Equilibrium Existence in Games with Ties

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May 2022

Abstract

We provide conditions that simplify applying Reny's (1999) better-reply security to Bayesian games, and use these conditions to prove the existence of equilibria for classes of games in which payoff discontinuities arise only at "ties." These games include a general version of all-pay contests, first-prize auctions with common values, and Hotelling models with incomplete information.

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[†]We thank the audiences of the Bern Workshop on Contest Theory, in Bern, Switzerland; the Fifth Annual Conference on Contests: Theory and Evidence in Norwich, UK; and the Nineteenth Annual SAET Conference at Ischia, Italy. We are especially grateful to Blake Allison, Guilherme Carmona, Srihari Govindan, Idione Meneghel, Philip Reny, Nicholas Yannelis, and the TE editor and referees for useful discussions and comments.

1 Introduction

Games with discontinuous payoffs arise naturally in various settings. In a standard auction, if several bidders submit the same highest bid the prize is allocated according to a lottery but each tying bidder can obtain the prize with certainty by increasing her bid slightly. In a Hotelling model, firms situated at the same location split their set of customers but each firm can typically increase its share discretely by changing its location slightly. Additional examples abound.

The existence of equilibria in many games with discontinuous payoffs, especially ones with incomplete information, is still an open question.[1] Proving general existence results is difficult because the presence of discontinuous payoffs precludes a direct use of most fixed-point theorems, such as Kakutani's theorem and its generalizations. In addition, the existence of an equilibrium may depend on subtle details of the setting. For example, in first-price auctions and all-pay auctions equilibrium may not exist if some bidder's valuation for the prize is 0 with positive probability.[2] But this is the only reason that an equilibrium may not exist in a general class of contests that includes all-pay auctions, as we show in Section 6.1, whereas other distributional assumptions are required for equilibrium existence even in private-value first-price auctions (Olszewski, Reny, and Siegel (2022)).

An obvious approach to proving the existence of equilibria is to approximate the original game by a sequence of games with a finite number of actions and types, for which the existence follows from the Kakutani fixed-point theorem. However, any particular sequence of equilibria of the approximating games, or even all such sequences, need not converge to an

[1] For example, existence of equilibrium in first-price auctions with interdependent values and correlated signals is still an open question. Special cases have been studied, including bidders with affiliated signals (Milgrom and Weber (1982) and Reny and Zamir (2004)) and only two bidders (Govindan and Wilson (2010)) among others.

[2] To see this, suppose there are two bidders. Bidder 1's valuation is 0 and bidder 2's valuation is drawn uniformly from the interval $[0, 1]$. If bidder 1 bids 0, then bidder 2 does not have a best response, both in a first-price and in an all-pay auction. But if bidder 1 bids more than 0 in a first-price auction she must lose with certainty, so low types of bidder 2 are not best responding. (This example is due to Lebrun (1996)). And in an all-pay auction bidder 1 must bid 0.

equilibrium of the original game, even if the original game has an equilibrium. Reny (1999) suggested a somewhat different approach,[3] and obtained a result that subsumed most existence results previously available for games with a continuum of actions and discontinuous payoffs. Instead of sequences of games with a finite number of actions and types, he studied sequences of games with continuous payoffs that approximate the original game with discontinuous payoffs. Reny's (1999) results have been used subsequently by many authors. The two approaches complement one another, and which approach is more useful may depend on the details of the setting.[4]

This paper provides tools for establishing equilibrium existence in distributional strategies[5] for a general class of Bayesian "games with ties," in which payoff discontinuities may arise only when players "tie," for example, when winning bidders submit the same bid in an auction or when firms choose the same location in a Hotelling model. We first introduce a condition, *improving deviations*, and prove that this condition guarantees equilibrium existence. The condition is similar to a combination of a weaker version of Dasgupta and Maskin's (1986) *reciprocal upper semi-continuity* (RUSC) and a version of Reny's (1999) *payoff security*. We then specialize our condition to games with ties to obtain two other conditions, *favorable tie breaking* and *favorable tie breaking on average*. These two conditions say, roughly, that by breaking ties unilaterally players can increase the sum of their payoffs above the value of an upper semi-continuous envelope of the sum of payoffs. Favorable tie breaking requires this increase for every strategy profile in which ties whose resolution affects at least one player's payoff arise with positive probability, whereas favorable tie breaking on average requires the increase only for strategy profiles in which such ties arise "on average." The former condition is more demanding but easier to check.

Our result that improving deviations implies equilibrium existence follows from Reny's

[3]See also the work of McLennan, Monteiro, and Tourky (2011), Barelli and Meneghel (2014), and Reny (2016), which generalizes Reny's (1999) earlier result.

[4]The two approaches can also be unified for mixed strategies (see Bich and Laraki (2017), Section 4.1).

[5]Distributional strategies were introduced by Milgrom and Weber (1985). A player's distributional strategy is a distribution over the product of the player's types space and action space whose marginal on the type space coincides with the player's type distribution. We discuss the choice of this concept in Section 2.

(1999) main theorem. Our mathematical contribution is therefore limited. Instead, our goal is to "extract" from Reny's result tools for proving equilibrium existence in some applications, especially ones with incomplete information. Reny's (1999) result requires *better-reply security*, a property of the graph of the mapping from strategy profiles to payoff profiles. Verifying better-reply security is a demanding task in incomplete-information settings in which players use distributional strategies because the graph of the mapping from strategy profiles to payoff profiles is an infinite-dimensional object. Perhaps for this reason, various authors have used stronger but simpler conditions to prove equilibrium existence in various applications. For example, Carbonell-Nicolau and McLean (2018) used the RUSC of Dasgupta and Maskin (1986) to show existence in common-value first-prize auctions, and He and Yannelis (2016) provided their own condition, which they applied to all-pay auctions. This required strong, and sometimes restrictive, modelling assumptions. (We discuss the details in the literature review below.) By extracting what is needed from Reny's (1999) result for games with ties, we are able to relax some of these assumptions and prove equilibrium existence for general contests with many heterogeneous prizes and interdependent values, and for general common-value first-prize auctions. In addition, we show that our conditions also apply to Hotelling's models with incomplete information.[6]

1.1 Literature review

Early existence results for complete-information games with discontinuous payoffs were obtained by Dasgupta and Maskin (1986) and Simon (1987) by approximating the original game with a sequence of finite games. They were later subsumed by Reny's (1999) results.[7] Recently, several authors established equilibrium existence in various incomplete-information settings. This line of research includes Lebrun (1996, 1999) for private-value first-price auc-

[6]The Supplementary Appendix of our working paper, Olszewski and Siegel (2022), contains three additional applications, a two-firm Hotelling setting with private production costs, independent private-value first-price auctions with costly bids, and a two-player contest model in which players' bids determine not only who wins but which prize the winner obtains.

[7]There is also more recent research on the existence of equilibria in games of complete information (for example, Carmona (2009)), which is only remotely related to the present paper.

tions, Jackson and Swinkels (2005), Monteiro and Page (2007), Prokopovych and Yannelis (2014), He and Yannelis (2016), Carbonell-Nicolau and McLean (2018), Carmona and Podczeck (2018), and Reny (2016). Reny (2019) surveys the literature on equilibrium existence in discontinuous games.

Carmona and Podczeck (2018) seems the most closely related to our paper. They study games with sharing rules (in particular, first-price auctions and contests). They focus on providing conditions for the existence of a common equilibrium set for all possible sharing rules. Their results imply the existence of equilibria in some incomplete-information games with ties, for example, in a version of contests with *private* values (see their Example 5). The most important added value of our favorable tie breaking condition, which guarantees equilibrium existence, in the context of contests and first-price auctions is that it applies to these games with *interdependent* and *common* values, respectively. Carmona and Podczeck's (2018) results apply to a more restricted class of games with interdependent values, because their Φ-strong indeterminacy condition is violated in many such contests and first-price auctions.

