Asset Bubbles and Product Market Competition

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February 28, 2023

Abstract

This paper studies the interplay between asset bubbles and product market competition. It offers two main insights. The first is that imperfect competition creates a wedge between interest rates and the marginal product of capital. This makes rational bubbles possible even when there is no overaccumulation of capital. The second is that, when providing a production subsidy, bubbles stimulate competition and reduce monopoly rents. I show that bubbles can destroy efficient investment and have ambiguous welfare consequences. However, when they stimulate competition, they can have crowding-in effects on capital.

JEL Classification: E44, L13, L16

Key words: rational bubbles, overaccumulation, competition, market power

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1 Introduction

"With valuations based on multiples of revenue, there's ample incentive to race for growth, even at the cost of low or even negative gross margins."

('Dotcom history is not yet repeating itself, but it is starting to rhyme', *Financial Times, 12/March/2015)*

Stock markets often experience fluctuations that seem too large to be driven entirely by fundamentals. Major historical events include the Mississippi and the South Sea bubbles of 1720 or the British railway mania of the 1840s. A more recent example is that of the US stock market during the dotcom bubble: between October 1995 and March 2000, the NASDAQ Composite index increased by almost sixfold to then collapse by 77% in the following two years. One common aspect among these episodes is that they appear to be concentrated on a particular industry, and to bring about a surge in competition.[1] The dotcom bubble constitutes a good example in this regard. In a period characterized by soaring prices of technology stocks, many internet firms had an IPO and entered the stock market. Furthermore, as the valuation of firms is typically based on metrics of size (revenues or market shares) and not on earnings, some of these firms sought rapid growth and engaged in aggressive commercial practices, such as unusually low penetration prices. For example, some online companies offered their services for free (e.g. *Kozmo.com* or *UrbanFetch*) or made money payments to consumers (e.g. *AllAdvantage.com*).[2]

The idea that the dotcom bubble was associated with a more competitive market structure is corroborated by indicators of market power. Figure 1 shows average price-cost markups for four high-tech industries that were at the center of the bubble. These are shown against the Shiller CAPE ratio, which is a popular measure for stock market overvaluation. A common pattern can be detected in these four industries — average markups decline from 1995 until the peak of the bubble in 2000/2001, and start increasing after the stock market crash. These patterns could be observed for both the full sample of firms (green line) and for the set of firms already active in 1995 (dashed red line).

[1] The Mississippi and the South Sea bubbles involved two trading companies (the *Compagnie d'Occident* in France and the *South Sea Company* in Great Britain) that engaged in innovative financial schemes; the *railway mania* involved the British railway industry; the dotcom bubble was concentrated on a group of internet and high-tech industries.

[2] Even if following unsustainable business models, the new *dotcoms* often posed a threat to incumbents, which were in many cases forced to react. Some well-known examples involve GE or Microsoft (Queirós (2021)).

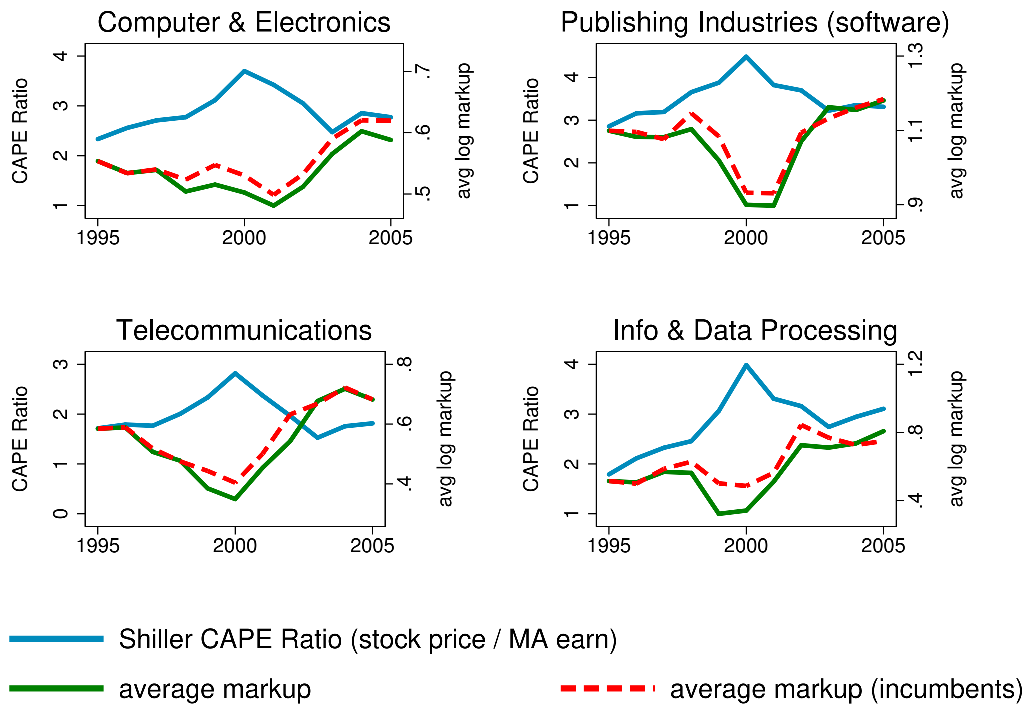


Figure 1: Average markups in the dotcom bubble

This figure shows the Shiller CAPE ratio and average (unweighted) markups for four industries during 1995-2005: 'Computer & Electronic Product Manufacturing' (NAICS 334), 'Publishing Industries (software)' (NAICS 511), 'Telecommunications' (NAICS 517), and 'Information & Data Processing' (NAICS 518-519). The CAPE ratio is the ratio of total stock market capitalization to a 10–year moving average of past earnings (EBITDA); the ratio is in logs and is measured at the beginning of the year. Markups are the ratio of sales to variable operating costs ('Cost of the Goods Sold'); the ratio is in logs. Incumbents are firms that were active in 1995.

Motivated by these observations, I investigate the interactions between asset bubbles and product market competition. I present a model featuring imperfect competition and rational bubbles, which builds upon the classical OLG economy of Diamond (1965). Individuals live for two periods. They work when young and have to decide how much to consume and save for retirement. They can save by investing in capital, or by purchasing bubbles. The production side of the model consists of a multi-industry economy. In every industry, there is a productive firm that faces competition from a fringe of unproductive competitors. This firm can charge a (limit) price above its marginal cost, hence enjoying market power and making some monopoly rents.

Individuals can trade two types of bubbly assets. One is an asset that is issued outside the corporate sector, which I label *government debt*. This asset will sustain a set of intergenerational transfers, and will not have an impact on the industry market structure. The second is an asset that is issued by firms, which I label *bubbly stocks*. Importantly, the rents that firms can obtain when issuing bubbly stocks depend on their size, so that larger firms can issue a larger amount of bubbles.[3] These assets will thus provide firms with the incentives to increase production, at the expense of markups and monopoly rents. Insofar as they reduce monopoly rents in a given industry, bubbly stocks will have a *pro-competitive effect* and correct a market failure. However, I show that if they are sufficiently large, these bubbles can generate situations of excessive production, with firms charging prices below their marginal cost. The model can thus explain the prevalence of low markups exhibited by high-tech firms in the dotcom bubble (as suggested by Figure 1), or examples of overinvestment in the British railway mania (Campbell and Turner (2015)).

Considering the general equilibrium properties of the model, the existence of a price-cost markup creates a wedge between factor prices and marginal products. In particular, interest rates will be below the (aggregate) marginal product of capital. This has two main consequences. First, rational bubbles can be traded even when there is no overaccumulation of capital. Under certain conditions, bubbles may crowd-out efficient investment, and be detrimental for welfare. Thus, the classical equivalence result of Tirole (1985) — between the existence of a bubbly equilibrium, overaccumulation of capital and Pareto inefficiency — is not necessarily satisfied in this economy. Second, the economy can be characterized by underinvestment. Since interest rates do not reflect the efficiency of investment, individuals may opt to save too little — and consume too much — when young. In such a case, the equilibrium can be shown to feature excessive first-period consumption and insufficient investment. However, if issued by the corporate sector and being *pro-competitive*, bubbles can increase the aggregate demand for investment; they can reduce first-period consumption and lead to a welfare-improving increase in the capital stock.

While in the baseline model I consider a simple game of limit pricing, I also provide an extension where firms compete via quantities and are subject to fixed costs. In this context, bubbly stocks can provide an entry subsidy and have an impact on the extensive margin of firms (and again result in more intense competition).

[3] This assumption is meant to capture one aspect of valuation techniques, namely the fact that they are often based on metrics of size (such as market shares) and not on profits. See section 2 for a discussion.

Related Literature This paper is mostly related to the literature that forms the theory of rational bubbles. The seminal contributions of Samuelson (1958) and Tirole (1985) explore the role of bubbles as a store of value, and show that they can be Pareto improving. Being a store of value, bubbles can also be a liquidity instrument as in Farhi and Tirole (2012), Miao and Wang (2012), Hirano and Yanagawa (2017) and Xavier (2022). A different strand of the literature has put an emphasis on the appearance of new bubbles: the formation of a new pyramid scheme provides a rent that can have economic consequences. In this category, Olivier (2000) shows that, if attached to R&D firms, bubbles stimulate growth. Martin and Ventura (2012, 2016) argue that the creation of new bubbles allows credit-constrained entrepreneurs to expand investment. Tang and Zhang (2022) study how bubbles affects the firm productivity distribution.[4]

My theory can be related to the class of models studying rational bubbles in the presence of financial frictions (Farhi and Tirole (2012), Martin and Ventura (2012, 2016), Hirano and Yanagawa (2017), Ikeda and Phan (2019)). This literature has shown that credit constraints can create a wedge between interest rates and the marginal product of capital, and that bubbles can increase investment. My model differs from this strand of the literature in two main ways. First, my focus is on frictions in product markets, not in financial markets. Second, previous models fail to explain how overvaluation can generate overinvestment and negative profits. These were important aspects of the dotcom bubble and other episodes (Haacke (2004)).

Finally, this paper is related to the literature studying the aggregate consequences of market power (Chatterjee et al. (1993), Jaimovich and Floetotto (2008), Ferrari and Queirós (2023) among others). One insight of my model is that markups create a wedge between interest rates and the marginal product of capital. A similar result has been contemporaneously shown by Eggertsson et al. (2019), Ball and Mankiw (2021) and Aguiar et al. (2021). In addition to this, a contribution of my paper is to show that bubbles can stimulate competition and have crowding-in effects on capital accumulation. I also show that bubbles can generate situations of excessive investment when they are large and/or the degree of market power is low.

The rest of the paper is organized as follows. Section 2 describes the model and characterizes the equilibrium of a single industry. Section 3 discusses the general equilibrium and the conditions for the existence of bubbles. Section 4 concludes.

[4] Recent contributions include the quantitative models of Larin (2020) and Guerron-Quintana et al. (2022). Galí (2014), Biswas et al. (2020) and Asriyan et al. (2021) study the interactions between bubbles and monetary policy.

2 The Model

2.1 Demographics and Preferences

Time is discrete and runs forever: $t = 0, 1, 2, \ldots$. The economy is populated by two overlapping generations, each of which has measure one. Each individual $j \in [0, 1]$ born at t has utility

$$U_{jt} = c^y_{jt} + \beta\, c^o_{j,t+1}, \tag{1}$$

where c^y_{jt} and $c^o_{j,t+1}$ represent young-age and old-age consumption, and β is the discount factor. The linear utility function is chosen for analytical convenience. In Appendix A, I show that the central results of this paper hold under more generic preferences. I will also assume that $\beta > 1$.[5]

Assumption 1. *The discount factor satisfies $\beta > 1$.*

Individuals supply one unit of labor when young, and earn a wage W_t. They can save by purchasing financial assets (such as government debt) which deliver a gross return R_{t+1}. When old, they run a firm in the corporate sector and earn income $\chi^o_{j,t+1}$ (which includes the profits that they make as entrepreneurs and lump sum transfers that they may receive from the government). They thus face the budget constraint

$$c^o_{j,t+1} = R_{t+1}\left(W_t - c^y_{jt}\right) + \chi^o_{j,t+1}. \tag{2}$$

The problem of each young individual i is to maximize (1) subject to (2). Denoting her savings level by $s_{jt} := W_t - c^y_{jt}$, the solution to this problem yields

$$s_{jt} \begin{cases} = W_t & \text{if } R_{t+1} > \dfrac{1}{\beta} \\ \in [0,\, W_t] & \text{if } R_{t+1} = \dfrac{1}{\beta} \end{cases}. \tag{3}$$

If the interest rate is greater than the inverse of the discount factor, young individuals save all their income. When the two are identical, the young are indifferent between saving and consuming in their first period of life. As it will be clear below, the equilibrium interest rate cannot be lower than the inverse of the discount factor.

[5] As shown below, this implies that rational bubbles can be traded when young individuals have positive consumption in a steady-state $\left(c^{y*}_j > 0\right)$.

2.2 Technology

There is a final good Y_t, which is a CES composite of different varieties

$$Y_t = \left(\int_0^1 y_{it}^{\rho} \, di \right)^{1/\rho} \quad , \; 0 < \rho < 1, \tag{4}$$

where y_{it} is the quantity of variety $i \in [0,1]$ and $\sigma := (1-\rho)^{-1} > 1$ is the elasticity of substitution. The final good is produced in a competitive sector and is chosen as the *numeraire*. The demand for each variety i is given by

$$p_{it} = \left(\frac{Y_t}{y_{it}} \right)^{1-\rho}. \tag{5}$$

Entrepreneur $j \in [0,1]$ can produce variety $i \in [0,1]$ by means of a Cobb-Douglas technology $F_{ij}(k,l) = z_{ij} \, A \, k^{\alpha} \, l^{1-\alpha}$. The term z_{ij} represents the idiosyncratic productivity of individual j in the production of variety i, while A is a common productivity component. Labor is hired at the competitive wage W_t. Capital needs to be invested one period ahead and fully depreciates in production. Each unit of capital used at t therefore costs R_t. Given these assumptions, an entrepreneur with productivity $z_{ij} \, A$ can produce variety i with unit cost $\theta_t / \left(z_{ij} \, A \right)$, where

$$\theta_t := \left(\frac{R_t}{\alpha} \right)^{\alpha} \left(\frac{W_t}{1-\alpha} \right)^{1-\alpha}$$

is the factor price frontier for a Cobb-Douglas technology with unit productivity.

