Optimal assignment mechanisms with imperfect verification*

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Abstract

Objects of different quality are to be allocated to agents. Agents can receive at most one object, and there are not enough high-quality objects for every agent. The value to the social planner from allocating objects to any given agent depends on that agent's private information. The social planner is unable to use transfers to give incentives for agents to convey their private information. Instead, she is able to imperfectly verify their reports through signals that are positively affiliated with each agent's type. We characterize mechanisms that maximize the social planner's expected payoff. In the optimal mechanism, each agent chooses one of various tracks, which are characterized by two thresholds. If the agent's signal exceeds the upper threshold of the chosen track, the agent receives a high-quality object, if it is in-between the two thresholds, he receives a low-quality object and if it is below the lower threshold, he receives no object.

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1 Introduction

Hylland and Zeckhauser (1979) and the ensuing literature study mechanisms that efficiently allocate objects to agents with private preferences. Agents are not allowed to make or receive monetary transfers and cannot be assigned more than one object. Classic examples include the allocation rule of public houses to families that cannot afford one, dorms to college students, offices to workers and seats at colleges/schools to students.

In some of these examples, it is natural to think that the agents' ordinal preferences are quite similar. In general, everyone prefers houses in better neighborhoods, offices with a window and schools with better reputations.[1] While finding the set of **ex-post** (Pareto) efficient allocation rules in settings where objects have different quality is straightforward (any non-wasteful allocation rule is ex-post efficient), the problem of finding **ex-ante** efficient allocation rules is more complicated. Even though agents have the same ordinal preferences over objects, the fact that they have different cardinal preferences leads to them having different preferences over lotteries of objects. Building on this insight, several authors have used mechanism design techniques to study the problem of finding welfare maximizing incentive compatible allocation rules in settings where agents have common ordinal preferences (Miralles, 2012, Hafalir and Miralles, 2015, Dogan and Uyanik, 2020, and Ortoleva, Safonov and Yariv, 2021).

We observe that, in many of these settings, even though the agents' (cardinal) preferences may be private, there is correlated information available. For example, in the college assignment setting, it seems natural to think that a student's grade on his SAT or GRE is positively correlated with that student valuation for college quality. Presumably, a student who values college quality a lot will put more effort and resources into his preparation, which is likely to lead to a better grade. Another example is the problem the State faces when administering natural disaster funds. In general, the State is able to obtain information about the damages suffered by each person interested in receiving help.[2] This information is correlated with how each person values the objects being assigned by the State, as one would think that those who have suffered more damages would have higher valuations. Similarly, in the public

[1] For the school assignment problem, there is empirical evidence that parents' ordinal preferences are indeed correlated (Abdulkadiroglu, Che and Yasuda, 2011) and that parents value similar things (Bosetti, 2004).

[2] For example, part of the process of obtaining housing assistance from the Federal Emergency Management Agency (FEMA) involves a house inspection (see https://www.fema.gov/fact-sheet/individual-and-households-program-exterior-and-remote-inspections).

housing problem, the State routinely goes through a screening process of the applicants that may involve requiring references, interviews and background checks.[3]

In this paper, we consider the mechanism design problem of finding the mechanism that maximizes total welfare when agents value quality differently, and the designer is able to costlessly observe a signal that is correlated with those valuations. We assume that there is a continuum of agents and a continuum of objects of high and low quality such that the measure of agents exceeds the measure of high-quality objects. Even though all agents prefer high-quality objects over low-quality objects, different agents might have different valuations over quality (their valuations are their type). The designer is not able to observe each agent's valuation; instead she observes a positively affiliated exogenous signal.

We characterize optimal incentive compatible allocation rules and show that they can be implemented as follows. Initially, agents are asked to choose one of many "tracks". After that, signals, which are correlated with the agents' types, are realized. The object the agent is awarded, if any, depends on the track he chose and on the signal that is realized. Specifically, each track is characterized by two thresholds for the signal: an upper threshold and a lower threshold. If the agent's signal exceeds the upper threshold, the agent is assigned a high-quality object; if his signal is in-between the two thresholds, he is assigned a low-quality object; finally, if his signal is below the lower threshold, he is not assigned any object. Different tracks have different pairs of thresholds; for some tracks, the two thresholds are very close, while for some others, they are very far apart. Figure 1 illustrates.[4]

By the revelation principle, the problem of finding an optimal mechanism reduces to finding the set of optimal incentive compatible allocation rules. In our setting, an allocation rule is simply a probability distribution for each agent over the objects being assigned given his private type and signal. An allocation rule is incentive compatible if every agent, knowing only his type (but not his signal), prefers to report truthfully. Different types may have different incentives over what to report due to having different beliefs over which signals will be realized.

