t)$. Combining (5) and (6), and using $\frac{dp^H(t)}{dt} = \frac{dP^H(\kappa^H(t))}{d\kappa^H}\frac{d\kappa^H(t)}{dt}$, $P^H(\cdot)$ solves:

$$\forall \kappa \in [\hat{v}, \overline{v}], \quad \frac{dP^H(\kappa)}{d\kappa} = \big(r(\kappa - P^H(\kappa)) + \lambda(\underline{v} - P^H(\kappa))\big)\frac{f(\kappa)}{F(\kappa)}\frac{(P^H(\kappa) - \underline{v})}{r\underline{v}}, \quad (8)$$

with $P^H(\hat{v}) = c_H + \frac{\lambda}{r+\lambda}\underline{v}$.

Equation (8) is a decoupled ODE, giving us the price that a high-cost seller charges in the frequent-offer limit to each buyer type. Besides being an object of interest in its own right, solving for $P^H(\kappa)$ allows one to solve for $\kappa^H(t)$ in (6) using $p^H(t) = P^H(\kappa^H(t))$ (and this, in turn, allows one to solve for $p^H(t)$ in (5)).

Comparative statics. I now use Theorem 2 and equation (8) to study equilibrium properties and derive several comparative statics. For each $\kappa \in [\hat{v}, \overline{v}]$, define $q(\kappa) \equiv -\frac{d\kappa^H(t)}{dt}\frac{f(\kappa^H(t))}{F(\kappa^H(t))}\Big|_{\kappa^H(t)=\kappa}$ to be the speed with which buyer and seller trade in the frequent-offers limit when the marginal type is κ.[13] My next result shows how prices and speed of trade change with changes in (i) value distribution F, (ii) cost c_H, and (iii) rate λ at which costs fall.

Proposition 3. *(i) As F increases in terms of its reverse hazard rate, price $P^H(\kappa)$ increases for all $\kappa > \hat{v}$, and the speed of trade $q(\kappa)$ falls for all $\kappa > \hat{v}$.*

[13]Indeed, $-\frac{d\kappa^H(t)}{dt}$ measures how fast the cutoff $\kappa^H(t)$ falls, while $\frac{f(\kappa^H(t))}{F(\kappa^H(t))}$ denotes the conditional density of the buyers' value, evaluated at the cutoff $\kappa^H(t)$.

(ii) As c_H increases, price $P^H(\kappa)$ increases for all $\kappa > \hat{v}$, and the speed of trade $q(\kappa)$ falls for all $\kappa > \hat{v}$.

(iii) As λ increases, price $P^H(\kappa)$ increases for all $\kappa \in [\hat{v}, \tilde{v}]$ for some $\tilde{v} > \hat{v}$, and the speed of trade $q(\kappa)$ falls for all $\kappa \in [\hat{v}, \tilde{v}]$.

The first part of Proposition 3 shows that the prices $P^H(\kappa)$ at which the different buyer types trade when costs are high increase when F increases in terms of its reverse hazard rate $\frac{f(v)}{F(v)}$. Since prices are now higher, by equation (7) the speed of trade $q(\kappa)$ must be adjusted downwards to deter a low cost seller from pretending to have a high cost. To understand why, note when F increases in terms of its reverse hazard rate: (i) the right-hand side of (8) increases, and (ii) the boundary condition $P^H(\hat{v}) = c_H + \frac{\lambda}{r+\lambda}\underline{v}$ remains unchanged. As a result, function $P^H(\cdot)$ now takes larger values for all $\kappa > \hat{v}$. Intuitively, when F increases in terms of its reverse hazard rate, there is a larger mass of high-value buyers.[14] Hence, belief cutoff $\kappa^H(t)$ must now fall at a lower rate to prevent a low-cost seller from pretending she has a high cost. Since a high-cost seller now takes longer to sell, she can charge higher prices.

The second and third parts of Proposition 3 establish similar results for changes in the high cost and in the rate at which the seller's cost falls.

The last result in this section studies equilibrium outcomes as the buyer's lowest value $\underline{v}$ becomes small.

Proposition 4. *In the limit as $\underline{v} \to 0$, trade under the limiting welfare maximizing separating equilibrium only occurs when the seller's costs are low, at a price of zero.*

Proposition 4 follows from equation (7): as $\underline{v} \to 0$, the speed at which buyer and seller trade while costs are c_H must converge to zero to deter a low cost seller from pretending to have a high cost. Hence, in the limit trade occurs only when the seller's costs fall to c_L.

[14]Indeed, recall that hazard rate dominance implies first-order stochastic dominance.

An implication of Proposition 4 is that inefficiencies may grow in the limit as the lowest valuation goes to zero. Indeed, consider a family of distributions $\{F_{\underline{v}}\}$ indexed by the lowest point in their support $\underline{v}$, with the property that $\mathbb{E}_{F_{\underline{v}}}[v] = \mu$ for all $\underline{v}$. By Proposition 4, as $\underline{v} \to 0$ the total equilibrium surplus converges to $(q\frac{\lambda}{r+\lambda}+1-q)\mathbb{E}_{F_{\underline{v}=0}}[v] = (q\frac{\lambda}{r+\lambda}+1-q)\mu$. In contrast, for any $\underline{v} > 0$, total equilibrium surplus will be strictly larger than $(q\frac{\lambda}{r+\lambda}+1-q)\mu$, since a high cost seller makes sales with positive probability each period.

Relation to previous literature. Theorem 2 and Proposition 4 allow for a comparison between the current model and previous models in the literature. Consider first models with two-sided private information. Cho (1990) shows that, in such models, separating equilibria satisfy a version of the Coase conjecture: bargaining outcomes are efficient if and only if gains from trade are common knowledge (i.e., the seller's highest cost is lower than the buyer's lowest value).

The results in the current model are consistent with those in Cho (1990), with some subtle differences. Recall from Section 2 that the counterpart of Assumption 1 in a setting with time-invariant costs is $c_H \in (\underline{v}, \overline{v})$ (i.e., gains from trade are not common-knowledge). Hence, from Cho (1990) we would expect equilibrium outcomes to be inefficient when Assumption 1 holds. The difference, however, is that when costs are time-varying, Assumption 1 is consistent with common-knowledge gains from trade.

Proposition 4 allows for further comparisons between the current model and the previous literature. When the seller's production cost is fixed and publicly known, the seller's profits converge to zero as the buyer's lowest valuation $\underline{v}$ converges to zero (Fudenberg et al., 1985, Gul et al., 1985). But the limiting equilibrium outcome is efficient: all buyers trade immediately at price equal to marginal cost.

For models with two-sided private information and with time-invariant costs, the results in Cho (1990) imply that, in any separating stationary equilibrium, the seller's profits also converge to zero as the buyer's lowest value converges to zero. However, inefficiencies "explode" in this limit: only the seller with the lowest possible cost makes sales.[15]

Proposition 4 illustrates how these results generalize when the seller is privately informed about her time-varying production cost. As in the two cases described above, the seller's profits go to zero as the buyer's lowest value $\underline{v}$ goes to zero. Moreover, as in Cho (1990), inefficiencies also grow in this "gapless" limit. The difference, however, is that seller and buyer eventually trade with probability 1 in this model, when costs fall to c_L.

5 An Efficient Mechanism

This section shows that, under certain conditions, the environment that I study admits a mechanism satisfying IC, IR, and budget balance that attains the first best.

Consider the following direct mechanism, which I denote M^{FB}. At $t = 0$, buyer reports her type $v \in [\underline{v}, \overline{v}]$ and seller reports her initial cost $c_0 \in \{c_L, c_H\}$. If seller reports $c_0 = c_L$, then buyer and seller trade at $t = 0$ at a price of $\underline{v}$, regardless of the buyer's report.

If seller instead reports $c_0 = c_H$, then at $t = 0$: (i) if buyer reported $v \in [v^*(\Delta), \overline{v}]$, she trades at price $c_H + \rho(\Delta)\underline{v}$; (ii) if buyer reported $v \in [\underline{v}, v^*(\Delta))$, she pays the seller a price $\rho(\Delta)\underline{v}$ at $t = 0$ but doesn't trade yet. Then, at each period $t \in T(\Delta), t > 0$, seller reports her cost $c_t \in \{c_L, c_H\}$. If at $t > 0$ the seller reports $c_t = c_H$, nothing happens. The first period $t > 0$ at which seller reports $c_t = c_L$, a buyer who reported $v \in [\underline{v}, v^*(\Delta))$ trades, and pays price $c_L (= 0)$ to the seller at this point.

Note that mechanism M^{FB} is budget balance, and implements the efficient outcome if players report truthfully. The following result shows that, under certain conditions, mechanism M^{FB} satisfies IC and IR.

[15] Ausubel and Deneckere (1992) establish a related result.

Proposition 5. *Suppose that* $(1-\rho(\Delta))\underline{v} \geq (1-F(v^*(\Delta)))c_H$*. Then, mechanism* M^{FB} *satisfies IC and IR.*

Proposition 5 establishes existence of an efficient mechanism, provided $\underline{v} \geq (1 - F(v^*(\Delta)))\frac{c_H}{1-\rho(\Delta)} = (1-F(v^*(\Delta)))v^*(\Delta)$. This inequality guarantees that truthful reporting is optimal for a seller with initial cost c_L. Indeed, a seller with $c_0 = c_L$ obtains a payoff of $\underline{v}$ from reporting truthfully under mechanism M^{FB}, and obtains $(1 - F(v^*(\Delta)))(c_H + \rho(\Delta)\underline{v}) + F(v^*(\Delta))\rho(\Delta)\underline{v} = (1 - F(v^*(\Delta)))c_H + \rho(\Delta)\underline{v}$ from reporting $c_0 = c_H$.

The existence of an efficient mechanism satisfying IC, IR and budget balance distinguishes the current model from prior bargaining games with two-sided private information. For instance, separating equilibria in Cho (1990) are inefficient only when the distribution of buyer values and the distribution of seller costs overlap. But we know from Myerson and Satterthwaite (1983) that such a framework does not admit an efficient mechanism satisfying IC, IR and budget balance. In contrast, separating equilibria in this model are always inefficient, regardless of whether the conditions in Proposition 5 hold.

Proposition 5 and Theorem 2 show that the inefficiencies that arise in equilibrium are in some cases strictly larger than those implied by feasibility. This is reminiscent of Deneckere and Liang (2006), who study a bargaining model with correlated values and show that equilibria fail to be second-best whenever the first-best is not implementable. The difference, however, is that in Deneckere and Liang (2006) equilibria are first-best efficient whenever the first best is implementable (provided values are positively correlated).

6 Non-separating Equilibria

Throughout the paper, I focused on separating equilibria, under which the seller's price each period reveals her current cost realization. There are several reasons behind this choice. First, such equilibria are intuitive, tractable, and help rationalize observed pricing dynamics in markets for new durable goods (Conlon, 2012). Second, such equilibria represent a natural point of comparison to prior papers in the literature, like Cho (1990) and Ortner (2017). Third, as Theorem 1 shows, without loss separating equilibria can be taken to be weakly stationary, permitting further comparisons with the classic papers in the literature (Gul et al., 1985, Fudenberg et al., 1985).

The game admits many other equilibria. For instance, the game admits semi-separating equilibria, in which a seller with high cost $c_t = c_H$ posts price p_t^H, and a seller whose cost fell to c_L posts price p_t^H with probability $1 - \alpha_t$ and price $p^L(\kappa_t^H)$ with probability $\alpha_t \in (0, 1)$. The seller thus reveals that her cost is c_L when she posts price $p^L(\kappa_t^H)$, and the continuation equilibrium is as in Gul et al. (1985), Fudenberg et al. (1985). To keep a low-cost seller indifferent between prices p_t^H and $p^L(\kappa_t^H)$, under such an equilibrium inequality (2) holds with equality at all periods t in which $\alpha_t \in (0, 1)$. Hence, such equilibria also feature inefficient delays.