Imperfect competition will arise because of an unequal distribution of idiosyncratic productivities. In particular, I assume that

$$z_{ij} = \begin{cases} 1 & \text{if} \quad j = i \qquad \text{(leader)} \\ \gamma \in [\rho, 1] & \text{if} \quad j \neq i \qquad \text{(followers)} \end{cases}. \tag{6}$$

I refer to entrepreneur j as the *leader* of industry $i = j$ and to all other entrepreneurs $j \neq i$ as the *followers*. The crucial aspect of (6) is that, in every industry $i \in [0,1]$, there is only one individual with access to the best technology.[6]

[6] The idiosyncratic productivity level z_{ij} cannot be transferred across individuals. This assumption, and the fact that there are no costs to set up firms, together imply that current entrepreneurs cannot sell claims to profits that will be generated by other entrepreneurs in the future.

2.3 Bubbleless Industry Equilibrium

I assume that firms compete à la Bertrand. Given that $\gamma \geq \rho$, the leader must set a limit price equal to the followers' marginal cost[7]

$$p_{it} = \frac{1}{\gamma} \frac{\theta_t}{A}. \tag{7}$$

The parameter γ is equal to the inverse of the markup, and can thus be seen as a measure of competition. In the particular case of $\gamma = 1$, the model features perfect competition in product markets. The price in (7) is the equilibrium price of good i when firms are not overvalued. As I show below, when firms have the possibility of issuing overvalued stocks, they may find it optimal to charge a price below (7). To conclude the discussion of the bubbleless industry equilibrium, I shall briefly characterize optimal policy interventions.

Optimal Policy A regulatory authority intervening in industry i would like to ensure that the leader produces a quantity consistent with marginal cost pricing, i.e. $p_{it} = \theta_t / A$. There are different ways of implementing this outcome. One possibility is to grant the followers with an *ad valorem* subsidy equal to $\phi^F := 1/\gamma - 1$, so that the followers effectively obtain $p_{it}/\gamma \geq p_{it}$ per each unit they sell. Given this subsidy, the followers will produce at any price $p_{it} > (\theta_t / A)$ and so the leaders will be forced to charge $p_{it} = (\theta_t / A)$.

2.4 Asset Bubbles

In this economy, agents can trade two types of rational bubbles. The first is an asset that is issued outside the corporate sector. As an illustration, I will consider a government debt scheme that is rolled over forever. The second is an asset that is issued by the corporate sector. The assumptions we make about how this asset is issued and distributed across firms can change the equilibrium in goods markets, and in particular the limit price chosen by the leaders.

Government Debt Suppose that there is a government that can issue one-period debt, to be rolled over forever. Let D_t be the funds raised by the government in period t. I assume that

$$D_t \;=\; R_t \, D_{t-1} + d_t \quad \text{with} \quad d_t \geq 0. \tag{8}$$

[7]When $\gamma < \rho$, the leaders charge the desired monopoly price $p_{it}^M = \rho^{-1} (\theta_t / A) < \gamma^{-1} (\theta_t / A)$.

According to this formulation, the government is capable of issuing an amount of debt D_t that is sufficient to cover previous debt repayments $R_t\, D_{t-1}$. I assume that $d_t \geq 0$ (i.e. the funds raised in excess of debt repayments) are distributed to the old generation as a lump sum transfer.

Bubbly Stocks Bubbles can also be initiated by the corporate sector. As an example of a bubble issued by firms, one can think of a stock that never pays any dividend or cash-flow (but which is still traded at a positive price). Let B_{it} be the value of all bubbly stocks issued by firms in industry i until time t. I assume that it evolves according to

$$B_{it} = \underbrace{R_t\, B_{i,t-1}}_{\substack{\text{return on} \\ \text{past bubbles}}} + \underbrace{b_{it}}_{\substack{\text{new} \\ \text{bubbles}}} \quad \text{with} \quad b_{it} \geq 0. \tag{9}$$

According to equation (9), the time t value of all bubbly stocks issued in industry i has two components. The first is the return on bubbly stocks that were issued in the past $(R_t\, B_{i,t-1})$; no arbitrage implies that, in equilibrium, this rate of return coincides with the rate of return on capital. The second represents the value of new stocks issued by firms in industry i at time t (b_{it}). An important assumption to make concerns how these new bubbles are distributed across firms. One can assume, for example, that b_{it} is equally split across firms (leader and followers), independently of whether they produce or not. In such a case, the industry equilibrium would be unchanged.[8] This assumption seems however unsatisfactory, since it implies that the followers can issue stocks, even if producing nothing. I will be therefore making two assumptions.

1. The total amount of new stocks that industry i as a whole can issue is exogenously determined by financial markets and equal to $b_{it} \geq 0$.[9]

2. This value is split across firms according to market shares, so that firm j issues an amount of bubbly stocks equal to $b_{ijt} = (y_{jit}/y_{it})\, b_{it}$.

According to this formulation, investors have a demand $b_{it} \geq 0$ for new stocks issued by industry i. Thus, current firms can sell a total amount of securities that exceeds their fundamental value by b_{it}.[10] Furthermore, larger firms can issue a greater amount of bubbly stocks. This second

[8]This is no longer the case when firms are subject to fixed costs, as in the extension considered in Appendix B.

[9]I assume free disposal of bubbles, which rules out $b_{it} < 0$.

[10]Since a firm is only active for one period, its fundamental value is equal to the profits that it currently generates. Firms can also sell claims to these one-period profits, but have no incentives to do so.

assumption captures one aspect of financial markets — namely, the fact that valuation models are often based on multiples of revenues or market shares, and not on profits.[11] For instance, Hong et al. (2007) provide evidence that equity analysts offering valuations for Amazon in the 1997-1999 period tended to emphasize its sales growth and highly disregarded operating margins. A well-known consequence of these valuation methods is that they induce firms to boost revenues and market shares, at the expense of profits (Aghion and Stein (2008)).

These assumptions have consequences on the industry equilibrium. Since firms get $p_{it} + b_{it}/y_{it}$ per each unit that they sell, the leader must charge a limit price such that

$$p_{it} + \frac{b_{it}}{y_{it}} = \frac{\theta_t}{\gamma A}. \tag{10}$$

This limit price can be shown to be decreasing in b_{it}, as stated in Proposition 1.[12]

Proposition 1. *(Limit Price with Bubbles) The limit price in* (10) *decreases in* b_{it} *(for fixed* Y_t *and* θ_t*).*

Proof. From (5), y_{it} decreases in p_{it}. Using (10), it follows that p_{it} decreases in b_{it}. ∎

A corollary of Proposition 1 is that bubbles stimulate competition: they induce the leader to reduce the markup $\mu_{it} := p_{it}/(\theta_t/A)$ and the level of production profits, or monopoly rents, $\pi_{it} := (p_{it} - \theta_t/A)\, y_{it}$. Figure 2 shows the industry price, quantity and production profits as a function of b_i (time subscripts are omitted). There is a value of b_i for which the leader sets a price equal to marginal cost ($p_i = \theta/A$). In this case, the market provides a substitute for the optimal production subsidy discussed above. However, if b_i becomes larger, the price p_i will fall short of the leader's marginal cost, making (production) profits negative. Therefore, sufficiently large bubbles can lead to a situation of excessive production and profit losses, as it was documented in the British railway mania, the dotcom bubble and other episodes. For example, as noted in the context of the recent Silicon Valley boom: "*With valuations based on multiples of revenue, there's ample incentive to race for growth, even at the cost of low or even negative gross margins. The many taxi apps and instant delivery services (...) are facing huge pressure to cut prices*".[13]

[11] These valuation techniques are especially used for young firms: typically they start with low or even negative earnings, which makes it difficult to project future cash flows from current earnings. See Damodaran (2006).

[12] If industry bubbles are distributed according to capital or employment shares (and not sales shares), the limit price satisfies $p_{it} + b_{it}/(\gamma y_{it}) = \theta_t/(\gamma A)$.

[13] "Dotcom history is not yet repeating itself but it is starting to rhyme" (03/12/2015), Financial Times

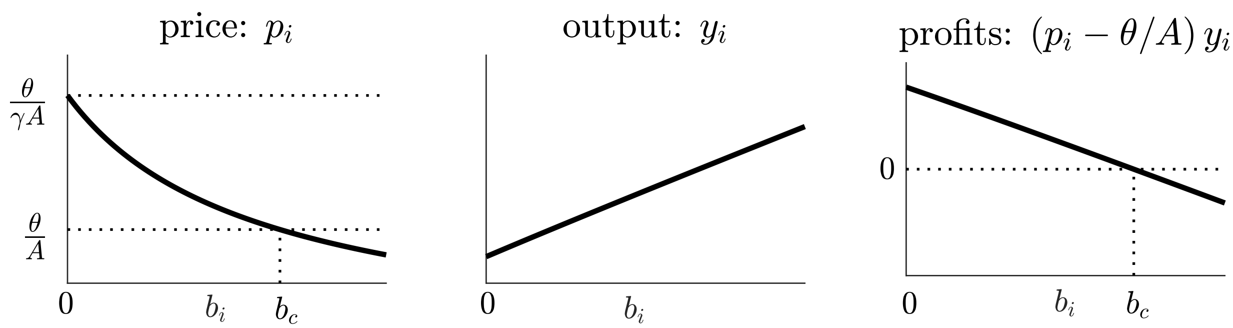


Figure 2: Industry equilibrium with bubbly stocks

In the present model, bubbles can have an impact on production variables because their distribution across firms depends on size/market shares. This forces the leaders to set a lower price, in order to keep a high market share. Appendix B considers an extension where firms also have fixed costs. In this case, bubbles can provide firms with an entry subsidy, and thus affect the industry equilibrium even when they are not distributed according to size.

3 General Equilibrium

In this section, I solve for the general equilibrium. I will focus on symmetric equilibria in which all industries are characterized by identical levels of new bubble issuance $b_{it} = b\ \forall t$. This ensures that industries will be identical and characterized by the same prices $p_{it} = p_t$ and quantities $y_{it} = y_t$. Denoting by L_t and K_t the aggregate stocks of labor and capital, we have $L_t := \int_0^1 l_{it}\, di = l_{it}$ and $K_t := \int_0^1 k_{it}\, di = k_{it}$. Similarly, denoting by B_t the aggregate value of bubbly stocks, we have $B_t := \int_0^1 B_{it}\, di = B_{it}$. I also assume that there is no uncertainty.

Definition. An equilibrium consists of a non-negative sequence for aggregate bubbles (government debt and stocks), capital, labor and consumption $\{ d_t, b_t, D_t, B_t, K_t, L_t, C_t^y, C_t^o \}_{t=0}^{\infty}$ and prices $\{ W_t, R_t, p_t \}_{t=0}^{\infty}$ such that (i) individuals optimize, (ii) the leaders set prices given by (10), (iii) government debt and bubbly stocks evolve according to (8) and (9) (where d_t and b_t are exogenous), and (iv) labor and capital markets clear, i.e.

$$L_t = 1 \tag{11}$$

and

$$K_{t+1} = W_t - \left(D_t + B_t + C_t^y \right). \tag{12}$$

To facilitate the exposition, in subsection 3.1, I start by characterizing the general equilibrium without bubbles. I characterize the conditions for the existence of a bubbly equilibrium and of capital overaccumulation. Then, in subsection 3.2, I describe the aggregate equilibrium with government debt and with bubbly stocks.

3.1 General Equilibrium without Bubbles

Aggregate Output Using the fact that all industries are symmetric and that $L_t = 1$, aggregate output can be written as

$$Y_t \;=\; A\,K_t^{\alpha} \;:=\; f\left(K_t\right). \tag{13}$$

Equilibrium Factor Prices When no bubbles are traded, equilibrium factor prices are given by

$$\begin{aligned} W_t &= \gamma\,\left(1-\alpha\right)\,A\,K_t^{\alpha}, \\ R_t &= \gamma\,\alpha\,A\,K_t^{\alpha-1}. \end{aligned} \tag{14}$$

The parameter γ is the inverse of the markup, and corresponds to the aggregate factor share (i.e. the ratio of total labor and interest payments to aggregate output).

Capital Dynamics and Steady-State When no bubbles are traded, equilibrium in the capital market requires that aggregate investment is equal to aggregate savings,

$$K_{t+1} = W_t - C_t^y. \tag{15}$$

Combining the previous equation with equations (3) and (14), we find an expression for the dynamics of capital

$$K_{t+1} = \min\left\{\gamma\left(1-\alpha\right)AK_t^{\alpha},\, \left(\beta\gamma\alpha A\right)^{1/(1-\alpha)}\right\}. \tag{16}$$

To understand (16), note that the equilibrium interest rate cannot fall short of $1/\beta$. In the first region, K_t is low enough so that the young save all their wage, convert it into capital and obtain a return $R_{t+1} \geq 1/\beta$. In the second region, K_t is sufficiently high so that one would observe $R_{t+1} < 1/\beta$ if the young were to convert all their labor income into capital. Therefore, when no bubbles are traded, the economy converges to a steady-state

$$K^{*}=\left(\gamma A\min\left\{\beta\alpha,\,1-\alpha\right\}\right)^{1/(1-\alpha)},\tag{17}$$

with an associated interest rate

$$R^{*}=\max\left\{\frac{\alpha}{1-\alpha},\,\frac{1}{\beta}\right\}.\tag{18}$$

When $\alpha/\left(1-\alpha\right)>1/\beta$, the steady-state features $R^{*}>1/\beta$ and the young save all their wage. When $\alpha/\left(1-\alpha\right)<1/\beta$, the economy converges to a steady-state with $R^{*}=1/\beta$ where the young only save part of their wage. There are two aspects that are worth highlighting about equations (17) and (18). First, K^{*} increases in the degree of competition γ. To understand this result, note that γ represents the aggregate factor share. When $R^{*}>1/\beta$, the young save all their wage; a higher γ implies that a greater fraction of output is distributed to the young as wages. When $R^{*}=1/\beta$, a higher γ allows the economy to keep the same interest rate with a greater capital stock K^{*}. Second, the steady-state interest rate is independent of γ. This happens because the degree of competition γ has a dual role on interest rates, as equation (14) highlights. On the one hand, a higher γ results in a higher capital share and hence a greater R_t for the same K_t. On the other hand, it also results in a greater K^{*}, which implies a lower R^{*} (because of decreasing returns). With linear preferences, these two effects exactly cancel out. However, as shown in Appendix A.1, under general CRRA utility, the steady-state interest rate depends on γ.

Negative Interest Rates The bubbleless equilibrium described above is unique when rational bubbles cannot be traded. Rational bubbles can be traded whenever the steady-state net interest rate is negative ($r^{*}:=R^{*}-1<0$). As the next proposition states, this happens if and only if the capital elasticity α is sufficiently low.

Proposition 2. *($r^{*}<0$) The bubbleless steady-state features a negative net interest rate $r^{*}:=R^{*}-1$ if and only if*

$$\alpha<\frac{1}{2}.$$

Proof. Using (18) and given that $\beta>1$, it follows that $r^{*}:=R^{*}-1<0$ if and only if $\alpha<1/2$. ■

Capital Overaccumulation In Tirole (1985), the condition for $r^{*}<0$ coincides with the conditions for capital overaccumulation and Pareto inefficiency. As I show below, such an equivalence holds in this model when markets are characterized by perfect competition ($\gamma=1$). However, under imperfect competition ($\gamma < 1$), the equivalence result of Tirole (1985) may not hold. For example, one may have $r^* < 0$ even when there is no overaccumulation of capital. Additionally, the equilibrium may be Pareto inefficient even when there is no overaccumulation of capital.