Carbonell-Nicolau and McLean (2018) and He and Yannelis (2016) also apply their results to auctions. Carbonell-Nicolau and McLean (2018) show existence in some common-value auctions,[8] but those auctions must also satisfy a somewhat involved condition (Assumption C), which - as the authors point out - excludes settings in which whether a player prefers to win or lose a tie depends on other players' types. He and Yannelis (2016) prove existence in all-pay auctions in which bidders assign a common value to the single prize. Our favorable tie breaking condition applies to a general class of contests with multiple prizes, which includes all-pay auctions, in which the prizes are assigned possibly different (and interdependent) values by different bidders.

We are able to cover a range of applications due to several factors. First, Carbonell-Nicolau and McLean (2018) and He and Yannelis (2016) find conditions on the primitives of incomplete-information games which imply that behavioral or distributional strategies satisfy both Reny's (1999) payoff security and RUSC; these conditions in turn imply better-reply

[8] Their paper contains a summary of the extensive previous literature on the existence of equilibria in auctions.

security (according to one of Reny's (1999) results). In contrast, our conditions on games with ties imply better-reply security but are weaker than payoff security and RUSC. Second, we noticed that (a) a player's deviation (tie-breaking) actions can be allowed to depend on other players' strategies; moreover, (b) tie breaking is not required for every strategy profile in which ties whose resolution affects at least one player's payoff arise with positive probability; it is required only for strategy profiles in which such ties arise in expectation with respect to the other players' strategies. The former observation allows to include common-value auctions in which whether a player prefers to win or lose a tie depends on other players' types, and the latter observation allows to include natural extensions of the Hotelling model to incomplete information.

The paper by Allison, Bagh, and Lepore (2018) is also closely related to ours. Similarly to Carmona and Podczeck (2018), Allison, Bagh, and Lepore (2018) are concerned with the invariance of equilibria in some classes of games. They provide a sufficient condition for the invariance of equilibria, which they call *superior payoff matching*, and which is very similar to the existence of our tie breakers. Their condition turns out to be sufficient for the existence of pure-strategy equilibria in some games of *complete* information with discontinuous payoffs. These games include some contests and oligopolies with endogenous choices of product qualities. Allison, Bagh, and Lepore (2018) also refer directly to Reny's (1999) better-reply security and show existence even in some settings in which better-reply security fails.

Reny and Zamir (2004) and Prokopovych and Yannelis (2019) address the problem of equilibrium existence in first-price auctions with affiliated types and interdependent values. They provide sufficient conditions for the existence of sequences of monotone approximate equilibria whose limits are pure-strategy Bayes-Nash equilibria. Since these authors are concerned with monotone equilibria, their conditions are naturally more restrictive than ours.

2 Main existence theorem

Consider a (possibly) incomplete-information game among n players whose type spaces are $X_1, \ldots, X_n$, action spaces are $B_1 = \cdots = B_n = B$, and payoff functions are $u_i(x_1, ..., x_n, b_1, ..., b_n)$

for $i = 1, ..., n$. We assume that $X_1, ..., X_n$ and B are compact metric spaces and

$$0 \leq u_i(x_1, ..., x_n, b_1, ..., b_n) \leq \beta$$

for some $\beta > 0$ and all i and $x_1, ..., x_n, b_1, ..., b_n$. (This last assumption is equivalent to assuming that payoffs are bounded.) We denote by F_i the distribution of player i's type x_i and assume that players' types are independent. Our aim is to show the existence of equilibria for a class of such games.

We will show the existence of equilibria in *distributional strategies* (introduced by Milgrom and Weber (1985)), which are defined as follows. A distributional strategy of player i, which we will denote by μ_i or σ_i, is a probability measure on $X_i \times B_i$ whose marginal on X_i coincides with the distribution F_i of player i's type.

Equilibrium existence in distributional strategies implies equilibrium existence in behavioral strategies because there is a many-to-one payoff-preserving mapping from behavioral strategies to distributional strategies. That is, each distributional strategy corresponds to an equivalence class of behavioral strategies.[9] An additional, mathematical advantage of working with distributional strategies is that the strategy spaces have a convenient linear topological structure, which we will introduce next.[10]

We endow each player's set of (distributional) strategies with the weak* topology and observe that the set of strategies is compact.[11] We define the operations of addition and multiplication by a scalar in the usual way,[12] so the strategy sets are also convex. Given a strategy profile $\mu = (\mu_1, \ldots, \mu_n)$, we also denote by $\mu = \mu_1 \times \cdots \times \mu_n$, with some abuse

[9] See page 624 of Milgrom and Weber (1985). Our assumptions imply those required by Milgrom and Weber (1985).

[10] Another alternative, which we did not explore, would be to study distributional Bayesian equilibria. Balder and Rustichini (1994), Kim and Yannelis (1997), and Balbus et al. (2015) prove existence results for this concept and games with continuous payoffs, even when the number of players is infinite. This approach naturally dispenses with the independent type assumption.

[11] The set of probability measures over $X_i \times B$ is compact, and it is straightforward to verify that the set of strategies is a closed subset of the set of probability measures over $X_i \times B$. In addition, the weak* topology is metrizable.

[12] Given a constant λ, two measures μ and ν, and a measurable set S, we let $(\mu + \nu)(S) = \mu(S) + \nu(S)$, and $(\lambda\mu)(S) = \lambda\mu(S)$.

of notation, the product measure on the product space $X_1 \times \cdots \times X_n \times B^n$. Player i's payoff given a strategy profile μ is $U_i(\mu) = \int u_i(x_1, .., x_n, b_1, ..., b_n) d\mu$, that is, the expected value of u_i taken with respect to all players' strategies. (Throughout the paper we endow product spaces with the product topology, use the Borel σ-algebras, and assume that the payoff functions u_i are measurable.)

We first identify a sufficient condition for equilibrium existence, and later show that this condition is satisfied by many games in which payoff discontinuities arise only when multiple players take the same action. The condition says, roughly, that at any strategy profile with payoff discontinuities, the sum of players' payoffs can be increased more by players deviating unilaterally than by simultaneously changing players actions and types slightly. We formalize this condition, which we call *improving deviations*, with the following definitions.

Definition 1 *A* deviation plan *specifies for every profile of strategies μ, every player i, and every $\varepsilon > 0$ a measurable function $\tau_i^{\mu,\varepsilon} : X_i \times B_i \to B_i$ such that the* deviation strategy *μ_i^{ε}, which prescribes for type x_i action $\tau_i^{\mu,\varepsilon}(x_i, b_i)$ whenever μ_i prescribes action b_i,*[13] *satisfies*

$$U_i(\mu) - \varepsilon \leq U_i(\mu_i^{\varepsilon}, \mu_{-i}). \tag{1}$$

A continuous deviation plan *is a deviation plan such that every player i's payoff U_i is continuous at $(\mu_i^{\varepsilon}, \mu_{-i})$ as a function of the strategies of all players other than i.*

The existence of a continuous deviation plan is closely related to the game with distributional strategies being payoff secure (Reny (1999)).[14]

Definition 2 *A* payoff envelope *is an upper semi-continuous function $W(x_1, ..., x_n, b_1, ..., b_n)$ such that*

$$\sum_{i=1}^{n} u_i(x_1, ..., x_n, b_1, ..., b_n) \leq W(x_1, ..., x_n, b_1, ..., b_n). \tag{2}$$

The existence of a payoff envelope can be viewed as a weakening of the requirement that the sum of players' payoffs is upper semi-continuous in the game with distributional strategies (Dasgupta and Maskin (1986)).

[13] In particular, for any set $A = S_i \times Y_i$, where $S_i \subseteq X_i$ and $Y_i \subseteq B_i$ are measurable, $\mu_i^{\varepsilon}(A) = \mu_i((S_i \times B_i) \cap ((\tau_i^{\mu,\varepsilon})^{-1}(Y_i)))$.

[14] More precisely, the game is payoff secure if a continuous deviation plan exists.

Definition 3 *A game has* improving deviations *if there is a payoff envelope W and a continuous deviation plan such that if the payoff U_i of some player i is discontinuous at μ as a function of all players' strategies, then for some $\varepsilon > 0$,*

$$\int W(x_1, ..., x_n, b_1, ..., b_n) d\mu < \sum_{j=1}^{n} U_j(\mu_j^\varepsilon, \mu_{-j}), \tag{3}$$

where μ_j^ε is player j's deviation strategy corresponding to the deviation plan.