The main challenge of finding optimal incentive compatible allocation rules is that

[3]See, for example, the link to apply for public housing in Providence (https://provhousing.org/housing/public-housing-admissions/public-housing-application/) and Oklahoma (http://www.oakha.org/Residents/Housing%20choice-voucher-residents/Pages/Eligibility-Screening.aspx).

[4]Not being assigned any object can alternatively be interpreted as receiving an object with even lesser quality, provided there is enough supply of that object. Throughout the paper, we use dotted lines to represent the allocation of high-quality objects, dashed lines to represent the allocation of low-quality objects, and solid lines to represent not being allocated any object just like in Figure 1.

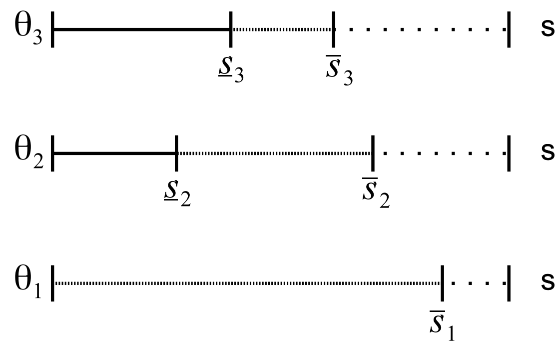

Figure 1: The optimal mechanism when there are only three types. In equilibrium, if the agent's type is the highest one (θ_3), he chooses the upper track; if it is second highest one (θ_2), he chooses the middle track; if it is smallest one (θ_1), he chooses the bottom track. Once a track has been chosen, the agent is assigned the high-quality object if his signal s lands on the dotted line, a low-quality object if it lands on the dashed line, and no object if it lands on the solid line.

the standard Myerson approach of mechanism design with transfers does not apply (Myerson, 1981, Myerson and Satterthwaite, 1983). In particular, the incentive constraints alone are not sufficient to determine which types receive larger probabilities of receiving either objects. For example, it is easy to construct incentive compatible allocation rules where lower types - types who value quality less - have a larger probability of receiving a high quality object than higher types.

In order to find the set of optimal allocation rules, we start by defining a family of allocation rules which we call *ordered* allocation rules. Ordered allocation rules (represented in Figure 1) are induced by the mechanism described above, and are such that i) conditional on each agent's type and signal, there is no randomness in which object is assigned to which agent and ii) the quality of the object received by each agent is increasing with his signal. Ordered allocation rules are convenient because they can simply be characterized by two thresholds for each type - an upper threshold and a lower threshold. Moreover, in a similar way to Myerson (1981), Myerson and Satterthwaite (1983), and the papers that follow, incentive compatibility alone is sufficient to determine how the two thresholds vary according to each agent's type; specifically, higher types, who are more confident that their signal will be high, are associated with lower upper thresholds, while lower types, who believe that lower signals are more likely, are associated with lower lower thresholds. This property allows us to find the optimal ordinal allocation rule using "local" incentive constraints.

The final part of the argument is to show that ordered allocation rules are indeed optimal. In order to do that, we first demonstrate that the binding incentive constraints are the ones that prevent lower types from mimicking higher types. The intuition comes from the designer's first best allocation rule: if the designer knew the agents' types, she would simply ignore the signals and assign the higher quality objects to the agents with the higher types, because these agents value quality more (Becker, 1974). The first best allocation rule is clearly not incentive compatible because lower types would rather claim to having larger valuations. Then, we prove that ordered allocation rules minimize the deviation payoffs of low types who mimic higher types. The intuition for that is that the positive affiliation between each agent's type and his signal means that it is relatively more likely for a higher type to generate a higher signal than it is for a lower type. As a result, the best way to deter deviations from lower types is to stack the better rewards at the top of the signal distribution, i.e., for each type, the higher the signal, the higher should be the reward.