The game also admits pooling equilibria, in which both types of sellers post the same price at times $t = 0, ..., \tau$; and buyer and seller play a continuation equilibrium in $\Sigma^{\mathsf{S}}(\Delta)$ from time $\tau + \Delta$ onwards. For instance, one particularly simple such continuation equilibrium has a high cost seller posting high prices that are rejected with probability 1 from time $\tau + \Delta$ onwards, and a low cost seller playing the continuation strategy of the one-sided private information game. Note that inequality (2) need not hold during the pooling period. As a result, pooling equilibria can be more efficient than the separating equilibria I study in the main text. However, Appendix D shows that outcomes under this simple class of pooling equilibria are still bounded away from the first-best outcome whenever $q = \text{prob}(c_0 = c_H)$ is strictly below 1.[16]

The next Proposition establishes a stronger result: the first-best outcome cannot be attained by any PBE, or by any limiting PBE as $\Delta \to 0$. Hence, any equilibrium

[16]When $\text{prob}(c_0 = c_H) = 1$, the game initially features one-sided private information. As in bargaining games with one-sided private information (Gul et al., 1985, Fudenberg et al., 1985), in this case the game does admit efficient limiting equilibria as $\Delta \to 0$.

(or any limiting equilibrium as $\Delta \to 0$) must be (at least slightly) inefficient.

Let $\Sigma(\Delta)$ denote the set of PBE of the game with time-period $\Delta > 0$. Note that any $(\sigma, \mu) \in \Sigma(\Delta)$ (or any limit of equilibria $(\sigma^n, \mu^n) \in \Sigma(\Delta^n)$ with $\Delta^n \to 0$) induces an outcome $\tau : [\underline{v}, \overline{v}] \times \{c_L, c_H\} \to \mathbb{R}_+$ and $p : [\underline{v}, \overline{v}] \times \{c_L, c_H\} \to \mathbb{R}_+$, where $\tau(v, c_0)$ (resp. $p(v, c_0)$) is the random time (resp. expected price) at which a buyer with value v buys when the seller's initial cost is c_0.

Recall that the first-best outcome $\tau^{FB} : [\underline{v}, \overline{v}] \times \{c_L, c_H\} \to \mathbb{R}_+$ has $\tau^{FB}(v, c_L) = 0$ for all v and $\tau^{FB}(v, c_H) = \mathbf{1}_{v<v^*}\tau_L$, where $\tau_L = \inf\{t : c_t = c_L\}$.

Proposition 6. *Let $\tau : [\underline{v}, \overline{v}] \times \{c_L, c_H\} \to \mathbb{R}_+$ and $p : [\underline{v}, \overline{v}] \times \{c_L, c_H\} \to \mathbb{R}_+$ be an outcome induced by a PBE $(\sigma, \mu) \in \Sigma(\Delta)$, or the pointwise limit of outcomes induced by a sequence of PBE $(\sigma^n, \mu^n) \in \Sigma(\Delta^n)$ with $\Delta^n \to 0$. Then, $\tau \neq \tau^{FB}$.*

To establish Proposition 6, I start by arguing that if equilibrium outcome (τ, p) is efficient, a seller whose initial cost is c_L must earn profits equal to $\underline{v}$. The reason this holds is that, in any equilibrium, the seller never offers a price lower than $\underline{v}$.[17]

In addition, to satisfy incentive compatibility, under efficient equilibrium outcome (τ, p) a seller whose initial cost is c_H must sell to all buyer types with $v \geq v^*$ immediately, at price $c_H + \rho\underline{v} = (1-\rho)v^* + \rho\underline{v}$. But then, a seller with initial cost c_L can obtain profits of $(1 - F(v^*))(c_H + \rho\underline{v}) + e^{-r\Delta}F(v^*)\underline{v}$ by pretending to have a high initial cost at $t = 0$, and then playing her continuation strategy. Since $c_H + \rho\underline{v} > \underline{v}$ (Assumption 1), such a deviation is strictly profitable whenever $\Delta > 0$ is small.

Appendix

[17] Indeed, by the same arguments as in Lemma 1 in Gul et al. (1985), in any equilibrium all buyer types accept an offer of $p = \underline{v}$ with probability 1.

A Proofs of Theorem 1 and Proposition 2

In any PBE in Σ^{S}, when costs falls to c_L, continuation play coincides with equilibrium play in the one-sided incomplete information game in Fudenberg et al. (1985) and Gul et al. (1985).[18] Hence, I focus on characterizing equilibrium behavior at periods t with $c_t = c_H$.

By the skimming property, any PBE in Σ^{S} induces a decreasing sequence of belief cutoffs $\{\kappa_t^H\}$ such that along the path of play, at any time t with $c_t = c_H$, (i) the seller believes that the buyer's type lies in $[\underline{v}, \kappa_t^H]$, and (ii) the buyer buys at time t if and only if her valuation lies in $[\kappa_{t+\Delta}^H, \kappa_t^H)$.

Lemma A.1. *Fix a PBE $(\sigma, \mu) \in \Sigma^{\mathsf{S}}$. Consider a seller history h_t^S with $c_s = c_H$ for all $s < t$ such that the seller's belief cutoff κ_t at time t is strictly larger than $\underline{v}$. Let p_t^H be the price that the seller charges under (σ, μ) at history h_t^S if $c_t = c_H$, and let $\kappa_{t+\Delta}$ be the highest consumer type that buys at time t when $c_t = c_H$. Then, κ_t and $\kappa_{t+\Delta}$ satisfy*

$$p_t^H \frac{F(\kappa_t) - F(\kappa_{t+\Delta})}{F(\kappa_t)} \leq U^L(\kappa_t) - e^{-r\Delta} \frac{F(\kappa_{t+\Delta})}{F(\kappa_t)} U^L(\kappa_{t+\Delta}). \tag{9}$$

Proof. Consider a seller whose cost changed from c_H to c_L after history h_t^S. The profits that this seller obtains by revealing her cost are $U^L(\kappa_t)$. The profits that this seller would make by posting price p_t^H that she would have posted if $c_t = c_H$, and then from $t + \Delta$ onwards playing the continuation strategy with common knowledge cost c_L and belief cutoff $\kappa_{t+\Delta}$ are $p_t^H \frac{F(\kappa_t) - F(\kappa_{t+\Delta})}{F(\kappa_t)} + e^{-r\Delta} \frac{F(\kappa_{t+\Delta})}{F(\kappa_t)} U^L(\kappa_{t+\Delta})$. A seller whose cost changed to c_L at period t has an incentive to reveal her cost only if (9) holds. ■

Recall that $\rho = \frac{e^{-r\Delta}(1-e^{-\lambda\Delta})}{1-e^{-(r+\lambda)\Delta}}$. Fix a PBE in Σ^{S}, and consider a seller history h_t^S with $c_s = c_H$ for all $s < t$ leading to belief cutoff $\kappa_t^H = \kappa$. Note that at such a history, a seller with cost $c_t = c_H$ can obtain a payoff equal to $\rho U^L(\kappa)$ by posting

[18] For ease of exposition, throughout Appendix A I drop the dependence on time period Δ.

prices higher than κ at all periods until her costs fall to c_L, and then playing her continuation strategy. Hence, the seller's continuation profits at this history under (σ, μ) cannot be lower than $\rho U^L(\kappa)$.

Lemma A.2. *Fix a PBE $(\sigma, \mu) \in \Sigma^S$, and consider a seller history h_t^S with belief cutoff κ_t. If $c_t = c_H$, then $\kappa_{t+\Delta} \geq \min\{\kappa_t, v^*\}$. In particular, if $\kappa_t \leq v^*$ and $c_t = c_H$, the seller makes a sale with probability zero at time t (i.e., $\kappa_{t+\Delta} = \kappa_t$).*

Proof. Towards a contradiction, suppose that $c_t = c_H$ and $\kappa_{t+\Delta} < \min\{\kappa_t, v^*\} \leq v^*$. Let $\{\kappa_{t+\tau\Delta}\}_{\tau=0}^{\infty}$ be a weakly decreasing sequence such that for all $\tau \geq 0$, if the seller's cost is c_H at time $t + \tau\Delta$, under (σ, μ) the seller sells to the buyer when her valuation is in $[\kappa_{t+(\tau+1)\Delta}, \kappa_{t+\tau\Delta})$. Let $\{p_{t+\tau\Delta}^H\}_{\tau=0}^{\infty}$ denote the sequence of prices that the seller charges at each time $t + \tau\Delta$ if $c_{t+\tau\Delta} = c_H$. Recall that $p^L(\kappa)$ is the price that the seller charges if her cutoff belief is κ and her costs are c_L. By Fudenberg et al. (1985) and Gul et al. (1985), $p^L(\kappa)$ is weakly increasing in κ.

Note first that, for all $\tau \geq 0$, it must be that

$$\kappa_{t+(\tau+1)\Delta} - p_{t+\tau\Delta}^H \geq \rho(\kappa_{t+(\tau+1)\Delta} - p^L(\kappa_{t+(\tau+1)\Delta})). \tag{10}$$

Indeed, a buyer with value $\kappa_{t+(\tau+1)\Delta}$ can guarantee a payoff of at least $\rho(\kappa_{t+(\tau+1)\Delta} - p^L(\kappa_{t+(\tau+1)\Delta}))$ by delaying her purchase until the seller's cost falls to c_L. Note further that,

$$\kappa_{t+(\tau+1)\Delta} - c_H < \rho\kappa_{t+(\tau+1)\Delta},$$

where the inequality follows since $\kappa_{t+(\tau+1)\Delta} \leq \kappa_{t+\Delta} < v^*$ and since $v^* - c_H = \rho v^*$. Combining this inequality with inequality (10),

$$p_{t+\tau\Delta}^H \leq (1-\rho)\kappa_{t+(\tau+1)\Delta} + \rho p^L(\kappa_{t+(\tau+1)\Delta}) < c_H + \rho p^L(\kappa_{t+(\tau+1)\Delta}) \tag{11}$$

Equation (11) implies that the profit margin $p_{t+\tau\Delta}^H - c_H$ that the seller earns from selling to consumers with value $v \in [\kappa_{t+(\tau+1)\Delta}, \kappa_{t+\tau\Delta})$ when her costs are c_H is strictly lower than the expected discounted profit margin $\rho p^L(\kappa_{t+(\tau+1)\Delta})$ that the seller would earn if she waited until her costs fell to $c_L = 0$ and then charged price of $p^L(\kappa_{t+(\tau+1)\Delta})$.

For all $s \in T(\Delta)$, let U^H_s denote the seller's on-path continuation payoff at time s if $c_s = c_H$ under equilibrium (σ, μ). For all $\kappa \in [\underline{v}, \overline{v}]$, recall that $U^L(\kappa)$ is the seller's continuation payoff under (σ, μ) at a history with belief cutoff κ and at which her costs are c_L, respectively. I now use equation (11) to show that $U^H_t < \rho U^L(\kappa_t)$. This implies that (σ, μ) cannot be an equilibrium, since at time t the seller can earn $\rho U^L(\kappa_t)$ by waiting until her costs fall to c_L and then playing the continuation equilibrium from that point onwards.