Intuitively, we typically think of W as the lowest possible, analytically tractable upper semi-continuous envelope of the sum of payoffs.[15] Similarly, if the payoff of player i is discontinuous at (x_i, b_i) and μ_{-i}, we typically think of $\tau_i^{\mu,\varepsilon}(x_i, b_i)$ as an action player i of type x_i that is close to b_i and resolves a payoff discontinuity in her favor, possibly leaving the action b_i unchanged, that is, setting $\tau_i^{\mu,\varepsilon}(x_i, b_i) = b_i$ for (x_i, b_i) and μ_{-i} such that the payoff of player i is continuous.

We will see that in many applications, W and $\tau_i^{\mu,\varepsilon}(x_i, b_i)$ are easy to find. The key condition, (3), says that when each player resolves the discontinuity in her favor the resulting sum of payoffs is higher than W, that is, higher than when players actions and types are changed slightly in a way that maximizes the sum of payoffs. Definitions 7 and 8 below adapt the condition to games in which payoff discontinuities arise only when multiple players take the same action.[16]

Theorem 1 *Every game with improving deviations has a Nash equilibrium in distributional strategies.*

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[15] We can in fact define the payoff envelope W as the lowest upper semi-continuous envelope of the sum of payoffs, that is,

$$W(x_1, ..., x_n, b_1, ..., b_n) = \inf_{O \varepsilon \mathcal{O}} \sup_{\bar{x}_1, ..., \bar{x}_n, \bar{b}_1, ..., \bar{b}_n \varepsilon O} . \sum_{i=1}^{n} u_i(\bar{x}_1, ..., \bar{x}_n, \bar{b}_1, ..., \bar{b}_n),$$

where O is the set of open neighborhoods of $x_1, ..., x_n, b_1, ..., b_n$. However, in some settings a different W may be easier to work with.

[16] Continuous deviation plans can be though of as an expression of discontinuities not being *prevalent*, a key concept in proving equilibrium existence in discontinuous games, which goes back at least to Dasgupta and Maskin's (1986) finitely many "lines" of discontinuity, and more recently to Barelli, Govindan and Wilson's (2014) "mild-discontinuities."

In Section 4, we use Theorem 1 to prove equilibrium existence results for games in which payoff discontinuities occur only when players choose the same action. But Theorem 1 can also be applied to other discontinuous games. We provide an example in Section 7.

Remark 1 *We can also formulate a closely related result for general normal-form games in which the set of pure strategies of player i is A_i and player i's payoff function is u_i.*[17] *Assume that: (i) (a version of payoff security) For every i, every $a \in A = A_i \times A_{-i}$ and every $\varepsilon > 0$, there exist some $d_i^{\varepsilon} \in A_i$ and an open neighborhood V_{-i} of a_{-i} such that $u_i(d_i^{\varepsilon}, a'_{-i}) > u_i(a) - \varepsilon$ for every $a'_{-i} \in V_{-i}$; and (ii) (a version of weak RUSC): There exists an upper semi-continuous function $Z : A \to \mathbb{R}$ such that $\sum_{i=1}^{n} u_i(a) \leq Z(a)$ for all $a \in A$, and such that for every discontinuity point a of some u_i there is an $\varepsilon > 0$ such that $\sum_{i=1}^{n} u_i(d_i^{\varepsilon}, a_{-i}) > Z(a)$. Then, there exists a Nash equilibrium in pure strategies.*

To use this result, however, one would need to show that the payoff security and weak RUSC conditions are satisfied for the distributional strategies, whereas our conditions refer to the extent possible to the underlying actions and reduce the requirements that need to be checked with respect to distributional strategies.

3 Proof of Theorem 1

To prove Theorem 1, we will use Reny's (1999) Theorem 3.1. Since the game with distributional strategies is obviously compact and quasi-concave according to Reny's (1999) terminology, his Theorem 3.1 guarantees that an equilibrium exists if the game is *better-reply secure.* To define better-reply security, we denote by Γ the closure of the graph of the function that maps each profile of strategies to the vector of players' payoffs. A game is better-reply secure if for every strategy profile $\mu^* = (\mu_1^*, \ldots, \mu_n^*)$ that is not an equilibrium and every vector $u^* = (u_1^*, \ldots, u_n^*)$ such that (μ^*, u^*) is in Γ, there is a player i, a strategy μ_i, and a number $\eta > 0$ such that player i's payoff from playing μ_i exceeds $u_i^* + \eta$ for every profile of strategies of the other players in some neighborhood of μ_{-i}^*.

Consider μ^* and u^* such that μ^* is not an equilibrium and (μ^*, u^*) is in Γ. Suppose first that the payoff of every player is continuous at μ^*, so $u_i^* = U_i(\mu^*)$ for every player i.

[17] We thank a referee for suggesting this result.

Take a player i that has a profitable deviation μ_i, so $U_i(\mu_i, \mu^*_{-i}) > U_i(\mu^*) + 3\eta$ for some $\eta > 0$. Because the game has improving deviations, there is a continuous deviation strategy μ^ε_i of player i such that $U_i(\mu^\varepsilon_i, \mu^*_{-i}) > U_i(\mu^*) + 2\eta$.[18] Since U_i is continuous at $(\mu^\varepsilon_i, \mu^*_{-i})$ as a function of the strategies of all players other than i, player i's payoff from playing μ^ε_i exceeds $u^*_i + \eta$ for every profile of strategies of the other players in some neighborhood of μ^*_{-i}.

Now suppose that the payoff of some player i is discontinuous at μ^*. By definition of Γ, $\mu^k \to_k \mu^*$ and $u^k \to_k u^*$ for some sequence $(\mu^k, u^k)^\infty_{k=1}$ from the graph of the function that maps each profile of strategies to the vector of players' payoffs (so $u^k_j = U_j(\mu^k)$ for every player j).

Since the payoff envelope W is upper semi-continuous,

$$\limsup \int W d\mu^k \leq \int W d\mu^*.$$

(See Billingsley (1995), Problem 29.1.) Thus, by (2),

$$u^*_1 + \cdots + u^*_n = \lim \int (u_1 + \cdots + u_n)\, d\mu^k \leq \int W d\mu^*.$$

By (3) for $\mu_{-j} = \mu^*_{-j}$, each player j has a deviation strategy μ^ε_j (for some $\varepsilon > 0$) such that

$$\int W d\mu^* < \sum_{j=1}^{n} U_j(\mu^\varepsilon_j, \mu^*_{-j}),$$

so

$$u^*_1 + \cdots + u^*_n < \sum_{j=1}^{n} U_j(\mu^\varepsilon_j, \mu^*_{-j}).$$

The last inequality implies that there is a player j such that $U_j(\mu^\varepsilon_j, \mu^*_{-j}) > u^*_j + 2\eta$ for some $\eta > 0$. Since U_j is continuous at $(\mu^\varepsilon_j, \mu^*_{-j})$ as a function of all players other than j, player j's payoff from playing μ^ε_j exceeds $u^*_j + \eta$ for every profile of strategies of the other players in some neighborhood of μ^*_{-j}.

4 Games with ties

Our focus is on games in which payoff discontinuities arise only when two or more players choose the same action. We refer to such actions as *ties* and to such games as *games with ties*.

[18] This follows from the definition of a deviation strategy for $\varepsilon = \eta$.