In general, the designer does strictly better in the optimal mechanism than by simply ignoring the agents and assigning the objects based exclusively on the signals because, even though the optimal mechanism does not perfectly discriminate in favor of agents with higher types (as the first best mechanism would), it does so partially. Higher types, who are more confident that their signal will be high, select tracks with lower upper thresholds, which increases their chances of receiving a high quality object compared to the tracks that lower types choose, which have lower lower thresholds but higher upper thresholds (see Figure 1).

The paper proceeds as follows. In the next section, we discuss the related literature. In Section 3, we discuss the model. In Section 4, we characterize the optimal mechanism and in Section 5 we discuss several extensions. All proofs are in the Appendix.

2 Related Literature

As discussed in the Introduction, the paper considers an allocation problem without transfers similar to the problem considered by the literature that followed Hylland and Zeckhauser (1979). This literature has focused on finding mechanisms with various desirable properties like efficiency, incentive compatibility or fairness (Zhou, 1990, Svensson, 1999, Papai, 2000). A branch of this literature concentrates on ordinal mechanisms, which only require that the agents provide their ordinal preferences, which are public in our paper (Abdulkadiroglu and Sonmez, 1998, Bogomolnaia and Moulin, 2001, McLennan, 2002, Kojima and Manea, 2010). A separate branch of this literature studies pseudo-market mechanisms, where agents are endowed with points and have to assign them to "buy" probabilities of getting each object (Hylland and Zechauser, 1979, Miralles and Pycia, 2021).[5]

Our work is different not only because we allow the designer to observe signals that are correlated with the agents' private information but also because of a different approach. Rather than specifying what properties are desirable and then looking for mechanisms with those properties, we find the optimal mechanism relative to an exogenously given objective function. Indeed, we find that several of the properties considered desirable by this literature are violated in the optimal mechanism. Most importantly, we find that the optimal mechanism might not even be ex-post efficient, because, there might be low-quality objects left unassigned and agents who are not given any object.

There are a few papers which have followed a similar mechanism design approach to allocation problems. Miralles (2012) considers a setting with two objects and two agents and finds that, in the optimal non-wasteful allocation rule, objects are randomly allocated when the agents' reported (ordinal) preferences are the same. We find the analogous result in our model with signals: if one imposes that all agents be assigned one object, the best the designer can do is to assign objects based exclusively on the signals. Dogan and Uyanik (2020) study a simplified version of Miralles (2012) and, like us, find that wasteful allocation rules might be optimal. Hafalir and Miralles (2015) and Ortoleva, Safonov and Yariv (2021) consider models with a continuum of agents and find optimal mechanisms that have similar features to ours. Specifically, agents who have high valuations receive lotteries over objects where there is a large probability of receiving high-quality objects but also a high probability of receiving low-quality objects, while agents with low valuations receive lotteries with a high probability of receiving average quality objects. We obtain the same qualitative results, because not receiving an object is equivalent to receiving an object of even lower quality.

The main difference to our work is the addition of exogenous public signals and the characterization of how to best use these signals to maximize welfare. In this

[5]There is also a related literature, where agents have initial property rights, which follows the same approach of finding mechanisms with desirable properties (Shapley and Scarf, 1974, Abdulkadiroglu and Sonmez, 1998, Sonmez and Unver, 2010, Pycia and Unver, 2017).

literature, the reason why different types choose different lotteries is because they have different cardinal preferences over objects, which leads to evaluating lotteries over objects of different quality differently. Instead, in our model, what leads to the separation is the fact that different types have different beliefs about which signals are more likely. Indeed, the optimal mechanism described is still optimal when agents are assumed to have the same preferences over lotteries of objects, as we illustrate in the next section. Moreover, adding signals makes the mechanism design problem fundamentally different as it prevents a direct use of Myerson-like techniques that are standard in mechanism design (Myerson, 1981, Myerson and Satterthwaite, 1983) as discussed in the Introduction.

Our paper is not the first to consider a mechanism design problem where the principal has access to exogenous signals correlated with the agents' private information. In settings with transfers, the availability of correlated signals generally allows the principal to implement the first best (Cremer and McLean, 1988, Riordan and Sappington, 1988, McAfee and Reny, 1992, Bose and Zhao, 2007) unless transfers are bounded (Demougin and Garvie, 1991, Kessler et al., 2005, Gary-Bobo and Spiegel, 2006). Indeed, if transfers were allowed in our model, an example of an optimal mechanism would be the first price auction. In equilibrium, agents with higher valuations would bid more for the higher quality objects and the first best allocation would be implemented. More recently, this literature has expanded to consider settings without transfers. In most of these papers, the focus has been on showing how exogenous signals help the principal even when the agents have type independent preferences (see Kattwinkel, 2020, Silva, 2019a, 2021, and Siegel and Strulovici, 2021). Closest to our paper is Silva (2019b), who considers a setting with multiple agents similar to ours but without a resource constraint (instead, agents' types are correlated).[6]