To clarify these points, I start by providing a definition of capital overaccumulation. The *golden rule* capital stock, or the capital stock that maximizes aggregate consumption in a steady-state, $C^* = f(K^*) - K^*$, is given by

$$f'(K_{GR}) = 1$$

or, equivalently,

$$K_{GR} := (\alpha A)^{1/(1-\alpha)}.$$

The economy will be characterized by overaccumulation of capital when it converges to a steady-state characterized by $f'(K^*) < 1$.

Definition. (Overaccumulation of Capital) The bubbleless steady-state K^* features overaccumulation of capital if $f'(K^*) < 1$.

If the steady-state is characterized by overaccumulation, the equilibrium is Pareto inefficient, since it is possible to increase aggregate consumption by reducing the capital stock. A Pareto improvement can be obtained by a policy that implements a set of transfers from the young to the old. Proposition 3 states the conditions for capital overaccumulation.

Proposition 3. *(Overaccumulation of Capital) The bubbleless steady-state features overaccumulation of capital if and only if*

$$\gamma > \max\left\{\frac{\alpha}{1-\alpha}, \frac{1}{\beta}\right\}.$$

Proof. From the steady-state equation (17), we have $f'(K^*) = (\beta\gamma)^{-1}$ when $\alpha/(1-\alpha) < 1/\beta$, and $f'(K^*) = \alpha/[\gamma(1-\alpha)]^{-1}$ otherwise. ■

Underinvestment The existence of market power can result in underinvestment. Old-age income has two components: the return on savings $R_{t+1}K_{t+1}$ plus an additional component that is distributed in a lump-sum fashion (namely, monopoly rents π_{t+1}). When young individuals make their consumption and savings decisions, they only take into account the market return on capital, $R_{t+1} = \gamma f'(K_{t+1})$. They do not internalize the fact that, by increasing capital K_{t+1}, they also raise total output and hence total profits for all individuals, $\pi_{t+1} = (1-\gamma) f(K_{t+1})$. As a result, the aggregate level of investment can be suboptimal, as stated in the next proposition.[14]

Proposition 4. *(Underinvestment) The bubbleless steady-state features underinvestment if and only if*

$$\frac{2\gamma-1}{\gamma} < \alpha < \frac{1}{1+\beta}.$$

In this case, the welfare of any given generation t can be increased if all the young individuals of that generation increase investment (K_{t+1}).

Proof. First, note that the bubbleless steady-state features young-age consumption if and only if $\alpha < 1/(1+\beta)$. Total utility for a young individual is $U_t = W_t - K_{t+1} + \beta\left[1-\gamma\left(1-\alpha\right)\right] AK_{t+1}^{\alpha}$. If all the young individuals at t marginally increase investment K_{t+1}, the change in utility is

$$-1+\beta\left[1-\gamma\left(1-\alpha\right)\right]\alpha AK_{t+1}^{\alpha-1} = \gamma^{-1} - \left(2-\alpha\right),$$

where $R_{t+1} = \gamma\alpha AK_{t+1}^{\alpha-1} = 1/\beta$ was used. This is positive if and only if $\left(2\gamma-1\right)/\gamma < \alpha$. ■

Note that the conditions of Propositions 3 and 4 can be simultaneously satisfied. In other words, raising the capital stock can generate a Pareto improvement also when $f'\left(K^*\right) < 1$. This happens because the savings decisions of the young have a positive externality on all individuals of the same generation. Thus, when the conditions of Propositions 3 and 4 are jointly satisfied, a Pareto improvement can be obtained in two alternative ways. In one, all the young increase capital accumulation. In the other, the young reduce capital accumulation, but receive a transfer in the following period (from the new young). Proposition 10 in Appendix A shows that underinvestment also arises under more generic preferences.

Pareto Efficiency Proposition 5 states the conditions under which the equilibrium is Pareto inefficient. As the proposition shows, the decentralized and bubbleless equilibrium is inefficient whenever (i) there is overaccumulation of capital or (ii) when there is first period consumption in a steady-state.

[14] I assume that all the profits $\pi_{t+1} = \left(1-\gamma\right) f\left(K_{t+1}\right)$ accrue to the old generation. However, underinvestment can still arise if only a fraction $\tau \in (0,1)$ of total profits are part of old-age income.

Proposition 5. *(Pareto Inefficiency) The bubbleless equilibrium is Pareto inefficient if and only if*

$$\gamma > \max\left\{\frac{\alpha}{1-\alpha}, \frac{1}{\beta}\right\}$$

or

$$\frac{\alpha}{1-\alpha} < \frac{1}{\beta}.$$

Proof. If $\gamma > \max\{\alpha/(1-\alpha), 1/\beta\}$, we have $f'(K^*) < 1$ and the bubbleless steady-state is above the *golden rule* $(K^* > K_{GR})$. Starting from K^*, it is possible to reduce the capital stock and increase the aggregate level of consumption.

If $\alpha/(1-\alpha) < 1/\beta$, we have $R^* = 1/\beta$. Thus, there is first-period consumption in the steady-state ($C^{y*} > 0$). Suppose that the young give away their consumption level C^{y*} to old. This does not change the capital stock and, given $\beta > 1$, results in a Pareto improvement.

If $\alpha/(1-\alpha) \geq \max\{\gamma, 1/\beta\}$, we have $f'(K^*) \geq 1$ and $R^* \geq 1/\beta$. The steady-state is such that $K^* \leq K_{GR}$ and $C^{y*} = 0$. Any transfer from the old to the young will reduce the welfare of the old. Any transfer from the young to the old will reduce the capital stock and hurt the welfare of some generation. Let the economy start at some K_0 such that $f'(K_0) \geq 1$ and $C_0^y = 0$ (which must be reached in finite time). Suppose that the young give $\lambda > 0$ to the old. Let $\tilde{\Delta}X_t := \tilde{X}_t - X_t$ be the difference of X_t between the new and the old allocation. We have $C_t^o = Y_t - K_{t+1}$ and must impose $\tilde{\Delta}C_{t+1}^o \geq 0$ $\forall t$, which implies $\tilde{\Delta}K_{t+1} \leq \tilde{\Delta}Y_t$. We have $-\tilde{\Delta}K_1 = \tilde{\Delta}C_0^o = \lambda$ and $f'\left(\tilde{K}_1\right) > 1$, which implies $\tilde{\Delta}Y_1 < \tilde{\Delta}K_1 < 0$. Combining the last inequality with $\tilde{\Delta}K_2 \leq \tilde{\Delta}Y_1$, it follows that $\tilde{\Delta}K_2 < \tilde{\Delta}K_1$. This implies that the capital stock will eventually reach a value of zero, which makes this plan unfeasible. ■

Region	Net Interest Rate	MPK	Underinvestment	Pareto Efficiency
I.1	$r^* < 0$	$f'(K^*) < 1$	No	No
I.2	$r^* < 0$	$f'(K^*) < 1$	No	No
I.3	$r^* < 0$	$f'(K^*) < 1$	Yes	No
II.1	$r^* < 0$	$f'(K^*) > 1$	Yes	No
II.2	$r^* < 0$	$f'(K^*) > 1$	No	Yes
III	$r^* > 0$	$f'(K^*) > 1$	No	Yes

Table 1: Characterization of the bubbleless steady-state

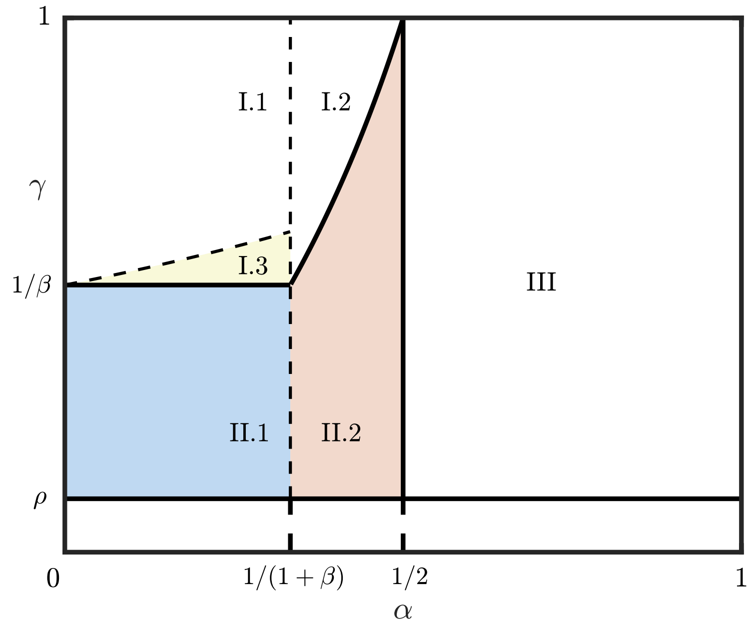


Figure 3: Characterization of the bubbleless steady-state

Propositions 2, 3 and 5 are equivalent if and only if $\gamma = 1$ (perfect competition). Under $\gamma < 1$, it is possible to have $r^* < 0$ even if there is no overaccumulation and the equilibrium is Pareto efficient. Additionally, it is possible to have an equilibrium without overaccumulation and which is Pareto inefficient. To clarify these points, Figure 3 illustrates these propositions in the (α, γ) space. In region III, we have $\alpha > 1/2$: bubbles cannot emerge $(r^* > 0)$, there is no overaccumulation and the equilibrium is Pareto efficient. Bubbles can appear in regions I and II, where $\alpha < 1/2$. In region I, the conditions of Propositions 3 and 5 are also satisfied: there is overaccumulation and the equilibrium is Pareto inefficient. In region II, we have a more interesting case: bubbles can emerge even if $f'(K^*) > 1$. There are two subregions to distinguish. In II.2, we have $\alpha > 1/(1+\beta)$: the young save all their income and the equilibrium is Pareto efficient. In II.1, we have $\alpha < 1/(1+\beta)$, which implies that young individuals consume a strictly positive amount. Even if there is no overaccumulation of capital in II.1, the equilibrium is Pareto inefficient — the young can give their consumption to the contemporaneous old, which generates a Pareto improvement (since $\beta > 1$). However, this region is also characterized by underinvestment — the young can enjoy higher welfare also by investing more. Thus, in this region, bubbles can result in greater welfare through two channels: by providing a store of value that allows individuals to defer consumption from young to old-age, and by increasing investment.[15] This can be achieved by bubbly stocks, as I show in the following section.

[15] This is also possible in region I.3, even though it features $f'(K^*) < 1$.

Discussion As Proposition 5 shows, the decentralized and bubbleless equilibrium can be Pareto efficient even when $\gamma < 1$. This result depends on the fact that labor supply is inelastic (hence market power does not distort labor supply decisions). Two other aspects should be highlighted. First, in region II.2, we have $r^* < 0$ even when the bubbleless equilibrium is Pareto efficient. A consequence of this fact is that, in this economy, bubbles may not always be Pareto improving. Second, in regions I.3 and II.1, there is underinvestment. Thus, when raising investment demand, bubbles can lead to a welfare-improving increase in capital accumulation. This will be shown with the discussion of bubbly stocks in general equilibrium.

3.2 General Equilibrium with Bubbles

I next discuss the aggregate consequences of asset bubbles. I will assume that $\alpha < 1/2$, so that rational bubbles can be traded.

3.2.1 Government Debt

I start by discussing the general equilibrium effects of the government debt scheme introduced in (8). In the OLG model of Tirole (1985) with competitive markets, such a debt scheme would (i) not be expansionary, but (ii) would be Pareto-improving. In the current model, as in Tirole (1985), this debt scheme will never increase capital accumulation. However, it may or may not be Pareto-improving. These results are established in Propositions 6 and 7.

Proposition 6. *(Contractionary Debt) When $C_t^y = 0$, a marginal increase in the stock of government debt D_t leads to a reduction in K_{t+1}. When $C_t^y > 0$, a marginal increase in D_t does not change K_{t+1}.*

Proof. The aggregate resource constraint implies that $K_{t+1} + D_t + C_t^y = W_t$. Note that W_t is predetermined at t (from (14)). When $C_t^y > 0$, the interest rate is $R_{t+1} = 1/\beta$ and young individuals are indifferent between saving or consuming; an infinitesimal increase in D_t crowds out C_t^y. When $C_t^y = 0$, a marginal increase in D_t crowds out K_{t+1}. ■

Thus, government debt does not increase capital accumulation. When the economy is characterized by $C_t^y = 0$, current D_t crowd-outs future capital K_{t+1}. When $C_t^y > 0$, a marginal increase in D_t has no impact on future capital K_{t+1}.[16]

[16]This last case depends on linear preferences. Proposition 12 in Appendix A shows that, under concave preferences, government debt always reduces the stock of capital that the economy achieves on a stable steady-state.

Contrarily to Tirole (1985), bubbly government debt may not always be Pareto improving. To illustrate this point, suppose that the economy starts at its bubbleless steady-state defined by (17) and that the government makes a one time debt issuance such that

$$d_0 > 0,$$

$$d_t = 0 \, \forall \, t \geq 1.$$

Suppose that d_0 is the maximum level of debt that the economy can sustain. In this case, the economy will converge to a steady-state with interest rate $R^{**} = 1$, capital stock

$$K^{**} = (\gamma \, \alpha \, A)^{1/(1-\alpha)} \tag{19}$$

and debt level

$$D^{**} = (\gamma \, \alpha \, A)^{1/(1-\alpha)} \, \frac{1-2\alpha}{\alpha}. \tag{20}$$

As the next proposition highlights, such a government debt scheme will be Pareto improving when the economy is characterized by a high level of competition.

