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Distance on matchings: An axiomatic approach

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Matchings in a market may have varying degrees of compromise from efficiency, fairness, and or stability. A distance function allows to quantify such concepts or the (dis)similarity between any two matchings. There are a few attempts to propose such functions; however, these are tailored for specific applications and ignore the individual preferences completely. In this paper, we construct a normative framework to quantify the difference between outcomes of market mechanisms in matching markets, while endogenizing the preferences of the individuals into the distance concept. Several conditions are introduced to capture natural and appealing behavior of such functions. We find a class of distance functions called *scaled Borda distances*, which is the only class of distance functions that satisfies these conditions simultaneously.

Keywords. Matching markets, distance function, metrics.

JEL classification. C0, C78, D61, D63.

1. Introduction

Matching theory analyzes markets where agents (e.g., buyers and sellers, hospitals and interns, high schools and students) are matched according to their preferences, and thereby conduct some transactions within the relevant context. These mechanisms pro-

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duce matchings with various normative features,[1] e.g., stability, Pareto efficiency, fairness, and computational complexity.[2] Given the different features of matching mechanisms, it is natural to ask how different two matchings are. By means of a distance function, or for short a *distance*, one could quantify the dissimilarity of a mechanism to a particular solution concept, or select/refine from a set of matchings. One can pick the matching(s) from the core with the minimal total distance to all other stable matchings (as a tool to find the median stable matching(s)[3]) or analyze the similarity of a chosen outcome to the men(women)-optimal stable matchings, (as a tool for a fairness analysis).

General-purpose distances on matchings can be useful as descriptive summary statistics in various different markets. For instance, consider the house allocation problem, Abdulkadiroğlu and Sönmez (1999), which is a one-sided market. In these markets, each agent has an initial endowment and a preference over all endowments. In case a distance function is used to compare the outcome of an individually rational mechanism and the initial endowment, the result can be interpreted as the social welfare improvement of implementing that particular mechanism. See the example below.

Example 1 (Housing allocation). Let $N = \{1, 2, \dots, 6\}$ be the set of agents and $H = \{h_1, \dots, h_6\}$ be the set of houses. Let the initial endowments σ of the agents be $\sigma(i) = h_i$ for all $i \in \{1, \dots, 6\}$. The preference of each agent is shown in Figure 1. It can be verified that after applying the top trading cycle algorithm, the final allocation μ^{TTC} will be as follows: $\mu^{\mathrm{TTC}}(1) = h_1$, $\mu^{\mathrm{TTC}}(2) = h_3$, $\mu^{\mathrm{TTC}}(3) = h_4$, $\mu^{\mathrm{TTC}}(4) = h_2$, $\mu^{\mathrm{TTC}}(5) = h_5$ and $\mu^{\mathrm{TTC}}(6) = h_6$. $\Diamond$

There have been some methods proposed in the literature to address this quantification problem for particular domains and particular features. The most general and

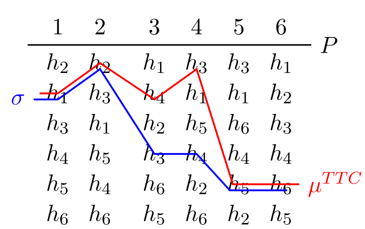

Figure 1. House allocation problem.

[1]For comparisons of some of these methods, see Abdulkadiroglu and Sönmez (2003), Abdulkadiroğlu, Pathak, Roth, and Sönmez (2005), Ergin and Sönmez (2006), Chen and Sönmez (2006), Erdil and Ergin (2008), Kesten (2010), Abdulkadiroğlu, Che, and Yasuda (2011), Kesten and Ünver (2015).

[2]See Irving (1985), Irving, Manlove, and Scott (2000), Manlove, Irving, Iwama, Miyazaki, and Morita (2002).

[3]See Chen, Egesdal, Pycia, and Yenmez (2016) and Klaus and Klijn (2006) for median stable matching. Note that this method is analogous to the use of the Kemeny distance (Kemeny (1959), and Can and Storcken (2018)) in the Kemeny–Young method (Kemeny (1959), and Young and Levenglick (1978)).

intuitive way to compare two matchings is by simply looking at the number of individuals who are matched differently. This method[4] would assign zero if the matchings were identical in all pairs, and would be maximal if they had nothing in common. Such a method can compare any two matchings, and hence, can be used to conceptualize "closeness" to any desired feature. However, this neglects individuals' preferences in the market. Partially addressing this, Biró, Iñarra, and Molis (2014) use the number of blocking pairs for a given matching as a concept "closeness to stability," while Niederle and Roth (2007) propose to count the number of "disruptive" blocking pairs only for the same concept. Although both attempts somewhat endogenize preferences, they also focus on quantifying closeness to a particular feature, i.e., stability.

In this paper, rather than proposing varying definitions for "closeness to stability" or "closeness to efficiency" in different matching problems, we propose a systematic way to quantify "closeness" to any desired feature with a domain agnostic framework. This paper explores metric (distance) functions[5] on matchings. We introduce intuitive conditions and endogenize individual preferences in quantifying the dissimilarity (distance) between two matchings, and hence, between two mechanisms or between a mechanism and a solution concept (a desired feature) in roommate markets.[6] We formulate our result on the domain of roommate markets since we are also interested in markets that are not necessarily solvable, i.e., markets in which there are no stable matchings. In addition, since roommate markets are generic one-to-one matching problems, the results apply to other well-known two-sided markets, e.g., the marriage markets. This creates richness in the way these distances can be employed under different interpretations.

The conditions we propose characterize an intuitive class of positional distances that behave like Borda scoring rules in the context of voting. They assign distances based on the ranks of agents' partners. We refer to this class as scaled Borda distances. Given a market, these distances scale the sum of absolute differences in Borda scores[7] of agents' partners in two matchings.

The paper proceeds as follows. In Section 2, we present the basic notation for the model. Section 3 introduces the model, which is a metric framework and the conditions on distance functions. Section 4 is devoted to the analysis of the structure of distance functions satisfying those conditions and eventually bringing forth a complete characterization.

2. Notation

We consider a countable and infinite set of potential individuals, denoted by $\mathcal{N}$, with a nonempty and finite subset $N \subsetneq \mathcal{N}$ interpreted as a set of agents. For each $i \in N$, let R_i

[4]To the best of our knowledge, the first reference to such method can be found in Klaus, Klijn, and Walzl (2010) for stochastic markets.

[5]A metric is a function, which satisfies nonnegativity, identity of indiscernibles, symmetry, and triangular inequality.

[6]A roommate market is a one-sided one-to-one matching market.

[7]The Borda score of a matching for an individual is the number of alternatives that are ranked strictly below the partner of the individual in that matching.

denote the preference of agent i, that is a complete, transitive, and antisymmetric binary relation over N, while $R \equiv (R_i)_{i\in N}$ is the preference profile. We say agent j is "at least as good as" agent k for agent i whenever $j\, R_i\, k$. We denote the position of agent j in the preference R_i, by $\text{rank}(j, R_i) = |\{k \in N : k\, R_i\, j\}|$. A generic market (also referred to as a roommate problem) is denoted by $P = (N, R)$, and the set of all possible roommate problems over a particular set of agents N by $\mathcal{P}(N)$. We denote the domain of all roommate problems by $\mathcal{D} = \langle \mathcal{P}(N)\rangle_{N \subsetneq \mathcal{N}}$, i.e., the set of all possible roommate problems over all possible sets of agents.

A *matching* μ is a permutation on N such that for all $i, j \in N$, $\mu(i) = j$ if and only if $\mu(j) = i$. We refer to j as the *partner* (roommate) of i at matching μ, and in case $\mu(i) = i$, i is said to be *single* at matching μ. A matching in which every agent is single is referred to as the *identity matching* and is denoted by μ^I. We denote the set of all possible matchings on N by $\mathcal{M}(N)$. Given any problem $P = (N, R)$ and any two matchings $\mu, \bar{\mu} \in \mathcal{M}(N)$, the set of agents that are preferred (nested) between $\mu(i)$ and $\bar{\mu}(i)$ according to R_i forms an *interval* denoted by $[\mu, \bar{\mu}]_{R_i}$. Formally,

$$[\mu, \bar{\mu}]_{R_i} = \big\{j \in N : \mu(i)\, R_i\, j\, R_i\, \bar{\mu}(i) \text{ or } \bar{\mu}(i)\, R_i\, j\, R_i\, \mu(i)\big\}.$$

The *length* of an interval is denoted by $|\mu, \bar{\mu}|_{R_i} = \#[\mu_1, \mu_2]_{R_i} - 1$, i.e., the cardinality of the interval minus 1.

We say a matching $\bar{\mu}$ is *between* matchings μ and $\bar{\bar{\mu}}$, if $\bar{\mu}(i) \in [\mu, \bar{\bar{\mu}}]_{R_i}$ for all $i \in N$. Given any sequence of matchings $\mu^1, \dots, \mu^t$ in $\mathcal{M}(N)$, we say $\mu^1, \dots, \mu^t$ are "on a line," denoted by $[\mu^1 - \mu^2 - \cdots - \mu^t]$, if μ^j is between μ^i and μ^k for all $1 \le i \le j \le k \le t$. We say a matching μ is *weakly above* $\bar{\mu}$ whenever $\mu(i)\, R_i\, \bar{\mu}(i)$ for all $i \in N$. In addition, we say μ and $\bar{\mu}$ are *adjacent* whenever $|\mu, \bar{\mu}|_{R_i} = 1$ for all $i \in N$, we say μ and $\bar{\mu}$ are *disjoint* whenever $\mu(i) \neq \bar{\mu}(i)$ for all $i \in N$. Figure 2 demonstrates the concepts of interval, length, the identity matching, and betweenness.

Consider problem $P = (N, R)$. Let π be a permutation over the set of agents N. We denote the *permuted preference profile* by R^π where for all $i, j, k \in N$, $j\, R_i\, k$ if and only if $\pi(j)\, R^\pi_{\pi(i)}\, \pi(k)$. Define the *permuted problem* $P^\pi = (N, R^\pi)$ accordingly. Given a matching $\mu \in \mathcal{M}(N)$, we denote the *permuted matching* by μ^π where for all $i, j \in N$, $\mu(i) = j$ if and only if $\mu^\pi(\pi(i)) = \pi(j)$.[8] The permutations are denoted by the cycle notation, e.g.,

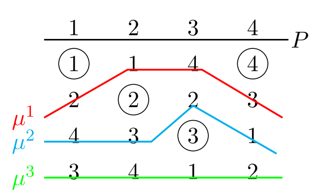

FIGURE 2. In the problem above, an example for an interval is $[\mu^1, \mu^3]_{R_1} = \{2, 4, 3\}$ and the length of this interval is $|\mu^1, \mu^3|_{R_1} = 2$. The matching μ^2 is between μ^1 and μ^3 while the identity matching is denoted with circles.

[8]This is a typical definition for permutations in roommate markets. For examples of this, see Klaus (2017), Özkal-Sanver (2010), Sasaki and Toda (1992).

$\pi = (123)(45)$ denotes $\pi(1) = 2$, $\pi(2) = 3$, $\pi(3) = 1$, $\pi(4) = 5$, $\pi(5) = 4$, and $\pi(i) = i$ for all $i \in N \setminus \{1, 2, 3, 4, 5\}$.

Let N be a set of agents and consider a *newcomer* $a \in \mathcal{N} \setminus N$. A problem $P^* = (N \cup \{a\}, R^*)$ is called an *extension* of the problem $P = (N, R)$ by a whenever preferences of agents in N over agents in N does not change from R to R^*, and the *newcomer* is ranked at the bottom in R^* by everyone in N. Formally:

(i) $\text{rank}(j, R_i) = \text{rank}(j, R_i^*)$, for all $i, j \in N$,

(ii) $\text{rank}(a, R_i^*) = \#N + 1$, for all $i \in N$.

Similarly, we say $\bar{\mu} \in \mathcal{M}(N \cup \{a\})$, is the extension of a matching $\mu \in \mathcal{M}(N)$ by agent $a \in \mathcal{N} \setminus N$, whenever $\bar{\mu}(i) = \mu(i)$ for all $i \in N$, and $\bar{\mu}(a) = a$. In such extensions, we call $a \in \mathcal{N} \setminus N$, an *irrelevant newcomer*.

Finally, let $A = \{a_1, a_2, \dots, a_k\}$, be a set of agents such that $N \cap A = \emptyset$. Consider the sequence $P^0, P^1, P^2, \dots, P^k$ of problems such that $P^0 = P$ and P^t is an extension of P^{t-1} by agent $a_t \in A$. Then we say P^k is an extension of P by the set of agents A. Similarly, we can define the extension of a matching by a set of agents. It should be noted that, the order of adding agents results in different problems.

3. Model

We use metric functions as our main framework for comparing matchings. Given a set of agents N, and a problem $P \in \mathcal{P}(N)$, a function on matchings $\delta_P : \mathcal{M}(N) \times \mathcal{M}(N) \to \mathbb{R}$ is called a metric (or a distance function) function if and only if it satisfies the regular metric conditions.[9] Hence, for a given problem P, a distance function δ_P assigns every pair of matchings $\mu, \bar{\mu} \in \mathcal{M}(N)$ a nonnegative real number depending on the structure of P. Note that according to this framework, as P changes, so does the distance between matchings. Therefore, we define *distances* on matchings as "collections of distance functions on all possible problems in the domain," denoted by

$$\delta = \langle \delta_P \rangle_{P \in \mathcal{D}}.$$

Next, we propose some conditions on distance functions on matchings. When three points are on a line, i.e., they are aligned, distances typically have an additive feature. In the case of individuals, this alignment can be thought of as a prospective partner being ranked in between another two. Hence, the change from the worst to the middle partner and the change from the middle to the best partner should add up to the change from the worst to the best partner. In the case of matchings, this alignment can be thought of as a matching being ordered by every individual in between another two. Therefore, this feature, which Kemeny (1959) calls *betweenness*,[10] requires that when three matchings are ordered "on a line," the distance function should be additive on these matchings.

[9] (i) Nonnegativity: $\delta_P(\mu, \bar{\mu}) \geq 0$, (ii) identity of indiscernibles: $\delta_P(\mu, \bar{\mu}) = 0$ if and only if $\mu = \bar{\mu}$, (iii) symmetry: $\delta_P(\mu, \bar{\mu}) = \delta_P(\bar{\mu}, \mu)$, and (iv) triangular inequality: $\delta_P(\mu, \bar{\bar{\mu}}) \leq \delta_P(\mu, \bar{\mu}) + \delta_P(\bar{\mu}, \bar{\bar{\mu}})$.

[10] This is a standard additivity condition, which strengthens the triangular inequality condition for cases where the weak inequality becomes equality, e.g., when three points are on a line in the Euclidian sense.

Condition 1 (Betweenness). δ satisfies *betweenness* if for all problems $P = (N, R) \in \mathcal{D}$ and for all matchings $\mu, \bar{\mu}, \bar{\bar{\mu}} \in \mathcal{M}(N)$ such that $\bar{\mu}$ is between $\mu, \bar{\bar{\mu}}$,

$$\delta_P(\mu, \bar{\bar{\mu}}) = \delta_P(\mu, \bar{\mu}) + \delta_P(\bar{\mu}, \bar{\bar{\mu}}).$$

Anonymity condition is straightforward and requires that the relabeling of the agents should not matter.

Condition 2 (Anonymity). δ satisfies *anonymity* if for all problems $P = (N, R) \in \mathcal{D}$ and for all matchings $\mu, \bar{\mu} \in \mathcal{M}(N)$ and permutation $\pi : N \to N$,

$$\delta_P(\mu, \bar{\mu}) = \delta_{P^\pi}\big(\mu^\pi, \bar{\mu}^\pi\big).$$

The monotonicity condition requires that if from one problem to another, the two matchings fall further apart from one another, then the distance function should reflect that by an increase in the distance.

Condition 3 (Monotonicity). δ satisfies *monotonicity* if for all problems $P = (N, R) \in \mathcal{D}$ and $\widehat{P} = (N, \widehat{R}) \in \mathcal{D}$ and all matchings $\mu, \bar{\mu} \in \mathcal{M}(N)$ such that $[\mu, \bar{\mu}]_{R_i} \subseteq [\mu, \bar{\mu}]_{\widehat{R}_i}$ for all $i \in N$,

$$\delta_P(\mu, \bar{\mu}) \leq \delta_{\widehat{P}}(\mu, \bar{\mu}).$$

Remark 1. An immediate implication of monotonicity is that if for two matchings μ and $\bar{\mu}$, the intervals remain the same across two problems on the same set of agents, then the distance should not change. Furthermore, changing the relative order of $\mu, \bar{\mu}$ in individual preferences does not alter the distance as long as the intervals remain the same.