The addition of these exogenous signals to the mechanism design problem allows the principal to (imperfectly) verify the agents' reports. The costly verification literature considers similar settings but assumes that the verification is perfect but costly (Ben-Porath, Dekel and Lipman, 2014, Li, 2020, Chua, Hu and Liu, 2019). In these papers, the principal assigns objects with the same quality. If, in our model of costless imperfect verification, we were also to assume that all objects were homogeneous, the optimal mechanism would be for the principal to ignore the agents and simply assign the objects

[6]Recently, we have become aware of Bloch, Dutta and Dziubinski (2021) who study a similar model to Silva (2019b) with a resource constraint and independent types but assume only two types for each agent who each generate only binary signals. In such a model, if there were many agents (or a continuum in our case), the optimal mechanism would be for the designer to ignore the agents and make decisions using the signals exclusively.

based on the public signals.[7]

Finally, this paper is also related to the literature on disclosure, where agents may provide evidence for the statements they make. Earlier work has assumed that agents were able to provide "hard" evidence (Dye, 1985, Green and Laffont, 1986, Lipman and Seppi, 1995, Glazer and Rubinstein, 2004 and 2006, Bull and Watson, 2007, Deneckere and Severinov, 2008, Sher, 2011, Hart, Kremer and Perry, 2017, Ben-Porath, Dekel and Lipman, 2019). Recently, models of probabilistic verification have emerged, where the evidence provided by the agents may not be perfectly accurate (Caragiannis et al., 2012, Ball and Kattwinkel, 2019, Silva, 2020). Our model is similar to Ball and Kattwinkel (2019) and to Silva (2020) except that the technology to verify the agents' reports is independent of each agent's actions.[8] We expand on these papers by characterizing optimal mechanisms (Ball and Kattwinkel, 2019, focuses on discussing versions of the revelation principle, while Silva, 2020a, discusses the extent to which commitment power is important when disclosure is probabilistic).

3 An illustrative example

There is a continuum of agents who have preferences over two objects, a high-quality object (h) and a low-quality object (l). The principal only has enough high quality objects for 50% of the agents, but has enough low quality objects for everyone (this is generalized in the text). Each agent's preferences over objects depend on the quality of the object and also on a privately observed random variable θ (the agent's type). In this example, we assume that $\theta \in \{\theta_L, \theta_H\}$ with $0 < \theta_L < \theta_H$ and where each θ is equally likely. Each agent's willingness to pay for each object is denoted by $u(\theta, j)$ for $j = l, h$, where

	$u(\cdot, l)$	$u(\cdot, h)$
θ_L	1	2
θ_H	2	4

.

[7] Related costly verification papers include Epitropou and Vohra (2019), Erlanson and Kleiner (2019), Halac and Yared (2019) and Li (2021). Mylovanov and Zapechelnyuk (2017) is also related even though the verification is costless, because of the restrictions on punishments for those agents who are caught lying.

[8] Specifically, it is as if i) the set of tests (in Ball and Kattwinkel, 2019) and documents (in Silva, 2020) is a singleton and ii) agents always have to make effort. Conditions i) and ii) ensure that the distribution of the principal's signal only depends on the agents' type and not on their possible report (in particular, ii) ensures that agents cannot have a "bad" signal on purpose). We note, however, that a constraint that forces agents to exert effort would not bind as in the optimal mechanism each agent has an incentive to always have as high of a signal as possible.

Notice that not only does every agent have the same ordinal preferences over the two objects, it is also the case that every agent has the same preferences over lotteries of the two objects, because $\frac{u(\cdot,h)}{u(\cdot,l)}$ is constant. Indeed, contrary to what is standard in the literature on allocation problems, each agent's preferences are type independent.[9] Nevertheless, the two types of agents value having these two objects differently; high-type agents value quality more than low-type agents in the sense that their willingness to pay for quality is higher (a high-type agent is willing to pay 2 for each increase in quality, while a low-type agent is only willing to pay 1). Indeed, if transfers were possible, a principal who wishes to maximize the sum of the agents' payoffs could achieve her first best in a number of ways: she could run a first price auction or simply charge a price of 4 for object h and of 1 for object l and then share the revenue equally among the agents. In either case, in equilibrium, high type agents would be assigned object h and low type agents would be assigned object l. The ex-ante expected payoff of each agent would be 2.5.