Note that, for all $\tau \geq 0$,

$$\begin{aligned}
U^H_{t+\tau\Delta} &= (p^H_{t+\tau\Delta} - c_H)\frac{F(\kappa_{t+\tau\Delta}) - F(\kappa_{t+(\tau+1)\Delta})}{F(\kappa_{t+\tau\Delta})} + e^{-(r+\lambda)\Delta}\frac{F(\kappa_{t+(\tau+1)\Delta})}{F(\kappa_{t+\tau\Delta})}U^H_{t+(\tau+1)\Delta} \\
&\quad + e^{-r\Delta}\frac{F(\kappa_{t+(\tau+1)\Delta})}{F(\kappa_{t+\tau\Delta})}(1 - e^{-\lambda\Delta})U^L(\kappa_{t+(\tau+1)\Delta}) \\
&< \rho p^L(\kappa_{t+(\tau+1)\Delta})\frac{F(\kappa_{t+\tau\Delta}) - F(\kappa_{t+(\tau+1)\Delta})}{F(\kappa_{t+\tau\Delta})} + e^{-(r+\lambda)\Delta}\frac{F(\kappa_{t+(\tau+1)\Delta})}{F(\kappa_{t+\tau\Delta})}U^H_{t+(\tau+1)\Delta} \\
&\quad + e^{-r\Delta}\frac{F(\kappa_{t+(\tau+1)\Delta})}{F(\kappa_{t+\tau\Delta})}(1 - e^{-\lambda\Delta})U^L(\kappa_{t+(\tau+1)\Delta}) \\
&= \rho\left(p^L(\kappa_{t+(\tau+1)\Delta})\frac{F(\kappa_{t+\tau\Delta}) - F(\kappa_{t+(\tau+1)\Delta})}{F(\kappa_{t+\tau\Delta})} + \frac{F(\kappa_{t+(\tau+1)\Delta})}{F(\kappa_{t+\tau\Delta})}U^L(\kappa_{t+(\tau+1)\Delta})\right) \\
&\quad - e^{-(r+\lambda)\Delta}\rho\frac{F(\kappa_{t+(\tau+1)\Delta})}{F(\kappa_{t+\tau\Delta})}U^L(\kappa_{t+(\tau+1)\Delta}) + e^{-(r+\lambda)\Delta}\frac{F(\kappa_{t+(\tau+1)\Delta})}{F(\kappa_{t+\tau\Delta})}U^H_{t+(\tau+1)\Delta}
\end{aligned} \tag{12}$$

where the strict inequality follows from (11), and the last equality uses $\rho = e^{-r\Delta}(1 - e^{-\lambda\Delta}) + e^{-(r+\lambda)\Delta}\rho$. Note next that, for all $\tau \geq 0$,

$$U^L(\kappa_{t+\tau\Delta}) \geq p^L(\kappa_{t+(\tau+1)\Delta})\frac{F(\kappa_{t+\tau\Delta}) - F(\kappa_{t+(\tau+1)\Delta})}{F(\kappa_{t+\tau\Delta})} + \frac{F(\kappa_{t+(\tau+1)\Delta})}{F(\kappa_{t+\tau\Delta})}U^L(\kappa_{t+(\tau+1)\Delta}). \tag{13}$$

Indeed, a seller with cost $c = c_L$ and with belief cutoff $\kappa_{t+\tau\Delta}$ can earn the right-hand side of (13) by posting price $p^L(\kappa_{t+(\tau+1)\Delta})$ and then playing her continuation strategy.[19] Combining (13) with (12), for all $\tau \geq 0$,

$$U^H_{t+\tau\Delta} < \rho\left(U^L(\kappa_{t+\tau\Delta}) - e^{-(r+\lambda)\Delta}\frac{F(\kappa_{t+(\tau+1)\Delta})}{F(\kappa_{t+\tau\Delta})}U^L(\kappa_{t+(\tau+1)\Delta})\right) + e^{-(r+\lambda)\Delta}\frac{F(\kappa_{t+(\tau+1)\Delta})}{F(\kappa_{t+\tau\Delta})}U^H_{t+(\tau+1)\Delta}. \tag{14}$$

Using equation (14) repeatedly for all $\tau \geq 0$ yields

$$U^H_t < \sum_{\tau=0}^{\infty} e^{-(r+\lambda)\tau\Delta}\rho\left(\frac{F(\kappa_{t+\tau\Delta})}{F(\kappa_t)}U^L(\kappa_{t+\tau\Delta}) - e^{-(r+\lambda)\Delta}\frac{F(\kappa_{t+(\tau+1)\Delta})}{F(\kappa_t)}U^L(\kappa_{t+(\tau+1)\Delta})\right) = \rho U^L(\kappa_t).$$

But this cannot be, since a seller whose cost is c_H at time t can obtain $\rho U^L(\kappa_t)$ by waiting until her costs fall to $c_L = 0$ and then playing her continuation strategy. ■

For any equilibrium $(\sigma, \mu) \in \Sigma^{\mathsf{S}}$, let

$$\kappa^{(\sigma,\mu)} = \inf\{\kappa \in [\underline{v}, \overline{v}] : \exists \text{ on-path history } (h^S_t \sqcup c_H) \text{ at which type } \kappa \text{ buys under } (\sigma, \mu).\}$$

Note that $\kappa^{(\sigma,\mu)}$ is the lowest valuation at which the buyer buys when costs are c_H under equilibrium (σ, μ). By Lemma A.2, $\kappa^{(\sigma,\mu)} \geq v^*$ for all $(\sigma, \mu) \in \Sigma^{\mathsf{S}}$.

Fix a PBE $(\sigma, \mu) \in \Sigma^{\mathsf{S}}$. Let $\{\kappa^H_t\}$ be the sequence of belief cutoffs induced by (σ, μ) at histories at which the seller's costs are c_H. Under (σ, μ), a high cost seller stops selling whenever her cutoff beliefs about the buyer's valuation reach $\kappa^{(\sigma,\mu)}$, so $\kappa^H_t \geq \kappa^{(\sigma,\mu)}$ for all t.

Let $\hat{t}$ denote the time at which a high cost seller sells to a buyer with valuation $\kappa^{(\sigma,\mu)}$, provided that $\hat{t}$ is finite, and let $\kappa^H_{\hat{t}+\Delta} = \kappa^{(\sigma,\mu)}$. Note that, for all periods $t \geq \hat{t} + \Delta$ a high cost seller does not make sales. Hence $\kappa^H_t = \kappa^H_{\hat{t}+\Delta}$ for all $t \geq \hat{t} + \Delta$ (if $\hat{t}$ is infinite, this is vacuous).

Let $\{p^H_t\}^{\hat{t}}_{t=0}$ be the prices that the seller charges at times $t \leq \hat{t}$ under (σ, μ) at

[19] This follows since the equilibrium of the game with one-sided incomplete information is weakly stationary (Gul et al., 1985).

histories at which her cost is high. Note that, for all $t \leq \hat{t} - \Delta$, it is without loss to consider prices $\{p_t^H\}$ satisfying:

$$\kappa_{t+\Delta}^H - p_t^H = e^{-(r+\lambda)\Delta}(\kappa_{t+\Delta}^H - p_{t+\Delta}^H) + e^{-r\Delta}(1 - e^{-\lambda\Delta})(\kappa_{t+\Delta}^H - p^L(\kappa_{t+\Delta}^H)). \tag{15}$$

To see why, note that equality (15) must hold for all t such that $\kappa_{t+2\Delta}^H > \kappa_{t+\Delta}^H > \kappa_t^H$. Indeed, if buyer and seller trade with positive probability at times t and $t + \Delta$ when costs are c_H, then the marginal buyer type who trades at t (i.e., type $\kappa_{t+\Delta}^H$) must be indifferent between buying at time t or waiting and buying at period $t+\Delta$. If there is no trade at time t (i.e., if $\kappa_{t+\Delta}^H = \kappa_t^H$), we can set price p_t^H so that (15) holds without changing the equilibrium outcome. Similarly, if there is no trade at time $t + \Delta$ (i.e., if $\kappa_{t+2\Delta}^H = \kappa_{t+\Delta}^H$), we can again set price $p_{t+\Delta}^H$ so that (15) holds without changing the equilibrium outcome.

For all $\kappa \in [\underline{v}, \overline{v}]$, define $\hat{p}(\kappa) \equiv \kappa(1 - \rho) + \rho p^L(\kappa)$. Price $\hat{p}(\kappa)$ is such that a buyer with valuation κ is indifferent between buying at $\hat{p}(\kappa)$ when costs are c_H and waiting until costs fall to c_L and buying at price $p^L(\kappa)$. Note that $\hat{p}(\kappa)$ is increasing in κ (since $p^L(\kappa)$ is increasing in κ). Note further that, if $\hat{t}$ is finite, it must be that $p_{\hat{t}}^H = \hat{p}(\kappa^{(\sigma,\mu)}) = \kappa^{(\sigma,\mu)}(1 - \rho) + \rho p^L(\kappa^{(\sigma,\mu)})$. If $\hat{t}$ is finite, it is without loss to set $p_t^H = p_{\hat{t}}^H$ for all $t \geq \hat{t} + \Delta$.

Given sequences $\{p_t^H, \kappa_t^H\}$, for all times s let $U_s^H(\{p_t^H, \kappa_t^H\})$ be continuation profits that a seller obtains if $c_s = c_H$, when play is given by $\{p_t^H, \kappa_t^H\}$:

$$\begin{aligned} U_s^H(\{p_t^H, \kappa_t^H\}) = (p_s^H - c_H)\frac{F(\kappa_s^H) - F(\kappa_{s+\Delta}^H)}{F(\kappa_s^H)} + e^{-(r+\lambda)\Delta}\frac{F(\kappa_{s+\Delta}^H)}{F(\kappa_s^H)}U_{s+\Delta}^H(\{p_t^H, \kappa_t^H\}) \\ + e^{-r\Delta}(1 - e^{-\lambda\Delta})\frac{F(\kappa_{s+\Delta}^H)}{F(\kappa_s^H)}U^L(\kappa_{s+\Delta}^H). \end{aligned}$$

If an equilibrium $(\sigma, \mu) \in \Sigma^{\mathsf{S}}$ induces sequences $\{p_t^H, \kappa_t^H\}$, it must be that

$$\forall s, \quad U_s^H(\{p_t^H, \kappa_t^H\}) \geq \rho U^L(\kappa_s^H). \tag{16}$$

Indeed, a seller whose cost is high by time s and whose belief cutoff is κ_s^H can obtain a payoff of $\rho U^L(\kappa_s^H)$ by waiting until her costs fall to c_L and then playing the continuation equilibrium from that point onwards.

To prove Theorem 1, I first establish the following result.

Theorem A.1. *(i) Suppose sequences $\{p_\tau^H, \kappa_\tau^H\}$ are induced by an equilibrium $(\sigma, \mu) \in \Sigma^{\mathsf{S}}$. Then, $\{\kappa_\tau^H\}$ is decreasing, and for all t, $\{p_\tau^H, \kappa_\tau^H\}$ satisfy equations* (2) *and* (3)*. Moreover, there exists $\{\tilde{p}_\tau^H\}$, with $\tilde{p}_\tau^H = p_\tau^H$ for all τ with $\kappa_{\tau+\Delta}^H > \kappa_\tau^H$, such that $\{\tilde{p}_\tau^H, \kappa_\tau^H\}$ satisfy* (1)*.*

(ii) There exists $\overline{\Delta} > 0$ such that, if $\Delta \leq \overline{\Delta}$, for any sequences $\{p_\tau^H, \kappa_\tau^H\}$ satisfying (1)-(3) *with $\{\kappa_\tau^H\}$ decreasing, there exists an equilibrium $(\sigma, \mu) \in \Sigma^{\mathsf{S}}$ that induces $\{p_\tau^H, \kappa_\tau^H\}$.*

Proof. The arguments above imply that conditions (2) and (3) must hold in any $(\sigma, \mu) \in \Sigma^{\mathsf{S}}$; and that there exists $\{\tilde{p}_\tau^H\}$, with $\tilde{p}_\tau^H = p_\tau^H$ for all τ with $\kappa_{\tau+\Delta}^H > \kappa_\tau^H$, such that $\{\tilde{p}_\tau^H, \kappa_\tau^H\}$ satisfy (1).

I now turn to the proof of part (ii) of the Theorem. Fix sequences $\{p_\tau^H, \kappa_\tau^H\}$, with $\{\kappa_\tau^H\}$ decreasing, satisfying conditions (1)-(3). I now show show that there exists $\overline{\Delta} > 0$ such that, for all $\Delta \leq \overline{\Delta}$, there exists a PBE $(\sigma, \mu) \in \Sigma^{\mathsf{S}}$ that induces $\{p_\tau^H, \kappa_\tau^H\}$.

Let $\underline{\kappa} = \lim_{t\to\infty} \kappa_t^H$. By Lemma A.2, $\underline{\kappa} \geq v^*$. For all $\kappa \in [\underline{\kappa}, \overline{v}]$, let $\overline{p}^H(\kappa)$ denote the price at which a buyer with type κ buys under $\{p_t^H, \kappa_t^H\}$. For all $\kappa \in [\underline{v}, \overline{\kappa})$, let $\overline{p}^H(\kappa) = \overline{p}^L(\kappa)$, where $\overline{p}^L(\kappa)$ is the price that a buyer with type κ is willing to pay in the game with one-sided private information. The buyer's strategy under the proposed equilibrium (σ, μ) is as follows. For all histories $h_t^B \sqcup p_t$ with $\text{prob}(c_t = c_H | h_t^B \sqcup p_t) = 1$, a buyer with type κ buys iff $p_t \leq \overline{p}^H(\kappa)$. For all other histories, a buyer with type κ buys iff $p_t \leq \overline{p}^L(\kappa)$.

Buyer's beliefs under (σ, μ) are as follows. If at all periods $s \leq t$ the seller offered price p_s^H, the buyer at time t believes that the seller's cost is c_H with probability 1. In any other case, the buyer at time t believes that the seller's cost is c_L with probability 1.