Definition 4 *Given an action b and an action profile $(b_1, ..., b_n)$ with $b_i = b$ for two or more players i, we say that action b is a* tie *and that the players i for whom $b_i = b$ tie at b.*

Definition 5 *A* game with ties *is a game in which every player i's payoff u_i is continuous at $(x_1, .., x_n, b_1, ..., b_n)$ if player i does not tie at b_i.*

Not all ties necessarily lead to a payoff discontinuity. We refer to those ties that do as *essential ties on average.*

Definition 6 *(i) Given a strategy profile μ_{-i}, a bid $b_i = b$ is an* essential tie on average for player *i at (x_i, b_i) if $\int u_i(x_1, .., x_n, b_1, ..., b_n) d\nu_{-i}$ is discontinuous at μ_{-i} (as a function of strategy profile ν_{-i}).*

(ii) A strategy profile μ has essential ties on average for player *i if μ_i assigns a positive probability to the set T_i^* of type-bid pairs (x_i, b_i) at which player i has essential ties on average given μ_{-i}.*

(iii) A strategy profile μ has essential ties on average *if it has essential ties on average for some player.*

We will show that for games with ties any deviation plan that avoids essential ties on average is continuous and the payoff of every player is continuous at any strategy profile that does not have essential ties on average. We can therefore identify a class of games with ties that satisfies the assumptions of Theorem 1 as follows.

Definition 7 *A* deviation plan without essential ties on average *is a deviation plan τ such that for every profile of strategies μ, every player i, and every $\varepsilon > 0$, strategy profile $(\mu_i^\varepsilon, \mu_{-i})$, where μ_i^ε is the corresponding deviation strategy, does not have essential ties on average for player i.*

Definition 8 *A game satisfies* favorable tie breaking on average *if there is a payoff envelope W and a deviation plan without essential ties on average such that for any strategy profile μ with essential ties on average (3) holds for some $\varepsilon > 0$.*

We can now state our equilibrium existence result for games with ties.

Theorem 2 *A game with ties that satisfies favorable tie breaking on average has improving deviations and therefore has a Nash equilibrium in distributional strategies.*

Identifying essential ties on average may be somewhat involved because Definition 6 refers to the discontinuity of expected payoffs with respect to profiles of distributional strategies. It is often easier to consider the continuity of the payoff functions directly. This leads to the following stronger concept of essential ties.

Definition 9 *(i) A tie $b_i = b$ is* essential for player i *at* $(x_1, .., x_n, b_1, ..., b_n)$ *if* u_i *is discontinuous at* $(x_1, .., x_n, b_1, ..., b_n)$ *(as a function of all players' types and bids).*

(ii) Strategy profile μ *has* essential ties for player i *if* μ *assigns positive probability to the set* T_i^* *of profiles* $(x_1, .., x_n, b_1, ..., b_n)$ *at which player* i *has essential ties.*

(iii) Strategy profile μ *has* essential ties *if it has essential ties for some player.*

A strategy profile that has essential ties on average for player i also has essential ties for player i, but the reverse is not necessarily true. With two players, for example, starting from a tie $b_i = b$, by bidding slightly more than b player i's utility may discontinuously increase if the type of player $j \neq i$ is high and discontinuously decrease if player j's type is low, but change continuously in expectation over player j's type. By replacing essential ties on average with essential ties in Definition 7 and Definition 8, we define a *deviation plan without essential ties* and favorable tie breaking and obtain the following corollary of Theorem 2.

Corollary 1 *A game with ties that satisfies favorable tie breaking has improving deviations and therefore has a Nash equilibrium in distributional strategies.*

For example, all-pay auctions in which higher types have a higher value for the prize satisfy favorable tie breaking: W can be defined as the sum of payoffs when the fair tie-breaking rule is replaced with the rule that gives the prize to the tying player with the highest value, and the bid $\tau_i^{\mu,\varepsilon}(x_i, b_i)$ can be defined as a bid slightly higher than b_i. By resolving a tie in her favor each player wins the prize, whereas only one player wins the prize when the tie is resolved simultaneously for all players, so (3) holds. We formalize and generalize this example in Section 6.1. We will see later that in other examples, different types x_i of player i may prefer breaking the same tie in different ways, which in addition can depend on the strategy profile μ.

Favorable tie breaking requires (3) to be satisfied for all strategy profiles with essential ties. If (3) fails for such a strategy profile but holds for all strategy profiles at which players' payoff are discontinuous, then Corollary 1 fails but Theorem 2 may hold. This is the case in Section 6.3 below.

5 Proof of Theorem 2

By the definition of a game with improving deviations, to prove Theorem 2 it suffices to prove that (i) in a game with ties a deviation plan without essential ties on average is a continuous deviation plan, and (ii) any strategy profile at which some player's payoff is discontinuous as a function of all players' strategies is a strategy profile with essential ties on average. Both (i) and (ii) immediately follow from the following result, whose proof is in the appendix.

Lemma 1 *Player i's payoff is continuous as a function of all players' strategies at any strategy profile μ that does not have essential ties on average for player i.*

6 Applications

We now demonstrate equilibrium existence in three settings of games with ties. Corollary 1 suffices for the first two settings but not for the third setting, for which Theorem 2 is needed.[19]

The first setting is a model of perfectly discriminating contests, which generalizes multi-prize all-pay auctions with complete and incomplete information and allows for heterogeneous prizes and private, common, and interdependent values. The key properties of this model

[19] The Supplementary Appendix of our working paper, Olszewski and Siegel (2022), contains three additional applications, a two-firm Hotelling setting with private production costs, independent private-value first-price auctions with costly bids, and a two-player contest model in which players' bids determine not only who wins but which prize the winner obtains. Theorem 1 applies to the third application but Corollary 1 and Theorem 2 do not.

are that (i) every player strictly prefers to win any tie at which not all tying players get the same prize, and (ii) the highest allowable bid is strictly dominated by some lower bid. This leads to a simple deviation plan in which every player increases her bid slightly if possible, and the highest possible bid is replaced with a dominating lower bid. Unilateral deviations at an essential tie correspond to all tying players winning the best prize associated with the tie, whereas simultaneously breaking the tie cannot award the best prize to all tying players, so (3) holds.[20]

The second setting is a common-value first-price auction in which the value of the prize strictly increases in all players' types. The key properties of this model are that (i) for any bid and any strategy profile of the players other than i, at most one type of player i can be indifferent between winning and losing, and (ii) the sum of payoffs is continuous (it is the value of the prize minus the highest bid), so the payoff envelope can be chosen to be the sum of payoffs. This leads to a simple deviation plan in which every player increases or decreases her bid slightly depending on whether in expectation, given her type and the other players' strategies, she prefers to win or to lose the prize at that bid. Unilateral deviations at any essential tie strictly increase the payoff of at least one player (by property (i)) and do not decrease the payoff of any other player, so (3) holds by property (ii). Property (ii) is lost by relaxing common values to interdependent or private values, and this can prevent unilateral deviations from being better than simultaneous slight changes in players' actions and types, and even lead to equilibrium non-existence.

The third setting is a Hotelling location model on the interval $[0,1]$ in which prices are exogenous and identical across firms, and firms have private information about their continuous location cost function. The key property of this model is that the sum of payoffs is continuous (it is the mass of consumers minus firms' location costs), so the payoff envelope

[20] We point out that Reny's (1999) RUSC fails here, so his Proposition 3.2 cannot be applied. To see this, consider an all-pay auction with two players. Each player's valuation is either 1 or 3, with equal probability. Consider the following sequence of strategies for some $a < b$. Player 1 bids $a - 1/n$ when his value is 1 and b when his value is 3. Player 2 bids $b - 1/n$ when his value is 1 and a when his value is 3. Then, compared to the limit payoffs along the sequence, the expected payoff of both players jumps down in the limit strategy profile. On the other hand, weak RUSC and payoff security (with respect to distributional strategies) are satisfied, and can be used to show equilibrium existence (see Remark 1).

can be chosen to be the sum of payoffs. Corollary 1 does not apply to this setting because players' payoffs may be continuous at a strategy profile with essential ties for which no deviation plan satisfies (3).[21] However, if a player has an essential tie on average at some location, then a slightly higher or lower location strictly increases her payoff. This allows us to apply Theorem 2 by using the simple deviation plan in which every player increases or decreases her location slightly if in expectation, given the other players' strategies, this strictly increases her payoff. Unilateral deviations at any strategy profile with essential ties on average strictly increase the payoff of at least one player and do not decrease the payoff of any other player, so (3) holds by the fact that the payoff envelope is equal to the sum of payoffs.