The next condition is an invariance axiom, which states that if a problem and two matchings are extended by a dummy (single) agent, that essentially does not change the matchings, the distance between these matchings should be the same in the extended problem.

Condition 4 (Independence of irrelevant newcomers). δ satisfies *independence of irrelevant newcomers* if for all problems $P = (N, R) \in \mathcal{D}$ and any extension $P^* = (N^*, R^*) \in \mathcal{D}$ and all matchings $\mu, \bar{\mu} \in \mathcal{M}(N)$ with the extension $\mu^*, \bar{\mu}^* \in \mathcal{M}(\widehat{N})$ by some agent $a \in \mathcal{N} \setminus N$,

$$\delta_P(\mu, \bar{\mu}) = \delta_{P^*}\big(\mu^*, \bar{\mu}^*\big).$$

Remark 2. An immediate implication of independence of irrelevant newcomers is that if $P^*, \mu^*, \bar{\mu}^*$ are extensions of $P, \mu, \bar{\mu}$, by a set of agents $\mathcal{A}$, then $\delta_P(\mu, \bar{\mu}) = \delta_{P^*}(\mu^*, \bar{\mu}^*)$.

Often when dealing with matching problems, one has to compare matchings for different populations and sizes. Therefore, it is common to propose conditions to ensure consistency across these populations and and their subgroups.[11] As the set of agents enlarge, it is natural to expect the distances between matchings to increase, ceteris paribus. The simplest way to ensure consistency in such variable population scenarios, would be to set the minimal nonzero distance on disjoint matchings equal to the population size, i.e., $|N|$, to allow comparison of populations of varying cardinality. To allow richness in possible solutions, we propose a weaker version of this method, which sets this distance to be a function of the population instead.

Condition 5 (Standardization). δ satisfies *standardization* if there exists a function $\kappa : 2^{\mathcal{N}} \to \mathbb{R}$ such that for all N and for all disjoint matchings $\mu, \bar{\mu} \in \mathcal{M}(N)$,

$$\min_{P \in \mathcal{P}(N)} \delta_P(\mu, \bar{\mu}) = \kappa(N).$$

4. Results

In what follows, we focus on distance functions that satisfy the five conditions laid out in Section 3, i.e., Betweenness, Anonymity, Monotonicity, Independence of Irrelevant Newcomers, and Standardization. We also explicitly mention the necessary condition(s) for each lemma, proposition, and theorem as a prerequisite. We first introduce Lemma 1 (decomposition lemma), which proves that the distance functions we seek decompose the distance into sums of distances between pairs of matchings that look like components of the original matchings (see Figure 3). Thereafter, the results are presented in two subsections. In Section 4.1, we analyze the behavior of these distance functions specifically when they compare a matching with the identity matching, i.e., the matching where every agent is single, and in Section 4.2, the results are extended to cases where any two matchings are compared.

In Section 4.1, we first use Lemmas 2, 3, and 4 to show the distances between one-couple matchings (matchings in which everyone except one couple is single) and the identity matching is the same across all problems as long as the interval lengths are the same. These lemmas also quantify how different interval lengths relate to one another. Proposition 1 shows that the distances of such one-couple matchings to the identity matching should be based on positions of the partners. Finally, Theorem 1 combines the

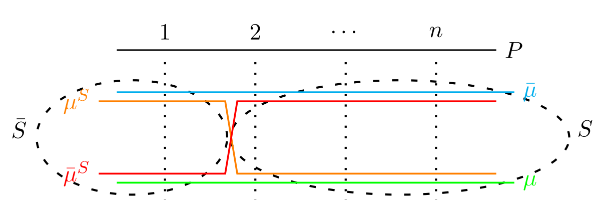

Figure 3. The general view of the decomposition lemma.

[11] Thomson (2011) provides an extensive survey on consistency and converse consistency in various domains while Karakaya and Klaus (2017) focus on population sensitivity properties in coalition formation games, e.g., roommate markets. Similar to our approach, Kemeny (1959) also uses normalization to set the minimal nonzero distance on rankings.

aforementioned results and extends Proposition 1 to conclude that the distance functions we seek is equivalent to a class of positional distances.

In Section 4.2, we extend the findings of Section 4.1 to any two matchings using two more building blocks, i.e., Propositions 3 and 4, to generalize Theorem 1 for any two matchings. Hence, Theorem 2 provides a complete characterization of a class of positional distances, which we refer to as scaled Borda distances. In Appendix D, we also demonstrate that the conditions in the characterization results are logically independent.

To state the first lemma, let $\mu, \bar{\mu} \in \mathcal{M}(N)$ be two matchings and $S \subseteq N$ be a subset of agents that are matched among themselves in μ and $\bar{\mu}$, i.e., $\mu(i), \bar{\mu}(i) \in S$ for all $i \in S$. Based on the set S, we define two matchings, μ^S and $\bar{\mu}^S$, as follows:

(i) for all $i \in S$, let $\mu^S(i) = \mu(i)$ and for all $i \in N \setminus S$, let $\mu^S(i) = \bar{\mu}(i)$,

(ii) for all $i \in S$, let $\bar{\mu}^S(i) = \bar{\mu}(i)$ and for all $i \in N \setminus S$, let $\bar{\mu}^S(i) = \mu(i)$.

In the following lemma, we show that the distance between μ, $\bar{\mu}$ can be decomposed into the sum of the distances from μ^S and $\bar{\mu}^S$ to μ (or $\bar{\mu}$). Figure 3 shows a demonstration of this decomposition.

Lemma 1 (Decomposition lemma). *Let δ be a distance function, which satisfies betweenness. Let $\mu, \bar{\mu} \in \mathcal{M}(N)$. Then, for all $S \subseteq N$ such that $\mu(i), \bar{\mu}(i) \in S$ for all $i \in S$, we have*

$$\delta_P(\mu, \bar{\mu}) = \delta_P(\mu, \bar{\mu}^S) + \delta_P(\mu, \mu^S) = \delta_P(\bar{\mu}^S, \bar{\mu}) + \delta_P(\mu^S, \bar{\mu}).$$

Proof. For a proof, see Appendix A. □

4.1 *Comparing any matching with the identity matching*

We now focus on the distance between any matching and the identity matching. By monotonicity, as long as the intervals between the two matchings remain the same, the distance will remain unchanged. Therefore, in order to keep the figures simple, we draw the identity matching below the other matchings and denote the matchings as straight lines whenever possible.

Consider a matching in which everyone is single except one couple, say $\mu(i) = j$ with $i \neq j$. We call such a matching a *one-couple matching* (see Figure 4) and denote it by μ^{ij}. Given a problem $P = (N, R)$, we say a one-couple matching μ^{ij} is of length (x, y) whenever $(|\mu^{ij}(i), i|_{R_i}, |\mu^{ij}(j), j|_{R_j}) = (x, y)$.

Remark 3. Consider any matching μ with k distinct couples. Then, by the decomposition lemma, and letting $S = \{i, j\}$ and $\bar{S} = N \setminus S$ for each couple of μ, the distance between μ and μ^I can be decomposed as the sum of distances of each of these k one-couple matchings, and the identity matching.

According to Remark 3, to compute the distance between any matching and the identity matching, we only need to focus on the distance between a one-couple matching and the identity matching. The total distance then equals the sum of each of these

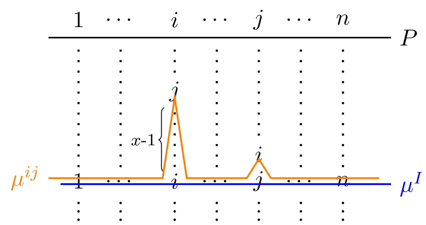

FIGURE 4. A one-couple matching μ^{ij} of length $(x, 1)$.

one-couple matchings. In the sequel, we show that the distance between a one-couple matching and the identity matching is the same for all problems whenever the interval lengths are the same. In Lemma 2, we demonstrate the case where the interval length is $(x, 1)$. Then in Lemma 3, we extend this to any interval length (x, y).

LEMMA 2. *Let δ be a distance function, which satisfies anonymity, monotonicity, and independence of irrelevant newcomers. Consider any finite N, $N' \subsetneq \mathcal{N}$ and a strictly positive integer x. Consider any one-couple matching $\mu^{ij} \in \mathcal{M}(N)$, and any $P \in \mathcal{P}(N)$ such that μ^{ij} is of length $(x, 1)$ in P. Similarly, consider any one-couple matching $\mu^{i'j'} \in \mathcal{M}(N')$, and any $P' \in \mathcal{P}(N')$ such that $\mu^{i'j'}$ is of length $(x, 1)$ in P'. Let μ^I and $\mu^{I'}$ denote the identity matchings in corresponding problems, then*

$$\delta_P(\mu^{ij}, \mu^I) = \delta_{P'}(\mu^{i'j'}, \mu^{I'}).$$

PROOF. For a proof, see Appendix B.1. □

REMARK 4. Given a distance function δ satisfying the conditions in Section 3, Lemma 2 shows that the distance between the identity matching and any one-couple matching of length $(x, 1)$ is the same across all the problems in the domain, i.e., regardless of the set of agents. Therefore, each distance function δ is associated with a constant for such one-couple matchings, denoted by α^{δ}_{x1}. Note, however, that throughout the rest of the paper, we omit the superscript to simplify the notation whenever it is clear and denote this constant simply as α_{x1} for δ.

The next lemma extends Lemma 2 to any one-couple matching of length (x, y).

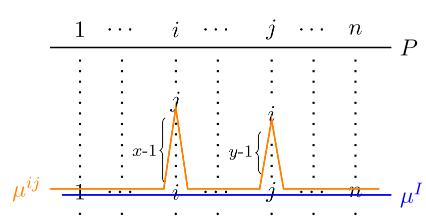

FIGURE 5. A one-couple matching μ^{ij} of length (x, y).

Lemma 3. *Let δ be a distance function, which satisfies betweenness, anonymity, monotonicity, and independence of irrelevant newcomers. Consider any finite $N, N' \subsetneq \mathcal{N}$, and two strictly positive integers x and y. Consider any one-couple matching $\mu^{ij} \in \mathcal{M}(N)$, and any $P \in \mathcal{P}(N)$ such that μ^{ij} is of length (x, y) in P. Similarly, consider any one-couple matching $\mu^{i'j'} \in \mathcal{M}(N')$, and any $P' \in \mathcal{P}(N')$ such that $\mu^{i'j'}$ is of length (x, y) in P'. Let μ^I and $\mu^{I'}$ denote the identity matchings in corresponding problems, then*

$$\delta_P(\mu^{ij}, \mu^I) = \delta_{P'}(\mu^{i'j'}, \mu^{I'}) = \alpha_{x1} + \alpha_{y1} - \alpha_{11}.$$

Proof. For a proof, see Appendix B.2. □

Lemma 3 shows that the distance between the identity matching and any one-couple matching of length (x, y) is the same across all the problems in the domain, i.e., regardless of the set of agents. To simplify the notation, we denote this distance by α_{xy}. Akin to Remark 4, each δ satisfying the conditions will have a unique α_{xy}.

Next, as a particular case of Lemma 3, we show that for any strictly positive integer x, $\alpha_{xx} = x\alpha_{11}$, i.e., a one-couple matching of length (x, x) has x times the distance that a one-couple matching of length $(1, 1)$ has (to the identity matching).

Lemma 4. *Let δ be a distance function, which satisfies betweenness, anonymity, monotonicity, independence of irrelevant newcomers, and standardization. Consider any finite $N \subsetneq \mathcal{N}$ and a strictly positive integer x. Consider any one-couple matching $\mu^{ij} \in \mathcal{M}(N)$, and any problem $P \in \mathcal{P}(N)$ such that μ^{ij} is of length (x, x) in P. Let μ^I denote the identity matching, then*

$$\delta_P(\mu^{ij}, \mu^I) = x \times \alpha_{11}.$$

Proof. For a proof, see Appendix B.3. □

Now we introduce Proposition 1. When all five conditions in Section 3 are imposed on a distance function δ, Proposition 1 states that the distance between the identity matching and a one-couple matching μ^{ij} must equal to a scalar function of the sum of absolute difference in the position of each agents' partners in these matchings.

Proposition 1. *Let δ be a distance function which satisfies betweenness, anonymity, monotonicity, independence of irrelevant newcomers, and standardization. Then, for any problem $P = (N, R)$ and any one-couple matching $\mu^{ij} \in \mathcal{M}(N)$, we have*

$$\delta_P(\mu^{ij}, \mu^I) = \frac{1}{2}\alpha_{11} \sum_{k \in \{i,j\}} \left|\operatorname{rank}(\mu^{ij}(k), R_k) - \operatorname{rank}(\mu^I(k), R_k)\right|. \tag{1}$$

Proof. For a proof, see Appendix B.4. □

Proposition 1 is fundamental in that it compares one-couple matchings and their distance to the identity matching. It also provides a clear picture of how the class of distance functions we are looking for should behave. In fact, the right-hand side of equation (1) in the proof above is very similar to a positional voting concept known as the

Borda rule in voting literature.[12] For each candidate in a voting problem, Borda rule defines a score that candidates get from each voter as follows:

$$\text{BordaScore}(j, R_i) = |N| - \text{rank}(j, R_i), \tag{2}$$

which is interpreted as the score candidate j gets from voter i. A very straightforward application of this scoring concept to comparing two matchings is one where we compare the Borda scores of partners of an agent i, in these two matchings (in absolute value):

$$\left|\text{BordaScore}\big(\mu(i), R_i\big) - \text{BordaScore}\big(\bar{\mu}(i), R_i\big)\right|$$

Finally, adding all the Borda score differentials for each individual's partners in the two matchings would properly define a new distance function, which we call the Borda distance.

Borda distance: A distance function is called the *Borda distance*, denoted by δ^{Borda}, if for all $P = (N, R) \in \mathcal{D}$, and for all matchings $\mu, \bar{\mu} \in \mathcal{M}(N)$,

$$\begin{aligned}\delta_P^{\text{Borda}}(\mu, \bar{\mu}) &= \sum_{i \in N}\left|\text{BordaScore}\big(\mu(i), R_i\big) - \text{BordaScore}\big(\bar{\mu}(i), R_i\big)\right|. \\ &= \sum_{i \in N}\left|\text{rank}\big(\mu(i), R_i\big) - \text{rank}\big(\bar{\mu}(i), R_i\big)\right|. \end{aligned} \tag{3}$$

Remark that equation (3) shows a clear resemblance to equation (1) in Proposition 1. In fact, the latter is just a scalar transformation of the former with some constant. In what follows, we formally define these scalar transformations of the Borda distance as scaled Borda distances. Formally, we have the following.

Scaled Borda distances: A distance function is called a *scaled Borda distance*, denoted by $\delta^{\sigma-\text{Borda}}$, if for some $\sigma \in \mathbb{R}_{++}$, for all $P = (N, R) \in \mathcal{D}$, and for all matchings $\mu, \bar{\mu} \in \mathcal{M}(N)$,

$$\delta_P^{\sigma-\text{Borda}}(\mu, \bar{\mu}) = \sigma \times \delta_P^{\text{Borda}}(\mu, \bar{\mu}) \tag{4}$$

We first show that all the scaled Borda distances satisfy the conditions introduced in Section 3. Hence, the lemmas and propositions introduced above are also applicable to scaled Borda distances.

Proposition 2. *Scaled Borda distances satisfy betweenness, anonymity, monotonicity, independence of irrelevant newcomers, and standardization.*

Proof. For a proof, see Appendix B.5 □

We can now introduce our first theorem, which expands Proposition 1 from one-couple matchings to all matchings when compared with the identity matchings. Formally, we have the following.

[12] See Borda (1781), Saari (1990).

Theorem 1. *A distance function δ satisfies betweenness, anonymity, monotonicity, independence of irrelevant newcomers, and standardization if and only if δ equals a scaled Borda distance when comparing the identity matching with others. That is, there exists $\sigma > 0$ such that for any problem $P = (N, R)$ and any $\mu \in \mathcal{M}(N)$, we have*

$$\delta_P(\mu, \mu^I) = \delta_P^{\sigma-\text{Borda}}(\mu, \mu^I).$$

Proof. For a proof, see Appendix B.6. □

Note that Theorem 1 is restricted to cases comparing matchings with the identity matching, although Proposition 2 is more general, i.e., the distances satisfy these conditions for any two matchings. The next section discusses the generalization of Theorem 1 and provides a complete characterization of the scaled Borda distances on any two matchings.