The seller's strategy is as follows. On the equilibrium path, for all t with $c_t = c_H$, she charges price p_t^H. For all off-path histories $h_t^S \sqcup c_H$, the seller posts a price higher than $\overline{v}$ (and no buyer type buys). For all t with $c_t = c_L$, the seller plays the continuation equilibrium of the game with one-sided private information.

Since $\{p_\tau^H, \kappa_\tau^H\}$ satisfies (15), optimal buyer behavior induces belief cutoffs $\{\kappa_\tau^H\}$, given the seller's strategy. Hence, the buyer's strategy is sequentially rational at histories at which she believes that the seller's cost is high. Moreover, buyer's strategy is sequentially rational at histories at which she believes that the seller's cost is low (since, at such histories, buyer uses the equilibrium strategy of the game with one-sided private information; and since seller uses the equilibrium strategy of the game with one-sided private information whenever her cost is c_L).

I now show that, for Δ small enough, the seller's strategy is also sequentially rational. Note first that, since $\{p_t^H, \kappa_t^H\}$ satisfy (9), the seller does not find it optimal to deviate at a period t such that $c_{t-\Delta} = c_H$ and $c_t = c_L$. Moreover, she doesn't find it optimal to deviate at a period t with $c_{t-\Delta} = c_L$ and $c_t = c_L$ (since, at such histories, buyer and seller are using the equilibrium strategy of the game with one-sided private information).

By the Coase conjecture (Gul et al., 1985), for every $\eta > 0$ there exists $\overline{\Delta}_\eta > 0$ such that, for all $\Delta \leq \overline{\Delta}_\eta$, price $p^L(\kappa)$ that the seller charges when costs are $c = c_L = 0$ is strictly smaller than $\underline{v} + \eta$ for all κ. Pick $\eta' > 0$ such that $\underline{v} + \eta' - c_H < \rho\underline{v}$; since $\underline{v} < v^* = \frac{c_H}{1-\rho}$ (by Assumption 1), such an η' exists. Let $\overline{\Delta} = \overline{\Delta}_{\eta'}$, and suppose $\Delta \leq \overline{\Delta}$. Note that if at a period s with $c_s = c_H$ the seller posts a price different from p_s^H, the highest profit she can obtain is $\rho U^L(\kappa_s^H)$.[20] Since $\{p_t^H, \kappa_t^H\}$ satisfies (16), the

[20]This follows since $p^L(\kappa) \in [\underline{v}, \underline{v}+\eta']$ for all $\kappa \in [\underline{v}, \overline{v}]$ whenever $\Delta \leq \overline{\Delta}$, and since $\underline{v}+\eta'-c_H < \rho\underline{v}$. Hence, the seller's profit margin $p-c_H$ from any sale she makes while costs are high following such a deviation is strictly smaller than $\rho\underline{v}$. Since $p^L(\kappa) \geq \underline{v}$ for all κ, the seller's most profitable deviation is to wait until costs fall to c_L and then play the continuation equilibrium, obtaining a payoff of $\rho U^L(\kappa_s^H) \geq \rho\underline{v}$.

seller finds it optimal to post price p_s^H. ■

Proof of Theorem 1. Note first that, for all $\Delta > 0$, there always exist sequences $\{p_\tau^H, \kappa_\tau^H\}$ satisfying conditions (1)-(3). For instance, sequences $\{p_\tau^H, \kappa_\tau^H\}$ with $\kappa_\tau^H = \overline{v}$ and $p_\tau^H = p$ for all τ, with p satisfying

$$\overline{v} - p = e^{-(r+\lambda)\Delta}(\overline{v} - p) + e^{-r\Delta}(1 - e^{-\lambda\Delta})(\overline{v} - p^L(\overline{v}))$$

satisfy (1)-(3). Hence, by Theorem A.1(ii), for all $\Delta \leq \overline{\Delta}$, Σ^{S} is non-empty.

Finally, note that the arguments in the proof of Theorem A.1 imply that, for all $\Delta \leq \overline{\Delta}$ and for any $(\sigma, \mu) \in \Sigma^{\mathsf{S}}$, there exists a weakly stationary equilibrium $(\sigma^{\mathsf{ws}}, \mu^{\mathsf{ws}}) \in \Sigma^{\mathsf{S}}$ that induces the same outcome as (σ, μ). ■

Proof of Proposition 2. Follows from Lemma A.2. ■

Mixed strategy equilibria. Theorem A.1 characterizes equilibria under which the seller uses a pure action while her costs are c_H.

The game also admits separating equilibria under which the seller mixes while her costs are c_H. In any such equilibrium, the (now random) sequence $\{p_t^H, \kappa_t^H\}$ must still satisfy (9) and (15). Indeed, Lemma A.1 applies to mixed strategy separating equilibria as well. And inequality (16) must hold in any separating equilibrium, pure or mixed. In addition to these conditions, if the seller mixes at some period t with $c_t = c_H$, she must be indifferent among any price that she posts with positive probability.

Welfare maximizing equilibria. Let (σ, μ) be an equilibrium in Σ^{S} that delivers the largest social surplus (among all equilibria in Σ^{S}). Under (σ, μ), constraint (9) must be satisfied with equality at all times t with $\kappa_{t+\Delta}^H > \kappa_t^H$. As a result, there exists a finite period $\hat{t}$ at which, under (σ, μ), a buyer with value $\kappa^{(\sigma,\mu)}$ buys if $c_{\hat{t}} = c_H$; (and so $\kappa^H_{\hat{t}+\Delta} = \kappa^{(\sigma,\mu)}$).

Moreover, under (σ, μ), the price $p^H_{\hat{t}}$ at which the seller sells at time $\hat{t}$ if $c_{\hat{t}} = c_H$ must be equal to $\hat{p}(\kappa^{(\sigma,\mu)}) = (1-\rho)\kappa^{(\sigma,\mu)} + \rho p^L(\kappa^{(\sigma,\mu)})$. Indeed, if the buyer rejects price $p^H_{\hat{t}}$, buyer and seller don't trade until costs fall to c_L. Price $\hat{p}(\kappa^{(\sigma,\mu)})$ is the price that leaves consumer $\kappa^{(\sigma,\mu)}$ indifferent between buying at time $\hat{t}$ with $c_{\hat{t}} = c_H$, or waiting until costs fall to c_L and buying at that point (at price $p^L(\kappa^{(\sigma,\mu)})$).

B Proof of Theorem 2

For each $\Delta > 0$, let $(\sigma^\Delta, \mu^\Delta)$ be an equilibrium in $\Sigma^{\mathsf{S}}(\Delta)$ achieving the largest social welfare. Let $\{p^H_t(\Delta), \kappa^H_t(\Delta)\}_{t\in T(\Delta)}$ denote the prices and belief cutoffs induced by $(\sigma^\Delta, \mu^\Delta)$ at periods at which the seller's costs are c_H, and let $\kappa^{(\sigma^\Delta,\mu^\Delta)}$ be the lowest value buyer who trades while costs are c_H under $(\sigma^\Delta, \mu^\Delta)$.

Lemma B.1. $\kappa^{(\sigma^\Delta,\mu^\Delta)} - v^*(\Delta) \to 0$ *as* $\Delta \to 0$.

Proof. By Proposition 2, for all $\Delta \geq 0$ we have $\kappa^{(\sigma^\Delta,\mu^\Delta)} \geq v^*(\Delta)$. Towards a contradiction, suppose the result is false. Hence, there exists a sequence $\{\Delta^n\} \to 0$ and an $\epsilon > 0$ such that $\lim_{n\to\infty} \kappa^{(\sigma^{\Delta^n},\mu^{\Delta^n})} - v^*(\Delta^n) > \epsilon$.

For each n, let $\hat{t}_n$ be the time at which a buyer with value $\kappa_n \equiv \kappa^{(\sigma^{\Delta^n},\mu^{\Delta^n})}$ buys under $(\sigma^{\Delta^n}, \mu^{\Delta^n})$ if $c_t = c_H$ for all $t \leq \hat{t}_n$. The price at which a buyer with value κ_n buys under $(\sigma^{\Delta^n}, \mu^{\Delta^n})$ when costs are c_H is $\hat{p}(\kappa_n) = (1-\rho(\Delta^n))\kappa_n + \rho(\Delta^n)p^L(\kappa_n)$.

For each n, fix $\hat{\kappa}_n \in (v^*(\Delta^n), \kappa_n)$ such that

$$\hat{p}(\hat{\kappa}_n)\left(\frac{F(\kappa_n) - F(\hat{\kappa}_n)}{F(\kappa_n)}\right) \leq U^L(\kappa_n) - e^{-r\Delta^n}\frac{F(\hat{\kappa}_n)}{F(\kappa_n)}U^L(\hat{\kappa}_n).$$

Let $\{\tilde{\kappa}^H_t(\Delta^n)\}$ be such that, for all $t \leq \hat{t}_n + \Delta^n$, $\tilde{\kappa}^H_t(\Delta^n) = \kappa^H_t(\Delta^n)$, (where $\{\kappa^H_t(\Delta^n)\}$ is the sequence of belief cutoffs under $(\sigma^{\Delta^n}, \mu^{\Delta^n})$) and for all $t \geq \hat{t}_n + 2\Delta^n$, $\tilde{\kappa}^H_t(\Delta^n) = \hat{\kappa}_n$. Let $\{\tilde{p}^H_t(\Delta^n)\}$ be such that $\tilde{p}^H_t(\Delta^n) = \hat{p}(\hat{\kappa}_n)$ for all $t \geq \hat{t}_n + \Delta^n$, and such that, for all $t < \hat{t}_n + \Delta^n$,

$$\tilde{\kappa}^H_{t+\Delta^n}(\Delta^n) - \tilde{p}^H_t(\Delta^n) = e^{-(r+\lambda)\Delta^n}(\tilde{\kappa}^H_{t+\Delta^n}(\Delta^n) - \tilde{p}^H_{t+\Delta^n}(\Delta^n)) \\ + e^{-r\Delta^n}(1-e^{-\lambda\Delta^n})(\tilde{\kappa}^H_{t+\Delta^n}(\Delta^n) - p^L(\tilde{\kappa}^H_{t+\Delta^n}(\Delta^n))). \quad (17)$$

That is, $\{\tilde{p}^H_t(\Delta^n), \tilde{\kappa}^H_t(\Delta^n)\}$ satisfies (15). Note that the inefficiencies under $\{\tilde{p}^H_t(\Delta^n), \tilde{\kappa}^H_t(\Delta^n)\}$ are smaller than under $\{p^H_t(\Delta^n), \kappa^H_t(\Delta^n)\}$, since trade is delayed by less under the former. The rest of the proof shows that, for n large enough, $\{\tilde{p}^H_t(\Delta^n), \tilde{\kappa}^H_t(\Delta^n)\}$ can be supported by an equilibrium in $\Sigma^{\mathsf{S}}(\Delta^n)$. This leads to a contradiction, since $(\sigma^{\Delta^n}, \mu^{\Delta^n})$ was assumed to be a welfare maximizing equilibrium in $\Sigma^{\mathsf{S}}(\Delta^n)$.

As a first step, I show that $\tilde{p}^H_t(\Delta^n) < p^H_t(\Delta^n)$ for all $t \leq \hat{t}_n$. Since sequences $\{\kappa^H_t(\Delta^n), p^H_t(\Delta^n)\}$ satisfy (9) for all $t \leq \hat{t}_n$, and since $\tilde{\kappa}^H_t(\Delta_n) = \kappa^H_t(\Delta_n)$ for all $t \leq \hat{t}_n + \Delta^n$, $\tilde{p}^H_t(\Delta^n) < p^H_t(\Delta^n)$ for all $t \leq \hat{t}_n$ implies that sequences $\{\tilde{\kappa}^H_t(\Delta^n), \tilde{p}^H_t(\Delta^n)\}$ satisfy (9).