Proposition 7. *(Aggregate Consequences of Government Debt) Suppose that the economy starts at the bubbleless steady-state given by* (17) *and that, at $t = 0$, the government issues the largest amount of bubbly debt that the economy can sustain. This bubble generates a Pareto improvement if and only if*

$$\gamma > \begin{cases} \dfrac{1-\beta^{-\alpha/(1-\alpha)}}{1-\alpha\beta^{-\alpha/(1-\alpha)}-(1-\alpha)\,\beta^{-1}} & \textit{if} \quad \dfrac{\alpha}{1-\alpha} < \dfrac{1}{\beta} \\ \dfrac{(1-\alpha)^{\alpha/(1-\alpha)}-\alpha^{\alpha/(1-\alpha)}}{(1-\alpha)^{1/(1-\alpha)}-\alpha^{1/(1-\alpha)}} & \textit{if} \quad \dfrac{\alpha}{1-\alpha} \geq \dfrac{1}{\beta} \end{cases}.$$

Proof. When $\alpha/\left(1-\alpha\right) < 1/\beta$, utility in a bubbleless steady-state is $U^* = W^* + \beta \, \pi^* = A \left(\beta \, \gamma \, \alpha \, A\right)^{\alpha/(1-\alpha)} \left[\gamma \left(1-\alpha\right) + \beta \left(1-\gamma\right)\right]$ where (14) and (17) have been used. When $\alpha/\left(1-\alpha\right) \geq 1/\beta$, utility in a bubbleless steady-state is $U^* = R^* W^* + \pi^* = A \left[\gamma \left(1-\alpha\right) A\right]^{\alpha/(1-\alpha)} \left[1-\gamma \left(1-\alpha\right)\right]$ where (14) and (17) have been used. Instead, in a steady-state with the government debt level (20), total welfare is $U^{**} = R^{**} W^{**} + \pi^{**} = A \left(\gamma \alpha A\right)^{\alpha/(1-\alpha)} \left(1-\gamma\alpha\right)$, where (14) and (19) have been used. Thus, $U^{**} > U^*$ if and only if the condition in Proposition 7 satisfied.

I now show that generations born during the transition have welfare greater than U^{**} (hence, they also benefit from the bubble when $U^{**} > U^{*}$). First note that, during the transition, $K_{t+1} < K_t$, $R_t < R_{t+1} < 1$ and $D_{t+1} = R_{t+1} D_t < D_t$. Second, the new steady-state features $R^{**} = 1 > 1/\beta$ and $C^{y**} = 0$. Third, as I show below, individuals born during the transition have $C_t^y = 0$ and welfare equal to $U_t = \beta \left(Y_{t+1} - W_{t+1} + D_{t+1}\right) = \beta \left(\left[1 - \gamma \left(1 - \alpha\right)\right] A K_{t+1}^{\alpha} + D_{t+1}\right)$. Since K_{t+1} and D_{t+1} decrease during the transition, it follows that $U_t > U_{t+1} > U^{**}$.

To prove that $C_t^y = 0$ for generations born during the transition, note the following. If the bubbleless steady-state features $R^* > 1/\beta$ and $C^{y*} = 0$, then $C_t^y = 0$ since $R_{t+1} > R^*$. If the bubbleless steady-state features $R^* = 1/\beta$ and $C^{y*} > 0$, it must be the case that $C_0^y = 0$ when the bubble starts at $t = 0$. Otherwise, $R_1 = 1/\beta$, $K_1 = K^*$, $W_1 = W^*$ and $D_1 = (1/\beta) D_0 < D_0$, implying that the economy would remain with positive first period consumption forever. ■

Proposition 7 is illustrated in Figure 4. If the economy converges to a steady-state with the maximum debt level, this leads to Pareto improvement when γ is large (green region). However, a Pareto improvement does not take place when γ is low (yellow region). In this case, there is a large wedge between interest rates and the marginal product of capital — hence, the economy can be characterized by $r^* < 0$ even if the marginal product of capital is high. Bubbles crowd-out efficient investment and result in a reduction in welfare for some generations.[17]

The previous proposition shows that bubbles can have negative welfare consequences. Proposition 8 characterizes the level of new debt issuance that maximizes steady-state welfare.

Proposition 8. *(Golden Rule Level of Government Debt Issuance) The level of new government debt issuance that maximizes steady-state welfare is*

$$d^{**} = \begin{cases} (1-\gamma)\left(\alpha A\right)^{1/(1-\alpha)}\left(\frac{1-\alpha}{\alpha}\gamma - 1\right) & \text{if} \quad \gamma > \max\left\{\frac{\alpha}{1-\alpha}, \frac{1}{\beta}\right\} \\ \frac{\beta-1}{\beta}\left(\beta\gamma\alpha A\right)^{1/(1-\alpha)}\left(\frac{1-\alpha}{\beta\alpha} - 1\right) & \text{if} \quad \frac{1}{\beta} > \max\left\{\frac{\alpha}{1-\alpha}, \gamma\right\} \\ 0 & \text{otherwise} \end{cases} . \tag{21}$$

When $\gamma = 1$ and the conditions of Proposition 3 are satisfied, we have $d^{**} = 0$, while the golden rule stock of debt is given by (20).

[17]This can also happen in region I, which is characterized by overaccumulation in a bubbleless steady-state. In this region, a sufficiently small bubble reduces overaccumulation and increases aggregate consumption. However, the largest possible bubble eliminates efficient investment and hurts the welfare of some generations.

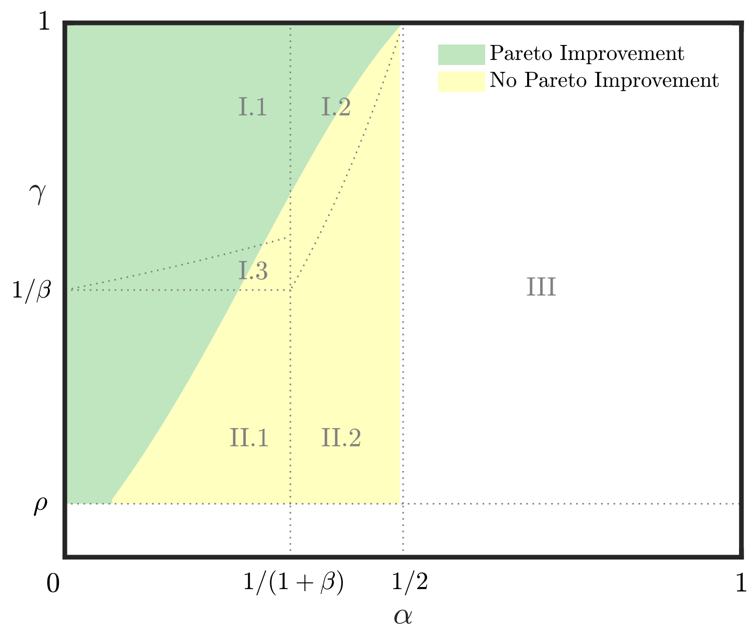


Figure 4: Welfare consequences of the maximum government debt level

Proof. If $\gamma > \max\{\alpha/(1-\alpha), 1/\beta\}$, there is overaccumulation of capital. Steady-state welfare is maximized when the capital stock satisfies $f'(K^{**}) = 1$ and there is no first-period consumption. This is achieved with $K^{**} = K_{GR} = (\alpha A)^{1/(1-\alpha)}$, implying an interest rate $R^{**} = \gamma$ and a wage $W^{**} = \gamma(1-\alpha)(\alpha A)^{\alpha/(1-\alpha)}$. Given this interest rate, there is no first-period consumption. If $1/\beta > \max\{\alpha/(1-\alpha), \gamma\}$, there is no overaccumulation of capital, but there is first-period consumption and $R^{**} = 1/\beta < 1$. The optimal level of government debt must absorb all young-age consumption, i.e. $C^{y**} = 0$. Both the resulting interest rate, capital stock and wage are unchanged, i.e. $R^{**} = 1/\beta$, $K^{**} = (\beta\gamma\alpha A)^{1/(1-\alpha)}$, and $W^{**} = \gamma(1-\alpha)(\beta\gamma\alpha A)^{\alpha/(1-\alpha)}$. Combining these expressions with capital market clearing $W^{**} = K^{**} + d^{**}/(1-R^{**})$, (21) obtains. If $\alpha/(1-\alpha) \geq \max\{\gamma, 1/\beta\}$, the bubbleless equilibrium is Pareto efficient. ■

Proposition 8 characterizes the *golden rule* level of new debt issuance d^{**}. Note that each level of d^{**} will be associated with a steady-state stock of debt equal to $D^{**} = d^{**}/(1-R^{**})$, where R^{**} is characterized in the proof of the proposition. When $\gamma > \max\{\alpha/(1-\alpha), 1/\beta\}$, there is overaccumulation of capital. Steady-state welfare is maximized when the capital stock satisfies the *golden rule* $f'(K^{**}) = 1$. The government can ensure that the transition takes place in one period if it initially issues the steady-state level of debt $d_0 = D^{**}$ and levies a lump sum tax equal to $\tau_0 = W^* - D^{**} - K^{**}$ to the young (which can be distributed to the old). Then, every period, it must issue a new debt level d^{**} according to (21) to sustain this steady-state.

When $1/\beta > \max\{\alpha/(1-\alpha), \gamma\}$, there is no overaccumulation and debt will simply be absorbing first-period consumption. The steady-state level of capital will remain unchanged. The government can issue the steady-state level of debt $d_0 = D^{**}$ and the transition to the new steady-state equilibrium will take place within a period. Every period, the government must issue a new debt level d^{**} according to (21) to sustain this steady-state.

3.2.2 Bubbly Stocks

A distinctive feature of bubbly stocks, which makes them different from government debt, is that they can result in higher aggregate investment. As I show below, this can happen when (i) there is new bubbly stock issuance and (ii) the economy is characterized by positive young-age consumption. Note that, when all industries are identical and characterized by the same amount of new stock issuance $b_{it} = b_t$, they will feature the same price $p_{it} = 1$ and output $y_{it} = y_t$. Thus, aggregate output is still given by (13). From (10), we obtain an expression for the aggregate factor share (which corresponds to the inverse of the markup)

$$\omega_t := \frac{R_t\,K_t + W_t}{Y_t} = \left(1 + \frac{b_t}{Y_t}\right)\gamma. \tag{22}$$

Equilibrium factor prices are equal to

$$\begin{aligned} W_t = &\ \omega_t\,(1-\alpha)\,A\,K_t^{\alpha} &&= \gamma\,(1-\alpha)\,(A\,K_t^{\alpha} + b_t)\,, \\ R_t = &\ \omega_t\,\alpha\,A\,K_t^{\alpha-1} &&= \gamma\,\alpha\,(A\,K_t^{\alpha} + b_t)\,K_t^{-1}. \end{aligned} \tag{23}$$

As (23) shows, a larger amount of bubbly stock issuance b_t results in higher factor prices (for fixed K_t). One observation should be made. Even if leading to a higher wage W_t (and hence higher income for the young), a larger b_t will not lead to higher K_{t+1}. To see this, combine (12) with (9) and (23) to write

$$K_{t+1} + R_t B_{t-1} + C_t^y \ = \ \gamma\,(1-\alpha)\,A\,K_t^{\alpha} - b_t\,[1 - \gamma\,(1-\alpha)]\,.$$

Intuitively, even if a higher b_t leads to a higher wage W_t, these bubbles have to be purchased by the young, using the same wage income W_t. Even if K_{t+1} does not increase with b_t, it can however increase with b_{t+1}. A higher b_{t+1} results in a higher interest rate R_{t+1} (through (23)), and may induce young agents to reduce consumption C_t^y and increase investment K_{t+1}. Thus, bubbly stocks can be expansionary only when young-age consumption takes place.

Suppose then that all industries issue the same amount of bubbly stocks $b_t = b$ and that the economy is in an equilibrium with $C_t^y > 0$. Combining $R_{t+1} = 1/\beta$ with (22) and (23), we can find an equation that implicitly defines K_{t+1} as a function of $b_{t+1} = b$,

$$K_{t+1}\left[(\gamma\beta\alpha)^{-1} - AK_{t+1}^{\alpha-1}\right] = b. \tag{24}$$

It immediately follows from (24) that, when $C_t^y > 0$, K_{t+1} is increasing in b. These results are formally established in the next proposition.

Proposition 9. *(Expansionary Bubbly Stocks) Suppose that all industries feature a constant amount of new bubbly stock issuance $b_t = b \geq 0$ $\forall t$. These bubbles result in higher aggregate investment at all periods if and only if $\alpha < 1/(1+\beta)$ and $b \leq \overline{b} := \eta^{-1}\left[(A\beta\alpha\gamma)^{-1} - (A\eta)^{-1}\right]^{-1/(1-\alpha)}$ where $\eta := \left[(\beta-1)(1-\alpha(1+\beta))/\left(\beta^2\alpha\right)\right]^{-1}$. For any $b \leq \overline{b}$, the aggregate capital stock satisfies (24). In particular, the capital stock associated with the maximum expansionary bubble $b = \overline{b}$ is equal to $\overline{K} = \eta\,\overline{b}$.*

Proof. The aggregate resource constraint requires that $K_{t+1} + B_t + C_t^y = W_t$. Bubbly stocks increase K_{t+1} only if $C_t^y > 0$ in a bubbleless steady-state, which happens if and only if $\alpha < 1/(1+\beta)$. Note the the left-hand side of (24) increases in K_{t+1}; hence this equation establishes that K_{t+1} increases in b. The largest expansionary bubble is such that (i) $C^{y**} = 0$, (ii) $R^{**} = 1/\beta$ and (iii) $B^{**} = b/(1-R^{**})$. Combining these conditions with the aggregate resource constraint and equations (23) and (24), $\overline{b}$ and $\overline{K}$ obtain. ■

Bubbles can increase the capital stock if first-period consumption takes place in a bubbleless steady-state (regions I.1, I.3 and II.1). In that case, as firms issue bubbly stocks, their demand for capital increases. Bubbles reduce young-age consumption and crowd-in capital. Note that $\overline{b}$ is the maximum amount of new bubbly stocks that firms can issue on a symmetric equilibrium with $R^{**} = 1/\beta$. Therefore, the capital stock is increasing in b provided that $b < \overline{b}$. An analysis of the expression defining $\overline{b}$ shows that the largest expansionary bubble is increasing in the level of aggregate TFP (A) and in the degree of competition (γ).[18] As shown in section 2.4, sufficiently large bubbles induce firms to charge a price below their unit cost of production. That result was, however, obtained in partial equilibrium. I conclude by showing that the economy

[18]Since capital is increasing in both A and γ (from (24)), the economy can accommodate larger bubbles.

can sustain a stationary equilibrium where all industries are characterized by negative earnings. To see this, let us focus on an equilibrium in which the economy is characterized by the largest expansionary bubble $\overline{b}$, so that capital is equal to $\overline{K} = \eta\,\overline{b}$. Combining the expressions for $\overline{b}$ and $\overline{K}$ (from Proposition 9) with (22), we obtain an aggregate factor share $\omega > 1$ if and only if

$$\gamma > \left[1 + (\beta - 1)\,\frac{1 - \alpha\,(1 + \beta)}{\beta}\right]^{-1}. \tag{25}$$

Therefore, when the degree of competition γ is large (so that the aggregate factor share is already high in a bubbleless equilibrium), the economy can experience a bubble-driven expansion which makes firms expand too much and exhibit negative production profits.

3.3 Discussion and Extensions

This model provides two main insights. The first is that, when firms charge a price-cost markup, interest rates will be below the marginal product of capital. In this case, rational bubbles can be traded even when there is no overaccumulation of capital. The second is that, when the issuance of bubbles depends on firm size, bubbles can stimulate production and result in lower markups.