4.2 *Comparing any two nonidentity matching*

In this section, we generalize Theorem 1 to any two matchings. That is, under the imposed conditions, given any problem and any two matchings, a distance function, which compares these two matchings, must be equivalent to a scalar Borda distance. To do so, first we propose two propositions for four-agents' problems, then use these two propositions as building blocks to construct Theorem 2.

Proposition 3. *Let δ be a distance function which satisfies betweenness, anonymity, monotonicity, independence of irrelevant newcomers, and standardization. Consider a problem P over four agents with the preference profile and the matchings shown in Figure 6. Note that one singleton is nested between μ^1 and μ^2 and another is nested between μ^2 and μ^3. In such specific cases,*

(i) $\delta_P(\mu^1, \mu^2) = \delta_P^{\sigma-\text{Borda}}(\mu^1, \mu^2)$ *for* $\sigma = \frac{1}{2}\alpha_{11}$,

(ii) $\delta_P(\mu^2, \mu^3) = \delta_P^{\sigma-\text{Borda}}(\mu^2, \mu^3)$ *for* $\sigma = \frac{1}{2}\alpha_{11}$.

Proof. For a proof, see Appendix C.1. □

Proposition 4. *Let δ be a distance function, which satisfies betweenness, anonymity, monotonicity, independence of irrelevant newcomers, and standardization. Consider a*

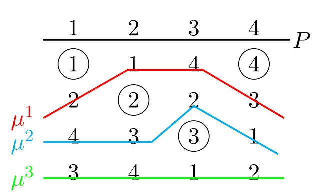

Figure 6. A problem over four agents with one singleton agent between the matchings.

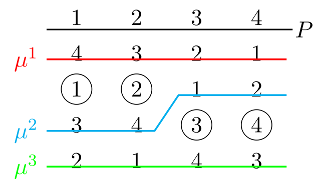

FIGURE 7. Problem P over four agents with two singleton agents between the matchings.

problem P over four agents with the preference profile and the matchings shown in Figure 7. Note that two singletons are nested between μ^1 and μ^2 and another two are nested between μ^2 and μ^3. In such specific cases,

(i) $\delta_P(\mu^2, \mu^3) = \delta_P^{\sigma-\text{Borda}}(\mu^2, \mu^3)$ *for* $\sigma = \frac{1}{2}\alpha_{11}$,

(ii) $\delta_P(\mu^1, \mu^2) = \delta_P^{\sigma-\text{Borda}}(\mu^1, \mu^2)$ *for* $\sigma = \frac{1}{2}\alpha_{11}$.

PROOF. For a proof, see Appendix C.2. □

Next, we propose our main characterization.

THEOREM 2. *A distance function δ satisfies betweenness, anonymity, monotonicity, independence of irrelevant newcomers, and standardization if and only if δ equals a scaled Borda distance. That is, there exists $\sigma > 0$ such that for any problem $P = (N, R)$ and any $\mu, \tilde{\mu} \in \mathcal{M}(N)$, we have*

$$\delta_P(\mu, \tilde{\mu}) = \delta_P^{\sigma-\text{Borda}}(\mu, \tilde{\mu}).$$

PROOF. For a proof, see Appendix C.3. □

APPENDIX A: PROOFS OF LEMMA 1

LEMMA 1 (Decomposition lemma). *Let δ be a distance function, which satisfies betweenness. Let $\mu, \bar{\mu} \in \mathcal{M}(N)$. Then, for all $S \subseteq N$ such that $\mu(i), \bar{\mu}(i) \in S$ for all $i \in S$, we have*

$$\delta_P(\mu, \bar{\mu}) = \delta_P(\mu, \bar{\mu}^S) + \delta_P(\mu, \mu^S) = \delta_P(\bar{\mu}^S, \bar{\mu}) + \delta_P(\mu^S, \bar{\mu}).$$

PROOF. By definition of $\bar{\mu}^S$ and μ^S, both are between μ and $\bar{\mu}$; hence, betweenness yields

$$\delta_P(\mu, \bar{\mu}) = \delta_P(\mu, \bar{\mu}^S) + \delta_P(\bar{\mu}^S, \bar{\mu}) \quad \text{and,} \tag{5}$$

$$\delta_P(\mu, \bar{\mu}) = \delta_P(\mu, \mu^S) + \delta_P(\mu^S, \bar{\mu}). \tag{6}$$

Since μ and $\bar{\mu}$ are both between μ^S and $\bar{\mu}^S$ betweenness results in

$$\delta_P(\mu^S, \bar{\mu}^S) = \delta_P(\mu^S, \mu) + \delta_P(\mu, \bar{\mu}^S) \quad \text{and,}$$

$$\delta_P(\mu^S, \bar{\mu}^S) = \delta_P(\mu^S, \bar{\mu}) + \delta_P(\bar{\mu}, \bar{\mu}^S).$$

The four equations above yield

$$\delta_P(\mu, \bar{\mu}^S) + \delta_P(\bar{\mu}^S, \bar{\mu}) = \delta_P(\mu, \mu^S) + \delta_P(\mu^S, \bar{\mu}) \quad \text{and,}$$
$$\delta_P(\mu^S, \mu) + \delta_P(\mu, \bar{\mu}^S) = \delta_P(\mu^S, \bar{\mu}) + \delta_P(\bar{\mu}, \bar{\mu}^S).$$

Subtracting the latter equation from the former, we have $\delta_P(\bar{\mu}^S, \bar{\mu}) - \delta_P(\mu^S, \mu) = \delta_P(\mu, \mu^S) - \delta_P(\bar{\mu}, \bar{\mu}^S)$, which reduces to

$$\delta_P(\bar{\mu}^S, \bar{\mu}) = \delta_P(\mu^S, \mu). \tag{7}$$

Plugging equation (7) into equation (5), and as δ is a symmetric function, yields $\delta_P(\mu, \bar{\mu}) = \delta_P(\mu, \bar{\mu}^S) + \delta_P(\mu, \mu^S)$, and plugging equation (7) into equation (6) results in $\delta_P(\mu, \bar{\mu}) = \delta_P(\bar{\mu}^S, \bar{\mu}) + \delta_P(\mu^S, \bar{\mu})$. □

Appendix B: Proofs of Section 4.1

B.1 *Proof of Lemma 2*

Lemma 2. *Let δ be a distance function, which satisfies anonymity, monotonicity, and independence of irrelevant newcomers. Consider any $N, N' \subsetneq \mathcal{N}$ and a strictly positive integer x. Consider any one-couple matching $\mu^{ij} \in \mathcal{M}(N)$, and any $P \in \mathcal{P}(N)$ such that μ^{ij} is of length $(x, 1)$ in P. Similarly, consider any one-couple matching $\mu^{i'j'} \in \mathcal{M}(N')$, and any $P' \in \mathcal{P}(N')$ such that $\mu^{i'j'}$ is of length $(x, 1)$ in P'. Let μ^I and $\mu^{I'}$ denote the identity matchings in corresponding problems, then*

$$\delta_P(\mu^{ij}, \mu^I) = \delta_{P'}(\mu^{i'j'}, \mu^{I'}).$$

Proof. Consider an extension $\bar{P}$ of P and the extension $\bar{\mu}^{ij}$ and $\bar{\mu}^I$ of matchings μ^{ij} and μ^I by the set of agents $N' \setminus N$, respectively. By Remark 2, $\delta_P(\mu^{ij}, \mu^I) = \delta_{\bar{P}}(\bar{\mu}^{ij}, \bar{\mu}^I)$. For simplicity, we abuse the notation and write P, μ^{ij} and μ^I instead of $\bar{P}$, $\bar{\mu}^{ij}$ and $\bar{\mu}^I$, respectively. Also, consider an extension $\bar{P}'$ of P' and the extension $\bar{\mu}^{i'j'}$ and $\bar{\mu}^{I'}$ of matchings $\mu^{i'j'}$ and $\mu^{I'}$ by the set of agents $N \setminus N'$, respectively. By Remark 2, $\delta_{P'}(\mu^{i'j'}, \mu^{I'}) = \delta_{\bar{P}'}(\bar{\mu}^{i'j'}, \bar{\mu}^{I'})$. For simplicity, we abuse the notation and write P', $\mu^{i'j'}$, and $\mu^{I'}$ instead of $\bar{P}'$, $\bar{\mu}^{i'j'}$, and $\bar{\mu}^{I'}$, respectively. Note that now both P and P' (as well as the matchings) are defined on the same set of agents $\bar{N} = N' \cup N$.

Let $Z = \{z_1, \dots, z_{x-1}\}$ be the set of other agents nested between j and i in R_i, and $Z' = \{z'_1, \dots, z'_{x-1}\}$ be the set of other agents nested between j' and i' in $R'_{i'}$. There are two possible situations: either $Z = Z'$ or $Z \neq Z'$.

Case 1. $Z = Z'$: Consider permutation $\pi = (ii')(jj')$. Applying this permutation on P, and using anonymity yields $\delta_P(\mu^{ij}, \mu^I) = \delta_{P^\pi}((\mu^{ij})^\pi, (\mu^I)^\pi)$. Since by this permutation, $(\mu^{ij})^\pi = \mu^{i'j'}$ and $(\mu^I)^\pi = \mu^{I'}$, then $\delta_{P^\pi}((\mu^{ij})^\pi, (\mu^I)^\pi) = \delta_{P^\pi}(\mu^{i'j'}, \mu^{I'})$. Since $Z = Z'$ and both problems are defined on the same set of agents, monotonicity implies $\delta_{P^\pi}(\mu^{i'j'}, \mu^{I'}) = \delta_{P'}(\mu^{i'j'}, \mu^{I'})$. Therefore, $\delta_P(\mu^{ij}, \mu^I) = \delta_{P'}(\mu^{i'j'}, \mu^{I'})$.

Case 2. $Z \neq Z'$: In this case, we add the same set of irrelevant newcomers to both problems P and P', and map the agents in Z and Z' to these newcomers so that the set of agents that are nested between the two matchings in these two problems become the same, then Case 1 implies the result. Formally, let $A = \{a_1, \dots, a_{x-1}\}$ be a set of agents such that $\bar{N} \cap A = \emptyset$. Next, let $\widehat{P}$ and $\widehat{P'}$ be the extensions of P and P' by the set of agents A, respectively. Also, let $\widehat{\mu}^{ij}$ and $\widehat{\mu}^{I}$ be the extensions of μ^{ij} and μ^{I}, and $\widehat{\mu}^{i'j'}$ and $\widehat{\mu}^{I'}$ be the extensions of $\mu^{i'j'}$ and $\mu^{I'}$, respectively, all by the same set of agents A. By Remark 2, $\delta_P(\mu^{ij}, \mu^I) = \delta_{\widehat{P}}(\widehat{\mu}^{ij}, \widehat{\mu}^I)$ and $\delta_{P'}(\mu^{i'j'}, \mu^{I'}) = \delta_{\widehat{P'}}(\widehat{\mu}^{i'j'}, \widehat{\mu}^{I'})$. For simplicity, we abuse the notation and write P, μ^{ij}, and μ^I instead of $\widehat{P}$, $\widehat{\mu}^{ij}$, and $\widehat{\mu}^I$, and we write P', $\mu^{i'j'}$, and $\mu^{I'}$ instead of $\widehat{P'}$, $\widehat{\mu}^{i'j'}$, and $\widehat{\mu}^{I'}$, respectively.

Consider the permutation $\pi = (z_t a_t)$ for all $t \in \{1, \dots, x-1\}$. Applying π on P permutes the agents that are nested between j and i in R_i to the agents in A. Also, applying the permutation $\pi' = (z'_t a_t)$ for all $t \in \{1, \dots, x-1\}$ on P' permutes the agents nested between j' and i' in $R'_{i'}$ to the agents in A. In both problems, anonymity implies the distances to be unchanged. As the set of agents nested between the two matchings both in P and P' are now identical, a similar argument to the one in Case 1 implies the result. □

B.2 *Proof of Lemma 3*

Lemma 3. *Let δ be a distance function, which satisfies betweenness, anonymity, monotonicity, and independence of irrelevant newcomers. Consider any $N, N' \subsetneq \mathcal{N}$ and two strictly positive integers x and y. Consider any one-couple matching $\mu^{ij} \in \mathcal{M}(N)$, and any $P \in \mathcal{P}(N)$ such that μ^{ij} is of length (x, y) in P. Similarly, consider any one-couple matching $\mu^{i'j'} \in \mathcal{M}(N')$, and any $P' \in \mathcal{P}(N')$ such that $\mu^{i'j'}$ is of length (x, y) in P'. Let μ^I and $\mu^{I'}$ denote the identity matchings in corresponding problems, then*

$$\delta_P(\mu^{ij}, \mu^I) = \delta_{P'}(\mu^{i'j'}, \mu^{I'}) = \alpha_{x1} + \alpha_{y1} - \alpha_{11}.$$

Proof. Consider an extension $\bar{P} = (N \cup \{a, b\}, \bar{R})$ of P and extensions $\bar{\mu}^{ij}, \bar{\mu}^I \in \mathcal{M}(N \cup \{a, b\})$ of $\mu^{ij}, \mu^I \in \mathcal{M}(N)$, respectively, by the set of agents $A = \{a, b\}$. By Remark 2, $\delta_P(\mu^{ij}, \mu^I) = \delta_{\bar{P}}(\bar{\mu}^{ij}, \bar{\mu}^I)$. For simplicity, we abuse the notation and write P, μ^{ij}, and μ^I instead of $\bar{P}$, $\bar{\mu}^{ij}$, and $\bar{\mu}^I$, respectively (see Figure 8).

Consider any problem $\widehat{P} = (\widehat{N}, \widehat{R})$, shown in Figure 9, with $\widehat{N} = N$ and $\widehat{R}$ such that

- $\mathrm{rank}(a, \widehat{R}_i) = 1$ and $[\mu^{ij}, \mu^I]_{R_i} = [\mu^{ij}, \mu^I]_{\widehat{R}_i}$,
- $\mathrm{rank}(b, \widehat{R}_j) = 1$ and $[\mu^{ij}, \mu^I]_{R_j} = [\mu^{ij}, \mu^I]_{\widehat{R}_j}$,

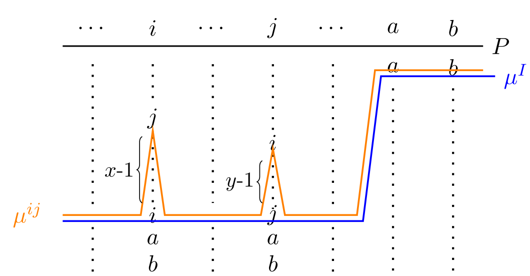

Figure 8. Problem P after adding the two newcomers a and b.

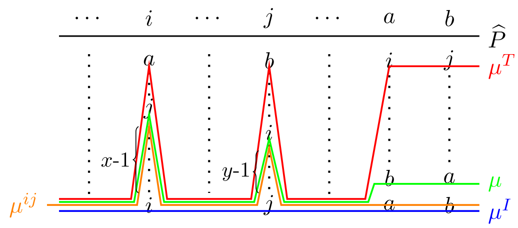

FIGURE 9. Problem $\widehat{P} = (\widehat{N}, \widehat{R})$.

- $\operatorname{rank}(i, \widehat{R}_a) = 1$, $\operatorname{rank}(b, \widehat{R}_a) = |\widehat{N}| + 1$, and $\operatorname{rank}(a, \widehat{R}_a) = |\widehat{N}| + 2$,
- $\operatorname{rank}(j, \widehat{R}_b) = 1$, $\operatorname{rank}(a, \widehat{R}_b) = |\widehat{N}| + 1$, and $\operatorname{rank}(b, \widehat{R}_b) = |\widehat{N}| + 2$.

By monotonicity, we have

$$\delta_P(\mu^{ij}, \mu^I) = \delta_{\widehat{P}}(\mu^{ij}, \mu^I). \tag{8}$$

Therefore, it is sufficient to prove that $\delta_{\widehat{P}}(\mu^{ij}, \mu^I) = \alpha_{x1} + \alpha_{y1} - \alpha_{11}$.

Consider the following two matchings in problem $\widehat{P}$ such that $\mu \in \mathcal{M}(\widehat{N})$ with $\mu(i) = j$, $\mu(a) = b$, and $\mu(t) = t$ for all other agent t and $\mu^T \in \mathcal{M}(\widehat{N})$ with $\mu^T(i) = a$, $\mu^T(j) = b$, and $\mu^T(t) = t$ for all other agent t.