Note that[21]

$$\begin{aligned}
\tilde{\kappa}^H_{\hat{t}_n+\Delta^n} - \tilde{p}^H_{\hat{t}_n} &= e^{-(r+\lambda)\Delta^n}(\tilde{\kappa}^H_{\hat{t}+\Delta^n} - \tilde{p}^H_{\hat{t}_n+\Delta^n}) + e^{-r\Delta^n}(1-e^{-\lambda\Delta^n})(\tilde{\kappa}^H_{\hat{t}_n+\Delta^n} - p^L(\tilde{\kappa}^H_{\hat{t}_n+\Delta^n})) \\
&> e^{-(r+\lambda)\Delta^n}(\tilde{\kappa}^H_{\hat{t}_n+\Delta^n} - \hat{p}(\tilde{\kappa}^H_{\hat{t}_n+\Delta_n})) + e^{-r\Delta^n}(1-e^{-\lambda\Delta^n})(\tilde{\kappa}^H_{\hat{t}_n+\Delta^n} - p^L(\tilde{\kappa}^H_{\hat{t}_n+\Delta^n})) \\
&= e^{-(r+\lambda)\Delta^n}\rho(\tilde{\kappa}^H_{\hat{t}_n+\Delta^n} - p^L(\tilde{\kappa}^H_{\hat{t}_n+\Delta^n})) + e^{-r\Delta^n}(1-e^{-\lambda\Delta^n})(\tilde{\kappa}^H_{\hat{t}_n+\Delta^n} - p^L(\tilde{\kappa}^H_{\hat{t}_n+\Delta^n})) \\
&= \rho(\tilde{\kappa}^H_{\hat{t}_n+\Delta^n} - p^L(\tilde{\kappa}^H_{\hat{t}_n+\Delta^n})),
\end{aligned}$$

where the strict inequality uses $\tilde{p}^H_{\hat{t}_n+\Delta^n} = \hat{p}(\tilde{\kappa}^H_{\hat{t}_n+2\Delta^n}) < \hat{p}(\tilde{\kappa}^H_{\hat{t}_n+\Delta^n})$, the second equality uses $\hat{p}(\tilde{\kappa}^H_{\hat{t}_n+\Delta^n}) = \tilde{\kappa}^H_{\hat{t}_n+\Delta^n}(1-\rho) + \rho p^L(\tilde{\kappa}^H_{\hat{t}_n+\Delta^n})$, and the last equality uses $\rho = e^{-r\Delta}(1-e^{-\lambda\Delta}) + \rho e^{-(r+\lambda)\Delta}$. Since $\tilde{\kappa}^H_{\hat{t}_n+\Delta^n} = \kappa^H_{\hat{t}_n+\Delta^n}$, and since $p^H_{\hat{t}_n} = \hat{p}(\kappa^H_{\hat{t}_n+\Delta^n}) = (1-\rho)\kappa^H_{\hat{t}_n+\Delta^n} + \rho p^L(\kappa^H_{\hat{t}_n+\Delta^n})$, it follows that $\tilde{p}^H_{\hat{t}_n} < p^H_{\hat{t}_n}$.

I now use this to show that $\tilde{p}^H_t(\Delta^n) < p^H_t(\Delta^n)$ for all $t < \hat{t}_n$. For all $t \leq \hat{t}_n$, prices

[21] In what follows, I drop the dependence on the time period Δ^n when there is no risk of confusion.

$\{p_t^H(\Delta^n)\}$ satisfy

$$\kappa^H_{t+\Delta^n}(\Delta^n) - p_t^H(\Delta^n) = e^{-(r+\lambda)\Delta^n}(\kappa^H_{t+\Delta^n}(\Delta^n) - p^H_{t+\Delta^n}(\Delta^n)) \\ + e^{-r\Delta^n}(1-e^{-\lambda\Delta^n})(\kappa^H_{t+\Delta^n}(\Delta^n) - p^L(\kappa^H_{t+\Delta^n}(\Delta^n))).$$

Combining this equation with (17), for all $t < \hat{t}_n$,

$$p_t^H(\Delta^n) - \tilde{p}_t^H(\Delta^n) = e^{-(r+\lambda)\Delta^n}(p^H_{t+\Delta^n}(\Delta^n) - \tilde{p}^H_{t+\Delta^n}(\Delta^n)),$$

where I used $\tilde{\kappa}_t^H(\Delta^n) = \kappa_t^H(\Delta^n)$ for all $t \leq \hat{t}_n + \Delta^n$. Since $\tilde{p}^H_{\hat{t}_n} < p^H_{\hat{t}_n}$, it follows that $p_t^H(\Delta^n) > \tilde{p}_t^H(\Delta^n)$ for all $t < \hat{t}_n$. Hence, $\{\tilde{\kappa}_t^H(\Delta^n), \tilde{p}_t^H(\Delta^n)\}$ satisfies (9).

I now show that, for n sufficiently large, $\{\tilde{\kappa}_t^H(\Delta^n), \tilde{p}_t^H(\Delta^n)\}$ also satisfies (16). I start by showing that $\tilde{p}_t^H(\Delta^n) > \tilde{p}^H_{t+\Delta^n}(\Delta^n)$ for all $t < \hat{t}_n + \Delta^n$, so prices $\tilde{p}_t^H(\Delta^n)$ are decreasing. This implies that $\tilde{p}_t^H(\Delta^n) > \tilde{p}^H_{\hat{t}_n+\Delta^n}(\Delta^n) = \hat{p}(\hat{\kappa}_n)$ for all $t \leq \hat{t}_n$. Since $\hat{p}(\hat{\kappa}_n) = (1-\rho(\Delta^n))\hat{\kappa}_n + \rho(\Delta^n)p^L(\hat{\kappa}_n)$, $\hat{\kappa}_n > v^*(\Delta^n) = \frac{c_H}{1-\rho(\Delta^n)}$, and $p^L(\hat{\kappa}_n) \geq \underline{v}$, this further implies that $\hat{p}(\hat{\kappa}_n) - c_H > \rho(\Delta^n)\underline{v}$. Hence, if prices $\tilde{p}_t^H(\Delta^n)$ are decreasing, then $\tilde{p}_t^H(\Delta^n) - c_H > \rho(\Delta^n)\underline{v}$ for all $t \leq \hat{t}_n + \Delta^n$.

Recall that

$$\begin{aligned} &\tilde{p}^H_{\hat{t}_n+\Delta^n} = \hat{p}(\tilde{\kappa}^H_{\hat{t}_n+2\Delta^n}) = (1-\rho(\Delta^n))\tilde{\kappa}^H_{\hat{t}_n+2\Delta^n} + \rho(\Delta^n)p^L(\tilde{\kappa}^H_{\hat{t}_n+2\Delta^n}) \\ \iff &\tilde{\kappa}^H_{\hat{t}_n+2\Delta^n} - \tilde{p}^H_{\hat{t}_n+\Delta^n} = \rho(\Delta^n)(\tilde{\kappa}^H_{\hat{t}_n+2\Delta^n} - p^L(\tilde{\kappa}^H_{\hat{t}_n+2\Delta^n})) \\ \iff &\tilde{\kappa}^H_{\hat{t}_n+2\Delta^n} - \tilde{p}^H_{\hat{t}_n+\Delta^n} = e^{-(r+\lambda)\Delta^n}(\tilde{\kappa}^H_{\hat{t}_n+2\Delta^n} - \tilde{p}^H_{\hat{t}_n+\Delta^n}) + e^{-r\Delta^n}(1-e^{-\lambda\Delta^n})(\tilde{\kappa}^H_{\hat{t}_n+2\Delta^n} - p^L(\tilde{\kappa}^H_{\hat{t}_n+2\Delta^n})), \end{aligned} \tag{18}$$

where the last line uses $\rho(\Delta) = \frac{e^{-r\Delta}(1-e^{-\lambda\Delta})}{1-e^{-(r+\lambda)\Delta}}$. Moreover, $\tilde{p}^H_{\hat{t}_n}$ satisfies (17), and so

$$\tilde{\kappa}^H_{\hat{t}_n+\Delta^n} - \tilde{p}^H_{\hat{t}_n} = e^{-(r+\lambda)\Delta^n}(\tilde{\kappa}^H_{\hat{t}_n+\Delta^n} - \tilde{p}^H_{\hat{t}_n+\Delta^n}) + e^{-r\Delta^n}(1-e^{-\lambda\Delta^n})(\tilde{\kappa}^H_{\hat{t}_n+\Delta^n} - p^L(\tilde{\kappa}^H_{\hat{t}_n+\Delta^n}))$$

Combining this with (18) yields

$$\tilde{p}^H_{\hat{t}_n} - \tilde{p}^H_{\hat{t}_n+\Delta^n} = (1-e^{-r\Delta^n})(\tilde{\kappa}^H_{\hat{t}_n+\Delta^n} - \tilde{\kappa}^H_{\hat{t}_n+2\Delta^n}) + e^{-r\Delta^n}(1-e^{-\lambda\Delta^n})(p^L(\tilde{\kappa}^H_{\hat{t}_n+\Delta^n}) - p^L(\tilde{\kappa}^H_{\hat{t}_n+2\Delta_n})) > 0,$$

where the strict inequality follows since $\tilde{\kappa}^H_{\hat{t}_n+\Delta^n} > \tilde{\kappa}^H_{\hat{t}_n+2\Delta^n}$ and $p^L(\cdot)$ is weakly increasing.

Towards an induction, suppose that $\tilde{p}^H_{t'} > \tilde{p}^H_{t'+\Delta^n}$ for all $t' = t+\Delta^n, ..., \hat{t}_n$. I now show that $\tilde{p}^H_t > \tilde{p}^H_{t+\Delta^n}$. Since $\tilde{p}^H_t$ and $\tilde{p}^H_{t+\Delta^n}$ satisfy (17), it follows that

$$\begin{aligned}\tilde{p}^H_t - \tilde{p}^H_{t+\Delta^n} &= (1-e^{-r\Delta^n})(\tilde{\kappa}^H_{t+\Delta^n} - \tilde{\kappa}^H_{t+2\Delta^n}) + e^{-(r+\lambda)\Delta^n}(\tilde{p}^H_{t+\Delta^n} - \tilde{p}^H_{t+2\Delta^n}) \\ &\quad + e^{-r\Delta^n}(1-e^{-\lambda\Delta^n})(p^L(\tilde{\kappa}^H_{t+\Delta^n}) - p^L(\tilde{\kappa}^H_{t+2\Delta^n})) > 0.\end{aligned}$$

By the Coase conjecture, for all κ, $U^L(\kappa) \to \underline{v}$ as $\Delta \to 0$; i.e., the seller earns a profit margin of $\underline{v}$ on each sale she makes when her costs are c_L. Since the profit margin $(\tilde{p}^H_t - c_H)$ that she earns on each sale when her cost is c_H is larger than $\rho\underline{v}$, in the limit as $n \to \infty$ the seller's profits from selling when her costs are c_H are larger than what she would get by waiting until her costs fall to c_L and then playing the continuation equilibrium. Hence, constraint (16) is satisfied under sequences $\{\tilde{p}^H_t(\Delta^n), \tilde{\kappa}^H_t(\Delta^n)\}$ when n is sufficiently large.

The arguments above show that, for n large enough, $\{\tilde{\kappa}^H_t(\Delta^n), \tilde{p}^H_t(\Delta^n)\}$ satisfies all the conditions in Theorem A.1(ii). Hence, for n large enough, there exists $(\sigma,\mu) \in \Sigma^{\mathsf{S}}(\Delta^n)$ that induces $\{\tilde{\kappa}^H_t(\Delta^n), \tilde{p}^H_t(\Delta^n)\}$. But this contradicts the fact that, for all n, $(\sigma^{\Delta^n}, \mu^{\Delta^n})$ is a welfare maximizing equilibrium in $\Sigma^{\mathsf{S}}(\Delta^n)$ (recall that inefficiencies under $\{\tilde{p}^H_t(\Delta^n), \tilde{\kappa}^H_t(\Delta^n)\}$ are smaller than under $\{p^H_t(\Delta^n), \kappa^H_t(\Delta^n)\}$). Therefore, $\kappa^{(\sigma^\Delta,\mu^\Delta)} - v^*(\Delta) \to 0$ as $\Delta \to 0$. ■

For all $\kappa \in [\underline{v}, \overline{v}]$ and $\Delta > 0$, let $U^L(\kappa;\Delta)$ be the seller's continuation profits when her cost is c_L and her belief cutoff is κ. Define $\pi^L(\kappa;\Delta) \equiv F(\kappa)U^L(\kappa;\Delta)$.