In the appendix, I provide two main extensions. In the first extension, I generalize preferences and assume that utility is described by a generic concave utility function. The assumption of linear utility is convenient for analytical purposes, but can be seen as restrictive. I show that the main results of this paper hold. In particular, rational bubbles can emerge when there is no overaccumulation of capital and they can reduce welfare. Furthermore, under CRRA preferences, bubbly stocks can be expansionary if the intertemporal elasticity of substitution is sufficiently high (a necessary condition is that it is greater than one).

In the second extension, I consider a different market structure. I assume that firms need to pay a fixed cost of production and compete à la Cournot (via quantities). There are two main differences in this alternative environment. First, contrarily to the previous setting with limit pricing, there can be variation in the number of firms. Second, bubbly stocks can boost entry even when their distribution across firms is not linked to size/market shares. If firms can issue a fixed amount of bubbly stocks upon entering, entry becomes more attractive. As a result, more firms decide to enter, pay the fixed cost and produce; output expands, while markups shrink.

4 Conclusion

Financial history shows that stock market boom/bust episodes are often an industry phenomenon that can be accompanied by changes in the market structure. Motivated by this observation, this paper developed a framework to investigate the interplay between asset bubbles and product market competition. The model shows that bubbles can reduce barriers to entry and force firms to expand, to the ultimate benefit of consumers. An interesting aspect of the theory is that asset bubbles may force incumbents to expand only when potential competitors can also get overvalued. This observation helps us think about different questions. For instance, how will a large company react to a bubble on its stock prices? Will Apple lower the price of its *iPhones* if investors suddenly become excited about the company alone and its market value doubles? This paper suggests that it will probably not. Instead, Apple is more likely to expand and cut its profit margins in the presence of a generalized boom in which potential competitors (perhaps smaller and less innovative) can also get overvalued. In such a case, as barriers to entry decrease, Apple may be forced to expand to preserve its market share.

The model developed in this paper gives a novel perspective on famous stock market overvaluation episodes. For instance, it provides a simple rationale for the low and negative profit margins reported by high-tech firms at the peak of the dotcom bubble. Rather than the realization of a negative technology shock (as argued by Pastor and Veronesi (2006)), this paper suggests that negative profits may have been a rational reaction to an environment characterized by high stock prices.

I conclude by pointing to some avenues for future research. The first concerns the role of policy. This paper provides a stylized model that connects financial and product markets. Its theoretical simplicity allowed me to uncover new mechanisms, but makes it unsuitable for a quantitative policy analysis. The second concerns bubbles and innovation. In the model explored in this paper, bubbles can be pro-competitive and correct a market failure. However, if market leaders could innovate (in order to increase their productivity advantage), such a pro-competitive effect might reduce firms' innovation incentives.

Appendix

A General Preferences

In this section, I depart from linear utility, and assume that preferences are described by

$$U\left(c_{jt}^{y}, c_{j,t+1}^{o}\right) = u\left(c_{jt}^{y}\right) + \beta v\left(c_{j,t+1}^{o}\right), \tag{26}$$

where $u(\cdot)$ and $v(\cdot)$ are two real-valued and twice-continuously differentiable functions, satisfying $u'(\cdot) > 0$, $v'(\cdot) > 0$ and $u''(\cdot) < 0$, $v''(\cdot) < 0$, as well as the Inada conditions.

Euler Equation The young maximize (26) subject to the budget constraint in (2). Their consumption and savings decisions satisfy the Euler equation

$$u'\left(c_{jt}^{y}\right) = \beta R_{t+1} v'\left(c_{j,t+1}^{o}\right). \tag{27}$$

General Equilibrium Except for the utility function, the model is as in section 2. Therefore, the previous equilibrium definition applies. In a bubbleless equilibrium, equations (13) to (15) hold. Combining these with the Euler equation (27), we obtain

$$u'\left(\gamma(1-\alpha) AK_t^{\alpha} - K_{t+1}\right) = \beta\left(\gamma\alpha AK_{t+1}^{\alpha-1}\right) v'\left(\left[1-\gamma(1-\alpha)\right] AK_{t+1}^{\alpha}\right). \tag{28}$$

As in the case of linear utility, the bubbleless equilibrium can feature underinvestment.

Proposition 10. *(Underinvestment) The bubbleless equilibrium features underinvestment if and only if $\gamma < (2-\alpha)^{-1}$. In this case, the welfare of any given generation t can be increased if all the young individuals of that generation increase investment (K_{t+1}).*

Proof. In a bubbleless equilibrium, we have $C_t^y = \gamma(1-\alpha) AK_t^{\alpha} - K_{t+1}$ and $C_{t+1}^o = Y_{t+1} - W_{t+1} = [1-\gamma(1-\alpha)] AK_{t+1}^{\alpha}$. If all the young individuals at t marginally increase investment K_{t+1}, the change in utility is $-u'\left(C_t^y\right) + \beta v'\left(C_{t+1}^o\right)[1-\gamma(1-\alpha)]\alpha AK_{t+1}^{\alpha-1} = \gamma^{-1} - (2-\alpha)$, where the Euler equation (27) and $R_{t+1} = \gamma\alpha AK_{t+1}^{\alpha-1}$ were used. ■

Steady-State I assume that (28) defines a concave law of motion $K_{t+1} = h(K_t)$, and that it has a (globally) stable steady-state. That is, I assume that $h'(\cdot) > 0$, $h''(\cdot) < 0$ with $h(0) = 0$ and that there is a unique $K^* \in \left(0, (\gamma(1-\alpha)A)^{1/(1-\alpha)}\right)$ such that $K^* = h(K^*)$. This must satisfy $h'(K^*) < 1$. The next proposition states the conditions for a bubbly equilibrium.

Proposition 11. *($r^* < 0$) The bubbleless steady-state features a negative net interest rate r^* if and only if $f'(K^*) < 1/\gamma$.*

Proof. Using (14), it follows that $r^* := R^* - 1 < 0$ if and only if $f'(K^*) < 1/\gamma$. ∎

If the bubbleless steady-state is characterized by $f'(K^*) < 1/\gamma$, rational bubbles can be traded. In this case, the government can run a bubbly debt policy satisfying equation (8). Proposition 12 states that government debt bubbles reduce the steady-state stock of capital.

Proposition 12. *(Contractionary Debt) Let D^{**} be the stock of government debt in a stable steady-state K^{**}. We have that $dK^{**}/dD^{**} < 0$.*

Proof. The law of motion of capital is $K_{t+1} = \gamma(1-\alpha)AK_t^{\alpha} - C_t^y - D_t := H(K_t, D_t)$. A steady-state is defined by $K^{**} = H(K^{**}, D^{**})$. Differentiating this last equation with respect to D^{**},

$$\frac{dK^{**}}{dD^{**}} = \frac{H_D(K^{**}, D^{**})}{1 - H_K(K^{**}, D^{**})} < 0$$

given that $H_D(\cdot) < 0$ and that, on a stable steady-state, $H_K(K^{**}, D^{**}) < 1$. ∎

A corollary of Proposition 12 is that debt increases the steady-state interest rate $(dR^{**}/dD^{**} > 0)$. Proposition 13 states the conditions under which debt increases steady-state welfare.[19]

Proposition 13. *(Welfare-Improving Government Debt) Let C^{**} be total consumption in a stable steady-state with a stock of government debt D^{**}. Let λ^{**} be the fraction of total consumption taking place in young-age. Let $R^{**} < 1$ be the steady-state interest rate. Steady-state welfare increases in D^{**} if and only if*

$$\frac{dC^{**}}{dD^{**}}\frac{D^{**}}{C^{**}}\left[\frac{1}{\lambda^{**}(1-R^{**})} - 1\right] > \frac{d\lambda^{**}}{dD^{**}}\frac{D^{**}}{\lambda^{**}}. \tag{29}$$

This condition is equivalent to

$$\left[(1-\alpha)(1-R^{**}) - \left(\frac{1}{\gamma} - 1\right)\right]\frac{dK^{**}}{dD^{**}} < \frac{1}{R^{**}} - 1. \tag{30}$$

[19] The steady-state stock of debt satisfies $D^{**} = d^{**}/(1-R^{**})$, where d^{**} is the value of new debt issuance, which is chosen by the government.

Proof. Steady-state utility is $U(C,\lambda) = u(\lambda C) + \beta v((1-\lambda)C)$. It increases in D if and only if

$$u'(\lambda C)\left\{\frac{d\lambda}{dD}C + \frac{dC}{dD}\lambda\right\} + \beta v'((1-\lambda)C)\left\{-\frac{d\lambda}{dD}C + \frac{dC}{dD}(1-\lambda)\right\} > 0$$

$$\Leftrightarrow \quad \frac{dC}{dD}(1-\lambda(1-R)) > \frac{d\lambda}{dD}C(1-R),$$

where the Euler equation (27) was used. In a bubbly steady-state, utility can also be written as $U(K,D) = u(\gamma(1-\alpha)AK^{\alpha} - K - D) + \beta v([1-\gamma(1-\alpha)]AK^{\alpha} + D)$. Totally differentiating U with respect to D, using the Euler equation (27) and the fact that $R = \gamma f'(K)$, (30) obtains. ■

As (29) shows, bubbles increase steady-state welfare when they increase aggregate consumption ($dC^{**}/dD^{**} > 0$) and/or when they improve the intertemporal allocation of consumption from young to old-age ($d\lambda^{**}/dD^{**} < 0$). From Proposition 12, we know that debt reduces the capital stock and increases the interest rate in a steady-state. The fact that debt crowds-out capital ($dK^{**}/dD^{**} < 0$) implies that aggregate consumption $C^{**} = f(K^{**}) - K^{**}$ increases if and only if $f(K^{**}) < 1$. The fact that interest rates increase ($dR^{**}/dD^{**} > 0$) implies that debt allows for a better intertemporal allocation of consumption. To understand this last result, suppose that the economy is in a steady-state such that $f(K^{**}) = 1$. In such a case, aggregate consumption does not increase when the stock of debt increases marginally ($dC^{**}/dD^{**} = 0$). From condition (29), if $R^{**} < 1$, welfare increases in D^{**} if and only if consumption is reallocated from young to old-age ($d\lambda^{**}/dD^{**} < 0$). Note that the steady-state Euler equation can be written as

$$\frac{u'(\lambda^{**}C^{**})}{v'\Big((1-\lambda^{**})C^{**}\Big)} = \beta R^{**},$$

where $\lambda^{**} := C^{y**}/C^{**}$ is the share of total consumption that takes place in young-age. Since interest rates increase in the level of debt, $u''(\cdot) < 0$ and $v''(\cdot) < 0$, it follows that the share of young-age consumption decreases in debt ($d\lambda^{**}/dD^{**} < 0$). This implies that (29) is satisfied. Therefore, when $f(K^{**}) = 1$ and $R^{**} < 1$, government debt bubbles can increase welfare not because they increase the level of aggregate consumption, but because they result in higher interest rates and allow for a better allocation of consumption across periods. Clearly, we can jointly have $f(K^{**}) = 1$ and $R^{**} < 1$ only when there is imperfect competition ($\gamma < 1$). The following corollaries characterize Proposition 13.

Corollary 1. *When $f'(K^{**}) < 1$, government debt bubbles increase steady-state welfare.*

Proof. From the resource constraint $C^{**} = f(K^{**}) - K^{**}$ and Proposition 12, it follows that $dC^{**}/dD^{**} > 0$ if and only if $f'(K^{**}) < 1$. If $dC^{y**}/dD^{**} > 0$ and $dC^{o**}/dD^{**} > 0$, utility necessarily increases and (29) is trivially satisfied. If $dC^{y**}/dD^{**} < 0$ and $dC^{o**}/dD^{**} > 0$, then $d\lambda^{**}/dD^{**} < 0$ and hence (29) is also satisfied. Note that $dC^{y**}/dD^{**} > 0$ and $dC^{o**}/dD^{**} < 0$ cannot jointly happen, since this would violate the Euler equation (27), as $dR^{**}/dD^{**} > 0$. ■

Corollary 2. *If markets are characterized by imperfect competition (i.e $\gamma < 1$), there exist*

(i) $\hat{K}$ satisfying $f'(\hat{K}) > 1$ such that, for all $K^{**} > \hat{K}$, steady-state welfare increases in D^{**} and

(ii) $\tilde{K}$ satisfying $\gamma f'(\tilde{K}) < 1$ such that, for all $K^{**} < \tilde{K}$, steady-state welfare decreases in D^{**}.

Proof. When $f'(K^{**}) = 1$, we have $dC^{**}/dD^{**} = 0$. Combining this with $dR^{**}/dD^{**} > 0$ and the Euler equation (27), it follows that $d\lambda^{**}/dD^{**} < 0$. Thus, (29) is satisfied when $f'(K^{**}) = 1$. Since dC^{**}/dD^{**}, $d\lambda^{**}/dD^{**}$ and R^{**} are all continuous in D^{**}, (29) must also be satisfied for a set of K satisfying $f'(K) > 1$. Combining this with Corollary 1, the first statement obtains.

When $R^{**} = \gamma f'(K^{**}) = 1$, condition (30) can be written as $(1-\gamma)(dK^{**}/dD^{**}) > 0$. In this case, steady-state welfare decreases in D^{**} if and only if $(1-\gamma)(dK^{**}/dD^{**}) < 0$. This condition is always satisfied when $\gamma < 1$, given that $dK^{**}/dD^{**} < 0$. Since R^{**} and dK^{**}/dD^{**} are continuous in D^{**}, this establishes that there is a set of K with $\gamma f'(K) < 1$ for which steady-state welfare also decreases in D^{**}. ■

Corollary 1 states that debt increases steady-state welfare when there is overaccumulation, i.e. $f'(K^{**}) < 1$. From Corollary 2, debt also increases welfare when $f'(K^{**}) > 1$, provided that $f'(K^{**})$ is not too high. In this case, aggregate consumption C^{**} declines in D^{**}, but the increase in interest rates allows for a better intertemporal allocation of consumption from young to old-age. However, Corollary 2 also establishes that government debt results in lower steady-state welfare when $f'(K^{**})$ is high and close to $1/\gamma$ — so that R^{**} is close to one. In this case, the reduction in total consumption C^{**} outweighs the gains from a higher interest rate R^{**}.

In this section, I considered a general utility function and showed that government debt bubbles are always contractionary (Proposition 12) and have mixed welfare consequences (Proposition 13 and Corollary 2). Under such a generic utility function, I cannot not give a sharp characterization of the general equilibrium consequences of bubbly stocks (in particular, the conditions under which they result in greater capital accumulation). This is done in the next subsection, where I consider the particular case of CRRA preferences.