In the paper, we study the same problem but, instead of allowing for transfers, we assume that the principal has access to a (verifiable) signal $s \in [0,1]$ for each agent, which is correlated with θ. For this example, we assume that

$$p(s|\theta) = \begin{cases} 2s \text{ if } \theta = \theta_H \\ 2(1-s) \text{ if } \theta = \theta_L \end{cases},$$

so that s and θ are positively affiliated (intuitively, it is more likely that s is high when $\theta = \theta_H$). The goal of the principal is to find the symmetric mechanism that maximizes the ex-ante payoff of each agent.

A simple alternative for the principal is to assign objects according to each agent's signal, relying on the positive correlation between signals and types. Specifically, the optimal mechanism that depends exclusively on the signals is such that each agent is assigned a high-quality object if $s \geq \frac{1}{2}$ and a low-quality object if $s < \frac{1}{2}$. This would return an ex-ante expected payoff of

$$\frac{1}{2} * \left(\frac{3}{4} * 4 + \frac{1}{4} * 2\right) + \frac{1}{2} * \left(\frac{1}{4} * 2 + \frac{3}{4} * 1\right) = \frac{19}{8}$$

for each agent. This mechanism is not optimal though as the principal is able to use the fact that different types of agents have different beliefs over the realization of the signals to partially discriminate in favor of high-type agents.

[9] In the text, we do allow agents to have type dependent preferences over lotteries. We chose an example where agents have type independent preferences to highlight one of the novelties of our work.

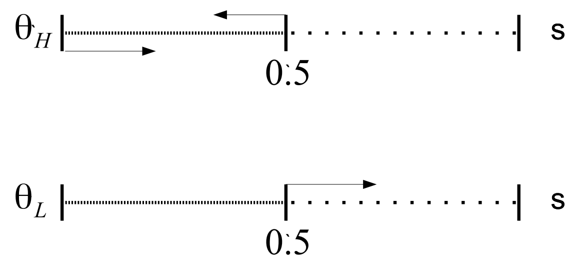

Figure 2: If objects are assigned based exclusively on the agents' signals, each agent is assigned a high-quality object if his signal is above the threshold at 0.5 and is assigned a low-quality object otherwise.

We find that the optimal mechanism is as follows: each agent may choose between tracks A and B. If track A is chosen, the agent guarantees that he will be assigned at least a low quality object and will be assigned a high quality object if $s \geq \overline{s}^A \simeq 0.64$. If track B is chosen, the agent is assigned a high quality object if $s \geq \overline{s}^A \simeq 0.36$ (so that it is easier to get the high quality object) but will not be assigned any object if $s < \underline{s}^B \equiv 0.15$.

The key reason why the optimal mechanism does strictly better than assigning objects according exclusively to signals has to do with the way agents self-select. By design, high-type agents, who are more confident that their signal will be high, prefer track B while low-type agents choose track A (despite being indifferent). As a result, the probability that each agent receives each object ultimately depends on both the agent's type and signal.

Figure 2 displays the allocation rule induced by the initial mechanism and the arrows represent what changes in the optimal mechanism. Two things happen. On the one hand, the threshold to obtain high-quality objects goes up for low-type agents (from 0.5 to 0.64) and down for high-type agents (from 0.5 to 0.36). The principal is made better off by this change because she prefers to assign the high-quality objects to the agents with higher valuations. On the other hand, the threshold to get low-quality objects goes up (from 0 to 0.15) for the high-type agents. While this second effect is necessary to prevent the low-type agents from mimicking the high-type agents and choosing track B, it makes the principal worse off (and is inefficient). Overall though, the ex-ante payoff of any agent is higher; it is equal to $2.415 > \frac{19}{8}$. Specifically, a high-type agent now has an 87% chance of receiving a high-quality object, an 11% chance of being assigned a low-quality object and only a 2% chance of receiving no object, while, for low-type agents, those numbers are 13%, 87% and 0% respectively.