6.1 Application 1: Multiprize contests with interdependent values

There are n players who compete for n prizes. Player i's signal $x_i \in X = [0,1]$ about the prize values is distributed according to a *cdf* F_i that does not have an atom at 0. We showed in footnote 1 that without this assumption equilibrium may not exist. The distributions need not be identical, but are commonly known and independent across players. Complete information is a special case, as are distributions that include atoms, gaps, and continuous components. Each prize is characterized by a number y, which represents its position in the prize ranking common to all players, and may be interpreted as an initial public signal about the prize's value, obtained before players learn their private signals. We order the n prizes so that $y_1 \leq y_2 \leq \cdots \leq y_n$.

Each player i chooses a bid $b_i \in B = [0,1]$, the player with the highest bid obtains prize y_n,

[21] To see this, suppose there are three players with complete information, consumers are distributed uniformly on $[0,1]$, and consider the strategy profile in which players 1 and 2 locate at $1/2$, and player 3 locates at 0 and 1 with equal probability. Then location $1/2$ is an essential tie for players 1 and 2 at the location profile $(1/2, 1/2, 0)$ and at the location profile $(1/2, 1/2, 1)$, so the strategy profile has essential ties. But each player's payoff continuously changes with her strategy. Thus, for sufficiently large (and continuous) costs of moving from players' chosen locations in the strategy profile, no deviation plan satisfies (3).

the player with the second-highest bid obtains prize y_{n-1}, and so on.[22] Ties are resolved by a fair lottery. The utility of player i from obtaining prize y is $\bar{u}_i(x_1, ..., x_n, b_1, \ldots, b_n, y)$, where $\bar{u}_i$ is a continuous function of $(x_1, ..., x_n) \in X^n$ and $(b_1, ..., b_n) \in B^n$ for all prizes y. In addition, $\bar{u}_i$ strictly increases in y for all $(x_1, ..., x_n)$ and $(b_1, ..., b_n)$,[23] with an exception for type $x_i = 0$ (type 0 is indifferent across all prizes).[24] We assume that bids close to 1 are strictly dominated by a lower bid, and thus irrelevant. More precisely, we assume that for every player i there is a bid $\bar{b}_i < 1$ such that $\bar{u}_i(x_1, ..., x_n, (1, b_{-i}), y_n) < \bar{u}_i\left(x_1, ..., x_n, (\bar{b}_i, b_{-i}), y_1\right)$ for all $(x_1, ..., x_n)$ and $b_{-i} = (b_1, \ldots, b_{i-1}, b_{i+1}, \ldots, b_n)$. Finally, $u_i(x_1, ..., x_n, b_1, ..., b_n)$ is the expected value of $\bar{u}_i(x_1, ..., x_n, b_1, ..., b_n, y)$ given the bid profile $(b_1, ..., b_n)$.

The strict monotonicity of $\bar{u}_i$ in y makes this a (generalized) contest model: regardless of players' types or bids it is better to win a higher prize. This is the case, for example, in an all-pay auction, but is not the case in a first-price auction (because at a bid higher than the prize's value losing is better than winning).[25] Notice that players' utilities need not monotonically decrease in their bids, as they do in an all-pay auction.[26] Notice also that the model accommodates private, common, and interdependent values and allows a player's utility from a given prize to depend on other players' bids.[27]

These contests are clearly games with ties. We now show that they also satisfy favorable tie breaking, which guarantees equilibrium existence by Corollary 1. A tie $b_i = b$ is essential for player i at $(x_1, \ldots, x_n, b_1, \ldots, b_n)$ if $x_i > 0$ and not all the players tying at b obtain equal prizes. Define the payoff envelope W as the sum of players' payoffs when ties

[22] Fu, Wu, and Zu (2022) prove equilibrium existence for a class of imperfectly discriminating multi-prize nested lottery contests.

[23] Notice that we do not assume monotonicity in types or bids (of the player or of other players).

[24] We could exclude type 0, and assume that $x_i \in [\underline{x}, 1]$ for some $\underline{x} > 0$. This would slightly simplify the analysis. We include type 0 because it usually appears in papers on contests with incomplete information.

[25] This property also fails in Section 6.2 below, which is why we restrict attention to common values there.

[26] For example, in a competition for a dominant market position based on advertising, a moderate level of advertising can increase the demand for the product, and thus the value of winning, by more than the cost of the advertising.

[27] Returning to the advertising example from the previous footnote, each firm's advertising may affect overall market demand, which in turn affects the winning firm's profit from a dominant market position.

are broken in a way that maximizes this sum. More precisely, let $W(x_1, ..., x_n, b_1, ..., b_n)$ be the sum of $\bar{u}_i(x_1, ..., x_n, b_1, \ldots, b_n, \widetilde{y}_i((x_1, ..., x_n), (b_1, \ldots, b_n)))$ across all players i, where $\widetilde{y}_i((x_1, ..., x_n), (b_1, \ldots, b_n))$ is determined as follows: (a) a player with a higher bid obtains a higher prize; and (b) if the bids of two or more players are equal, then among those players, prizes are allocated in any (measurable) way that maximizes the sum of the payoffs of the tying players.[28] This guarantees that W is upper semi-continuous, and also that (2) holds.[29]

We now describe a deviation plan. The idea is to increase bids slightly in order to win all essential ties, since winning a higher prize is always better. Bid $b = 1$ cannot be increased, but can be profitably replaced with a bid of $\bar{b}_i$ (or a bid close to $\bar{b}_i$ to avoid ties). Formally, we let the bid $\tau_i^{\mu,\varepsilon}(x_i, b_i)$ be a bid b_i' such that the marginal on B of each strategy μ_j, $j \neq i$, does not have an atom at b_i', and for $b_i < 1$ the bid b_i' satisfies $b_i' > b_i$ and

$$\bar{u}_i\left(x_1, ..., x_n, b_1, \ldots, b_n, y\right) - \bar{u}_i\left(x_1, ..., x_n, (b_i', b_{-i}), y\right) < \varepsilon \tag{4}$$

for all $(x_1, \ldots, x_n)$, b_{-i}, and $y = y_1, \ldots, y_n$. The existence of such a b_i' follows from the fact that the marginal on B of each strategy has at most a countable number of atoms and from the uniform continuity of $\bar{u}_i$ on $X^n \times B^{n-1} \times [b_i, 1] \times \{y\}$ for $y = y_1, \ldots, y_n$. For $b_i = 1$ let b_i' (in addition to the atom restriction) be such that $\bar{u}_i\left(x_1, ..., x_n, (b_i', b_{-i}), y_1\right) > \bar{u}_i\left(x_1, ..., x_n, (1, b_{-i}), y_n\right)$. (Recall that $\bar{u}_i\left(x_1, ..., x_n, (\bar{b}_i, b_{-i}), y_1\right) > \bar{u}_i\left(x_1, ..., x_n, (1, b_{-i}), y_n\right)$.) We then have (1) because $\bar{u}_i$ weakly increases in y for all $(b_1, \ldots, b_n)$ and $(x_1, \ldots, x_n)$. The deviation plan is without ties because, by Fubini's Theorem, T_i^* has a positive measure only if the marginals on B of player i's strategy and the strategy of another player have an atom at the same bid.

It remains to show that for any strategy profile μ with essential ties (3) holds for some $\varepsilon > 0$. For this, define function V_i as player i's payoff when ties are broken in her favor, that is, $V_i(x_1, ..., x_n, b_1, ..., b_n) = \bar{u}_i(x_1, ..., x_n, b_1, ..., b_n, \hat{y}_i(b_1, \ldots, b_n))$ where $\hat{y}_i(b_1, \ldots, b_n)$ is

[28] One way of doing this is as follows. Given a set of k tying players, order all the permutations $\pi_1, \pi_2, ..., \pi_m$ of this set. Then allocate the k relevant prizes to the k tying players according to permutation π_l that maximizes the sum of payoffs, where l is such that allocating the prizes according to any permutation $\pi_{l'}$ with $l' < l$ does not maximize the sum of payoffs.