A.1 CRRA Utility

Suppose that individuals born at t have utility

$$U\left(c_{jt}^{y},c_{j,t+1}^{o}\right)=\begin{cases}\dfrac{\left(c_{jt}^{y}\right)^{1-\theta}-1}{1-\theta}+\beta\dfrac{\left(c_{j,t+1}^{o}\right)^{1-\theta}-1}{1-\theta} & \text{if } \theta>0 \text{ and } \theta\neq 1\\[2ex] \log\left(c_{jt}^{y}\right)+\beta\log\left(c_{j,t+1}^{o}\right) & \text{if } \theta=1\end{cases}, \tag{31}$$

where $\theta > 0$ is the inverse of the intertemporal elasticity of substitution (IES). The model with linear utility obtains as the limit case of $\theta \to 0$. As before, young individuals face the budget constraint in (2). Denoting by $s_{jt} := W_t - c_{jt}^{y}$ their savings level, their optimal savings rate is

$$\frac{s_{jt}}{W_t}=\frac{1}{1+\beta^{-1/\theta}R_{t+1}^{(\theta-1)/\theta}}-\frac{\left(\beta R_{t+1}\right)^{-1/\theta}}{1+\beta^{-1/\theta}R_{t+1}^{(\theta-1)/\theta}}\frac{\pi_{j,t+1}}{W_t}.$$

Bubbleless Dynamics and Steady-State Absent the existence of bubbles, the economy is characterized by a law of motion

$$K_{t+1}=\frac{\gamma\left(1-\alpha\right)AK_t^{\alpha}-\left(\beta R_{t+1}\right)^{-1/\theta}\pi_{t+1}}{1+\beta^{-1/\theta}R_{t+1}^{(\theta-1)/\theta}}.$$

and converges to a steady-state that is defined by

$$G\left(K,\gamma\right):=\frac{A\beta^{1/\theta}\gamma\left(1-\alpha\right)-\left(\alpha A\gamma\right)^{-1/\theta}\left(1-\gamma\right)AK^{(1-\alpha)/\theta}}{\beta^{1/\theta}K^{1-\alpha}+\left(\alpha A\gamma\right)^{(\theta-1)/\theta}K^{(1-\alpha)/\theta}}-1=0. \tag{32}$$

The steady-state can be shown to be unique and increasing in γ, as stated below.

Proposition 14. *(Steady-State Capital Stock) There is a unique steady-state K^* defined by (32), which is increasing in the level of competition γ.*

Proof. The steady-states of the model are given by the values K^* such that $G\left(K^*,\gamma\right)=0$. We have $G_K\left(K,\gamma\right)<0$, with $G\left(0,\gamma\right)=\infty$ and $G\left(\infty,\gamma\right)=-1$ for $\gamma\in(0,1]$. This establishes that there is a unique K^* such that $G\left(K^*,\gamma\right)=0$. To prove that K^* is increasing in γ, we can use the implicit function theorem. We have that

$$\frac{\partial K^*}{\partial \gamma} = -\frac{G_\gamma (K, \gamma)}{G_K (K, \gamma)}.$$

Given that $G_K (K, \gamma) < 0$, it suffices to show that $G_\gamma (K, \gamma) > 0$. This happens if and only if

$$\beta^{1/\theta} (1 - \alpha) A + \frac{AK^{(1-\alpha)/\theta}}{\theta} (\alpha A \gamma)^{-1/\theta} \left[\gamma^{-1} + (\theta - 1) (1 - \alpha) \right] > 0,$$

which is always satisfied given that $\theta \geq 0$, $\gamma < 1$ and $\alpha < 1$. ■

The steady-state interest rate R^* can be obtained by combining the expression for R_t in (14) with (32). It is implicitly defined by[20]

$$\alpha (R^*)^{-1} + \left[\gamma^{-1} - (1 - \alpha) \right] (\beta R^*)^{-1/\theta} - (1 - \alpha) = 0. \tag{33}$$

Asset bubbles can be sustained when $R^* < 1$. The condition for investment efficiency can again be written as $f' (K^*) = \alpha A (K^*)^{\alpha - 1} > 1$ which can be restated in terms of the steady-state interest rate as $R^* > \gamma$. The next propositions state the conditions for $R^* < 1$ and for $f' (K^*) < 1$.

Proposition 15. *(Rational Bubbles) Rational bubbles can be traded if and only if*

$$\alpha < \frac{\beta^{1/\theta} - \gamma^{-1} + 1}{2\beta^{1/\theta} + 1}.$$

Proof. Using (33), we have $R^* < 1$ if and only if the previous inequality is satisfied. ■

Proposition 16. *(Capital Overaccumulation) The economy features overaccumulation of capital in a bubbleless steady-state if and only if*

$$\alpha < \frac{(\beta\gamma)^{1/\theta} - \gamma^{-1} + 1}{(\beta\gamma)^{1/\theta} (1 + \gamma^{-1}) + 1}.$$

Proof. Using (33), we have $R^* < \gamma$ if and only if the previous inequality is satisfied.

[20] The degree of competition γ has a dual role on interest rates: (i) it increases the capital share and hence R^* for given K^* and (ii) it results in larger K^* and lower R^* because of decreasing returns. In the present setting, the second effect dominates, and R^* declines in γ.

Figure 5 illustrates Propositions 15 and 16 in the (α, γ) space, for the particular case of $\theta = 1$ (logarithmic utility) and $\beta = 1$. Bubbles can appear only in regions I and II. In region I, $R < \gamma$ and there is overaccumulation. However, in region II, $R > \gamma$ and there is no overaccumulation. ■

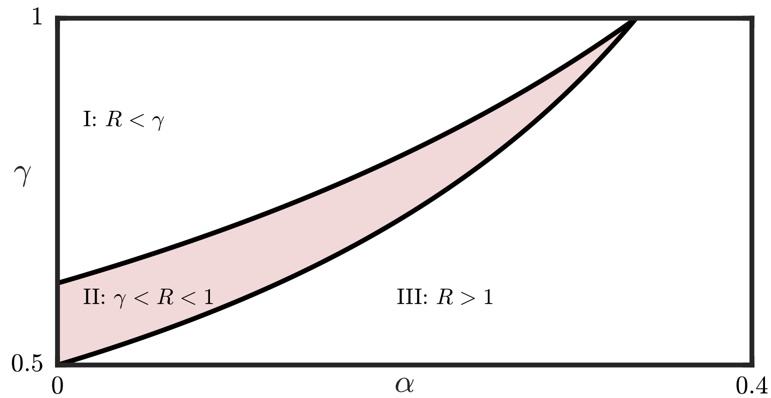


Figure 5: Bubbles and overaccumulation with $\theta = 1$ and $\beta = 1$

Bubbly Stocks When all industries issue $b_{it} = b$, equilibrium in the capital market requires

$$K_{t+1} + \underbrace{R_t B_{t-1} + b}_{B_t} = \frac{1}{1 + \beta^{-1/\theta} R_{t+1}^{(\theta-1)/\theta}} \underbrace{\left(1 + \frac{b}{AK_t^{\alpha}}\right) \gamma \left(1 - \alpha\right) AK_t^{\alpha}}_{W_t},$$

where B_t is the aggregate stock of bubbles at time t (issued at t and before). Equations (22) and (23) still describe the aggregate factor share and equilibrium factor prices, while (9) describes the aggregate bubble dynamics. Using these equations, steady-state capital is defined as

$$\frac{\gamma \left(1 - \alpha\right) \left(AK^{\alpha} + b\right) K^{-1}}{1 + \beta^{-1/\theta} \left[\gamma\alpha \left(AK^{\alpha} + b\right) K^{-1}\right]^{(\theta-1)/\theta}} - \frac{b}{K - \gamma\alpha \left(AK^{\alpha} + b\right)} - 1 = 0. \quad (34)$$

Under general CRRA preferences, it becomes harder to characterize the conditions under which bubbly stocks increase the steady-state capital stock. Figure 6 shows the set of values for θ under which this happens (for fixed α, γ and β). Bubbles are expansionary when the IES is high (θ is low). Intuitively, young individuals must be willing to postpone consumption to old-age. When a bubble appears, they must reduce first period consumption and increase savings. This additional savings must be allocated to both greater capital accumulation and bubble acquisitions. Proposition 17 says that bubbly stocks cannot be expansionary when the intertemporal elasticity of substitution is equal or lower than one.

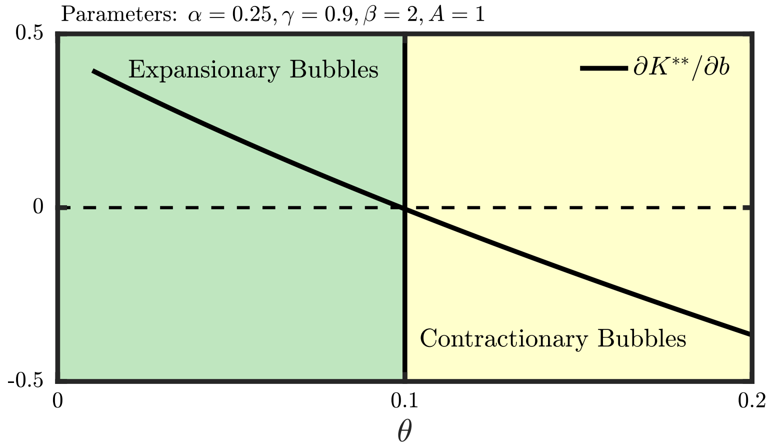


Figure 6: Condition for expansionary bubbly stocks

Proposition 17. *(Contractionary Bubbly Stocks) Let K^{**} be the capital stock on a steady-state where all firms issue a value of new bubbly stocks equal to $b \geq 0$. We have that $\left(\partial K^{**}/\partial b\right)|_{b=0} < 0$ if $1/\theta \leq 1$.*

Proof. A steady-state where firms issue a value of new bubbles b is implicitly defined by

$$\underbrace{K + \frac{b}{1-R} - \frac{W - (\beta R)^{-1/\theta}\pi}{1 + \beta^{-1/\theta} R^{(\theta-1)/\theta}}}_{F(K,b)} = 0. \tag{35}$$

We want to show that $\left(\partial K/\partial b\right)|_{b=0} = -F_b(K,0)/F_K(K,0) < 0$ when $\theta \geq 1$. Note that, when $b = 0$, $F(K,b)$ can be rewritten as

$$F(K,0) = K^{\alpha}\left(K^{1-\alpha} + \frac{(1-\gamma)(\beta\alpha\gamma A)^{-1/\theta} AK^{(1-\alpha)/\theta} - \gamma(1-\alpha)A}{1 + \beta^{-1/\theta}(\alpha\gamma A)^{(\theta-1)/\theta} K^{-(1-\alpha)(\theta-1)/\theta}}\right)$$

where $R = \gamma\alpha(AK^{\alpha} + b)K^{-1}$, $W = \gamma(1-\alpha)(AK^{\alpha} + b)$ and $\pi = (1-\gamma)(AK^{\alpha} + b)$ were used. When $\theta \geq 1$, $F(K,0)$ increases in K, implying $F_K(K,0) > 0$. We also have that

$$F_b(K,0) = (1-R)^{-1} - \gamma\left[\frac{1-\alpha}{\theta} + \frac{\theta-1}{\theta}\frac{\alpha}{R}\right]\left(1 + \beta^{-1/\theta} R^{(\theta-1)/\theta}\right)^{-1}.$$

When $\theta \geq 1$, the above expression is weakly declining in R. From (33), we have $R > \alpha/(1-\alpha)$. Thus, when $\theta \geq 1$, it follows that

$$F_b(K,0) > \underbrace{(1-R)^{-1}}_{>1} - \underbrace{\gamma(1-\alpha)\left(1 + \beta^{-1/\theta} R^{(\theta-1)/\theta}\right)^{-1}}_{<1} > 0.$$

This completes the proof. ∎

B Cournot Competition and Fixed Costs

In this section, I consider an alternative model of the market structure. Preferences and demographics are as in the baseline model, and so equations (1) to (3) hold. The only differences concern technology and the market structure.

Technology and Market Structure Firms are identical and have the same production function $F_{ij}(k,l) = A\,k^{\alpha}\,l^{1-\alpha}$ (i.e. there are no productivity differences). However, now production entails a fixed cost $f > 0$, which is in units of the final good. To give a role to fixed costs, and allow for a variable number of firms, I shall however depart from limit pricing. For this reason, I assume Cournot competition. Let $n_{it} \subseteq \mathbb{N}$ be the number of active firms in industry i. All firms $j \in \{1, \ldots n_{it}\}$ that entered and payed the fixed cost $f > 0$ solve

$$\max_{y_{jit}} \quad \left(p_{it} - \frac{\theta_t}{A}\right) y_{jit} \qquad \text{s.t.} \quad p_{it} = \left(\frac{Y_t}{y_{it}}\right)^{1-\rho}$$
$$y_{it} = \sum_{k=1}^{n_{it}} y_{kit}.$$

The solution to this problem yields a markup

$$\mu_{it} := \frac{p_{it}}{(\theta_t/A)} = \frac{n_{it}}{n_{it} - (1-\rho)},$$

which is decreasing in the number of active firms. In every period, each active firm can issue an amount of bubbly stocks $b_{jit} = b \geq 0$. This is assumed to be exogenous and independent of output.[21] The equilibrium number of firms n_{it} is determined by two conditions: (i) all active firms must break even, and (ii) no additional firm can profitably enter. Formally,

$$\left[\pi\left(n_{it}, \theta_t, Y_t\right) - (f-b)\right]\left[\pi\left(n_{it}+1, \theta_t, Y_t\right) - (f-b)\right] \leq 0, \tag{36}$$

where $\pi\left(n_{it}, \theta_t, Y_t\right) := \left(p_{it} - \theta_t/A\right) y_{jit}$ are production profits (revenues minus variable costs). The profit function $\pi\left(n_{it}, \theta_t, Y_t\right)$ can be shown to be decreasing in the number of active firms n_{it}, and to approach zero as $n_{it} \to \infty$, as stated in the following lemma.

[21] As before, one could assume that there is a fixed amount of bubbly stocks at the industry level, which is distributed according to market shares. This is considered in Queirós (2021).

Lemma 1. *The profit function* $\pi(n_{it}, \theta_t, Y_t)$ decreases in n_{it} and satisfies $\lim_{n_{it}\to\infty} \pi(n_{it}, \theta_t, Y_t) = 0$.