CLAIM. $\delta_{\widehat{P}}(\mu, \mu^I) = \alpha_{x1} + \alpha_{y1}$.

PROOF OF CLAIM. Consider a new problem $\widehat{P}^\pi$ shown in Figure 10. Problem $\widehat{P}^\pi$ is the permuted problem of $\widehat{P}$ with $\pi = (aj)$. By this permutation, the identity matching remains the same; hence, we write μ^I instead of $(\mu^I)^\pi$ in $\widehat{P}^\pi$. By anonymity, the following equation holds:

$$\delta_{\widehat{P}}(\mu^T, \mu) = \delta_{\widehat{P}^\pi}((\mu^T)^\pi, \mu^\pi). \tag{9}$$

Consider a new problem $\widetilde{P}$ shown in Figure 11. Problem $\widetilde{P}$ is almost identical to problem $\widehat{P}^\pi$ except that the position of the partners of each agent in $(\mu^T)^\pi$ and μ^π are swapped. By monotonicity for $\widehat{P}^\pi$ and $\widetilde{P}$, $\delta_{\widehat{P}^\pi}((\mu^T)^\pi, \mu^\pi) = \delta_{\widetilde{P}}((\mu^T)^\pi, \mu^\pi)$. Plugging this into equation (9), we have

$$\delta_{\widehat{P}}(\mu^T, \mu) = \delta_{\widetilde{P}}((\mu^T)^\pi, \mu^\pi). \tag{10}$$

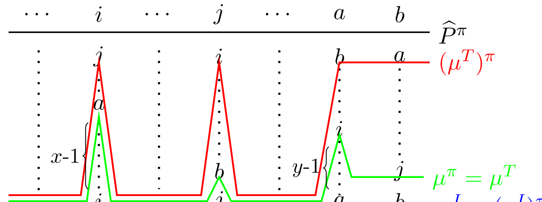

FIGURE 10. Problem $\widehat{P}^\pi$ after permuting problem $\widehat{P}$ in Figure 9 with $\pi = (aj)$.

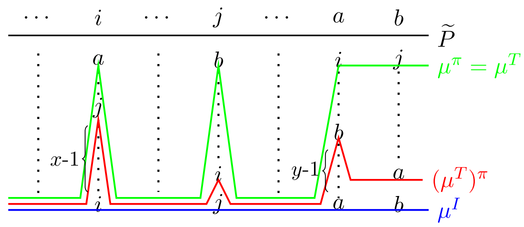

FIGURE 11. Problem $\widetilde{P}$, after swapping the positions of μ^π and $(\mu^T)^\pi$ in problem $\widehat{P}^\pi$ in Figure 10.

Since μ is between μ^T and μ^I in problem $\widehat{P}$, and $(\mu^T)^\pi$ is between μ^π and μ^I in problem $\widetilde{P}$, betweenness yields

$$\delta_{\widehat{P}}(\mu^T, \mu^I) = \delta_{\widehat{P}}(\mu^T, \mu) + \delta_{\widehat{P}}(\mu, \mu^I) \quad \text{and,} \tag{11}$$

$$\delta_{\widetilde{P}}(\mu^\pi, \mu^I) = \delta_{\widetilde{P}}(\mu^\pi, (\mu^T)^\pi) + \delta_{\widetilde{P}}((\mu^T)^\pi, \mu^I). \tag{12}$$

Note that by permutation π, $\mu^\pi = \mu^T$, hence $\delta_{\widetilde{P}}(\mu^\pi, \mu^I) = \delta_{\widetilde{P}}(\mu^T, \mu^I)$. Considering this and the monotonicity for problems $\widetilde{P}$ and $\widehat{P}$, we have $\delta_{\widetilde{P}}(\mu^T, \mu^I) = \delta_{\widehat{P}}(\mu^T, \mu^I)$. Therefore, the left-hand sides of equations (11) and (12) are equal, which yield

$$\delta_{\widehat{P}}(\mu^T, \mu) + \delta_{\widehat{P}}(\mu, \mu^I) = \delta_{\widetilde{P}}(\mu^\pi, (\mu^T)^\pi) + \delta_{\widetilde{P}}((\mu^T)^\pi, \mu^I).$$

Combining this with equation (10) results in $\delta_{\widehat{P}}(\mu, \mu^I) = \delta_{\widetilde{P}}((\mu^T)^\pi, \mu^I)$. Finally, by the decomposition lemma and Lemma 2, $\delta_{\widetilde{P}}((\mu^T)^\pi, \mu^I) = \alpha_{x1} + \alpha_{y1}$. Hence, $\delta_{\widehat{P}}(\mu, \mu^I) = \alpha_{x1} + \alpha_{y1}$, which concludes the claim. □

By the decomposition lemma, for matching μ in problem $\widehat{P}$, we have $\delta_{\widehat{P}}(\mu, \mu^I) = \delta_{\widehat{P}}(\mu^{ij}, \mu^I) + \delta_{\widehat{P}}(\mu^{ab}, \mu^I)$. By the claim proven above, $\delta_{\widehat{P}}(\mu, \mu^I) = \alpha_{x1} + \alpha_{y1}$, and by Lemma 2, we have $\delta_{\widehat{P}}(\mu^{ab}, \mu^I) = \alpha_{11}$. So, $\delta_{\widehat{P}}(\mu^{ij}, \mu^I) = \alpha_{x1} + \alpha_{y1} - \alpha_{11}$, and by equation (8), $\delta_P(\mu^{ij}, \mu^I) = \alpha_{x1} + \alpha_{y1} - \alpha_{11}$. Finally, by Lemma 2, the right-hand side of this equation is independent of the set of agents $N \subsetneq \mathcal{N}$. Therefore, for all $N \subsetneq \mathcal{N}$, for all problems $P \in \mathcal{P}(N)$ and for all one-couple matchings μ^{ij} of length (x, y) we conclude that

$$\delta_P(\mu^{ij}, \mu^I) = \alpha_{x1} + \alpha_{y1} - \alpha_{11}.$$ □

B.3 *Proof of Lemma 4*

LEMMA 4. *Let δ be a distance function, which satisfies betweenness, anonymity, monotonicity, independence of irrelevant newcomers, and standardization. Consider any $N \subsetneq \mathcal{N}$ and a strictly positive integer x. Consider any one-couple matching $\mu^{ij} \in \mathcal{M}(N)$, and any problem $P \in \mathcal{P}(N)$ such that μ^{ij} is of length (x, x) in P. Let μ^I denote the identity matching, then*

$$\delta_P(\mu^{ij}, \mu^I) = x \times \alpha_{11}.$$

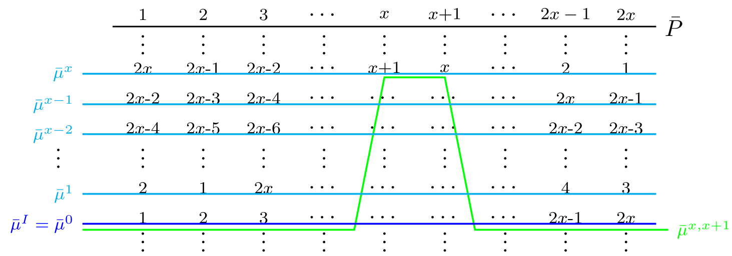

Figure 12. Problem $\bar{P} = (\bar{N}, \bar{R})$.

Proof. By Lemma 3, the distance between the identity matching and any one-couple matching of length (x, y) is the same across all problems in the domain. Therefore, it suffices to prove that the lemma holds for some problem $\bar{P}$, and some one-couple matching of length (x, x) in $\bar{P}$. Consider the following specific problem $\bar{P} = (\bar{N}, \bar{R})$ with $\bar{N} = \{1, 2, \ldots, 2x\}$ shown in Figure 12. Let $\bar{\mu}^{x,x+1}$ be the one-couple matching of length (x, x) in $\bar{P}$. Note that the structure of matchings $\bar{\mu}^x, \ldots, \bar{\mu}^1$ in problem $\bar{P}$ is such that

- $\bar{\mu}^x(1) = 2x$, $\bar{\mu}^x(2) = 2x - 1$, so on and so forth,
- for all $k \in \{2, \ldots, x\}$, and for all $i \in \bar{N}$, $\bar{\mu}^{k-1}(i) = \bar{\mu}^k((i+2) \bmod (2x))$, e.g., $\bar{\mu}^{x-1}(2x - 1) = \bar{\mu}^x(1) = 2x$,
- for all $k \in \{1, \ldots, x\}$, $\bar{\mu}^k$ and $\bar{\mu}^{k-1}$ are adjacent.

Next, we show that $\delta_{\bar{P}}(\bar{\mu}^{x,x+1}, \bar{\mu}^I) = x \times \alpha_{11}$, which in returns shows that $\alpha_{xx} = x \times \alpha_{11}$. To ease the notation in this problem, we denote the identity matching by $\bar{\mu}^0$.

Claim. $\delta_{\bar{P}}(\bar{\mu}^t, \bar{\mu}^{t+1}) = \frac{|\bar{N}|}{2}\alpha_{11}$ *for all* $t \in \{0, \ldots, x-1\}$.

Proof of claim. Note that by construction, the two matchings $\bar{\mu}^0$ and $\bar{\mu}^1$ in $\bar{P}$ are disjoint. By standardization, for $\bar{\mu}^0$, $\bar{\mu}^1$, there exists a problem $P' = (\bar{N}, R') \in \mathcal{P}(\bar{N})$ such that $\delta_{P'}(\bar{\mu}^0, \bar{\mu}^1) = \kappa(\bar{N})$ and is minimal. Note that for all $i \in \bar{N}$, as $\bar{\mu}^0$, $\bar{\mu}^1$ have the minimal possible intervals, we have $[\bar{\mu}^0, \bar{\mu}^1]_{\bar{R}_i} \subseteq [\bar{\mu}^0, \bar{\mu}^1]_{R'_i}$. Therefore, monotonicity implies that $\delta_{\bar{P}}(\bar{\mu}^0, \bar{\mu}^1) = \kappa(\bar{N})$. By the decomposition lemma, the distance between $\bar{\mu}^0$ and $\bar{\mu}^1$ can be decomposed as the sum of $\frac{|\bar{N}|}{2}$ one-couple matchings, each of the same length $(1, 1)$. Hence, $\delta_{\bar{P}}(\bar{\mu}^0, \bar{\mu}^1) = \frac{|\bar{N}|}{2}\alpha_{11}$. Together with the previous equation, we have $\kappa(\bar{N}) = \frac{|\bar{N}|}{2}\alpha_{11}$. Note that, by monotonicity the distance between $\bar{\mu}^t$, $\bar{\mu}^{t+1}$ for all $t \in \{1, 2, \ldots, x-1\}$, is also minimal, and by standardization this distance also equals $\kappa(\bar{N})$. Hence, $\delta_{\bar{P}}(\bar{\mu}^t, \bar{\mu}^{t+1}) = \frac{|\bar{N}|}{2}\alpha_{11}$ for all $t \in \{0, \ldots, x-1\}$, which completes the proof of the claim. $\square$

Next, we complete the proof of the lemma by showing $\delta_{\bar{P}}(\bar{\mu}^{x,x+1}, \bar{\mu}^I) = x \times \alpha_{11}$. Note that by construction of $\bar{P}$, the matchings $[\bar{\mu}^x - \bar{\mu}^{x+1} - \cdots - \bar{\mu}^1 - \bar{\mu}^0]$ are on a line. Therefore, betweenness—together with the claim above—yields

$$\delta_{\bar{P}}\big(\bar{\mu}^x, \bar{\mu}^0\big) = \sum_{t=0}^{x-1} \delta_{\bar{P}}\big(\bar{\mu}^t, \bar{\mu}^{t+1}\big) = x \times \frac{|\bar{N}|}{2}\alpha_{11}. \tag{13}$$

By the decomposition lemma, the distance between $\bar{\mu}^x$ and $\bar{\mu}^0$ can be decomposed as the sum of $\frac{|\bar{N}|}{2}$ one-couple matchings, each of the same length (x, x). Hence, $\delta_{\bar{P}}(\bar{\mu}^x, \bar{\mu}^0) = \frac{|\bar{N}|}{2}\alpha_{xx}$. Together with equation (13), $\frac{|\bar{N}|}{2}\alpha_{xx} = x \times \frac{|\bar{N}|}{2}\alpha_{11}$, which results in $\alpha_{xx} = x \times \alpha_{11}$. As α_{xx} is the same across all problems in the domain Lemma 3, this completes the proof of the lemma. □

B.4 *Proof of Proposition 1*

Proposition 1. *Let δ be a distance function, which satisfies betweenness, anonymity, monotonicity, independence of irrelevant newcomers, and standardization. Then, for any problem $P = (N, R)$ and any one-couple matching $\mu^{ij} \in \mathcal{M}(N)$, we have*

$$\delta_P\big(\mu^{ij}, \mu^I\big) = \frac{1}{2}\alpha_{11} \sum_{k\in\{i,j\}} \big|\text{rank}\big(\mu^{ij}(k), R_k\big) - \text{rank}\big(\mu^I(k), R_k\big)\big|.$$

Proof. Let μ^{ij} be any one-couple matching of length (x, y). By Lemma 3, $\alpha_{xy} = \alpha_{x1} + \alpha_{y1} - \alpha_{11}$. Also by Lemma 3, $\alpha_{xx} = \alpha_{x1} + \alpha_{x1} - \alpha_{11} = 2\alpha_{x1} - \alpha_{11}$, and by Lemma 4, $\alpha_{xx} = x\alpha_{11}$. Combining the two implies $\alpha_{x1} = \frac{(x+1)}{2}\alpha_{11}$. Setting $\alpha_{x1} = \frac{(x+1)}{2}\alpha_{11}$ and $\alpha_{y1} = \frac{(y+1)}{2}\alpha_{11}$ into $\alpha_{xy} = \alpha_{x1} + \alpha_{y1} - \alpha_{11}$ simplifies to

$$\alpha_{xy} = \frac{1}{2}\alpha_{11}(x + y).$$

Note that $x = |\mu^{ij}, \mu^I|_{R_i}$ and $y = |\mu^{ij}, \mu^I|_{R_j}$. Then we have $\alpha_{xy} = \frac{1}{2}\alpha_{11}(|\mu^{ij}, \mu^I|_{R_i} + |\mu^{ij}, \mu^I|_{R_j})$, which can be rearranged as

$$\alpha_{xy} = \frac{1}{2}\alpha_{11} \sum_{k\in\{i,j\}} \big|\text{rank}\big(\mu^{ij}(k), R_k\big) - \text{rank}\big(\mu^I(k), R_k\big)\big|. \tag{14}$$

□

B.5 *Proof of Proposition 2*

Proposition 2. *Scaled Borda distances satisfy betweenness, anonymity, monotonicity, independence of irrelevant newcomers, and standardization.*

Proof.