[29] Note that the sum of payoffs is not upper semi-continuous in this application, as would be required by Dasgupta and Maskin's (1986) RUSC. Moreover, contests are not RUSC according to Reny's definition.

the prize player i would win if she bids slightly above b_i while every player $j \neq i$ bids b_j. Let V be the sum of V_i across all players i. Then

$$V(x_1, ..., x_n, b_1, ..., b_n) \geq W(x_1, ..., x_n, b_1, ..., b_n),$$

with a strict inequality whenever a tie b is essential at $(x_1, .., x_n, b_1, ..., b_n)$ for two or more players (because $x_i > 0$ if the tie $b = b_i$ is essential for player i, and $\bar{u}_i$ strictly increases in y whenever $x_i > 0$). Now, take a strategy profile μ with essential ties. Since for each player i type $x_i = 0$ has probability 0, we have that

$$\alpha = \int V(x_1, ..., x_n, b_1, ..., b_n) d\mu - \int W(x_1, ..., x_n, b_1, ..., b_n) d\mu > 0.$$

By (4) and the definition of $\tau_i^{\mu,\varepsilon}(x_i, b_i)$ for $b_i = 1$, we have

$$\int V(x_1, ..., x_n, b_1, ..., b_n) d\mu - \sum_{j=1}^{n} U_j(\mu_j^{\varepsilon}, \mu_{-j}^{*}) < n\varepsilon,$$

where μ_j^{ε} is the deviation strategy associated with $\tau_j^{\mu,\varepsilon}$. Therefore, (3) holds for $\varepsilon < \alpha/n$.

6.2 Application 2: First-price auctions with common values

There are n players who compete for one prize. Player i's signal $x_i \in X = [0, 1]$ about the prize value is distributed according to a continuous cdf F_i.[30] The distributions need not be identical, but are commonly known and independent across players. Each player i submits a bid $b_i \in B = [0, 1]$, and the player with the highest bid wins the prize and pays her bid. Ties are resolved by a fair lottery. The value of the prize $v(x_1, ..., x_n)$ is common to all players, with $v(0, \ldots, 0) = 0$ and $v(1, \ldots, 1) < 1$,[31] strictly increasing in each signal, and continuous as a function of the entire profile of signals. The utility of player i is

$$\bar{u}_i(x_1, ..., x_n, b_i) = \begin{cases} v(x_1, ..., x_n) - b_i \text{ if } i \text{ wins the prize,} \\ \quad 0 \text{ if } i \text{ does not win the prize.} \end{cases}$$

[30] Reny (1999) also allows for multi-dimensional types, i.e., $x_i \in [0, 1]^m$ for $i = 1, ..., n$. Our existence result can be generalized to this setting.

[31] That $v(1, \ldots, 1) < 1$ guarantees that any equilibrium remains an equilibrium if bidders can place any non-negative bid.

Finally, $u_i(x_1, ..., x_n, b_1, ..., b_n)$ is the expected value of $\bar{u}_i(x_1, ..., x_n, b_i)$ given the bid profile $(b_1, ..., b_n)$.

Various versions of common-value auctions have been studied by several other authors, and as pointed out in the introduction, some existence results have already been established. The added value of our result in this context is that we do not make additional assumptions that restrict its range of applications. In particular, whether a player prefers to win or lose a particular tie may depend on the signals of the other players. Relatedly, this application illustrates the importance of the feature of our existence result that the deviation plan $\tau_i^{\mu,\varepsilon}$ is allowed to depend on the profile of strategies μ.

These auctions are clearly games with ties. We now show that they also satisfy favorable tie breaking. A tie $b_i = b$ is essential for player i at $(x_1, \ldots, x_n, b_1, \ldots, b_n)$ if b is the winning bid and $b_i \neq v(x_1, ..., x_n)$. Define the payoff envelope W as the sum of players' payoffs, that is,

$$W(x_1, ..., x_n, b_1, ..., b_n) = \sum_{i=1}^{n} u_i(x_1, ..., x_n, b_1, ..., b_n) = v(x_1, ..., x_n) - \max\{b_1, ..., b_n\},$$

so W is continuous and (2) holds as an equality.[32]

We now describe a deviation plan. The idea is at every essential tie b to either increase or decrease the bid slightly, depending on whether (in expectation, conditional on tying at b) winning is better than losing. Notice that whether winning is better than losing at a particular bid depends on the other players' strategies, so unlike with contests the deviation plan will depend on the profile of strategies. More precisely, consider for each player i and bid b_i the event $E_i(b_i) \subset X_{-i} \times B_{-i}$ in which b_i is a winning bid and at least one other player bids b_i. If $\mu_{-i}(E_i(b_i)) > 0$, let $\tau_i^{\mu,\varepsilon}(x_i, b_i)$ be a bid slightly higher or lower than b_i, depending on whether, conditional on $E_i(b_i)$, player i of type x_i prefers winning the prize and paying the winning bid, or losing and paying nothing.[33] Note that player i strictly prefers one of the two options, except possibly at a single type x_i, because $v(x_1, ..., x_n)$ strictly increases

[32] Because the sum of payoffs is continuous, existence of equilibrium can also be shown to follow from payoff security (for the game in distributional strategies) and Dasgupta and Maskin's (1986) RUSC (see Reny's Proposition 3.2).

[33] Of course, ex-post, which of the two options player i prefers also depends on the types of the other players, but $\tau_i^{\mu,\varepsilon}$ is allowed to depend only on the type and bid of player i. Thus, by a preferred option

in x_i. Let $\tau_i^{\mu,\varepsilon}(x_i, b_i) = b_i$ for the (at most one) type x_i that is indifferent between the two options, and for all types x_i when $\mu_{-i}(E_i(b_i)) = 0$. Notice that at $b_i = 1$ player i strictly prefers losing to winning because v increases in all players' signals and $v(1, \ldots, 1) < 1$.

By choosing $\tau_i^{\mu,\varepsilon}(x_i, b_i)$ sufficiently close to b_i (whenever $\tau_i^{\mu,\varepsilon}(x_i, b_i) \neq b_i$), we have $U_i(\mu) \leq U_i(\mu_i^\varepsilon, \mu_{-i})$, where μ_i^ε is the corresponding deviation strategy, with a strict inequality whenever μ has essential ties for player i. This is because if μ has essential ties for player i then (by Fubini's Theorem) the marginal of μ_i on B assigns positive probability to some b_i for which $E_i(b_i) > 0$, and by choosing $\tau_i^{\mu,\varepsilon}(x_i, b_i)$ sufficiently close to b_i the expected payoff of player i of every (except at most one) type x_i from playing $\tau_i^{\mu}(x_i, b_i)$ strictly increases relative to playing b_i. This guarantees that (1) holds, and also that (3) holds. In addition, whenever $\tau_i^{\mu,\varepsilon}(x_i, b_i) \neq b_i$ we choose $\tau_i^{\mu,\varepsilon}(x_i, b_i)$ to be different from any bid at which the marginal on B of any strategy μ_j, $j \neq i$, has an atom. Thus, the deviation plan is without essential ties, which completes the demonstration that these auctions are games with ties that satisfy favorable tie breaking.

6.3 Application 3: Hotelling models

A finite number of firms compete for a unit mass of customers, each with unit demand. The customers are distributed with a positive density on the interval $[0, 1]$. Each firm chooses a location in $[0, 1]$, and then each customer chooses the firm closest to his or her location. We assume that prices are fixed, equal across firms, and normalized to 1. Variable production costs are negligible, so each firm's payoff is equal to the share of customers buying from the firm minus the cost of locating at the firm's chosen location. Firms face different costs of choosing different locations. Firm i's type $x_i \in [0, 1]$ is distributed according to a cdf F_i, and the location cost function $c_i(x_i, b_i)$ is a continuous function of the firm's type x_i and location b_i.[34] We assume that each firm must choose some location.

The existence of mixed-strategy equilibria in the complete-information setting was es-

we mean the option that gives player i the higher expected payoff (over the strategies of the other players) conditional on $E_i(b_i)$.