To prove that the allocation rule induced by the mechanism described is optimal, we start by considering a relaxed problem, where the incentive constraint that prevents high-type agents from mimicking low-type agents is ignored. Then, we prove that any allocation rule which solves the relaxed problem is such that there are two thresholds $\overline{s}_H$ and $\underline{s}_H$ such that type θ_H receives the high-quality object if $s \geq \overline{s}_H$, the low-quality object if $s \in [\underline{s}_H, \overline{s}_H)$ and receives no object if $s < \underline{s}_H$. This has to do with distribution $p(s|\theta)$: by stacking the best rewards at the top of interval $[0,1]$, one minimizes the deviation payoff of lower types. There is more flexibility with respect to the distribution of rewards that type θ_L receives. What is important is that i) type θ_H does not mimic and ii) type θ_L is indifferent (i.e., the low-type incentive constraint binds). Therefore, it is optimal for the threshold structure to also apply to type θ_L with thresholds $\underline{s}_L$ and $\overline{s}_L$, provided $\underline{s}_L \leq \underline{s}_H \leq \overline{s}_H \leq \overline{s}_L$. The reason why type θ_L must be indifferent is because, if not, one could always rearrange vector $(\underline{s}_L, \overline{s}_L, \underline{s}_H, \overline{s}_H)$ in some way that increases the ex-ante utility of the agents (and, consequently, of the principal) without violating any incentive constraint and without more objects being assigned than those available (for example, if $\underline{s}_H > 0$ and type θ_L is not indifferent, one could lower $\underline{s}_H$ and increase welfare). Finally, one can find the optimal mechanism by simply choosing vector $(\underline{s}_L, \overline{s}_L, \underline{s}_H, \overline{s}_H)$ optimally given two constraints: that type θ_L is indifferent and that $\underline{s}_L \leq \underline{s}_H$. In this example, it follows that $\underline{s}_L = 0$ (which is always the case when there are enough low-quality objects for every agent), $\overline{s}_L = \overline{s}^A$, $\underline{s}_H = \underline{s}^B$ and $\overline{s}_H = \overline{s}^B$.

4 Model

4.1 Fundamentals

There is a continuum of agents of mass 1 and a continuum of objects to be allocated to the agents. Each object can be of high (h) or low (l) quality. There is a measure $\alpha_h \in (0,1)$ of high-quality objects and a measure $\alpha_l \in (0,1)$ of low-quality objects.

Each agent has a private type $\theta \in \Theta$, where $\Theta = \{\theta_1, .., \theta_J\} \subset \mathbb{R}$. Each θ is independent and identically distributed across agents and the prior probability of each type $\theta \in \Theta$ is denoted by $q(\theta) \in (0,1)$.[10] Without loss of generality, we assume that $\theta_{j+1} > \theta_j$ for all $j = 1, ..., J-1$. Each agent generates a public signal $s \in [0,1]$, which is only correlated with that agent's type θ. Denote the conditional density of s by $p(s|\theta)$ and assume it is continuous. Assume further that $p(s|\theta) > 0$ for all $s \in (0,1)$ and $\theta \in \Theta$, and that densities $\{p(\cdot, \theta) : \theta \in \Theta\}$ have the monotone likelihood ratio property (MLRP), i.e., for any $\theta, \theta' \in \Theta$ and $s, s' \in (0,1)$ such that $\theta' > \theta$ and $s' > s$, $\frac{p(s'|\theta')}{p(s|\theta')} > \frac{p(s'|\theta)}{p(s|\theta)}$. This guarantees that larger types are the ones that are more likely to generate larger signals.

Each agent's payoff depends on his type and on the quality of the object that he receives. When an agent of type θ receives a high-quality object, his payoff is denoted by $u(\theta, h)$; if he receives a low-quality object it is $u(\theta, l)$. If the agent does not receive any object, his payoff is $u(\theta, \emptyset) = 0$ for all $\theta \in \Theta$.

We make the following three assumptions. First, we assume that all agents have the same ordinal preferences: $u(\theta, h) > u(\theta, l) > 0$ for every $\theta \in \Theta$. [11] Second, we assume that the preference for lotteries that place a large probability on high-quality objects relative to low-quality objects is weakly increasing with each agent's type. Specifically, notice that each agent's expected payoff is given by

$$u(\theta, h) \Pr\{\text{receiving the } h \text{ object}\} + u(\theta, l) \Pr\{\text{receiving the } l \text{ object}\},$$

which is proportional to

$$\frac{u(\theta, h)}{u(\theta, l)} \Pr\{\text{receiving the } h \text{ object}\} + \Pr\{\text{receiving the } l \text{ object}\}.$$

Our second assumption is that $\frac{u(\theta,h)}{u(\theta,l)}$ is weakly increasing with θ. If $\frac{u(\theta,h)}{u(\theta,l)}$ is constant for all θ (as in the example), then agents have the same preferences over lotteries of objects.