Lemma B.2 (no atoms)**.** *Fix a sequence* $\{\Delta^n\} \to 0$. *For each* n, *let* $(\sigma^{\Delta^n}, \mu^{\Delta^n})$

be a welfare maximizing equilibrium in $\Sigma^S(\Delta^n)$, and let $\{\kappa_t^H(\Delta^n), p_t^H(\Delta^n)\}$ be the sequences of prices and belief cutoffs induced by $(\sigma^{\Delta^n}, \mu^{\Delta^n})$. There exists $B > 0$ such that, for all $t \in T(\Delta^n)$,

$$\limsup_{n\to\infty} \frac{F(\kappa_t^H(\Delta^n)) - F(\kappa_{t+\Delta^n}^H(\Delta^n))}{\Delta^n} \leq B.$$

Hence, for all $t \in T(\Delta^n)$, $\kappa_t^H(\Delta^n) - \kappa_{t+\Delta^n}^H(\Delta^n) \to 0$ as $n \to \infty$.

Proof. Note first that, for all n, there exists $\hat{t}_n$ such that $\frac{F(\kappa_t^H(\Delta^n)) - F(\kappa_{t+\Delta^n}^H(\Delta^n))}{\Delta^n} = 0$ for all $t > \hat{t}_n$; i.e., $\hat{t}_n$ is the last period at which the seller makes sales when costs are high.

Consider next $t \leq \hat{t}_n$. By Lemma A.1, and using $\pi^L(\kappa;\Delta) = F(\kappa)U^L(\kappa;\Delta)$,

$$\begin{aligned}(F(\kappa_t^H(\Delta^n)) - F(\kappa_{t+\Delta^n}^H(\Delta^n)))p_t^H(\Delta^n) \leq{}& \pi^L(\kappa_t^H(\Delta^n);\Delta^n)(1 - e^{-r\Delta^n}) \\ &+ e^{-r\Delta^n}(\pi^L(\kappa_t^H(\Delta^n);\Delta^n) - \pi^L(\kappa_{t+\Delta^n}^H(\Delta^n);\Delta^n)).\end{aligned} \tag{19}$$

Let $p^L(\kappa;\Delta)$ be the price that a low cost seller would charge when her cutoff beliefs are κ in a setting with time period Δ. Note that, since $p^L(\kappa;\Delta) \in [\underline{v}, p^L(\overline{v};\Delta)]$ for all κ,

$$\pi^L(\kappa_t^H(\Delta^n);\Delta^n) - \pi^L(\kappa_{t+\Delta^n}^H(\Delta^n);\Delta^n) \leq p^L(\overline{v};\Delta^n)(F(\kappa_t^H(\Delta^n)) - F(\kappa_{t+\Delta^n}^H(\Delta^n))).$$

Combining this with (19),

$$\frac{F(\kappa_t^H(\Delta^n)) - F(\kappa_{t+\Delta^n}^H(\Delta^n))}{\Delta^n}(p_t^H(\Delta^n) - e^{-r\Delta^n}p^L(\overline{v};\Delta^n)) \leq \pi^L(\kappa_t^H(\Delta^n);\Delta^n)\frac{1 - e^{-r\Delta^n}}{\Delta^n}. \tag{20}$$

Next, recall from the proof of Lemma B.1 that prices $p_t^H(\Delta^n)$ are decreasing: for all $t < \hat{t}_n$, $p_t^H(\Delta^n) > p_{\hat{t}_n}^H(\Delta^n) = \hat{p}(\kappa^{(\sigma^{\Delta^n},\mu^{\Delta^n})}) \geq \hat{p}(v^*(\Delta^n)) = (1 - \rho(\Delta^n))v^*(\Delta^n) + \rho(\Delta^n)p^L(v^*(\Delta^n);\Delta^n)$. Since $\lim_{\Delta\to 0}\rho(\Delta) = \frac{\lambda}{r+\lambda}$, $\lim_{\Delta\to 0} v^*(\Delta) = \frac{r+\lambda}{r}c_H$ and $\lim_{\Delta\to 0} p^L(\overline{v};\Delta) =$

$\underline{v}$, it follows that

$$\liminf_{n\to\infty} p_t^H(\Delta^n) - e^{-r\Delta^n} p^L(\overline{v};\Delta^n) \geq c_H + \frac{\lambda}{r+\lambda}\underline{v} - \underline{v} = c_H - \frac{r}{r+\lambda}\underline{v} > 0.$$

The strict inequality holds since, by Assumption 1, $v^*(\Delta) \in (\underline{v}, \overline{v})$, and so $\lim_{\Delta\to 0} v^*(\Delta) = \frac{r+\lambda}{r} c_H > \underline{v}$.

Using this in inequality (20)

$$\begin{aligned}\limsup_{n\to\infty} \frac{F(\kappa_t^H(\Delta^n)) - F(\kappa_{t+\Delta}^H(\Delta^n))}{\Delta^n} &\leq \limsup_{n\to\infty} \frac{1}{p_t^H(\Delta^n) - e^{-r\Delta^n} p^L(\overline{v};\Delta^n)} \pi^L(\kappa_t^H(\Delta^n);\Delta^n) \frac{1-e^{-r\Delta^n}}{\Delta^n} \\ &\leq \frac{r+\lambda}{(r+\lambda)c_H - r\underline{v}} r\underline{v},\end{aligned}$$

where the last inequality uses $\lim_{\Delta\to 0} \pi^L(\kappa;\Delta) = F(\kappa)\underline{v} \leq \underline{v}$. ■

Proof of Theorem 2. Note first that, by (15), sequences $\{\kappa_t^H(\Delta), p_t^H(\Delta)\}_{t\in T(\Delta)}$ are such that, for all $t < \hat{t}$,

$$\begin{aligned}\kappa_{t+\Delta}^H(\Delta) - p_t^H(\Delta) &= e^{-(r+\lambda)\Delta}(\kappa_{t+\Delta}^H(\Delta) - p_{t+\Delta}^H(\Delta)) \\ &\quad + e^{-r\Delta}(1-e^{-\lambda\Delta})(\kappa_{t+\Delta}^H(\Delta) - p^L(\kappa_{t+\Delta}^H(\Delta);\Delta)). \qquad (21)\end{aligned}$$

For each $t \in [0,\infty)$, let $p^H(t) = \lim_{\Delta\to 0} p_t^H(\Delta)$ and $\kappa^H(t) = \lim_{\Delta\to 0} \kappa_t^H(\Delta)$ (if needed, take a convergent subsequence, which exists by Helly's Selection Theorem). Dividing both sides of (21) by Δ and rearranging,

$$\begin{aligned}\frac{p_t^H(\Delta) - p_{t+\Delta}^H(\Delta)}{\Delta} &= \kappa_{t+\Delta}^H(\Delta)\frac{(1-e^{-r\Delta})}{\Delta} - p_{t+\Delta}^H(\Delta)\frac{(1-e^{-(r+\lambda)\Delta})}{\Delta} \\ &\quad + e^{-r\Delta}\frac{(1-e^{-\lambda\Delta})}{\Delta} p^L(\kappa_{t+\Delta}^H(\Delta);\Delta). \qquad (22)\end{aligned}$$

Taking limits on both sides of (22) as $\Delta \to 0$ and using $\lim_{\Delta\to 0} p^L(\kappa,\Delta) = \underline{v}$ and

$\lim_{\Delta\to 0}\kappa_t^H(\Delta)-\kappa_{t+\Delta}^H(\Delta)=0$ (Lemma B.2),

$$\lim_{\Delta\to 0}\frac{p_t^H(\Delta)-p_{t+\Delta}^H(\Delta)}{\Delta}=-\frac{dp^H(t)}{dt}=r\kappa^H(t)-(r+\lambda)p^H(t)+\lambda\underline{v}.$$

Under the most efficient equilibrium, it must be that inequality (9) holds with equality for almost all $t\in T(\Delta)$. Using $\pi^L(\kappa;\Delta)=F(\kappa)U^L(\kappa;\Delta)$,

$$\begin{aligned}p_t^H(\Delta)(F(\kappa_t^H(\Delta))-F(\kappa_{t+\Delta}^H(\Delta)))&=\pi^L(\kappa_t^H(\Delta);\Delta)-\pi^L(\kappa_{t+\Delta}^H(\Delta);\Delta)\\&\quad+(1-e^{-r\Delta})\pi^L(\kappa_{t+\Delta}^H(\Delta);\Delta).\end{aligned}\tag{23}$$

Note next that, for all $\kappa,\kappa'\in[\underline{v},\overline{v}]$ with $\kappa>\kappa'$, the following inequalities hold:

$$\begin{aligned}\pi^L(\kappa;\Delta)-\pi^L(\kappa';\Delta)&\geq\underline{v}(F(\kappa)-F(\kappa'))\\ \pi^L(\kappa;\Delta)-\pi^L(\kappa';\Delta)&\leq p^L(\overline{v};\Delta)(F(\kappa)-F(\kappa')).\end{aligned}$$

The inequalities follow since, for all belief cutoffs $\tilde{\kappa}$, $p^L(\tilde{\kappa};\Delta)\in[\underline{v},p^L(\overline{v};\Delta)]$. Combining these inequalities with (23), and dividing through by Δ, yields

$$\begin{aligned}&\underline{v}\frac{F(\kappa_t^H(\Delta))-F(\kappa_{t+\Delta}^H(\Delta))}{\Delta}+\frac{1-e^{-r\Delta}}{\Delta}\pi^L(\kappa_{t+\Delta}^H(\Delta);\Delta)\\ \leq&p_t^H(\Delta)\frac{F(\kappa_t^H(\Delta))-F(\kappa_{t+\Delta}^H(\Delta))}{\Delta}\\ \leq&p^L(\overline{v};\Delta)\frac{F(\kappa_t^H(\Delta))-F(\kappa_{t+\Delta}^H(\Delta))}{\Delta}+\frac{1-e^{-r\Delta}}{\Delta}\pi^L(\kappa_{t+\Delta}^H(\Delta);\Delta).\end{aligned}$$

Taking the limit as $\Delta\to 0$ and using $\lim_{\Delta\to 0}p^L(\overline{v};\Delta)=\underline{v}$ and $\lim_{\Delta\to 0}\pi^L(\kappa;\Delta)=\underline{v}F(\kappa)$,

$$\begin{aligned}p^H(t)\lim_{\Delta\to 0}\frac{F(\kappa_t^H(\Delta))-F(\kappa_{t+\Delta}^H(\Delta))}{\Delta}&=\underline{v}\lim_{\Delta\to 0}\frac{F(\kappa_t^H(\Delta))-F(\kappa_{t+\Delta}^H(\Delta))}{\Delta}+r\underline{v}F(\kappa^H(t))\\ \iff\lim_{\Delta\to 0}\frac{F(\kappa_t^H(\Delta))-F(\kappa_{t+\Delta}^H(\Delta))}{\Delta}&=-\frac{d\kappa^H(t)}{dt}f(\kappa^H(t))=\frac{r\underline{v}F(\kappa^H(t))}{p^H(t)-\underline{v}}.\end{aligned}$$

The boundary condition for $\kappa^H(\cdot)$ is $\kappa^H(0) = \overline{v}$. To derive the boundary condition for $p^H(\cdot)$, let $\hat{v} = \lim_{\Delta \to 0} v^*(\Delta) = \frac{r+\lambda}{r} c_H$. By Lemma B.1, belief cutoffs $\kappa^H(t)$ reaches $\hat{v} = \frac{r+\lambda}{r} c_H$ at finite time $\hat{t} = \inf\{t \geq 0 : \kappa^H(t) = \hat{v}\}$. The price at which the seller sells to a buyer with valuation $\hat{v}$ must be such that this buyer is indifferent between buying now, or waiting until costs fall to c_L and getting the good at price $\underline{v}$. Hence, $p^H(\hat{t}) = \frac{r}{r+\lambda}\hat{v} + \frac{\lambda}{r+\lambda}\underline{v} = c_H + \frac{\lambda}{r+\lambda}\underline{v}$. ■

C Proofs of Propositions 3, 4, 5 and 6

Proof of Proposition 3. I start by showing that $\kappa - P^H(\kappa) > \frac{\lambda}{r+\lambda}(\kappa - \underline{v})$ for all $\kappa > \hat{v}$. Since $P^H(\kappa^H(t)) = p^H(t)$ for all $t \leq \hat{t}$, this is equivalent to showing that $\kappa^H(t) - p^H(t) > \frac{\lambda}{r+\lambda}(\kappa^H(t) - \underline{v})$ for all $t < \hat{t}$, or that

$$\forall t < \hat{t}, \qquad D(t) \equiv r(\kappa^H(t) - p^H(t)) + \lambda(\underline{v} - p^H(t)) > 0.$$