[34]The additive separability of the customer share, which does not depend on x_i, and the cost, which depends on x_i, is important for the analysis.

tablished by Simon (1987), whose result allows for multi-dimensional locations (for example, customers can be distributed on, and firms can choose their locations from, the cube $[0, 1]^m$). We conjecture that our existence result generalizes to settings with multi-dimensional locations. However, finding sharp conditions for existence would probably require writing a separate paper, given the complexity of Simon's (1987) analysis of complete information settings.[35] Our objective here is not to generalize Simon's (1987) result but to demonstrate that Theorem 2 can be used to prove equilibrium existence in some Hotelling models with incomplete information.

The Hotelling model with incomplete information is clearly a game with ties. We now show that it also satisfies favorable tie breaking on average. Define the payoff envelope W as the sum of players' payoffs (shares minus location costs), that is,

$$W(x_1, ..., x_n, b_1, ..., b_n) = 1 - \sum_{i=1}^{n} c_i(x_i, b_i),$$

so W is continuous and (2) holds as an equality.

We now describe a deviation plan. The idea is to shift a player's location slightly if doing so leads to a discrete payoff increase. More precisely, consider for each player i and location b_i the event $E_i(b_i) \subset X_{-i} \times B_{-i}$ in which at least one other player locates at b_i. If $\mu_{-i}(E_i(b_i)) > 0$, let $\tau_i^{\mu,\varepsilon}(x_i, b_i)$ be a location slightly to the left or to the right of b_i (different from any location at which the marginal on B of any strategy μ_j, $j \neq i$, has an atom) if, conditional on E_i, an infinitesimal shift in this direction discretely increases player i's (expected) market share, that is, discretely increases player i's payoff.[36] Notice that whether an infinitesimal shift in a particular direction is beneficial for a player depends on

[35] Multi-dimensional versions of Hotelling's model are settings in which Reny's (1999) approach, and our Theorem 1, are able to guarantee the existence of an equilibrium, while existence may be difficult or impossible to establish by approximating the original game with a sequence of games with a finite number of actions and types. This is because profitable deviations of different types at different locations may require shifting locations in different and specific directions.

[36] By an infinitesimal shift from a location b in a direction we mean the same location b but with a different market sharing rule: Customers in the direction of the shift find the firm's shifted location closer than location b, and customers in the opposite direction of the shift find the firm's shifted location farther away than location b.

the strategies μ_{-i} of the other players but is independent of the player's type, since types only affect location costs.[37] Notice also that the improving infinitesimal shift must be to the right for $b_i = 0$ and to the left for $b_i = 1$. If no infinitesimal shift discretely increases player i's payoff let let $\tau_i^{\mu,\varepsilon}(x_i, b_i)$ be a location close to or at b_i different from any location at which the marginal on B of any strategy μ_j, $j \neq i$, has an atom. If $\mu_{-i}(E_i(b_i)) = 0$, let $\tau_i^{\mu,\varepsilon}(x_i, b_i) = b_i$ for all types x_i. Similarly to Section 6.2, (1) holds and the deviation plan is without essential ties.

Since the payoff envelope W is equal to the sum of players' payoffs, to apply Theorem 2 it suffices to show that if a strategy profile μ has essential ties on average for player i, then $U_i(\mu) < U_i(\mu_i^\varepsilon, \mu_{-i})$, where μ_i^ε is the corresponding deviation strategy. By construction, $U_i(\mu) < U_i(\mu_i^\varepsilon, \mu_{-i})$ whenever there is a positive μ_i-measure of bids b_i for which an infinitesimal shift in player i's location discretely increases player i's market share. Thus, to conclude the proof it suffices to show that if μ has essential ties on average for player i then there is a positive μ_i-measure of bids b_i for which an infinitesimal shift in player i's location discretely increases player i's market share. We will show that if no infinitesimal shift from location b_i discretely increases player i's market share, then location b_i is not an essential tie on average for player i (recall that whether a bid is an essential tie for a player is independent of the player's type).

Consider a strategy profile μ and a location b_i for which no infinitesimal shift discretely increases player i's market share. Suppose that player i bids b_i. If no other player has an atom at b_i, then player i's payoff is continuous as a function of the other players' strategies, because discontinuities arise only at ties. It cannot be that two or more other players have an atom at b_i, because then an infinitesimal shift in player i's location would discretely improve player i's payoff (she discretely increases her share on the side to which she is moving, so at least one side is strictly profitable). And if only one other player j has an atom at b_i, then if that player moves slightly (which is what it means for his strategy to be weak*-close to μ_j), that is very similar to player i moving slightly - but that does not increase player i's payoff

[37] Thus, unlike in Section 6.2, it is possible that for some locations b_i with $\mu_{-i}(E_i(b_i)) > 0$ no infinitesimal shift is profitable for player i. This prevents the use of Theorem 1, as discussed in the description of the Hotelling application at the beginning of Section 6.

because an infinitesimal shift does not change player i's payoff. We formalize this argument with the following two claims, whose proofs are in the appendix.

Claim 1 *If no infinitesimal shift from location b_i increases player i's market share, then the marginal on B of the strategy of only one player $j \neq i$ can have an atom at b_i.*

Claim 2 *If (a) for no player $j \neq i$ the marginal on B of the strategy of player j has an atom at b_i, or (b) the marginal on B of the strategy of only one player $j \neq i$ has an atom at b_i, and no infinitesimal shift from location b_i discretely increases player i's market share, then b_i is not an essential tie at any (x_i, b_i).*

7 Concluding remarks

This paper studies Bayesian games with ties and introduces improving deviations, favorable tie breaking, and favorable tie breaking on average, which are sufficient conditions for equilibrium existence in distributional strategies. We apply favorable tie breaking and favorable tie breaking on average to obtain novel equilibrium existence results for multiprize contests with interdependent values, first-price auctions with common values, and Hotelling models with private location costs. The Supplementary Appendix of our working paper, Olszewski and Siegel (2022), contains two additional applications, a two-firm Hotelling setting with private production costs, and independent private-value first-price auctions with costly bids. A third application, also in the Supplementary Appendix of our working paper, demonstrates that improving deviations can sometimes be used to prove equilibrium existence when favorable tie breaking on average fails.[38]

These results should be used with care, however, since slight changes to such games can lead to games with ties to which Reny's (1999) existence result (and therefore Theorems 1 and 2), as well as other equilibrium existence results in the literature, do not apply, even though an equilibrium exists. One example is first-price auctions with private values. The difficulty in this case is that while unilateral tie breaking can increase the sum of payoffs,

[38] A special case of that application is a two-player contest in which the winner obtains an extra bonus if the gap between players' bids exceeds some threshold, as may be the case in some sales competitions.

this increase may not exceed the increase generated by slightly and simultaneously changing all players' actions and valuations.[39] Olszewski, Reny, and Siegel (2022) provide relatively permissive sufficient conditions for equilibrium existence in such auctions.

8 Appendix

8.1 Proof of Lemma 1

Consider a sequence $(\mu^l)_{l=1}^{\infty}$ of strategy profiles that converges to μ. We will show that for any $\delta > 0$, if l is large enough, then $\left|U_i\left(\mu^l\right) - U_i\left(\mu\right)\right| < \delta$. Given any $\varepsilon > 0$, take an open subset T_i^{ε} of $X_i \times B$ such that $T_i^* \subset T_i^{\varepsilon}$ and $\mu_i(clT_i^{\varepsilon}) < \varepsilon$, where clT_i^{ε} denotes the closure of T_i^{ε}. We can in addition assume that $\mu_i^l(clT_i^{\varepsilon}) < \varepsilon$ for sufficiently large l (see Billingsley (1995), Theorem 29.1). For every (x_i, b_i) from the complement of T_i^{ε}, we have that $\int u_i(x_1, .., x_n, b_1, ..., b_n)d\nu_{-i}$ is a continuous function of ν_{-i} at μ_{-i}. So, there is an $\bar{l}$ such that

$$\left|\int u_i(x_1, .., x_n, b_1, ..., b_n)d\mu_{-i}^l - \int u_i(x_1, .., x_n, b_1, ..., b_n)d\mu_{-i}\right| < \delta/2 \tag{5}$$

if $l \geq \bar{l}$. Thus, there is also an $\bar{l}$ that is common for all (x_i, b_i) from a set $S \subset X_i \times B - clT_i^{\varepsilon}$ whose μ_i-measure is higher than $1 - \varepsilon$.[40] We can assume that the set S is closed, because all measurable sets contain closed subsets of arbitrarily close measure. By Theorem 29.1 of Billingsley (1995), we can also assume that $\mu_i^l(S) > 1 - \varepsilon$ for sufficiently large l. By (5),

$$\left|\int_S \left(\int u_i(x_1, .., x_n, b_1, ..., b_n)d\mu_{-i}^l\right) d\mu_i^l - \int_S \left(\int u_i(x_1, .., x_n, b_1, ..., b_n)d\mu_{-i}\right) d\mu_i^l\right| < \delta/2 \tag{6}$$