Finally, our third assumption is that higher types value jumps in quality more than lower types. Formally, we assume that $(u(\theta, l) - u(\theta, \emptyset))$ and $(u(\theta, h) - u(\theta, l))$ are both strictly increasing in θ.[12] Essentially, the reader might want to think of our

[10] We also interpret $q(\theta)$ as the fraction of agents of type θ. We rely on the argument of Judd (1985) in order to identify probabilities with fractions when the population is a continuum.

[11] A similar assumption is made in Troyan (2012), Chade, Lewis and Smith (2014), Hafalir and Mirales (2015), Lien, Zheng and Zhong (2017), Hafalir et al. (2018), Akin (2019), Akbarpour et al. (2022), Dogan and Uyanik (2020) and Ortoleva et al. (2021) among others.

[12] Notice that if $(u(\theta, l) - u(\theta, \emptyset))$ is strictly increasing and $\frac{u(\theta,h)}{u(\theta,l)}$ is weakly increasing,

setting as an auction setting without monetary transfers, where agents with higher types have higher valuations for quality. In particular, the third assumption implies that the symmetric allocation rule that maximizes the ex-ante payoff of the agents under complete information is to assign the better quality objects to the agents with the higher types (Becker, 1974).

4.2 Definitions

Our goal is to find the optimal symmetric mechanism that maximizes the ex-ante expected payoff of each agent. By the revelation principle, it is enough to consider only revelation mechanisms, i.e., allocation rules that are incentive compatible.[13] A symmetric allocation rule (henceforth, simply allocation rule) is a mapping $x = (x_h, x_l) : \Theta \times [0,1] \to [0,1]^2$ such that

$$x_h(\theta, s) + x_l(\theta, s) \leq 1$$

for all $\theta \in \Theta$ and $s \in [0,1]$, where $x_h(\theta, s)$ and $x_l(\theta, s)$ represent the probability that an agent with type θ and signal s is assigned object h and l respectively.

An allocation rule x is *feasible* if the measure of assigned objects does not exceed the measure of available objects, i.e., if

$$\sum_{\theta \in \Theta} q(\theta) \int_0^1 p(s|\theta)\, x_h(\theta, s)\, ds \leq \alpha_h \tag{1}$$

and

$$\sum_{\theta \in \Theta} q(\theta) \int_0^1 p(s|\theta)\, x_l(\theta, s)\, ds \leq \alpha_l. \tag{2}$$

An allocation rule x is *incentive compatible* (IC) if each agent prefers to report truthfully, i.e., for all $\theta \in \Theta$,

$(u(\theta, h) - u(\theta, l))$ is strictly increasing, so we only really need to assume the first two.

[13] In line with the more recent models on probabilistic verification (e.g., Ball and Kattwinkel, 2019, and Silva, 2020), the revelation principal applies. To see why that is, suppose that the agents play a game with the principal before signals are realized. At the end of that game, and as a function of everyone's strategy, each agent is assigned a lottery of rewards based on the possible future realizations of s. Therefore, the social planner could instead simply ask each agent to report their true type and then assign those lotteries accordingly.

$$\theta \in \arg\max_{\theta' \in \Theta} \; U\left(\theta, X\left(\theta'\right)\right),$$

where $X(\theta')$ is the function that, for each signal s, returns the probability with which an agent who reports type θ' is assigned each object. That is,

$$X\left(\theta'\right) \equiv x(\theta', \cdot) : [0,1] \to [0,1]^2$$

for each $\theta' \in \Theta$ and

$$U\left(\theta, X(\theta')\right) \equiv \int_0^1 p\left(s|\theta\right)\left(x_h\left(\theta', s\right) u\left(\theta, h\right) + x_l\left(\theta', s\right) u\left(\theta, l\right)\right) ds.$$

Notice that each agent reports his type *before* observing his signal. That is how incentives are given to the agents; different types have different beliefs over signal s.