Proof. We have $\pi(n_{it}, \theta_t, Y_t) = (1-\rho)[n_{it} - (1-\rho)]^{\rho/(1-\rho)} n_{it}^{(\rho-2)/(1-\rho)} (\theta_t/A)^{-\rho/(1-\rho)} Y_t$. Furthermore, $\partial\pi(\cdot)/\partial n_{it} < 0 \Leftrightarrow 2(n_{it}-1) + \rho > 0$, which is always satisfied for $n_{it} \geq 1$. This proves the first statement. To prove the second, note that $\lim_{n_{it}\to\infty} \mu_{it} = 1$. ■

It immediately follows from Lemma 1 that, if there is a value n_{it} satisfying (36), such a value is unique and increasing in b. Figure 7 shows some equilibrium variables as a function of b. Given the parameters chosen, absent the formation of bubbles ($b = 0$), the industry consists of a monopoly ($n_i = 1$). For sufficiently large values of b, more firms will enter, even if they make negative operating profits (third panel). Even when firms make an operating loss, their entry necessarily results in higher consumer welfare, since total output y_{it} increases. To assess the efficiency gains associated with the entry of additional firms, we must evaluate the change in the total industry surplus[22]

$$\Omega_{it} := \underbrace{\int_0^{y_{it}} \left[\left(\frac{Y_t}{x} \right)^{1-\rho} - p_{it} \right] dx}_{\text{consumer surplus}} + \underbrace{\left(p_{it} - \frac{\theta_t}{A} \right) y_{it} - n_{it} f}_{\text{producer surplus}}.$$

The last panel of Figure 7 shows Ω_i as a function of b. As b increases, output y_i increases and the price p_i decreases. This results in higher consumer welfare, but in a lower producer surplus. When b is small, few firms produce, and the increase in consumer welfare exceeds the decrease in producer surplus. As b becomes large, the increase in consumer welfare is outweighed by the reduction in the producer surplus. In this example, the total surplus is maximized for $n_i = 2$.

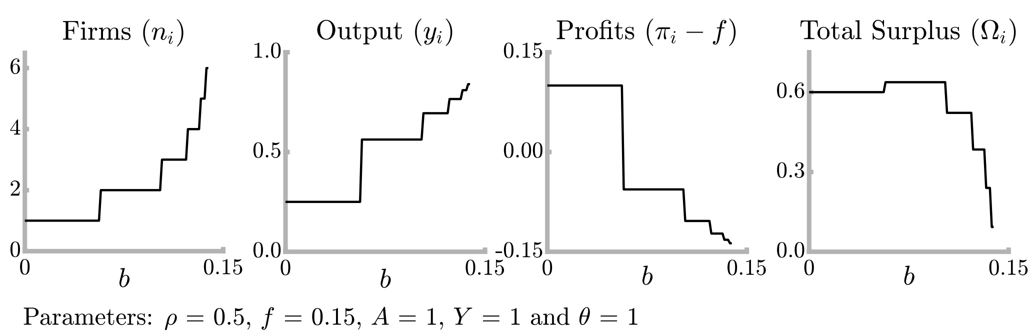


Figure 7: Industry equilibrium with exogenous firm bubbles

[22]This is a measure of economic efficiency that ignores the rents stemming from the issuance of bubbly stocks.

B.1 General Equilibrium

Definition. An equilibrium consists of a non-negative sequence for aggregate bubbles (government debt and stocks), capital, labor and consumption $\left\{D_t,\, B_t,\, K_t,\, L_t,\, C_t^y,\, C_t^o\right\}_{t=0}^{\infty}$, factor prices $\{W_t,\, R_t\}_{t=0}^{\infty}$, a set of active firms $\mathcal{I}_{it}$ and firm policies $\left\{p_{it},\, y_{it}^j,\, k_{it}^j,\, l_{it}^j,\, B_{it}^j\right\}_{t=0}^{\infty}$, $\forall\, i \in [0,1]$ and $j \in \mathcal{I}_{it}$, for all $t \geq 0$ such that (i) individuals optimize, (ii) all active firms maximize profits, (iii) the number of firms is given by (36), (iv) government debt and bubbly stocks evolve according to (8) and (9), and (v) labor and capital markets clear, i.e.

$$L_t = 1 \tag{37}$$

and

$$K_{t+1} \;=\; W_t - \left(\, D_t \,+\, B_t \,+\, C_t^y \,\right). \tag{38}$$

Next, I characterize the within-period equilibrium (in which I take K_t as given).

Within-Period Equilibrium The next proposition characterizes the equilibrium number of firms as a function of the aggregate capital stock (K_t). It states that, when $\rho \geq 1/2$, there is a unique equilibrium. Moreover, the aggregate number of firms (weakly) increases in K_t.

Proposition 18. *(Equilibrium Number of Firms) Let $\underline{K}(n)$ and $\overline{K}(n)$ be defined as*

$$\underline{K}(n) := \left(\frac{f}{A}\frac{n^2}{1-\rho}\right)^{1/\alpha},$$

$$\overline{K}(n) := \left[\frac{f}{A}\frac{(n+1)^2}{1-\rho}\left(\frac{n+1}{n}\frac{n-(1-\rho)}{n+\rho}\right)^{\rho/(1-\rho)}\right]^{1/\alpha}.$$

When $\rho \geq 1/2$ and no bubbles are traded, there is a unique within-period equilibrium. In particular, if

1. *$K_t \in \left[\underline{K}(n), \overline{K}(n)\right]$, then all industries have n firms.*
2. *$K_t \in \left[\overline{K}(n), \underline{K}(n+1)\right]$, then a fraction $\eta_t \in (0,1)$ of the industries features $n+1$ firms, and the remaining fraction $1-\eta_t$ features n firms. η_t is increasing in the aggregate capital stock K_t.*

Proof. In a symmetric equilibrium with aggregate capital K and n firms in every industry, each firm makes production profits equal to $\pi(n,K) = n^{-2}(1-\rho)AK^{\alpha}$. If one industry were to have $n+1$ firms, profits in this industry would be

$$\tilde{\pi}(n,K) = \frac{1-\rho}{(n+1)^2}\left(\frac{n}{n+1}\frac{n+\rho}{n-(1-\rho)}\right)^{\rho/(1-\rho)} AK^{\alpha}.$$

The thresholds $\underline{K}(n)$ and $\overline{K}(n)$ are such that $\pi\left(n, \underline{K}(n)\right) = f$ and $\tilde{\pi}\left(n, \overline{K}(n)\right) = f$. It follows that $\underline{K}(n) < \overline{K}(n)$, given $\rho < 1$. When $K \in \left(\overline{K}(n), \underline{K}(n+1)\right)$, a fraction $\eta \in (0,1)$ of the industries contains $n+1$ firms, while the rest has n firms. The equilibrium value of η is such that, in industries with $n+1$ firms, profits are zero. This zero profit condition is

$$\frac{1-\rho}{(n+1)^2}\,\nu(n)^{\rho}\;\underbrace{\frac{\left[1+\eta\left(\nu(n)^{\rho}-1\right)\right]^{(1-\rho)/\rho}}{1+\eta\left(\nu(n)-1\right)}}_{g(\eta)}\;AK^{\alpha} = f\,,$$

where

$$\nu(n) := \left(\frac{n}{n+1}\frac{n+\rho}{n-(1-\rho)}\right)^{1/(1-\rho)}.$$

If $g(\eta)$ is monotone in η, the previous equation defines a unique η as a function of K. Furthermore, if $g(\eta)$ decreases in η, η is increasing in K. We have $(dg/d\eta) < 0$ if and only if $\nu(n)^{\rho}(1-\rho) - 1 < (2\rho-1)\,\eta\left(\nu(n)^{\rho}-1\right)(\nu(n)-1)$, which is always satisfied if $\rho \geq 1/2$. ■

$\underline{K}(n)$ and $\overline{K}(n)$ correspond to the minimum and maximum values of K_t that are consistent with n in all industries. When $K_t < \underline{K}(n)$ or $K_t > \overline{K}(n)$, not all industries can have n firms. When $\rho < 1/2$, there can be multiple equilibria. In particular, for the same capital stock K_t, it is possible to have a symmetric equilibrium with n firms in every industry, and an asymmetric equilibrium with n firms in some industries, and $n-1$ firms in the other industries. In the first equilibrium, aggregate productivity is high and equal to A. In the second, aggregate productivity is lower than A (because industries are asymmetric and there is misallocation). Note that, when ρ is low, there are strong complementarities across industries.

Aggregate Output When a fraction $\eta_t \in (0,1)$ of the industries features $n+1$ firms, and the remaining fraction $1-\eta_t$ features n firms, aggregate output is equal to

$$Y_t \;=\; \underbrace{\frac{\left[1+\eta_t\left(\nu(n)^{\rho}-1\right)\right]^{1/\rho}}{1+\eta_t\left(\nu(n)-1\right)}}_{=\varphi_t}\,A\,K_t^{\alpha}.$$

For what follows, it will be useful to define net output, as the difference between aggregate output Y_t and the total value of resources spent in fixed costs.

Definition. Let $N_t := \int_0^1 n_{it}\,di$ denote the aggregate mass of firms. Net output is the difference between total output and the total mass of resources spent in fixed costs, $Y_t^{\text{net}} := Y_t - N_t \cdot f$.

Factor Shares and Factor Prices When a fraction $\eta_t \in (0,1)$ of the industries features $n+1$ firms, and the remaining fraction $1-\eta_t$ features n firms, the aggregate factor share is given by[23]

$$\sigma_t := \frac{R_t K_t + W_t L_t}{Y_t} = \frac{n-(1-\rho)}{n} \frac{1+\eta_t\,(\nu\,(n)-1)}{1+\eta_t\left(\nu\,(n)^{\rho}-1\right)}. \tag{39}$$

The interest rate is equal to

$$R_t \;=\; \alpha\,\frac{n-(1-\rho)}{n}\;\left[1+\eta_t\left(\nu\,(n)^{\rho}-1\right)\right]^{(1-\rho)/\rho}\;A\,K_t^{\alpha-1}. \tag{40}$$

K_t can have an ambiguous effect on R_t. On the one hand, a higher K_t leads to lower R_t because of decreasing returns. On the other hand, it leads to a higher η_t and hence a higher capital share; this leads to higher R_t. Lemma 2 states sufficient conditions for the interest rate to be monotonically decreasing in the capital stock.

Lemma 2. *(Equilibrium Interest Rate) If $\rho \geq 1/2$ and*

$$\alpha \leq 1 - \frac{1-\rho^2}{\rho}\frac{(1+\rho)^{\rho/(1-\rho)} - (2\,\rho)^{\rho/(1-\rho)}}{(1+\rho)^{1/(1-\rho)} - (2\,\rho)^{1/(1-\rho)}},$$

then the interest rate in (40) monotonically decreases in the capital stock K_t.

Proof. If $K_t \in \left[\underline{K}\,(n)\;,\,\overline{K}\,(n)\,\right)$, then $\eta_t = 0$ and R_t decreases in K_t. If $K_t \in \left[\,\overline{K}\,(n)\;,\,\underline{K}\,(n+1)\,\right)$, the no profit condition in industries with $n+1$ firms can be written as

$$\frac{1-\rho}{n+\rho}\frac{\nu\,(n)\,A}{\alpha}\;\left[\frac{K_t}{1+\eta_t\,(\nu\,(n)-1)}\right]\;R_t \;=\; f.$$

It suffices to show that the term in square brackets is increasing in K_t. This term can be written as

$$\left[1+\eta_t\,(\nu\,(n)-1)\right]^{(1-\alpha)/\alpha}\;\left[1+\eta_t\left(\nu\,(n)^{\rho}-1\right)\right]^{-(1-\rho)/(\alpha\rho)}.$$

Under $\rho \geq 1/2$, η_t is increasing in K_t. The previous expression is thus increasing in K_t if

$$1-\alpha > \frac{1-\rho}{\rho}\;\underbrace{\frac{1+\eta_t\,(\nu\,(n)-1)}{1+\eta_t\left(\nu\,(n)^{\rho}-1\right)}}_{g(\eta_t)}\;\frac{\nu\,(n)^{\rho}-1}{\nu\,(n)-1}.$$

[23] Aggregate gross profits are equal to $\Pi_t := (1-\sigma_t)\,Y_t$, while net profits are equal to $\Pi_t^{net} := (1-\sigma_t)\,Y_t - N_t \cdot f$.

$g(\eta_t)$ can be shown to be increasing in η_t. Thus, the previous condition is implied by

$$1-\alpha > \frac{1-\rho}{\rho}\frac{\nu(n)-\nu(n)^{1-\rho}}{\nu(n)-1}.$$

The left hand-side can be shown to be increasing in $\nu(n)$. Since $\nu(n)$ is decreasing in n, a sufficient condition is

$$\alpha \leq 1-\frac{1-\rho}{\rho}\frac{\nu(1)-\nu(1)^{1-\rho}}{\nu(1)-1}.$$

■

Let $R(K)$ describe the equilibrium interest rate as a unique function of K. Thus, under the conditions of Lemma 2, we have $R'(K)<0$.

Capital Dynamics and Steady-State (Bubbleless Equilibrium) The dynamics of the capital stock is described in Proposition 19. It states that under the conditions of Lemma 2, the capital stock follows a unique path and converges to a steady-state that can be shown to be weakly decreasing in fixed costs.

Proposition 19. *(Equilibrium Dynamics) Assume that the conditions of Lemma 2 are satisfied. Then, in a bubbleless equilibrium, the capital stock follows a unique path*

$$K_{t+1} = \min\left\{W_t\,,\,R^{-1}\left(\frac{1}{\beta}\right)\right\}. \tag{41}$$

Moreover, it converges to a unique steady-state K^* implicitly defined by

$$R(K^*) = \max\left\{\frac{\alpha}{1-\alpha},\frac{1}{\beta}\right\}. \tag{42}$$

The steady-state K^* is weakly decreasing in fixed costs f.

Proof. From Lemma 2, the interest rate is monotonically decreasing in K_t. Therefore, there is a unique $\widetilde{K}$ such that $R\left(\widetilde{K}\right)=1/\beta$ and $R(K)\geq 1/\beta$ if and only if $K<\widetilde{K}$. All savings are converted into capital $(K_{t+1}=W_t)$ provided that $R(W_t)\geq 1/\beta$. Otherwise, only a fraction of all savings are converted into capital $(K_{t+1}<W_t)$ and $R(K_{t+1})=1/\beta$, so that young individuals are indifferent between consuming when young and old. Then (19) obtains. Since $W_t = (1-\alpha)\alpha^{-1}R(K_t)\,K_t$, on a steady-state where all savings are converted into capital, we have $R(K^*)=\alpha/(1-\alpha)$. Then (42) obtains.