- Betweenness: Consider any problem $P = (N, R) \in \mathcal{D}$ and any three matchings $\mu, \bar{\mu}, \bar{\bar{\mu}} \in \mathcal{M}(N)$ such that $\bar{\mu}$ is between $\mu, \bar{\bar{\mu}}$. Note that by definition of the scaled Borda distances, for any $\sigma \in \mathbb{R}_{++}$, we have

$$\delta_P^{\sigma-\text{Borda}}(\mu, \bar{\mu}) + \delta_P^{\sigma-\text{Borda}}(\bar{\mu}, \bar{\bar{\mu}}) = \sigma\delta_P^{\text{Borda}}(\mu, \bar{\mu}) + \sigma\delta_P^{\text{Borda}}(\bar{\mu}, \bar{\bar{\mu}}).$$

The right-hand side is equivalent to

$$\sigma \sum_{i \in N} \left|\operatorname{rank}\left(\mu(i), R_i\right) - \operatorname{rank}\left(\bar{\mu}(i), R_i\right)\right| + \sigma \sum_{i \in N} \left|\operatorname{rank}\left(\bar{\mu}(i), R_i\right) - \operatorname{rank}\left(\bar{\bar{\mu}}(i), R_i\right)\right|.$$

This can then be merged as

$$\sigma \sum_{i \in N} \left(\left|\operatorname{rank}\left(\mu(i), R_i\right) - \operatorname{rank}\left(\bar{\mu}(i), R_i\right)\right| + \left|\operatorname{rank}\left(\bar{\mu}(i), R_i\right) - \operatorname{rank}\left(\bar{\bar{\mu}}(i), R_i\right)\right|\right).$$

As $\bar{\mu}$ is between μ and $\bar{\bar{\mu}}$, we have either $\operatorname{rank}(\mu(i), R_i) \leq \operatorname{rank}(\bar{\mu}(i), R_i) \leq \operatorname{rank}(\bar{\bar{\mu}}(i), R_i)$, or $\operatorname{rank}(\mu(i), R_i) \geq \operatorname{rank}(\bar{\mu}(i), R_i) \geq \operatorname{rank}(\bar{\bar{\mu}}(i), R_i)$. Therefore, the sign of each term in absolute values above must be the same. Hence, we can simplify the equation as follows:

$$\sigma \sum_{i \in N} \left(\left|\operatorname{rank}\left(\mu(i), R_i\right) - \operatorname{rank}\left(\bar{\bar{\mu}}(i), R_i\right)\right|\right),$$

which equals $\delta_P^{\sigma-\text{Borda}}(\mu, \bar{\bar{\mu}})$. Hence, we can conclude

$$\delta_P^{\sigma-\text{Borda}}(\mu, \bar{\mu}) + \delta_P^{\sigma-\text{Borda}}(\bar{\mu}, \bar{\bar{\mu}}) = \delta_P^{\sigma-\text{Borda}}(\mu, \bar{\bar{\mu}}).$$

- Anonymity: Note that relabeling the agents has no effect on the length of the intervals of relabeled matchings. Therefore, the scaled Borda distances satisfy anonymity.

- Monotonicity: Consider any two problems $P = (N, R) \in \mathcal{D}$ and $\widehat{P} = (N, \widehat{R}) \in \mathcal{D}$, and any two matchings $\mu, \bar{\mu} \in \mathcal{M}(N)$ as defined in the condition. Note that by definition of interval length, for all problems P and for all $i \in N$, we have $|\mu, \bar{\mu}|_{R_i} = |\operatorname{rank}(\mu(i), R_i) - \operatorname{rank}(\bar{\mu}(i), R_i)|$. By construction of P and $\widehat{P}$, we have $[\mu, \bar{\mu}]_{R_i} \subseteq [\mu, \bar{\mu}]_{\widehat{R}_i}$ for all $i \in N$, which implies $|\mu, \bar{\mu}|_{R_i} \leq |\mu, \bar{\mu}|_{\widehat{R}_i}$ for all $i \in N$. The latter is equivalent to $|\operatorname{rank}(\mu(i), R_i) - \operatorname{rank}(\bar{\mu}(i), R_i)| \leq |\operatorname{rank}(\mu(i), \widehat{R}_i) - \operatorname{rank}(\bar{\mu}(i), \widehat{R}_i)|$ for all $i \in N$. Note also that by definition of the scaled Borda distances, we have $\delta_P^{\sigma-\text{Borda}}(\mu, \bar{\mu}) = \sum_{i \in N} |\operatorname{rank}(\mu(i), R_i) - \operatorname{rank}(\bar{\mu}(i), R_i)|$, which in turn implies $\delta_P^{\sigma-\text{Borda}}(\mu, \bar{\bar{\mu}}) \leq \delta_{\widehat{P}}^{\sigma-\text{Borda}}(\mu, \bar{\bar{\mu}})$ for any $\sigma \in \mathbb{R}_{++}$.

- Independence of irrelevant newcomers: By construction, the newcomer in the new problem is single in both matchings. Therefore, the Borda scores of both matchings do not change in the new problem. This implies for any $\sigma \in \mathbb{R}_{++}$, the scaled Borda distances are unchanged.

- Standardization: Note that for any two disjoint matchings the minimal interval length for any individual is 1, i.e., the matches are adjacent in individuals preferences. Summing across all individuals leads to

$$\min_{P \in \mathcal{P}(N)} \delta_P^{\sigma-\text{Borda}}(\mu, \bar{\mu}) = \sigma \times |N|,$$

which concludes that $\kappa(N) = \sigma \times |N|$ for any $\sigma \in \mathbb{R}_{++}$. □

B.6 *Proof of Theorem 1*

Theorem 1. *A distance function δ satisfies betweenness, anonymity, monotonicity, independence of irrelevant newcomers, and standardization if and only if δ equals a scaled Borda distance when comparing the identity matching with others. That is, there exists $\sigma > 0$ such that for any problem $P = (N, R)$ and any $\mu \in \mathcal{M}(N)$, we have*

$$\delta_P(\mu, \mu^I) = \delta_P^{\sigma-\text{Borda}}(\mu, \mu^I).$$

Proof. Proposition 2 proves the "if part" for any two matchings. To show the "only-if" part for comparing any matching with the identity matching, consider the distance between any one-couple matching μ^{ij} of any length (x, y) and the identity matching μ^I. We can then plug equation (3) into Proposition 1, which yields

$$\delta_P(\mu^{ij}, \mu^I) = \alpha_{xy} = \frac{1}{2}\alpha_{11} \times \delta_P^{\text{Borda}}(\mu^{ij}, \mu^I).$$

Note that by the decomposition lemma, both for δ_P and δ_P^{Borda}, the distance between any μ and the identity matching μ^I is the sum of distances between the identity matching and all one-couple matchings induced by μ. Therefore,

$$\delta_P(\mu, \mu^I) = \frac{1}{2}\alpha_{11} \times \delta_P^{\text{Borda}}(\mu, \mu^I)$$

Finally, as δ_P is a metric function, $\alpha_{11} > 0$ (nonnegativity and the identity of indiscernibles). Setting $\sigma = \frac{1}{2}\alpha_{11} > 0$, we conclude the distance function is a scaled Borda distance with $\sigma = \frac{1}{2}\alpha_{11}$:

$$\delta_P(\mu, \mu^I) = \sigma \times \delta_P^{\text{Borda}}(\mu, \mu^I) = \delta_P^{\sigma-\text{Borda}}(\mu, \mu^I). \qquad \square$$

Appendix C: Proofs of Section 4.2

C.1 *Proof of Proposition 3*

Proposition 3. *Let δ be a distance function, which satisfies betweenness, anonymity, monotonicity, independence of irrelevant newcomers, and standardization. Consider a problem P over four agents with the preference profile and the matchings shown in Figure 6. Note that one singleton is nested between μ^1 and μ^2 and another is nested between μ^2 and μ^3. In such specific cases,*

(i) $\delta_P(\mu^1, \mu^2) = \delta_P^{\sigma-\text{Borda}}(\mu^1, \mu^2)$ *for* $\sigma = \frac{1}{2}\alpha_{11}$,

(ii) $\delta_P(\mu^2, \mu^3) = \delta_P^{\sigma-\text{Borda}}(\mu^2, \mu^3)$ *for* $\sigma = \frac{1}{2}\alpha_{11}$.

Proof. First, we show $\delta_P(\mu^1, \mu^2) = \delta_P(\mu^2, \mu^3)$, and then using this we prove the proposition. Consider the permutation $\pi = (23)$. Applying this permutation on P results in the problem P^π, which shown on the right-hand side of Figure 13.

Note that by anonymity we have $\delta_P(\mu^2, \mu^1) = \delta_{P^\pi}((\mu^2)^\pi, (\mu^1)^\pi)$. Furthermore, under the permutation π, $(\mu^1)^\pi = \mu^3$ and $(\mu^2)^\pi = \mu^2$, which implies $\delta_{P^\pi}((\mu^2)^\pi, (\mu^1)^\pi) =$

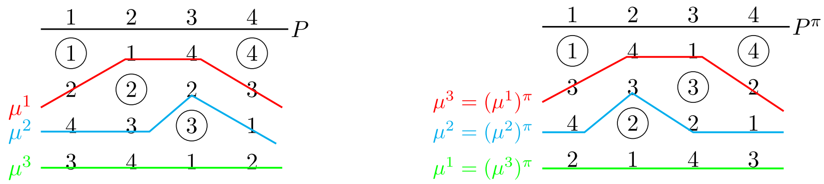

FIGURE 13. The original problem P in Proposition 3 (on the left) and the permuted problem P^π (on the right) after permuting with $\pi = (23)$.

$\delta_{P^\pi}(\mu^2, \mu^3)$. Note that by monotonicity for two problems P and P^π, we have $\delta_{P^\pi}(\mu^2, \mu^3) = \delta_P(\mu^2, \mu^3)$. Combining these equations and the fact that δ is a symmetric function, proves that $\delta_P(\mu^1, \mu^2) = \delta_P(\mu^2, \mu^3)$.

(i) Proving $\delta_P(\mu^1, \mu^2) = \delta_P^{\sigma-\text{Borda}}(\mu^1, \mu^2)$ for $\sigma = \frac{1}{2}\alpha_{11}$. Let $\bar{P}$, $\bar{\mu}^1$, $\bar{\mu}^2$, and $\bar{\mu}^3$ be extensions of P, μ^1, μ^2, and μ^3, respectively, by the set of agents $A = \{1', 2', 3', 4'\}$ (see Figure 14). By Remark 2, $\delta_P(\mu^1, \mu^2) = \delta_{\bar{P}}(\bar{\mu}^1, \bar{\mu}^2)$, $\delta_P(\mu^2, \mu^3) = \delta_{\bar{P}}(\bar{\mu}^2, \bar{\mu}^3)$, and $\delta_P(\mu^1, \mu^3) = \delta_{\bar{P}}(\bar{\mu}^1, \bar{\mu}^3)$. For simplicity, we abuse the notation and write P, μ^1, μ^2, and μ^3 instead of $\bar{P}$, $\bar{\mu}^1$, $\bar{\mu}^2$ and $\bar{\mu}^3$, respectively.

Consider also another problem $\widehat{P}$ shown in Figure 15.

Note that by monotonicity for two problems P and $\widehat{P}$ we have $\delta_P(\mu^1, \mu^2) = \delta_{\widehat{P}}(\mu^1, \mu^2)$, and $\delta_P(\mu^2, \mu^3) = \delta_{\widehat{P}}(\mu^2, \mu^3)$. Therefore, using the first part of the proposition, $\delta_{\widehat{P}}(\mu^1, \mu^2) = \delta_{\widehat{P}}(\mu^2, \mu^3)$. As in problem $\widehat{P}$, $[\mu^4 - \mu^1 - \mu^2 - \mu^3]$ are on a line, by betweenness $\delta_{\widehat{P}}(\mu^4, \mu^3) = \delta_{\widehat{P}}(\mu^4, \mu^1) + \delta_{\widehat{P}}(\mu^1, \mu^2) + \delta_{\widehat{P}}(\mu^2, \mu^3)$. Combining this with the previous equation implies

$$\delta_{\widehat{P}}(\mu^4, \mu^3) = \delta_{\widehat{P}}(\mu^4, \mu^1) + 2\delta_{\widehat{P}}(\mu^1, \mu^2)$$
$$\Rightarrow \quad \delta_{\widehat{P}}(\mu^1, \mu^2) = \frac{1}{2}\big(\delta_{\widehat{P}}(\mu^4, \mu^3) - \delta_{\widehat{P}}(\mu^4, \mu^1)\big) \tag{15}$$

Next, we show that the right-hand side of equation (15) equals $\delta_{\widehat{P}}^{\sigma-\text{Borda}}(\mu^1, \mu^2)$ for $\sigma = \frac{1}{2}\alpha_{11}$, i.e., $\sigma \times \delta_{\widehat{P}}^{\text{Borda}}(\mu^1, \mu^2)$. We do this by proving two claims for each of the terms on the right-hand side of equation (15).

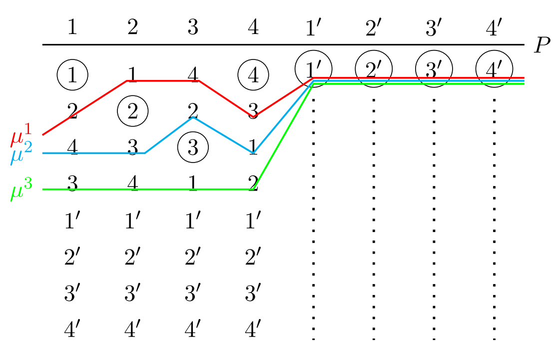

FIGURE 14. An extension of the problem P in Figure 6 by the set of agents $A = \{1', 2', 3', 4'\}$.

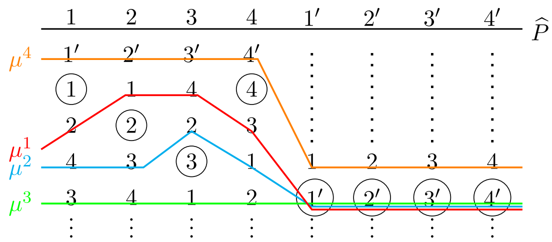

FIGURE 15. Problem $\widehat{P}$.

CLAIM 1. *$\delta_{\widehat{P}}(\mu^4, \mu^3) = \delta_{\widehat{P}}^{\sigma-\text{Borda}}(\mu^4, \mu^3)$ for $\sigma = \frac{1}{2}\alpha_{11}$.*

PROOF OF CLAIM 1. As in $\widehat{P}$, the identity matching is between μ^4 and μ^3, betweenness implies $\delta_{\widehat{P}}(\mu^4, \mu^3) = \delta_{\widehat{P}}(\mu^4, \mu^I) + \delta_{\widehat{P}}(\mu^I, \mu^3)$. Using Theorem 1, $\delta_{\widehat{P}}(\mu^4, \mu^I) = \sigma \times \delta_{\widehat{P}}^{\text{Borda}}(\mu^4, \mu^I)$ for $\sigma = \frac{1}{2}\alpha_{11}$ and $\delta_{\widehat{P}}(\mu^I, \mu^3) = \sigma \times \delta_{\widehat{P}}^{\text{Borda}}(\mu^I, \mu^3)$ for $\sigma = \frac{1}{2}\alpha_{11}$. With respect to this, and as δ^{Borda} satisfies betweenness, we have $\delta_{\widehat{P}}(\mu^4, \mu^3) = \delta_{\widehat{P}}^{\sigma-\text{Borda}}(\mu^4, \mu^3)$ where $\sigma = \frac{1}{2}\alpha_{11}$. □

CLAIM 2. *$\delta_{\widehat{P}}(\mu^4, \mu^1) = \delta_{\widehat{P}}^{\sigma-\text{Borda}}(\mu^4, \mu^1)$ for $\sigma = \frac{1}{2}\alpha_{11}$.*

PROOF OF CLAIM 2. To show this, consider the problem $\widetilde{P}$ shown in Figure 16. Note that, by monotonicity for two problems $\widehat{P}$ and $\widetilde{P}$ we have $\delta_{\widehat{P}}(\mu^4, \mu^1) = \delta_{\widetilde{P}}(\mu^4, \mu^1)$, and $\delta_{\widehat{P}}^{\text{Borda}}(\mu^4, \mu^1) = \delta_{\widetilde{P}}^{\text{Borda}}(\mu^4, \mu^1)$. Hence, it is sufficient to show $\delta_{\widetilde{P}}(\mu^4, \mu^1) = \delta_{\widetilde{P}}^{\sigma-\text{Borda}}(\mu^4, \mu^1)$ for $\sigma = \frac{1}{2}\alpha_{11}$.

To proceed, we show that $\delta_{\widetilde{P}}(\mu^4, \mu^1) = \delta_{\widetilde{P}}(\mu^1, \mu^5)$. Applying permutation $\pi = (12)(34)$, on $\widetilde{P}$ results in problem $\widetilde{P}^\pi$, which is shown in Figure 17.

Note that by anonymity we have $\delta_{\widetilde{P}}(\mu^1, \mu^4) = \delta_{\widetilde{P}^\pi}((\mu^1)^\pi, (\mu^4)^\pi)$. Furthermore, under the permutation π, $(\mu^1)^\pi = \mu^1$ and $(\mu^4)^\pi = \mu^5$, which implies $\delta_{\widetilde{P}^\pi}((\mu^1)^\pi, (\mu^4)^\pi) = \delta_{\widetilde{P}^\pi}(\mu^1, \mu^5)$. Note that by monotonicity for two problems $\widetilde{P}$ and $\widetilde{P}^\pi$ we have $\delta_{\widetilde{P}^\pi}(\mu^1, \mu^5) = \delta_{\widetilde{P}}(\mu^1, \mu^5)$, which shows $\delta_{\widetilde{P}}(\mu^1, \mu^4) = \delta_{\widetilde{P}}(\mu^1, \mu^5)$. Considering this and as in problem $\widetilde{P}$ matching μ^1 is between μ^4 and μ^5, we have

$$\delta_{\widetilde{P}}(\mu^4, \mu^1) = \delta_{\widetilde{P}}(\mu^1, \mu^5) = \frac{\delta_{\widetilde{P}}(\mu^4, \mu^5)}{2}. \tag{16}$$

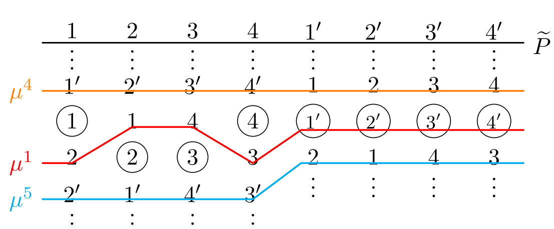

FIGURE 16. Problem $\widetilde{P}$.