[39] To see this, suppose there are two players with known valuations $0 < x_1 < x_2$, and consider the strategy profile in which both players bid $b = x_1$. The tie b is essential at (x_1, x_2, b, b) (player 2's payoff is discontinuous there), and by upper semi-continuity the value of any payoff envelope at (x_1, x_2, b, b) is at least $x_2 - x_1$, which corresponds to breaking the tie in favor of player 2. But $u_2(x_1, x_2, b, b_2) < x_2 - x_1$ for any bid b_2, and $u_1(x_1, x_2, b_1, b) \leq 0$ for any bid b_1, so there is no deviation plan for which (3) holds. It can also be shown that Reny's (1999) better-reply security and Reny's (2016) point security fail in this example.

[40] For every $m > 0$, denote by S^m the set of pairs (x_i, b_i) in $X_i \times B - clT_i^{\varepsilon}$ for which $\bar{l} < m$. The existence of S and a common l follows from $S^m \subseteq S^{m+1}$ and $\cup_{m>0} S^m = X_i \times B - clT_i^{\varepsilon}$.

if $l \geq \bar{l}$ for such a common $\bar{l}$. Actually, inequality (6) holds without restricting integration to S, provided that ε is sufficiently small compared to δ, because the measures μ_i^l and μ_i of the complement of S are smaller than ε and we assume that functions u_i are bounded.

Notice now that $\int u_i(x_1, .., x_n, b_1, ..., b_n) d\mu_{-i}$ is a continuous function of (x_i, b_i) on S. Therefore,

$$\left| \int \left(\int u_i(x_1, .., x_n, b_1, ..., b_n) d\mu_{-i} \right) d\mu_i^l - \int \left(\int u_i(x_1, .., x_n, b_1, ..., b_n) d\mu_{-i} \right) d\mu_i \right| < \delta/2 \tag{7}$$

for sufficiently large l. To see why, multiply $\int u_i(x_1, .., x_n, b_1, ..., b_n) d\mu_{-i}$ by a continuous function that takes value 1 on S and value 0 on clT_i^ε. This product is a continuous function, so by the definition of weak*-convergence, its expected value with respect to measures μ_i^l converges to its expected value with respect to measure μ_i. And since the measures μ_i^l and μ_i of the complement of S are smaller than ε, we obtain (7) provided that ε is sufficiently small compared to δ. (Recall that we assume that functions u_i are bounded.)

Inequalities (6) and (7) imply that $\left| U_i \left(\mu^l \right) - U_i \left(\mu \right) \right| < \delta$ for sufficiently large l.

8.2 Proofs from Section 6.3

Proof of Claim 1. Given a profile b_{-i} of locations of other players, let $\Delta_l(b_{-i})$ and $\Delta_r(b_{-i})$ be the difference in the market share of player i whose current location is b_i made by the infinitesimal shifts to the left and to the right, respectively. Notice that

$$\Delta_l(b_{-i}) + \Delta_r(b_{-i}) \geq 0$$

for any b_{-i}, and the inequality is strict if $b_j = b_i$ for more than one $j \neq i$. Indeed, an infinitesimal shift to each direction increases by a factor of $k + 1$ the share of customers in this direction, where k is the number of other players $j \neq i$ such that $b_j = b_i$. Thus, the sum of the expected values of $\Delta_l(b_{-i})$ and $\Delta_r(b_{-i})$ is positive if the marginals on B of the strategy of two or more other players have an atom at b_i.

Proof of Claim 2. Suppose that $b_i^k \to b_i$ and $\mu_j^k \to_k \mu_j$, $j \neq i$. (We disregard x_i, because it affects only the fixed costs of choosing a location, and the fixed costs are continuous in it.) Suppose first that condition (a) is satisfied. Then, the marginal of each μ_j assigns only an arbitrarily small probability to clW_i for a small neighborhood W_i of b_i, where clW_i is the closure of W_i. By Billingsley (1995), Theorem 29.1, the marginal of μ_j^k for sufficiently large k also assigns only an arbitrarily small probability to clW_i. Let V_i be another neighborhood of b_i such that $clV_i \subset W_i$. Thus, the only substantial difference between the market share of player i located at b_i^k when the other players play μ_{-i}^k and the market share of player i located at b_i when the other players play μ_{-i} can come from the strategies of other players contingent on the locations in the complement of clW_i. However, the share of player i is a continuous function of the locations selected by all players if player i is located in V_i, and the other players are located in the complement of V_i.

So, the difference between the market share of player i located at b_i^k when the other players play μ_{-i}^k and the market share of player i located at b_i when the other players play μ_{-i} , contingent on the locations in the complement of clW_i must also be small for sufficiently large k. To see why multiply the share of players i by a continuous function that is equal to 0 on clV_i, and is equal to 1 on the complement of W_i. The difference between the expected value of this product and the market share of player i contingent on the locations of the other players in the complement of clW_i is small for both μ_{-i} and μ_{-i}^k, because the marginals of all measures in μ_{-i} and μ_{-i}^k assign small probability to clW_i. So, we obtain the required property directly from the definition of weak*-convergence.

Suppose now that condition (b) is satisfied. Denote by j^* the j such that the marginal of μ_j has an atom at b_i. Then, the marginal of μ_j for all $j \neq i, j^*$ assigns only an arbitrarily small probability to clW_i for a sufficiently close neighborhood W_i of b_i, and so does the marginal of μ_j^k for sufficiently large k. Moreover, the marginal of $\mu_{j^*}^k$ for sufficiently large k assigns to clV_i, for a neighborhood V_i of b_i, a probability arbitrarily close to that assigned by μ_{j^*}. To see why, consider another neighborhood V_i' such that $clV_i \subset V_i'$, and such that the marginal of μ_{j^*} assigns to V_i' a probability arbitrarily close to that assigned by the marginal of μ_{j^*} to clV_i. Then, by Theorem 29.1 from Billingsley (1995), the probability assigned to V_i' by the marginal of μ_j^k for sufficiently large k cannot be much smaller that assigned by the marginal of μ_{j^*}, and the probability assigned to clV_i by the marginal of μ_j^k for sufficiently large k cannot be much greater that assigned by the marginal of μ_{j^*}.

With no loss of generality, assume that $clV_i \subset W_i$. Replacing W_i with a smaller W_i' such that $clV_i \subset W_i'$ if necessary, we can assume that μ_{j^*} and all $\mu_{j^*}^k$ (for sufficiently large k) assign an arbitrarily small probability to $clW_i - clV_i$. Thus, the only substantial difference between the market share of player i located at b_i^k when the other players play μ_{-i}^k and the market share of player i located at b_i when the other players play μ_{-i} can come from: (i) the strategies of all players $j \neq i$ choosing locations in the complement of clW_i, or (ii) the strategy of player j^* choosing a location in clV_i and the strategies of all players $j \neq i, j^*$ choosing locations in the complement of clW_i.

However, the market share of player i is a continuous function of the locations selected by all players if player i is located in V_i and the other players are located in the complement of V_i. So, the difference in the market shares of player i must also be small for sufficiently large k in case (i). It must also be small (for sufficiently large k) in case (ii), if clV_i is chosen sufficiently close to b_i by the assumption that the infinitesimal shifts from location b_i (to the right and to the left) have no effect on player i's market share.[41]

9 References

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[41] More precisely, we assume that infinitesimal shifts do not increase player i's expected payoff, but in this case a simple accounting argument shows that such shifts also do not decrease player i's expected payoff.

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