Using equation (5),

$$\begin{aligned} D'(t) &= r\frac{d\kappa^H(t)}{dt} - (r+\lambda)\frac{dp^H(t)}{dt} \\ &= r\frac{d\kappa^H(t)}{dt} + (r+\lambda)[r(\kappa^H(t) - p^H(t)) + \lambda(\underline{v} - p^H(t))] \\ &= r\frac{d\kappa^H(t)}{dt} + (r+\lambda)D(t). \end{aligned} \tag{24}$$

Note that $p^H(\hat{t}) = \hat{v} - \frac{\lambda}{r+\lambda}(\hat{v} - \underline{v}) = \kappa^H(\hat{t}) - \frac{\lambda}{r+\lambda}(\kappa^H(\hat{t}) - \underline{v})$, and so $D(\hat{t}) = 0$. Since $\frac{d\kappa^H(t)}{dt} < 0$ for all $t \leq \hat{t}$, it follows that $D'(\hat{t}) < 0$. Hence, $D(t) > 0$ for all $t < \hat{t}$ close to $\hat{t}$. Towards a contradiction, suppose there exists $t < \hat{t}$ with $D(t) \leq 0$, and let $\tilde{t} = \sup\{t < \hat{t} : D(t) \leq 0\}$. Since $D(t)$ is continuous, $D(\tilde{t}) = 0$. Moreover, since $D(t) > 0$ for all $t \in (\tilde{t}, \hat{t})$, it must be that $D'(\tilde{t}) \geq 0$. Using (24), and noting that

$\frac{d\kappa^H(t)}{dt}|_{t=\tilde{t}} < 0$ and $D(\tilde{t}) = 0$,

$$D'(\tilde{t}) = r\frac{d\kappa^H(t)}{dt}|_{t=\tilde{t}} + (r+\lambda)D(\tilde{t}) < 0,$$

a contradiction. Hence, $D(t) > 0$ for all $t < \hat{t}$. And so $\kappa - P^H(\kappa) > \frac{\lambda}{r+\lambda}(\kappa - \underline{v})$ for all $\kappa > \hat{v}$.

I now show part (ii). For each c_H, let $\hat{v}(c_H) = \frac{\lambda+r}{r}c_H$ be the efficient cutoff for cost c_H, and let $P^H(\kappa; c_H)$ denote the solution to (8) and boundary condition for cost c_H.

Fix $c_H' > c_H$, so $\hat{v}(c_H') > \hat{v}(c_H)$. Note that $\hat{v}(c_H) - P^H(\hat{v}(c_H); c_H) = \frac{\lambda}{r+\lambda}(\hat{v}(c_H) - \underline{v})$. By the arguments above, $\kappa - P^H(\kappa; c_H) > \frac{\lambda}{r+\lambda}(\kappa - \underline{v})$ for all $\kappa > \hat{v}(c_H)$; in particular, $\hat{v}(c_H') - P^H(\hat{v}(c_H'); c_H) > \frac{\lambda}{r+\lambda}(\hat{v}(c_H') - \underline{v}) = \hat{v}(c_H') - P^H(\hat{v}(c_H'); c_H')$, and so $P^H(\hat{v}(c_H'); c_H') > P^H(\hat{v}(c_H'); c_H)$.

I now show that $P^H(\kappa; c_H') > P^H(\kappa; c_H)$ for all $\kappa \in [\hat{v}(c_H'), \overline{v}]$. Towards a contradiction, suppose the result is not true, and let $\tilde{\kappa} = \inf\{\kappa \in [\hat{v}(c_H'), \overline{v}] : P^H(\kappa; c_H') \leq P^H(\kappa; c_H)\}$. Since $P^H(\kappa; c_H')$ and $P^H(\kappa; c_H)$ are continuous and since $P^H(\hat{v}(c_H'); c_H') > P^H(\hat{v}(c_H'); c_H)$, it must be that $\tilde{\kappa} > \hat{v}(c_H')$ and $P^H(\tilde{\kappa}; c_H') = P^H(\tilde{\kappa}; c_H)$. But then, $P^H(\cdot; c_H')$ and $P^H(\cdot; c_H)$ both solve ODE (8), with $P^H(\tilde{\kappa}; c_H') = P^H(\tilde{\kappa}; c_H)$; and so $P^H(\cdot; c_H') = P^H(\cdot; c_H)$, a contradiction. Hence, $P^H(\kappa; c_H') > P^H(\kappa; c_H)$ for all $\kappa \in [\hat{v}(c_H'), \overline{v}]$. Finally, by equation (6), the speed of trade falls when prices $p^H(t) = P^H(\kappa^H(t))$ increase.

I now turn to part (i). Fix distributions F_1 and F_0 such that F_1 dominates F_0 in terms of the reverse hazard rate. Let $P^H(\kappa; F_i)$ denote the solution to (8) and boundary condition under distribution F_i.

I start by showing that $P^H(\kappa; F_1) > P^H(\kappa; F_0)$ for all $\kappa > \hat{v}$. Note first that

$P^H(\hat{v}; F_i) = c_H + \frac{\lambda}{r+\lambda}\underline{v} = \hat{v} - \frac{\lambda}{r+\lambda}(\hat{v} - \underline{v})$ for $i = 0, 1$. Using (8), for $i = 0, 1$,

$$\frac{dP^H(\kappa; F_i)}{d\kappa}|_{\kappa=\hat{v}} = 0,$$
$$\frac{d^2P^H(\kappa; F_i)}{d\kappa^2}|_{\kappa=\hat{v}} = r\frac{f_i(\hat{v})}{F_i(\hat{v})}\frac{P^H(\hat{v}) - \underline{v}}{r\underline{v}}.$$

Since $\frac{f_1(v)}{F_1(v)} > \frac{f_0(v)}{F_0(v)}$ for all v, $\frac{d^2P^H(\kappa;F_1)}{d\kappa^2}|_{\kappa=\hat{v}} > \frac{d^2P^H(\kappa;F_0)}{d\kappa^2}|_{\kappa=\hat{v}}$. Hence, there exists $\tilde{v} > \hat{v}$ such that $P^H(\kappa; F_1) > P^H(\kappa; F_0)$ for all $\kappa \in (\hat{v}, \tilde{v})$.

Towards a contradiction, suppose that the result is not true, and let $\tilde{\kappa} = \inf\{\kappa > \hat{v} : P^H(\kappa; F_1) \leq P^H(\kappa; F_0)\}$. Since $P^H(\kappa; F_1)$ and $P^H(\kappa; F_0)$ are continuous, $P^H(\tilde{\kappa}; F_1) = P^H(\tilde{\kappa}; F_0)$. Since $P^H(\kappa; F_1) > P^H(\kappa; F_0)$ for all $\kappa \in (\hat{v}, \tilde{\kappa})$, it must be that $\frac{dP^H(\kappa;F_1)}{d\kappa}|_{\kappa=\tilde{\kappa}} \leq \frac{dP^H(\kappa;F_0)}{d\kappa}|_{\kappa=\tilde{\kappa}}$. But $P^H(\tilde{\kappa}; F_1) = P^H(\tilde{\kappa}; F_0)$ and $\frac{f_1(\tilde{\kappa})}{F_1(\tilde{\kappa})} > \frac{f_0(\tilde{\kappa})}{F_0(\tilde{\kappa})}$, together with ODE (8) implies $\frac{dP^H(\kappa;F_1)}{d\kappa}|_{\kappa=\tilde{\kappa}} > \frac{dP^H(\kappa;F_0)}{d\kappa}|_{\kappa=\tilde{\kappa}}$, a contradiction. Therefore, $P^H(\kappa; F_1) > P^H(\kappa; F_0)$ for all $\kappa > \hat{v}$. Lastly, since prices are higher under F_1 than under F_0, by equation (7) the rate at which the seller makes sales is slower under F_1 than under F_0.

Lastly, I turn to part (iii). For each λ, let $\hat{v}(\lambda) = \frac{\lambda+r}{r}c_H$, and let $P^H(\kappa; \lambda)$ denote the solution to (8) and boundary condition for λ. Note that $\frac{dP^H(\kappa;\lambda)}{d\kappa}|_{\kappa=\hat{v}(\lambda)} = 0$. Note further that

$$\frac{d}{d\lambda}P^H(\hat{v}(\lambda); \lambda) = \frac{\partial}{\partial\lambda}\left(c_H + \frac{\lambda}{r+\lambda}\underline{v}\right) = \frac{r}{(\lambda+r)^2}\underline{v} > 0.$$

Hence, for all $\lambda' > \lambda$ close enough to λ, it must that $P^H(\hat{v}(\lambda'); \lambda') > P^H(\hat{v}(\lambda'); \lambda)$. Since $P^H(\cdot; \lambda')$ and $P^H(\cdot; \lambda)$ are continuous, there exists $\tilde{\kappa} > \hat{v}(\lambda')$ such that $P^H(\kappa; \lambda') > P^H(\kappa; \lambda)$ for all $\kappa \in (\hat{v}(\lambda'), \tilde{\kappa})$. Next, note that by equation (6), the speed of trade falls when prices $p^H(t) = P^H(\kappa^H(t))$ increase. Hence, for all t with $\kappa^H(t) \in (\hat{v}(\lambda'), \tilde{\kappa})$, the speed of trade is lower under λ'. ■

Proof of Proposition 4. Equation (7) and the fact that, for all $t \leq \hat{t}$, $p^H(t) \geq$

$p^H(\hat{t}) = c_H + \frac{\lambda}{r+\lambda}\underline{v} > \underline{v}$,[22] together imply that $-\frac{d\kappa^H(t)}{dt}\frac{f(\kappa^H(t))}{F(\kappa^H(t))}$ goes to zero for all t as $\underline{v} \to 0$. Note then that, in the limit as $\underline{v} \to 0$, the seller only trades with the buyer once costs are c_L, at price $\underline{v} \to 0$. ■

Proof of Proposition 5. It is easy to check that mechanism M^{FB}: (a) is budget balance, (b) satisfies IC for the buyer, (c) satisfies IR for buyer and seller, and (d) implements the efficient outcome under truthful reporting. I now show that the mechanism also satisfies IC for the seller. Consider first a seller who reported $c_0 = c_H$ at $t = 0$. Then, for all $t > 0$, the seller strictly prefers to report $c_t = c_H$ if her current cost is c_H, while she is indifferent between reporting c_L or c_H if her cost is c_L. Hence, truthful reporting is (weakly) optimal.

Consider next time $t = 0$. A seller with initial cost c_H obtains a payoff of $\rho(\Delta)\underline{v}$ from reporting truthfully, and gets a payoff of $\underline{v} - c_H$ from reporting $c_0 = c_L$. Recall than $v^*(\Delta) = \frac{c_H}{1-\rho(\Delta)} > \underline{v}$, where the inequality follows from Assumption 1. Hence, $\rho(\Delta)\underline{v} > \underline{v} - c_H$, so a seller with initial cost c_H strictly prefers to report truthfully.

A seller with initial cost c_L gets a payoff of $\underline{v}$ if she reports truthfully. Her payoff from reporting $c_0 = c_H$ is $(1 - F(v^*(\Delta)))(c_H + \rho(\Delta)\underline{v}) + F(v^*(\Delta))\rho(\Delta)\underline{v}$. Reporting truthfully is optimal when $(1 - \rho(\Delta))\underline{v} \geq (1 - F(v^*(\Delta)))c_H$. ■

Proof of Proposition 6. Note first that, since the seller makes all the offers, prices $p(v, c)$ must satisfy $p(v, c) \geq \underline{v}$ for all $v \in [\underline{v}, \overline{v}]$ and $c \in \{c_L, c_H\}$: in any PBE, all buyer types accept a price $\underline{v}$ with probability 1.[23] This implies that, in any PBE, the profits of a seller with initial cost c_H are bounded below by $\rho\underline{v}$. Indeed, a seller with initial cost c_H can wait until her cost falls to c_L, charge price $\underline{v}$, and make a sale with probability 1, earning $\rho\underline{v}$.