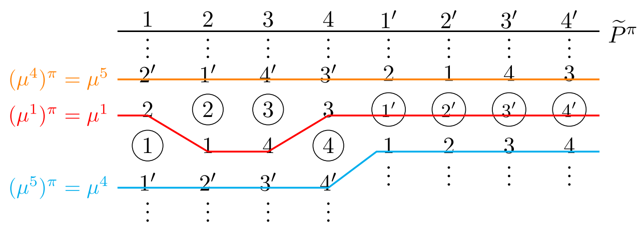

FIGURE 17. Problem $\widetilde{P}^\pi$ after permuting $\widetilde{P}$ in Figure 18 with $\pi = (12)(34)$.

Betweenness of μ^I in problem $\widetilde{P}$ yields $\delta_{\widetilde{P}}(\mu^4, \mu^5) = \delta_{\widetilde{P}}(\mu^4, \mu^I) + \delta_{\widetilde{P}}(\mu^I, \mu^5)$. By Theorem 1, $\delta_{\widetilde{P}}(\mu^4, \mu^I) = \sigma \times \delta^{\text{Borda}}_{\widetilde{P}}(\mu^4, \mu^I)$ for $\sigma = \frac{1}{2}\alpha_{11}$ and $\delta_{\widetilde{P}}(\mu^I, \mu^5) = \sigma \times \delta^{\text{Borda}}_{\widetilde{P}}(\mu^I, \mu^5)$ for $\sigma = \frac{1}{2}\alpha_{11}$. Therefore, $\delta_{\widetilde{P}}(\mu^4, \mu^5) = \frac{1}{2}\alpha_{11}(\delta^{\text{Borda}}_{\widetilde{P}}(\mu^4, \mu^I) + \delta^{\text{Borda}}_{\widetilde{P}}(\mu^I, \mu^5))$. Note that $\delta^{\text{Borda}}_{\widetilde{P}}(\mu^4, \mu^I) + \delta^{\text{Borda}}_{\widetilde{P}}(\mu^I, \mu^5) = 2\delta^{\text{Borda}}_{\widetilde{P}}(\mu^4, \mu^1)$, which with the previous equation implies $\delta_{\widetilde{P}}(\mu^4, \mu^5) = \alpha_{11}\delta^{\text{Borda}}_{\widetilde{P}}(\mu^4, \mu^1)$. Plugging this into equation (16) results in $\delta_{\widetilde{P}}(\mu^4, \mu^1) = \frac{1}{2}\alpha_{11}\delta^{\text{Borda}}_{\widetilde{P}}(\mu^4, \mu^1)$. Note that by monotonicity for two problems $\widehat{P}$ and $\widetilde{P}$ we have $\delta_{\widehat{P}}(\mu^4, \mu^1) = \delta_{\widetilde{P}}(\mu^4, \mu^1)$ and $\delta^{\text{Borda}}_{\widehat{P}}(\mu^4, \mu^1) = \delta^{\text{Borda}}_{\widetilde{P}}(\mu^4, \mu^1)$. Hence, $\delta_{\widehat{P}}(\mu^4, \mu^1) = \delta^{\sigma-\text{Borda}}_{\widehat{P}}(\mu^4, \mu^1)$ for $\sigma = \frac{1}{2}\alpha_{11}$, which completes the proof of the claim. □

Having proven the claims, we plug these back into equation (15), and

$$\begin{aligned}
\delta_{\widehat{P}}(\mu^1, \mu^2) &= \frac{1}{2}\big(\delta^{\sigma-\text{Borda}}_{\widehat{P}}(\mu^4, \mu^3) - \delta^{\sigma-\text{Borda}}\delta_{\widehat{P}}(\mu^4, \mu^1)\big) \\
&= \frac{1}{2}\left(\frac{1}{2}\alpha_{11}\delta^{\text{Borda}}_{\widehat{P}}(\mu^4, \mu^3) - \frac{1}{2}\alpha_{11}\delta^{\text{Borda}}\delta_{\widehat{P}}(\mu^4, \mu^1)\right) \\
&= \frac{1}{4}\alpha_{11}\big(\delta^{\text{Borda}}_{\widehat{P}}(\mu^4, \mu^3) - \delta^{\text{Borda}}\delta_{\widehat{P}}(\mu^4, \mu^1)\big) \\
&= \frac{1}{4}\alpha_{11}\big(\delta^{\text{Borda}}_{\widehat{P}}(\mu^1, \mu^3)\big) \\
&= \frac{1}{4}\alpha_{11}\big(2\delta^{\text{Borda}}_{\widehat{P}}(\mu^1, \mu^2)\big) \\
&= \frac{1}{2}\alpha_{11}\big(\delta^{\text{Borda}}_{\widehat{P}}(\mu^1, \mu^2)\big)
\end{aligned}$$

where the fourth and the fifth equations are due to betweenness of δ^{Borda}. Finally, by monotonicity for two problems P and $\widehat{P}$ we have $\delta_P(\mu^1, \mu^2) = \delta_{\widehat{P}}(\mu^1, \mu^2)$ and $\delta^{\sigma-\text{Borda}}_P(\mu^1, \mu^2) = \delta^{\sigma-\text{Borda}}_{\widehat{P}}(\mu^1, \mu^2)$. Therefore, $\delta_P(\mu^1, \mu^2) = \delta^{\sigma-\text{Borda}}_P(\mu^1, \mu^2)$ for $\sigma = \frac{1}{2}\alpha_{11}$, which completes the proof of the first part of the proposition.

(ii) Proving $\delta_P(\mu^2, \mu^3) = \delta^{\sigma-\text{Borda}}_P(\mu^2, \mu^3)$ for $\sigma = \frac{1}{2}\alpha_{11}$. As we proved $\delta_P(\mu^1, \mu^2) = \delta_P(\mu^2, \mu^3)$, and by the first part of this proposition and the fact that $\delta^{\text{Borda}}_P(\mu^1, \mu^2) = \delta^{\text{Borda}}_P(\mu^2, \mu^3)$, it can be easily concluded. □

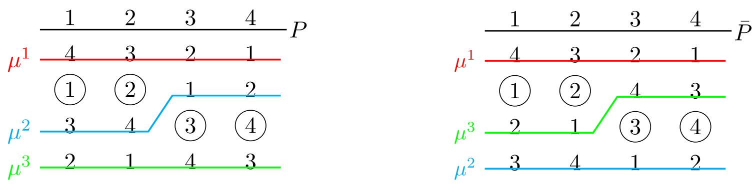

FIGURE 18. The original problem P of Proposition 4 (on the left) and problem $\bar{P}$ (on the right) after swapping the positions of μ^2 and μ^3 in problem P.

C.2 *Proof of Proposition 4*

PROPOSITION 4. *Let δ be a distance function, which satisfies betweenness, anonymity, monotonicity, independence of irrelevant newcomers, and standardization. Consider a problem P over four agents with the preference profile and the matchings shown in Figure 7. Note that two singletons are nested between μ^1 and μ^2 and another two are nested between μ^2 and μ^3. In such specific cases,*

(i) $\delta_P(\mu^2, \mu^3) = \delta_P^{\sigma-\text{Borda}}(\mu^2, \mu^3)$ *for* $\sigma = \frac{1}{2}\alpha_{11}$,

(ii) $\delta_P(\mu^1, \mu^2) = \delta_P^{\sigma-\text{Borda}}(\mu^1, \mu^2)$ *for* $\sigma = \frac{1}{2}\alpha_{11}$.

PROOF. Consider another problem $\bar{P}$ shown on the right-hand side of Figure 18. Note that by monotonicity for two problems P and $\bar{P}$ we have $\delta_P(\mu^2, \mu^3) = \delta_{\bar{P}}(\mu^2, \mu^3)$ and $\delta_P^{\text{Borda}}(\mu^2, \mu^3) = \delta_{\bar{P}}^{\text{Borda}}(\mu^2, \mu^3)$. Next, we show that for problem $\bar{P}$, $\delta_{\bar{P}}(\mu^3, \mu^2) = \delta_{\bar{P}}^{\sigma-\text{Borda}}(\mu^3, \mu^2)$ for $\sigma = \frac{1}{2}\alpha_{11}$.

CLAIM. $\delta_{\bar{P}}(\mu^3, \mu^2) = \delta_{\bar{P}}^{\sigma-\text{Borda}}(\mu^3, \mu^2)$ *for* $\sigma = \frac{1}{2}\alpha_{11}$.

PROOF OF CLAIM. Consider the permutation $\pi = (1324)$. Applying this permutation on $\bar{P}$ results in the problem $\bar{P}^\pi$, which is shown in Figure 19.

Note that by anonymity we have $\delta_{\bar{P}}(\mu^1, \mu^3) = \delta_{\bar{P}^\pi}((\mu^1)^\pi, (\mu^3)^\pi)$. Furthermore, under the permutation π, $(\mu^1)^\pi = \mu^2$ and $(\mu^3)^\pi = \mu^3$, which implies $\delta_{\bar{P}^\pi}((\mu^1)^\pi, (\mu^3)^\pi) = \delta_{\bar{P}^\pi}(\mu^2, \mu^3)$. Note that by monotonicity for two problems $\bar{P}$ and $\bar{P}^\pi$, we have $\delta_{\bar{P}^\pi}(\mu^2, \mu^3) = \delta_{\bar{P}}(\mu^2, \mu^3)$. Combining these equations and the fact that δ is a symmetric function proves that $\delta_{\bar{P}}(\mu^1, \mu^3) = \delta_{\bar{P}}(\mu^3, \mu^2)$. As in $\bar{P}$, μ^3 is between μ^1 and μ^2, betweenness implies $\delta_{\bar{P}}(\mu^1, \mu^2) = \delta_{\bar{P}}(\mu^1, \mu^3) + \delta_{\bar{P}}(\mu^3, \mu^2)$. This with the previous equation implies

$$\delta_{\bar{P}}(\mu^3, \mu^2) = \frac{\delta_{\bar{P}(\mu^1, \mu^2)}}{2} \tag{17}$$

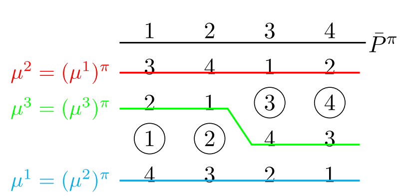

FIGURE 19. Permuted problem $\bar{P}^\pi$ after permuting problem $\bar{P}$ of Figure 18 with $\pi = (1324)$.

Now, as in problem $\bar{P}$ the identity matching is between μ^1 and μ^2, betweenness yields $\delta_{\bar{P}}(\mu^1,\mu^2)=\delta_{\bar{P}}(\mu^1,\mu^I)+\delta_{\bar{P}}(\mu^I,\mu^2)$. By Theorem 1, $\delta_{\bar{P}}(\mu^1,\mu^I)=\sigma\times\delta_{\bar{P}}^{\text{Borda}}(\mu^1,\mu^I)$ for $\sigma=\frac{1}{2}\alpha_{11}$ and $\delta_{\bar{P}}(\mu^I,\mu^2)=\sigma\times\delta_{\bar{P}}^{\text{Borda}}(\mu^I,\mu^2)$ for $\sigma=\frac{1}{2}\alpha_{11}$. Therefore,

$$\delta_{\bar{P}}(\mu^1,\mu^2)=\sigma\big(\delta_{\bar{P}}^{\text{Borda}}(\mu^1,\mu^I)+\delta_{\bar{P}}^{\text{Borda}}(\mu^I,\mu^2)\big)\quad\text{for }\sigma=\frac{1}{2}\alpha_{11}.\tag{18}$$

It can be verified that $\delta_{\bar{P}}^{\text{Borda}}(\mu^1,\mu^I)+\delta_{\bar{P}}^{\text{Borda}}(\mu^I,\mu^2)=\delta_{\bar{P}}^{\text{Borda}}(\mu^1,\mu^3)+\delta_{\bar{P}}^{\text{Borda}}(\mu^3,\mu^2)$, and that $\delta_{\bar{P}}^{\text{Borda}}(\mu^1,\mu^3)=\delta_{\bar{P}}^{\text{Borda}}(\mu^3,\mu^2)$. Replacing this into equation (18) implies $\delta_{\bar{P}}(\mu^1,\mu^2)=2\sigma\times\delta_{\bar{P}}^{\text{Borda}}(\mu^3,\mu^2)$ for $\sigma=\frac{1}{2}\alpha_{11}$. Plugging this into equation (17) yields $\delta_{\bar{P}}(\mu^3,\mu^2)=\sigma\times\delta_{\bar{P}}^{\text{Borda}}(\mu^3,\mu^2)$ for $\sigma=\frac{1}{2}\alpha_{11}$. This concludes the claim. □

(i) Proving that $\delta_P(\mu^2,\mu^3)=\delta_P^{\sigma-\text{Borda}}(\mu^2,\mu^3)$ for $\sigma=\frac{1}{2}\alpha_{11}$. Note that by monotonicity for two problems P and $\bar{P}$ we have $\delta_P(\mu^2,\mu^3)=\delta_{\bar{P}}(\mu^2,\mu^3)$. Replacing the latter using the above claim, we have $\delta_P(\mu^2,\mu^3)=\sigma\times\delta_{\bar{P}}^{\text{Borda}}(\mu^2,\mu^3)$ for $\sigma=\frac{1}{2}\alpha_{11}$. As $\delta_P^{\text{Borda}}(\mu^2,\mu^3)=\delta_{\bar{P}}^{\text{Borda}}(\mu^2,\mu^3)$, we have $\delta_P(\mu^2,\mu^3)=\sigma\times\delta_P^{\text{Borda}}(\mu^2,\mu^3)$ for $\sigma=\frac{1}{2}\alpha_{11}$, which concludes the first part of the proposition.