To prove that K^* is weakly decreasing in f note the following. If, on a steady-state, we have $K^* \in \left(\underline{K}(n), \overline{K}(n)\right)$ for some $n \subseteq \mathbb{N}$, then

$$K^* = \left(\frac{n-(1-\rho)}{n} A \min\{\alpha\beta, 1-\alpha\}\right)^{1/(1-\alpha)},$$

and a marginal change in f will not affect K^*. If the steady-state is such that $K^* \in \left[\overline{K}(n), \underline{K}(n+1)\right]$, for some $n \subseteq \mathbb{N}$, then K^* satisfies

$$\underbrace{\frac{1-\rho}{(n+1)^2}\left[1+\left(\frac{K^*}{f}\frac{R^* A \nu(n)}{\alpha}-1\right)\frac{\nu(n)^\rho-1}{\nu(n)-1}\right]^{(1-\rho)/\rho}(K^*)^{\alpha-1}}_{F(K^*,f)} = \frac{R^* \nu(n)^{1-\rho}}{\alpha}.$$

R^* is constant and independent of K^* and f. We have $\partial F(K^*, f)/\partial f < 0$. Thus, from the implicit function theorem, we have that K^* decreases in f if and only if $\partial F(K^*, f)/\partial K^* < 0$, which is equivalent to

$$1-\alpha > \frac{1-\rho}{\rho}\frac{1+\eta^*(\nu(n)-1)}{1+\eta^*\left(\nu(n)^\rho-1\right)}\frac{\nu(n)^\rho-1}{\nu(n)-1}.$$

This condition is satisfied under the conditions of Lemma 2 (see the proof of Lemma 2). ■

It is possible to give an analytical characterization of the steady-state capital stock K^* when it is characterized by full symmetry across industries. Proposition 20 says that, when fixed costs are within a certain interval, the steady-state will be characterized by n firms in all industries.

Proposition 20. *(Symmetric Steady-State) Let $\underline{f}^*(n)$ and $\overline{f}^*(n)$ be defined as*

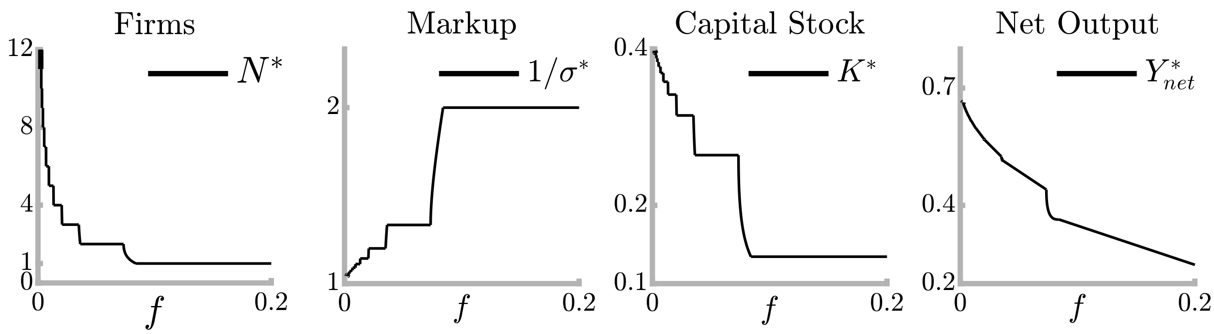
$$\underline{f}^*(n) := \frac{1-\rho}{(n+1)^2} A\left[\frac{n-(1-\rho)}{n} A \min\{\alpha\beta, 1-\alpha\}\right]^{\alpha/(1-\alpha)}\left(\frac{n}{n+1}\frac{n+\rho}{n-(1-\rho)}\right)^{\rho/(1-\rho)},$$

$$\overline{f}^*(n) := \frac{1-\rho}{n^2} A\left[\frac{n-(1-\rho)}{n} A \min\{\alpha\beta, 1-\alpha\}\right]^{\alpha/(1-\alpha)}.$$

When no bubbles are traded and the conditions of Lemma 2 are satisfied, the economy converges to a steady-state with n firms in all industries if and only if

$$f \in \left[\underline{f}^*(n), \overline{f}^*(n)\right]. \tag{43}$$

In this case, the steady-state capital stock is equal to



Parameters: $\rho = 0.5$, $\alpha = 0.4$, $\beta = 3$ and $A = 1$

Figure 8: Steady-state variables as a function of fixed costs (f)

$$K^* = \left(\frac{n - (1-\rho)}{n} A \min\{\alpha\beta, 1-\alpha\} \right)^{1/(1-\alpha)}. \tag{44}$$

Proof. When n firms are active in all industries, the aggregate factor share is $\sigma = [n - (1-\rho)]/n$. Combining the definitions of $\underline{K}(n)$ and $\overline{K}(n)$ from Proposition 18, with (41) and (42) from Proposition 19, (43) and (44) obtain. ∎

Equation (44) is the counterpart of (17). In an equilibrium with n firms in every industry, the aggregate factor share is $[n-(1-\rho)]/n$, while in the model of section 2 it was equal to γ.

Figure 8 shows the steady-state values of the average number of firms, the aggregate markup, the capital stock and net output as a function of fixed costs f. Higher fixed costs result in a (weakly) lower number of firms, a (weakly) higher markup, a (weakly) lower capital stock a (strictly) lower level of net output.

Overaccumulation of Capital With fixed costs, the definition of capital overaccumulation must be adjusted. It will reflect the impact of a marginal increase in investment on net output.

Definition. (Overaccumulation of Capital) The bubbleless steady-state K^* features overaccumulation of capital if

$$\left.\frac{\partial Y^{\text{net}}}{\partial K}\right|_{K=K^*} < 1.$$

Contrarily to the model of section 2, it is not possible to give a full analytical characterization of the conditions for capital overaccumulation. One can only characterize these conditions in the particular case of a steady-state with identical industries. Proposition 21 states that, if the economy converges to a steady-state with $n \in \mathbb{N}$ firms in all industries, it will be characterized by excessive capital when fixed costs are sufficiently low.

Proposition 21. *(Overaccumulation of Capital) Suppose that the economy converges to a steady-state where all industries are symmetric and have $n \in \mathbb{N}$ firms. Then, the bubbleless steady-state features overaccumulation of capital if*

$$\max\left\{\frac{1}{\beta}, \frac{\alpha}{1-\alpha}\right\} < 1 \tag{45}$$

and

$$f < A\left[1 - \max\left\{\frac{1}{\beta}, \frac{\alpha}{1-\alpha}\right\}\right]^2 \frac{(A\alpha)^{\alpha/(1-\alpha)}}{1-\rho}. \tag{46}$$

Proof. In a symmetric equilibrium with n firms per industry, we have $\partial Y/\partial K = \alpha A K^{\alpha-1}$. From (44), we have $\partial Y/\partial K|_{K=K^*} = \left[\left(1 - n^{-1}(1-\rho)\right)\min\{\beta, (1-\alpha)/\alpha\}\right]^{-1}$. It must be the case that $R^* < 1$, which is equivalent to $\max\{1/\beta, \alpha/(1-\alpha)\} < 1$ from (42). Then, the condition for capital overaccumulation becomes equivalent to

$$n > (1-\rho)\left[1 - \max\left\{\frac{1}{\beta}, \frac{\alpha}{1-\alpha}\right\}\right]^{-1} \equiv \underline{n}.$$

Combining this condition with $f < \overline{f}^*(\underline{n})$, the condition obtains. ∎

Let us analyze Proposition 21. First note that, when (45) is not satisfied, there is never overaccumulation of capital (since $R^* > 1$). Second, when (45) is satisfied, the steady-state features overaccumulation when fixed costs are low. Suppose, for example, that fixed costs take initially a value $f \in \left(\underline{f}^*(1), \overline{f}^*(1)\right)$, but then decrease to some $\tilde{f} \in \left(\underline{f}^*(2), \overline{f}^*(2)\right)$. The initial steady-state is characterized by a monopoly in all industries, but the new steady-state is characterized by a full set of duopolies. From Proposition 20, the new steady-state level of capital is greater than the previous one, i.e.

$$\underbrace{\left(\rho A \min\{\alpha\beta, 1-\alpha\}\right)^{1/(1-\alpha)}}_{K^*} < \underbrace{\left(\frac{1+\rho}{2} A \min\{\alpha\beta, 1-\alpha\}\right)^{1/(1-\alpha)}}_{\tilde{K}^*},$$

implying

$$\underbrace{\frac{1}{\rho}\max\left\{\frac{1}{\beta}, \frac{\alpha}{1-\alpha}\right\}}_{\partial Y/\partial K|_{K=K^*}} > \underbrace{\frac{2}{1+\rho}\max\left\{\frac{1}{\beta}, \frac{\alpha}{1-\alpha}\right\}}_{\partial Y/\partial K|_{K=\tilde{K}^*}}.$$

Proposition 21 gives the conditions for overaccumulation when industries are symmetric. When industries are not identical, it becomes harder to analytically characterize $\partial Y^{\text{net}}/\partial K|_{K=K^*}$.

One can however obtain a numerical characterization. Figure 9 shows $\partial Y^{\text{net}}/\partial K|_{K=K^*}$ as a function of f (for fixed values of ρ, α and β). Some aspects are worth mentioning. First, $\partial Y^{\text{net}}/\partial K|_{K=K^*}$ is not defined at $\left\{\underline{f}^*(n), \overline{f}^*(n)\right\}_{n\in\mathbb{N}}$ and is flat at $f \in \left(\underline{f}^*(n), \overline{f}^*(n)\right)$.[24] Second, as discussed, when fixed costs increase from some $f \in \left(\underline{f}^*(n+1), \overline{f}^*(n+1)\right)$ to some $\tilde{f} \in \left(\underline{f}^*(n), \overline{f}^*(n)\right)$, $\partial Y^{\text{net}}/\partial K|_{K=K^*}$ increases. Third, within $f \in \left(\overline{f}^*(n), \underline{f}^*(n+1)\right)$, $\partial Y^{\text{net}}/\partial K|_{K=K^*}$ happens to be decreasing in f. To understand this, note that an increase in f has a dual impact on $\partial Y^{\text{net}}/\partial K|_{K=K^*}$ in these intervals. On the one hand, as f increases, K^* declines; because of decreasing returns, this translates into a higher marginal product of capital. On the other hand, as f increases, more resources are absorbed in fixed costs whenever entry increases. Note that a marginal increase in K will boost entry when $f \in \left(\overline{f}^*(n), \underline{f}^*(n+1)\right)$; as a consequence, $\partial Y^{\text{net}}/\partial K|_{K=K^*}$ may actually decrease in these intervals.

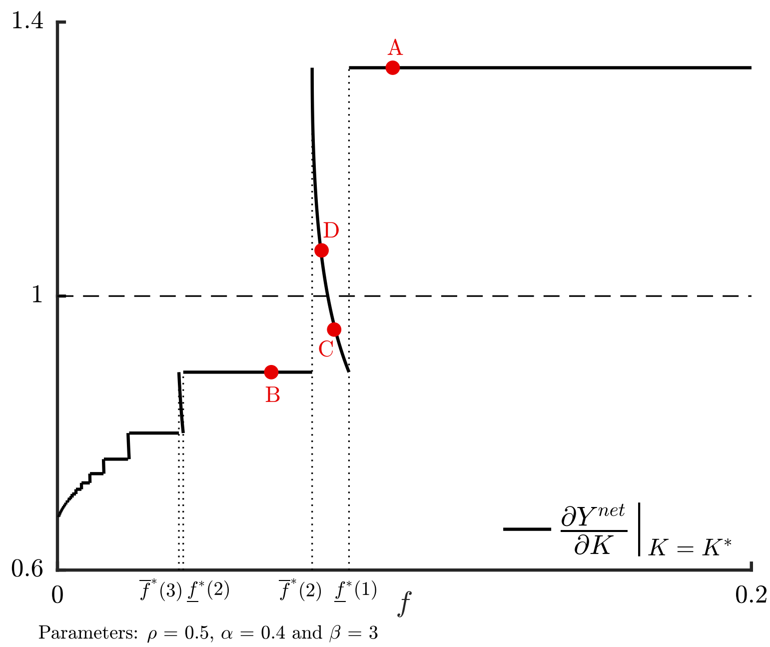


Figure 9: Condition for overaccumulation of capital

Consider again Figure 9. In point A, where all industries consist of a monopoly, we have $\partial Y^{\text{net}}/\partial K|_{K=K^*} > 1$. In point B, where all industries consist of a duopoly, $\partial Y^{\text{net}}/\partial K|_{K=K^*} < 1$. An increase of fixed costs from B to A will hence make the economy transition from a steady-state with overaccumulation of capital to a lower steady-state without overaccumulation.

Let us now consider points C and D. In both cases we have $f \in \left(\overline{f}^*(2), \underline{f}^*(1)\right)$; thus,

[24]For any $f \in \left(\underline{f}^*(n), \overline{f}^*(n)\right)$, the steady-state K^* is constant across f (and given by (44)). In this region, a marginal increase in K has always the same impact on Y^{net}, since it does not affect entry.

both economies converge to steady-states where some industries consist of a monopoly, while some others consist of a duopoly.[25] In point D, we have $\partial Y^{\text{net}}/\partial K|_{K=K^*} > 1$, while in point C, $\partial Y^{\text{net}}/\partial K|_{K=K^*} < 1$. Therefore, an increase in fixed costs from D to C will make the economy transition from a steady-state without capital overaccumulation, to a steady-state with overaccumulation. Note that, starting from any of the steady-states represented by C and D, a marginal increase in the capital stock will boost entry (contrarily to what happens in A and B). Since C is characterized by higher f, more resources will be spent in fixed costs as entry increases. Therefore, a marginal increase in the capital stock will have a lower impact on net output at C.

Rational Bubbles Proposition 22 characterizes the conditions for a rational bubble equilibrium. As in the model of section 2, this condition only depends on the capital elasticity α.

Proposition 22. *(Rational Bubbles) Assume that the conditions of Lemma 2 are satisfied. Then, rational bubbles can be traded if and only if $\alpha < 1/2$.*

Proof. Using (42) and given that $\beta > 1$, it follows that $R^* < 1$ if and only if $\alpha < 1/2$. ■

Bubbly stocks bubbles reduce capital accumulation when, in a bubbleless equilibrium, all savings are converted into capital $(K_{t+1} = W_t)$. However, as in the model of section 2, they can lead to higher investment if there is first period consumption $\left(C_t^y > 0\right)$. Table 2 shows the steady-state values of some variables for different levels of bubbly stock issuance b. Given the parameters chosen, there is always positive first period consumption $(C^{y*} > 0)$. The larger is the value of bubbly stock issuance (b), the higher is the steady-state number of firms (N^*), the stock of capital (K^*) and output (Y^*). For the particular parameters chosen, net output declines.

b	N^*	K^*	Y^*	Y^*_{net}	R^*	B^*	C^{y*}
0.00	1	0.23	0.69	0.59	0.5	0.00	0.12
0.04	2	0.31	0.75	0.55	0.5	0.08	0.08
0.02	3	0.34	0.76	0.46	0.5	0.16	0.01

Parameters: $\rho = 2/3$, $\alpha = 1/4$, $\beta = 2$, $A = 1$, $f = 0.1$

Table 2: Steady-state characterization with bubbly stock issuance

[25] The fraction of monopolies is higher in C, since fixed costs are also larger.

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