Consider first the case in which (τ, p) is the outcome induced by some equilibrium

[22] By Assumption 1, $\underline{v} < \hat{v} = \frac{r+\lambda}{r}c_H$.

[23] This follows from the arguments in Lemma 1 in Gul et al. (1985), or Lemma S10 in Ortner (2017).

in $\Sigma(\Delta)$. Suppose by contradiction that the result is not true, so $\tau = \tau^{FB}$. Let $U(v)$ be the utility that a buyer with type v gets under this outcome:

$$U(v) = \mathbb{E}[qe^{-r\tau^{FB}(v,c_H)}(v - p(v, c_H)) + (1 - q)e^{-r\tau^{FB}(v,c_L)}(v - p(v, c_L))].$$

By incentive compatibility, $U(v)$ satisfies:

$$U(v) = U(\underline{v}) + \int_{\underline{v}}^{v} \mathbb{E}[qe^{-r\tau^{FB}(x,c_H)} + (1 - q)e^{-r\tau^{FB}(x,c_L)}]dx \tag{25}$$

for all $v \in [\underline{v}, \overline{v}]$. Since $p(v, c) \geq \underline{v}$ for all v, $U(\underline{v}) = 0$.

Consider first $v < v^*$, and note that

$$\begin{aligned} U(v) &= q\rho(v - p(v, c_H)) + (1 - q)(v - p(v, c_L)) \\ &= q\rho(v - \underline{v}) + (1 - q)(v - \underline{v}), \end{aligned} \tag{26}$$

where the first equality uses the properties of $\tau^{FB}(v, c_0)$ and the second follows from equation (25), using $U(\underline{v}) = 0$. Since $p(v, c_0) \geq \underline{v}$ for $c_0 \in \{c_L, c_H\}$ and for all v, equation (26) implies $p(v, c_L) = p(v, c_H) = \underline{v}$ for all $v < v^*$.

Consider next $v \geq v^*$, and note that

$$\begin{aligned} U(v) &= q(v - p(v, c_H)) + (1 - q)(v - p(v, c_L)) \\ &= q[\rho(v^* - \underline{v}) + (v - v^*)] + (1 - q)(v - \underline{v}), \end{aligned} \tag{27}$$

where again the first equality uses the properties of $\tau^{FB}(v, c_0)$ and the second follows from equation (25), using $U(\underline{v}) = 0$. Equation (27) implies that, for all $v \geq v^*$,

$$qp(v, c_H) + (1 - q)p(v, c_L) = q[v^* - \rho(v^* - \underline{v})] + (1 - q)\underline{v} = q(c_H + \rho\underline{v}) + (1 - q)\underline{v},$$

where the last equality uses $v^* - c_H = \rho v^*$. Since $p(v, c_L) \geq \underline{v}$ for all v, it follows that $p(v, c_H) \leq c_H + \rho\underline{v}$. I now show that $p(v, c_L) = \underline{v}$ and $p(v, c_H) = c_H + \rho\underline{v}$ for almost all $v \geq v^*$. Suppose not, so there exists a positive measure of buyer types $v \geq v^*$ with $p(v, c_H) < c_H + \rho \underline{v}$. Since $p(v, c_H) = \underline{v}$ for all $v < v^*$, the profits of seller with $c_0 = c_H$ under outcome (τ, p) are

$$(1 - F(v^*))\mathbb{E}[p(v, c_H) - c_H | v \geq v^*] + F(v^*)\rho\underline{v} < \rho\underline{v}.$$

But this cannot be, since a seller with $c_0 = c_H$ can obtain $\rho\underline{v}$ by waiting until her costs fall to c_L and charging price $\underline{v}$. Hence, $p(v, c_H) = c_H + \rho\underline{v}$ and $p(v, c_L) = \underline{v}$ for almost all $v \geq v^*$.

By the arguments above, under outcome (τ, p) a seller with $c_0 = c_L$ earns profits $\underline{v}$. The profits that this seller can obtain by mimicking a seller with $c_0 = c_H$, and then playing as if her cost fell to c_L at time $s\Delta$ are $(1 - F(v^*))(c_H + \rho\underline{v}) + e^{-rs\Delta}F(v^*)\underline{v}$, which is strictly larger than $\underline{v}$ for all $s\Delta$ small enough (since, by Assumption 1, $v^* = \frac{c_H}{1-\rho} > \underline{v} \iff c_H + \rho\underline{v} > \underline{v}$), a contradiction. Hence, $\tau \neq \tau^{FB}$.

Consider next the case in which (τ, p) is the point-wise limiting outcome induced by some sequence of equilibria (σ^n, μ^n), with $(\sigma^n, \mu^n) \in \Sigma(\Delta^n)$ for all n and with $\Delta^n \to 0$. Note first that, by Dominated Convergence, (25) must hold under (τ, p). Hence, both (26) and (27) must also hold under (τ, p). And so, by the same arguments as above, we must have $p(v, c_L) = p(v, c_H) = \underline{v}$ for all $v < v^*$ and $p(v, c_H) = c_H + \rho\underline{v}$ and $p(v, c_L) = \underline{v}$ for almost all $v \geq v^*$.

Finally, fix $\epsilon > 0$ small. For n large enough, seller's profits at $t = 0$ under (σ^n, μ^n) are lower than $\underline{v}+\epsilon/2$ when $c_0 = c_L$. Similarly, for n large enough, a seller with $c_0 = c_L$ can obtain profits at least as large as $(1 - F(v^*))(c_H + \rho\underline{v}) + e^{-r\Delta_n}F(v^*)\underline{v} - \epsilon/2$ by mimicking a seller with $c_0 = c_H$ at time $t = 0$, and then playing as if her cost fell to c_L at time Δ_n. Since $c_H + \rho\underline{v} > \underline{v}$, such a deviation is profitable for $\epsilon > 0$ small and for all n sufficiently large. Hence, $\tau \neq \tau^{FB}$. ■

D Pooling Equilibria

This appendix studies pooling equilibria such that (i) the seller posts the same price at times $t = 0, ..., \tau$ regardless of her cost; and (ii) from time $\tau + \Delta$ onwards, a high cost seller posts high prices that are rejected with probability 1 (for instance, prices above $\overline{v}$), and a low cost seller plays the continuation strategy of the one-sided private information game. The goal is to show that outcomes under this class of pooling equilibria are bounded away from the first-best outcome whenever $q = \text{prob}(c_0 = c_H) < 1$.

Note first that, if $\Delta > 0$ is bounded away from zero, the outcome under such equilibria would be bounded away from the first-best outcome. Hence, I focus on showing that the equilibrium outcome under such equilibria is bounded away from the first-best outcome when Δ is small.

Consider such a pooling equilibrium. Let $\kappa \in [\underline{v}, \overline{v}]$ be the lowest value buyer who buys at the last pooling period $\tau \geq 0$, and let p denote the price the seller charges at τ. For each $t = 0, \Delta, 2\Delta, ...$, let $q_t \leq q$ be the probability that the seller's cost is c_H at period t: i.e., $q_t = q \times e^{-\lambda t} \leq q$. Note then that price p must satisfy

$$\begin{aligned}
&\kappa - p \geq q_{\tau+\Delta}\rho(\kappa - p^L(\kappa)) + (1 - q_{\tau+\Delta})e^{-r\Delta}(\kappa - p^L(\kappa)) \\
\iff &p \leq \kappa(1 - q_{\tau+\Delta}\rho - (1 - q_{\tau+\Delta})e^{-r\Delta}) + p^L(\kappa)(q_{\tau+\Delta}\rho + (1 - q_{\tau+\Delta})e^{-r\Delta}). \quad (28)
\end{aligned}$$

Indeed, under such an equilibrium, at period τ a buyer of type κ can obtain the payoff in the right-hand side of the first line by delaying trade until the seller charges price $p^L(\kappa)$. Since $\lim_{\Delta\to 0} \rho = \frac{\lambda}{r+\lambda}$ and $\lim_{\Delta\to 0} p^L(\kappa) = \underline{v}$, we have that for all $\eta > 0$ there exists $\overline{\Delta}_1 > 0$ such that, for all $\Delta < \overline{\Delta}_1$,

$$\begin{aligned} p &\leq \kappa\left(1 - q_{\tau+\Delta}\frac{\lambda}{r+\lambda} - (1-q_{\tau+\Delta})\right) + \underline{v}\left(q_{\tau+\Delta}\frac{\lambda}{r+\lambda} + (1-q_{\tau+\Delta})\right) + \eta \\ &= \kappa q_{\tau+\Delta}\frac{r}{r+\lambda} + \underline{v}\left(1 - q_{\tau+\Delta}\frac{r}{r+\lambda}\right) + \eta \\ &\leq \kappa q\frac{r}{r+\lambda} + \underline{v}\left(1 - q\frac{r}{r+\lambda}\right) + \eta, \end{aligned} \tag{29}$$

where the last inequality uses $\kappa \geq \underline{v}$ and $q \geq q_{\tau+\Delta}$.

The continuation profits a seller with cost c_H gets at the beginning of period τ under this equilibrium are $(p - c_H)\frac{F(\kappa_\tau)-F(\kappa)}{F(\kappa_\tau)} + \frac{F(\kappa)}{F(\kappa_\tau)}\rho\pi^L(\kappa)$, where $\kappa_\tau > \kappa$ is the seller's belief cutoff at the beginning of time τ.[24] Note then that

$$\begin{aligned} &(p - c_H)\frac{F(\kappa_\tau) - F(\kappa)}{F(\kappa_\tau)} + \frac{F(\kappa)}{F(\kappa_\tau)}\rho\pi^L(\kappa) \geq \rho\underline{v} \\ \Longleftrightarrow &(p - c_H)\frac{F(\kappa_\tau) - F(\kappa)}{F(\kappa_\tau)} \geq \rho\underline{v}\frac{F(\kappa_\tau) - F(\kappa)}{F(\kappa_\tau)} + \frac{F(\kappa)}{F(\kappa_\tau)}\rho(\underline{v} - \pi^L(\kappa)), \end{aligned} \tag{30}$$

where the first inequality follows since a high cost seller can always wait until her costs fall to c_L, charge a price equal to $\underline{v}$, and sell immediately at this price.[25] Since $\lim_{\Delta\to 0}\pi^L(\kappa) = \underline{v}$, and since $c_H = \frac{r}{r+\lambda}\hat{v}$, the inequality in (30) implies that, for all $\eta > 0$ there exists $\overline{\Delta}_2$ such that $p \geq c_H + \frac{\lambda}{r+\lambda}\underline{v} - \eta = \frac{r}{r+\lambda}\hat{v} + \frac{\lambda}{r+\lambda}\underline{v} - \eta$ whenever $\Delta < \overline{\Delta}_2$. Combining this with (29), it follows that for all $\Delta < \overline{\Delta} \equiv \min\{\overline{\Delta}_1, \overline{\Delta}_2\}$,

$$\begin{aligned} \kappa q\frac{r}{r+\lambda} + \underline{v}\left(1 - q\frac{r}{r+\lambda}\right) &\geq \frac{r}{r+\lambda}\hat{v} + \frac{\lambda}{r+\lambda}\underline{v} - 2\eta \\ \Longleftrightarrow \underline{v}\frac{r}{r+\lambda}(1-q) &\geq \frac{r}{r+\lambda}\hat{v} - \kappa q\frac{r}{r+\lambda} - 2\eta \end{aligned}$$

Finally, by Assumption 1, there exists $\gamma > 0$ such that $\underline{v} = \hat{v} - \gamma$. Using this in the

[24] For instance, if $\tau = 0$, so there is only one pooling period, then $\kappa_\tau = \overline{v}$.

[25] Indeed, since the seller makes all the offers, in any PBE all buyer types accept a price $\underline{v}$ with probability 1; this follows from the arguments in Lemma 1 in Gul et al. (1985), or Lemma S10 in Ortner (2017).

inequality above yields

$$\begin{aligned}(\hat{v}-\gamma)\frac{r}{r+\lambda}(1-q) &\geq \frac{r}{r+\lambda}\hat{v}-\kappa q\frac{r}{r+\lambda}-2\eta \\ \iff \kappa-\hat{v} &\geq \gamma\frac{1-q}{q}-\frac{r+\lambda}{r}\frac{2\eta}{q}.\end{aligned}$$

Since $\eta > 0$ is arbitrary, we get that for small enough Δ, κ must be bounded away from the first-best cutoff whenever $q \in (0, 1)$.

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