(ii) Proving that $\delta_P(\mu^1,\mu^2)=\delta_P^{\sigma-\text{Borda}}(\mu^1,\mu^2)$ for $\sigma=\frac{1}{2}\alpha_{11}$. As in problem P, the identity matching is between μ^1 and μ^3; betweenness implies $\delta_P(\mu^1,\mu^3)=\delta_P(\mu^1,\mu^I)+\delta_P(\mu^I,\mu^3)$. Using Theorem 1, $\delta_P(\mu^1,\mu^3)=\sigma\times\delta_P^{\text{Borda}}(\mu^1,\mu^3)$ for $\sigma=\frac{1}{2}\alpha_{11}$. As in problem P, μ^2 is between μ^1 and μ^3, betweenness implies $\delta_P(\mu^1,\mu^3)=\delta_P(\mu^1,\mu^2)+\delta_P(\mu^2,\mu^3)$. Together with the previous equation, we have $\sigma\times\delta_P^{\text{Borda}}(\mu^1,\mu^3)=\delta_P(\mu^1,\mu^2)+\delta_P(\mu^2,\mu^3)$ for $\sigma=\frac{1}{2}\alpha_{11}$. Replacing $\delta_P(\mu^2,\mu^3)$ with the first part of the proposition results in $\sigma\times\delta_P^{\text{Borda}}(\mu^1,\mu^3)=\delta_P(\mu^1,\mu^2)+\sigma\times\delta_P^{\text{Borda}}(\mu^2,\mu^3)$ for $\sigma=\frac{1}{2}\alpha_{11}$. Rearranging will result in $\delta_P(\mu^1,\mu^2)=\sigma\times\delta_P^{\text{Borda}}(\mu^1,\mu^3)-\sigma\times\delta_P^{\text{Borda}}(\mu^2,\mu^3)$ for $\sigma=\frac{1}{2}\alpha_{11}$, which by betweenness of the Borda distance equals $\delta_P(\mu^1,\mu^2)=\sigma\times\delta_P^{\text{Borda}}(\mu^1,\mu^2)$ for $\sigma=\frac{1}{2}\alpha_{11}$. This concludes the second part of the proposition. □

C.3 *Proof of Theorem 2*

Theorem 2. *A distance function δ satisfies betweenness, anonymity, monotonicity, independence of irrelevant newcomers, and standardization if and only if δ equals a scaled Borda distance. That is, there exists $\sigma>0$ such that for any problem $P=(N,R)$ and any $\mu,\tilde{\mu}\in\mathcal{M}(N)$, we have*

$$\delta_P(\mu,\tilde{\mu})=\delta_P^{\sigma-\text{Borda}}(\mu,\tilde{\mu}).$$

Proof. Proposition 2 proves the "if part" for any two matchings. To show the "only-if" part for comparing any two matchings, without loss of generality, let $N=\{1,2,\dots,n\}$ be the set of agents and consider any $P\in\mathcal{P}(N)$. In case $\mu=\tilde{\mu}$, as δ is a metric function, we have $\delta_P(\mu,\tilde{\mu})=0$, which equals $\delta_P^{\sigma-\text{Borda}}(\mu,\tilde{\mu})$ for any $\sigma>0$. In case $\mu=\mu^I$ (or $\tilde{\mu}=\mu^I$), by Theorem 1, we have $\delta_P(\mu,\tilde{\mu})=\delta_P^{\sigma-\text{Borda}}(\mu,\tilde{\mu})$ for $\sigma=\frac{1}{2}\alpha_{11}$. Next, we shall prove that

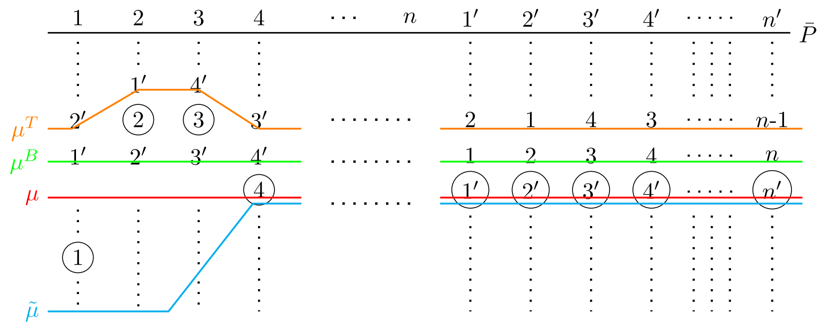

FIGURE 20. General structure for $\bar{P}$.

for any other possible pairs of matchings $\mu, \tilde{\mu} \in \mathcal{M}(N) \setminus \{\mu^I\}$ such that $\mu \neq \tilde{\mu}$, the distance also equals a scaled Borda distance with $\sigma = \frac{1}{2}\alpha_{11}$, i.e., $\delta_P(\mu, \tilde{\mu}) = \delta_P^{\sigma-\text{Borda}}(\mu, \tilde{\mu})$ for $\sigma = \frac{1}{2}\alpha_{11}$.

Note that if the number of agents is odd, we can use extensions of P, μ, and $\tilde{\mu}$ by one irrelevant newcomer. By independence of irrelevant newcomers, the distance would be unchanged. So, without loss of generality, we can assume that the number of agents to be even. Furthermore, by monotonicity, we can assume that μ is weakly above $\tilde{\mu}$.

Let $N' = \{1', 2', \dots, n'\}$ be a set of agents such that $|N| = |N'|$ and $N \cap N' = \emptyset$. Let $\bar{N} = N \cup N'$. Let P^*, μ^*, $\tilde{\mu}^*$ be an extension of P, μ, $\tilde{\mu}$ by the set N'. By Remark 2, $\delta_P(\mu, \tilde{\mu}) = \delta_{P^*}(\mu^*, \tilde{\mu}^*)$. For simplicity, we abuse the notation and write P, μ and $\tilde{\mu}$ instead of writing P^*, μ^*, and $\tilde{\mu}^*$, respectively. Let us define two additional matchings $\mu^B, \mu^T \in \mathcal{M}(\bar{N})$ such that: (1) for all $i \in N$, $\mu^B(i) = i' \in N'$, and (2) for all odd $i \in N$, $\mu^T(i) = (i+1)' \in N'$ and for all even $i \in N$, $\mu^T(i) = (i-1)' \in N'$.

Next, we construct another problem $\bar{P} = (\bar{N}, \bar{R})$ on the same set of agents $\bar{N}$ (see Figure 20 for a general view of the structure of this problem) such that:

(i) $[\mu, \tilde{\mu}]_{R_i} = [\mu, \tilde{\mu}]_{\bar{R}_i}$ for all $i \in \bar{N}$, i.e., the intervals of μ and $\tilde{\mu}$ in $\bar{P}$ are the same as those in P,

(ii) μ^T is weakly above μ^B, μ^B is weakly above μ (and they are adjacent), and μ is weakly above $\tilde{\mu}$,

(iii) if $i \in [\mu, \tilde{\mu}]_{\bar{R}_i}$, then $[\mu^T, \mu^B]_{\bar{R}_i} = \{\mu^T(i), \mu^B(i)\}$, i.e., if i is nested between $\mu(i)$ and $\tilde{\mu}(i)$, then no other agent is nested between $\mu^T(i)$ and $\mu^B(i)$,

(iv) if $i \notin [\mu, \tilde{\mu}]_{\bar{R}_i}$, then $[\mu^T, \mu^B]_{\bar{R}_i} = \{\mu^T(i), i, \mu^B(i)\}$, i.e., if i is not nested between $\mu(i)$ and $\tilde{\mu}(i)$, then i is the only other agent nested between $\mu^T(i)$ and $\mu^B(i)$.

Note that by monotonicity for problems P and $\bar{P}$, we have $\delta_P(\mu, \tilde{\mu}) = \delta_{\bar{P}}(\mu, \tilde{\mu})$. Also, $\delta_P^{\text{Borda}}(\mu, \tilde{\mu}) = \delta_{\bar{P}}^{\text{Borda}}(\mu, \tilde{\mu})$. Hence, it is sufficient to show $\delta_{\bar{P}}(\mu, \tilde{\mu}) = \sigma \times \delta_{\bar{P}}^{\text{Borda}}(\mu, \tilde{\mu})$ for $\sigma = \frac{1}{2}\alpha_{11}$.

Note that $[\mu^T - \mu^B - \mu - \tilde{\mu}]$ are on a line in problem $\bar{P}$, therefore, betweenness implies $\delta_{\bar{P}}(\mu^T, \tilde{\mu}) = \delta_{\bar{P}}(\mu^T, \mu^B) + \delta_{\bar{P}}(\mu^B, \mu) + \delta_{\bar{P}}(\mu, \tilde{\mu})$, and hence,

$$\delta_{\bar{P}}(\mu, \tilde{\mu}) = \delta_{\bar{P}}(\mu^T, \tilde{\mu}) - \delta_{\bar{P}}(\mu^T, \mu^B) - \delta_{\bar{P}}(\mu^B, \mu). \tag{19}$$

In the next three steps, we show that the distance between each of the three pairs of matchings on the right-hand side of equation (19) equals the scaled Borda distance for some $\sigma > 0$. By betweenness of scaled Borda distances, this in return shall imply $\delta_{\bar{P}}(\mu, \tilde{\mu}) = \sigma \times \delta_{\bar{P}}^{\text{Borda}}(\mu, \tilde{\mu})$ for some $\sigma > 0$.

Step 1. (Proving that $\delta_{\bar{P}}(\mu^T, \tilde{\mu})$ equals the scaled Borda distance for some $\sigma > 0$.) By construction of $\bar{P}$, $[\mu^T - \mu^I - \tilde{\mu}]$ are on a line. Then by betweenness and Theorem 1, $\delta_{\bar{P}}(\mu^T, \tilde{\mu}) = \sigma \times \delta_{\bar{P}}^{\text{Borda}}(\mu^T, \tilde{\mu})$ for $\sigma = \frac{1}{2}\alpha_{11}$.

Step 2. (Proving that $\delta_{\bar{P}}(\mu^B, \mu)$ equals the scaled Borda distance for some $\sigma > 0$.) By construction of $\bar{P}$, we can consider any problem $\bar{\bar{P}}$ where $[\mu^B - \mu - \mu^I]$ are on a line, and the intervals of μ^B and μ are unchanged, i.e., $[\mu^B, \mu]_{\bar{R}_i} = [\mu^B, \mu]_{\bar{\bar{R}}_i}$ for all $i \in \bar{N}$, therefore, by monotonicity the distance is unchanged. Then by betweenness and Theorem 1, $\delta_{\bar{P}}(\mu^B, \mu) = \sigma \times \delta_{\bar{P}}^{\text{Borda}}(\mu^B, \mu)$ for $\sigma = \frac{1}{2}\alpha_{11}$.

Step 3. (Proving that $\delta_{\bar{P}}(\mu^T, \mu^B)$ equals the scaled Borda distance for some $\sigma > 0$.) Consider the partition of $\bar{N}$ into the following subsets of agents $T_1 = \{1, 2, 1', 2'\}$, $T_2 = \{3, 4, 3', 4'\}, \dots, T_{\frac{n}{2}} = \{n-1, n, (n-1)', n'\}$ where $\bar{N} = \bigcup_{l=1}^{\frac{n}{2}} T_l$. Let μ^{T_l} denote a matching where $\mu^{T_l}(i) = \mu^T(i)$ for all $i \in T_l$, and $\mu^{T_l}(i) = \mu^B(i)$ for all $i \in \bar{N} \setminus T_l$. By construction, for all $l \in \{1, \dots, \frac{n}{2}\}$, μ^{T_l} is between μ^T and μ^B. By the decomposition lemma, we have

$$\delta_{\bar{P}}(\mu^T, \mu^B) = \sum_{l=1}^{\frac{n}{2}} \delta_{\bar{P}}(\mu^{T_l}, \mu^B). \tag{20}$$

To simplify notation, we denote a generic μ^{T_l} simply by μ^S. Based on the construction of μ^T and μ^B, each of these matchings, μ^S, will have one of the following three structures: (1) *no singleton* is nested between μ^S and μ^B (see Figure 21), or (2) *one singleton* is nested between μ^S and μ^B (see Figures 22 and 25), or (3) *two singletons* are nested between μ^S and μ^B (see Figure 26). In the sequel, we shall show that for each of the three possible structures, $\delta_{\bar{P}}(\mu^S, \mu^B) = \sigma \times \delta_{\bar{P}}^{\text{Borda}}(\mu^S, \mu^B)$ for $\sigma = \frac{1}{2}\alpha_{11}$, i.e., the distance is a scaled Borda distance.

- *Case 1. (no singleton)* Consider the case in which no singleton is nested between μ^S and μ^B (Figure 21). By construction of $\bar{P}$, we can consider any problem $\bar{\bar{P}}$ where $[\mu^S - \mu^B - \mu^I]$ are on a line, and the intervals of μ^S and μ^B are unchanged, i.e., $[\mu^S, \mu^B]_{\bar{R}_i} = [\mu^S, \mu^B]_{\bar{\bar{R}}_i}$ for all $i \in \bar{N}$, therefore, by monotonicity the distance is unchanged. Then by betweenness and Theorem 1, $\delta_{\bar{P}}(\mu^S, \mu^B) = \sigma \times \delta_{\bar{P}}^{\text{Borda}}(\mu^S, \mu^B)$ for $\sigma = \frac{1}{2}\alpha_{11}$.

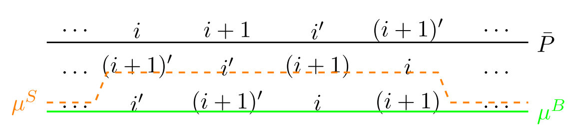

FIGURE 21. The no singleton structure.

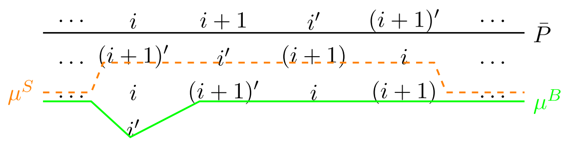

FIGURE 22. The one singleton structure with i as the singleton.

- *Case 2. (one singleton)* Consider the case in which one singleton is nested between μ^S and μ^B. By construction of μ^T and μ^B the singleton is either i or $i+1$. Therefore, two situations are plausible:

  I. i is the singleton nested (see Figure 22). Consider the four agent problem P in Proposition 3, and rename the agents as $2 = i$, $4 = i+1$, $3 = i'$, and $1 = (i+1)'$. Let $\widehat{P}$ be an extension of this problem P, by the set of agents $A = \bar{N} \setminus \{i, i', (i+1), (i+1)'\}$, and $\widehat{\mu}^1$ and $\widehat{\mu}^2$ be the extension of μ^1 and μ^2 by the set A, respectively (see Figure 23). By Remark 2,

$$\delta_P(\mu^1, \mu^2) = \delta_{\widehat{P}}(\widehat{\mu}^1, \widehat{\mu}^2). \tag{21}$$

  Now, consider another problem P' shown in Figure 24. Monotonicity implies

$$\delta_{\widehat{P}}(\widehat{\mu}^1, \widehat{\mu}^2) = \delta_{P'}(\widehat{\mu}^1, \widehat{\mu}^2). \tag{22}$$

  Note that the structure of the four matchings, $\widehat{\mu}^1$, $\widehat{\mu}^2$, $\mu^S\mu$, in problem P' corresponds to the four matchings in Figure 3 (to $\bar{\mu}$, $\bar{\mu}^S$, μ^S, μ^B, respectively). Therefore, by equation (7) in the decomposition lemma we have

$$\delta_{P'}(\widehat{\mu}^1, \widehat{\mu}^2) = \delta_{P'}(\mu^S, \mu^B). \tag{23}$$

  Putting equations (21), (22), and (23) together results in $\delta_P(\mu^1, \mu^2) = \delta_{P'}(\mu^S, \mu^B)$. Note that by monotonicity for problems P' and $\bar{P}$, we have $\delta_{P'}(\mu^S, \mu^B) = \delta_{\bar{P}}(\mu^S, \mu^B)$. Combining these two equations yield

$$\delta_P(\mu^1, \mu^2) = \delta_{\bar{P}}(\mu^S, \mu^B). \tag{24}$$

  By Proposition 3, we have $\delta_P(\mu^1, \mu^2) = \sigma \times \delta_P^{\text{Borda}}(\mu^1, \mu^2)$ for $\sigma = \frac{1}{2}\alpha_{11}$, which also equals $\sigma \times \delta_{\bar{P}}^{\text{Borda}}(\mu^S, \mu^B)$ for $\sigma = \frac{1}{2}\alpha_{11}$. Plugging the last term back into the left-hand side of equation (24) yields $\delta_{\bar{P}}(\mu^S, \mu^B) = \sigma \times \delta_{\bar{P}}^{\text{Borda}}(\mu^S, \mu^B)$ for $\sigma = \frac{1}{2}\alpha_{11}$.

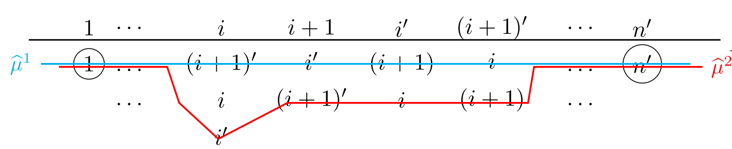

FIGURE 23. The four agents problem P of Proposition 3 after adding the set of agents $A = \bar{N} \setminus \{i, i', (i+1), (i+1)'\}$, as irrelevant newcomers.

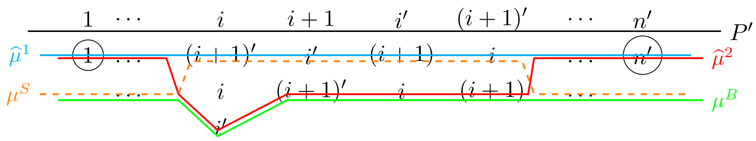

FIGURE 24. The problem P'.

II. $(i+1)$ is the singleton nested (see Figure 25). Renaming the agents in Proposition 3 as $4 = i$, $2 = i+1$, $1 = i'$, and $3 = (i+1)'$ and using a similar argument as above yield $\delta_{\bar{P}}(\mu^S, \mu^B) = \sigma \times \delta_{\bar{P}}^{\text{Borda}}(\mu^S, \mu^B)$ for $\sigma = \frac{1}{2}\alpha_{11}$.

- *Case 3. (two singletons)* Consider the case in which two singletons are nested between μ^S and μ^B. By the construction of μ^T and μ^B, only i and $i+1$ can be the singletons (see Figure 26). Renaming the agents in Proposition 4 as $1 = i$, $2 = i+1$, $3 = i'$, $4 = (i+1)'$, and using a similar argument as above, where only i was single, yield $\delta_{\bar{P}}(\mu^S, \mu^B) = \sigma \times \delta_{\bar{P}}^{\text{Borda}}(\mu^S, \mu^B)$ for $\sigma = \frac{1}{2}\alpha_{11}$.

Plugging the results of the three cases above into equation (20) yield

$$\delta_{\bar{P}}(\mu^T, \mu^B) = \sigma \times \sum_{l=1}^{\frac{n}{2}} \delta_{\bar{P}}^{\text{Borda}}(\mu^{T_l}, \mu^B) \quad \text{for } \sigma = \frac{1}{2}\alpha_{11}. \tag{25}$$

As Borda distance satisfies the conditions, and by the decomposition lemma the right-hand side of equation (25) can be rearranged as

$$\delta_{\bar{P}}(\mu^T, \mu^B) = \sigma \times \delta_{\bar{P}}^{\text{Borda}}(\mu^T, \mu^B) \quad \text{for } \sigma = \frac{1}{2}\alpha_{11}.$$

Finally, combining all three steps for $\delta_{\bar{P}}(\mu^T, \tilde{\mu})$, $\delta_{\bar{P}}(\mu^T, \mu^B)$, and $\delta_{\bar{P}}(\mu^B, \mu)$ into equation (19) yield

$$\delta_{\bar{P}}(\mu, \tilde{\mu}) = \sigma \times \left(\delta_{\bar{P}}^{\text{Borda}}(\mu^T, \tilde{\mu}) - \delta_{\bar{P}}^{\text{Borda}}(\mu^T, \mu^B) - \delta_{\bar{P}}^{\text{Borda}}(\mu^B, \mu)\right) \quad \text{for } \sigma = \frac{1}{2}\alpha_{11}.$$

By betweenness of scaled Borda distances and symmetry, the right-hand side of the equation above reduces to $\sigma \times \delta_{\bar{P}}^{\text{Borda}}(\mu, \tilde{\mu})$, and hence, $\delta_{\bar{P}}(\mu, \tilde{\mu}) = \sigma \times \delta_{\bar{P}}^{\text{Borda}}(\mu, \tilde{\mu})$ for $\sigma = \frac{1}{2}\alpha_{11}$.

Note that by monotonicity for problems P and $\bar{P}$, we have $\delta_P(\mu, \tilde{\mu}) = \delta_{\bar{P}}(\mu, \tilde{\mu})$. Also, $\delta_P^{\text{Borda}}(\mu, \tilde{\mu}) = \delta_{\bar{P}}^{\text{Borda}}(\mu, \tilde{\mu})$. Therefore, with respect to the previous equation, we have $\delta_P(\mu, \tilde{\mu}) = \sigma \times \delta_P^{\text{Borda}}(\mu, \tilde{\mu})$ for $\sigma = \frac{1}{2}\alpha_{11}$. □

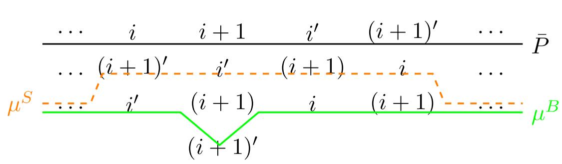

FIGURE 25. The one singleton structure with $i+1$ as the singleton.

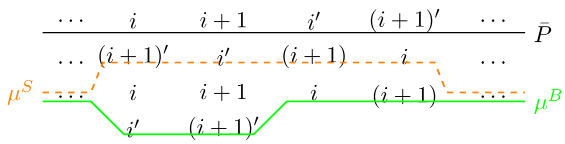

FIGURE 26. The two-singleton structure with both i and $i+1$ as the singleton agents.

APPENDIX D: LOGICAL INDEPENDENCE OF THE CONDITIONS

In this section, we discuss the logical independence of the conditions used in the characterization, by presenting different distances, which satisfy every condition except one of them.

D.1 *Betweenness*

For any N and for any $P \in \mathcal{P}(N)$, the following rule satisfies everything except betweenness:

$$\delta_P^B(\mu, \bar{\mu}) = \left|\left\{i \in N : \mu(i) \neq \bar{\mu}(i)\right\}\right|$$

The example in Figure 27 shows δ^B violates the betweenness condition.

It is easy to verify that $\delta_P^B(\mu, \bar{\mu}) = 2$, $\delta_P^B(\bar{\mu}, \bar{\bar{\mu}}) = 4$, and $\delta_P^B(\mu, \bar{\bar{\mu}}) = 4$; however, according to betweenness, we must have $\delta_P^B(\mu, \bar{\bar{\mu}}) = 6$. It is easy to see that this rule satisfies anonymity, monotonicity, independence of irrelevant newcomers, and standardization.

D.2 *Anonymity*

Let the set of potential agents $\mathcal{N}$ equal to natural numbers that is $\mathbb{N}$, then for any N and for any $P \in \mathcal{P}(N)$ the following rule satisfies everything except anonymity:

$$\delta_P^A(\mu, \bar{\mu}) = 2 \times \sum_{i \in O} |\mu, \bar{\mu}|_{R_i} + \sum_{i \in E} |\mu, \bar{\mu}|_{R_i},$$

where O denotes the set of odd numbered agents and E denotes the set of even numbered agents.

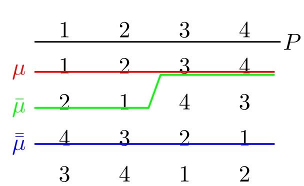

FIGURE 27. An example of violation of betweenness condition by δ^B.

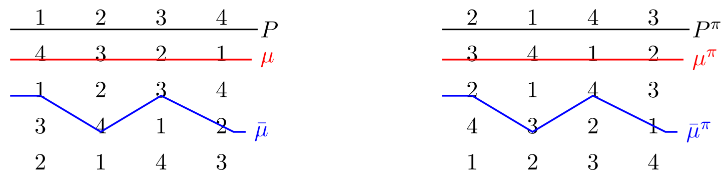

FIGURE 28. An example of violation of anonymity condition by δ^A.

The example shown in Figure 28 shows δ^A violates the anonymity condition.

It is easy to check that $\delta^M_P(\mu, \bar{\mu}) = 2(1+1) + (2+2) = 8$; however, after applying the permutation $\pi = (12)(34)$, we have $\delta^A_{P^\pi}(\mu, \bar{\mu}) = 2(2+2) + (1+1) = 10$. It is easy to see that this rule satisfies betweenness, monotonicity, independence of irrelevant newcomers, and standardization.

D.3 *Monotonicity*

For any N and for any $P \in \mathcal{P}(N)$, the following rule satisfies everything except monotonicity:

$$\delta^M_P(\mu, \bar{\mu}) = \sum_{i \in N} \left| 2^{\text{rank}(\mu(i), R_i)} - 2^{\text{rank}(\bar{\mu}(i), R_i)} \right|$$

The example shown in Figure 29 shows δ^M violates the monotonicity condition.

It is obvious that $[\mu, \bar{\mu}]_{R_i} \subseteq [\mu, \bar{\mu}]_{\widehat{R}_i}$, for $i \in \{1, 2, 3\}$; however, $\delta^M_P(\mu, \bar{\mu}) = 3 \times |2^2 - 2^3| = 12$ and $\delta^M_{\widehat{P}}(\mu, \bar{\mu}) = |2^1 - 2^3| + |2^1 - 2^2| + |2^1 - 2^2| = 10$, which violates monotonicity.

To see that δ^M satisfies betweenness, let P be a problem and $\bar{\mu}$ be such that it is between μ and $\bar{\bar{\mu}}$. We have

$$\begin{aligned}
\delta^M_P(\mu, \bar{\mu}) + \delta^M_P(\bar{\mu}, \bar{\bar{\mu}}) &= \sum_{i \in N} \left| 2^{\text{rank}(\mu(i), R_i)} - 2^{\text{rank}(\bar{\mu}(i), R_i)} \right| + \sum_{i \in N} \left| 2^{\text{rank}(\bar{\mu}(i), R_i)} - 2^{\text{rank}(\bar{\bar{\mu}}(i), R_i)} \right| \\
&= \sum_{i \in N} \left(\left| 2^{\text{rank}(\mu(i), R_i)} - 2^{\text{rank}(\bar{\mu}(i), R_i)} \right| + \left| 2^{\text{rank}(\bar{\mu}(i), R_i)} - 2^{\text{rank}(\bar{\bar{\mu}}(i), R_i)} \right| \right) \\
&= \sum_{i \in N} \left(\left| 2^{\text{rank}(\mu(i), R_i)} - 2^{\text{rank}(\bar{\bar{\mu}}(i), R_i)} \right| \right) \\
&= \delta^M_P(\mu, \bar{\bar{\mu}})
\end{aligned}$$

Note that the third equality is due to the fact that for each $i \in N$ we have either $\mu(i) R_i \bar{\mu}(i)$ or $\bar{\mu}(i) R_i \mu(i)$. Therefore, $\text{rank}(\mu(i), R_i) \leq \text{rank}(\bar{\mu}(i), R_i) \leq \text{rank}(\bar{\bar{\mu}}(i), R_i)$, or

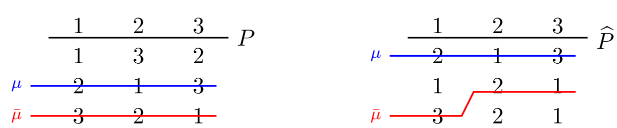

FIGURE 29. An example of violation of monotonicity condition by δ^M.

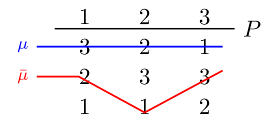

FIGURE 30. An example of violation of independence of irrelevant newcomers condition by δ^I.

$\text{rank}(\mu(i), R_i) \geq \text{rank}(\bar{\mu}(i), R_i) \geq \text{rank}(\bar{\bar{\mu}}(i), R_i)$. Hence, in the second equation both absolute values have the same sign, which allows to conclude the third equation. It is easy to see that this rule satisfies anonymity, independence of irrelevant newcomers, and standardization.

D.4 *Independence of irrelevant newcomers*

To show that the independence of irrelevant newcomers is logically independent from other conditions, we first define the set of matchings that are between two given matchings.

The following rule satisfies everything except independence of irrelevant newcomers:

$$\delta^I_P(\mu, \bar{\mu}) = \begin{cases} 3, & \text{if } |N| = 3 \text{ and } \mu, \bar{\mu} \text{ are disjoint and} \\ & \text{there is no matching between } \mu, \bar{\mu} \\ \delta^{\text{Borda}}_P(\mu, \bar{\mu}), & \text{otherwise} \end{cases}$$

The example shown in Figure 30 shows δ^I violates the independence of irrelevant newcomers condition.

As $|N| = 3$, and μ and $\bar{\mu}$ are disjoint and there is no other matching between them, we have $\delta^I_P(\mu, \bar{\mu}) = 3$. Now assume that an irrelevant newcomer joins, and hence, $|N| = 4$, which results in $\delta^I_P(\mu, \bar{\mu}) = \delta^{\text{Borda}}_P(\mu, \bar{\mu}) = 4$. It is easy to verify that the rule satisfies anonymity, monotonicity, betweenness, and standardization.

D.5 *Standardization*

Let N be any set of agents and $P \in \mathcal{P}(N)$. First, we define the following sets for any two matchings $\mu, \bar{\mu} \in \mathcal{M}(N)$. Let Γ be the set of agents such that they are single in only one of the matchings, i.e., $\Gamma(\mu, \bar{\mu}) = \{i \in N | [\mu(i) = i \text{ or } \bar{\mu}(i) = i] \text{ and } [\mu(i) \neq \bar{\mu}(i)]\}$. Also, let Ω be the set of agents that are single in the strict interval between μ and $\bar{\mu}$. Formally, $\Omega(\mu, \bar{\mu}) = \{i \in N | i \in [\mu, \bar{\mu}]_{R_i} \text{ and } i \neq \mu(i) \text{ and } i \neq \bar{\mu}(i)\}$. Consider the following rule:

$$\delta^S_P(\mu, \bar{\mu}) = \delta^{\text{Borda}}_P(\mu, \bar{\mu}) - \frac{1}{2}\big|\Gamma(\mu, \bar{\mu})\big| - \big|\Omega(\mu, \bar{\mu})\big|.$$

To show that δ^S violates standardization, let $N = \{1, 2, 3, 4\}$ and consider the matchings $\mu = \{(1, 4), (3, 2)\}$, $\bar{\mu} = \{(1, 2), (3, 4)\}$. By the above distance, $\min_{P \in \mathcal{P}(N)} \delta^S_P(\mu, \bar{\mu}) = 4$ and $\min_{P \in \mathcal{P}(N)} \delta^S_P(\mu, \mu^I) = 2$. Hence, standardization fails.

To show that δ^S satisfies betweenness, let $\mu, \bar{\mu}, \bar{\bar{\mu}} \in \mathcal{M}(N)$ be such that $\bar{\mu}$ is between μ and $\bar{\bar{\mu}}$. Let $\bar{s}$ be the number of agents that are single in $\bar{\mu}$ but not in μ or $\bar{\bar{\mu}}$. Then the following equation holds:

$$|\Omega(\mu, \bar{\bar{\mu}})| = \bar{s} + |\Omega(\mu, \bar{\mu})| + |\Omega(\bar{\mu}, \bar{\bar{\mu}})| \tag{26}$$

That is the number of agents that are single in the strict interval of μ and $\bar{\bar{\mu}}$ equals to the number of agents that are single in the strict interval of μ and $\bar{\mu}$ plus the number of agents that are single in the strict interval of $\bar{\mu}$ and $\bar{\bar{\mu}}$ plus those agents that are only single in $\bar{\mu}$. Also, it is straightforward to see that

$$|\Gamma(\mu, \bar{\mu})| + |\Gamma(\bar{\mu}, \bar{\bar{\mu}})| = |\Gamma(\mu, \bar{\bar{\mu}})| + 2\bar{s} \tag{27}$$

We have

$$\begin{aligned}
&\delta_P^S(\mu, \bar{\mu}) + \delta_P^S(\bar{\mu}, \bar{\bar{\mu}}) \\
&\quad = \delta_P^{\text{Borda}}(\mu, \bar{\mu}) - \frac{1}{2}|\Gamma(\mu, \bar{\mu})| - |\Omega(\mu, \bar{\mu})| + \delta_P^{\text{Borda}}(\bar{\mu}, \bar{\bar{\mu}}) - \frac{1}{2}|\Gamma(\bar{\mu}, \bar{\bar{\mu}})| - |\Omega(\bar{\mu}, \bar{\bar{\mu}})| \\
&\quad = \delta_P^{\text{Borda}}(\mu, \bar{\bar{\mu}}) - \frac{1}{2}|\Gamma(\mu, \bar{\mu})| - \frac{1}{2}|\Gamma(\bar{\mu}, \bar{\bar{\mu}})| - |\Omega(\mu, \bar{\mu})| - |\Omega(\bar{\mu}, \bar{\bar{\mu}})| \\
&\quad \overset{\text{by}}{\underset{(27)}{=}} \delta_P^{\text{Borda}}(\mu, \bar{\bar{\mu}}) - \frac{1}{2}|\Gamma(\mu, \bar{\bar{\mu}})| - \bar{s} - |\Omega(\mu, \bar{\mu})| - |\Omega(\bar{\mu}, \bar{\bar{\mu}})| \\
&\quad \overset{\text{by}}{\underset{(26)}{=}} \delta_P^{\text{Borda}}(\mu, \bar{\bar{\mu}}) - \frac{1}{2}|\Gamma(\mu, \bar{\bar{\mu}})| - \bar{s} - |\Omega(\mu, \bar{\bar{\mu}})| + \bar{s} \\
&\quad = \delta_P^{\text{Borda}}(\mu, \bar{\bar{\mu}}) - \frac{1}{2}|\Gamma(\mu, \bar{\bar{\mu}})| - |\Omega(\mu, \bar{\bar{\mu}})| \\
&\quad = \delta_P^S(\mu, \bar{\bar{\mu}})
\end{aligned}$$

where the second equation holds as δ^{Borda} satisfies betweenness. It is easy to see that δ^S satisfies anonymity, monotonicity, and independence of irrelevant newcomers.

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