An allocation rule x is *ordered* if, for all $\theta \in \Theta$, there is a pair $\underline{s}_\theta, \overline{s}_\theta$ such that $0 \leq \underline{s}_\theta \leq \overline{s}_\theta \leq 1$ and

$$x_h\left(\theta, s\right) = \begin{cases} 1 \text{ if } s \geq \overline{s}_\theta \\ 0 \text{ if } s < \overline{s}_\theta \end{cases} \text{ and } x_l\left(\theta, s\right) = \begin{cases} 1 \text{ if } s \in [\underline{s}_\theta, \overline{s}_\theta] \\ 0 \text{ if } s \notin [\underline{s}_\theta, \overline{s}_\theta] \end{cases}.$$

In an ordered allocation rule, the only randomness an agent of some type θ faces comes from the signal s, i.e., conditional on his type and on the signal, there is no randomization. Furthermore, the agent always prefers to have a larger signal than a lower signal; the rewards are at the top. Figure 3 presents an example of an ordered allocation rule. Notice that an ordered allocation rule is completely characterized by its thresholds $\{(\underline{s}_\theta, \overline{s}_\theta)\}_{\theta \in \Theta}$.

Finally, we let

$$W\left(x\right) \equiv \sum_{\theta \in \Theta} q\left(\theta\right) U\left(\theta, X\left(\theta\right)\right)$$

denote the ex-ante expected payoff of any agent given allocation rule x. An allocation rule x is **optimal** if it maximizes W.

5 Optimal allocation rules

In this section, we characterize optimal allocation rules. We find that ordered allocation rules are optimal and describe their properties in the following theorem. Moreover, we

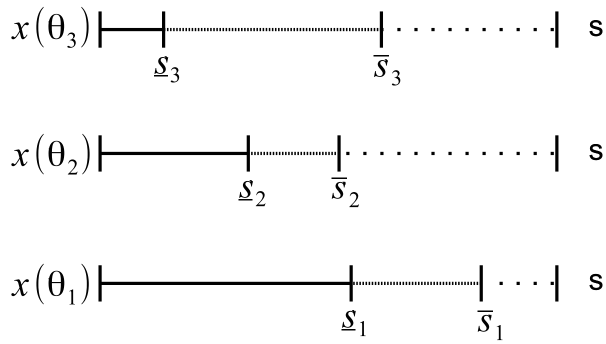

Figure 3: An example of an ordered allocation rule. This particular ordered allocation rule is not IC because type θ_1 would prefer to report to being type θ_2.

show that every optimal allocation rule is essentially an ordered allocation rule. By this we mean that any optimal allocation rule is equal to some optimal ordered allocation rule almost everywhere, except possibly for the allocation of type θ_1 (the lowest type), who still receives the same probability of being assigned any of the two objects.

Theorem. *(Optimality of ordered allocation rules)*

1. *There is an ordered allocation rule* $\{(\underline{s}_\theta, \overline{s}_\theta)\}_{\theta \in \Theta}$ *that is an optimal allocation rule. It has the following properties: i)* $\overline{s}_\theta$ *is weakly decreasing; ii)* $\underline{s}_\theta$ *is weakly increasing; iii) type* θ_j *is indifferent to reporting to being type* θ_{j+1} *for all* $j < J$.

2. *For any optimal allocation rule* x*, there is an optimal ordered allocation rule* $\widehat{x}$ *such that* $X(\theta) =^{a.e.} \widehat{X}(\theta)$ *for all* $\theta > \theta_1$ *and*

$$\int_0^1 p(s|\theta_1)\, x_k(\theta_1, s)\, ds = \int_0^1 p(s|\theta_1)\, \widehat{x}_k(\theta_1, s)\, ds$$

for $k = h, l$.

Figure 1 of the Introduction displays the optimal allocation rule when there are three types. In sections 5.1-5.3, we guide the reader through the argument of part 1 of the Theorem (the formal proofs are in the Appendix). In section 5.4 we prove part 2.

5.1 The single-crossing problem and ordered allocations

At first glance, the problem of finding optimal allocation rules might appear relatively standard. Recall that each agent's expected utility is given by

$$u(\theta, h) \Pr\{\text{receiving the } h \text{ object}\} + u(\theta, l) \Pr\{\text{receiving the } l \text{ object}\},$$

where the two goods - the probability of being assigned a high-quality object and the probability of being assigned a low-quality object - enter linearly. Furthermore, the condition that $\frac{u(\theta,h)}{u(\theta,l)}$ is increasing (we only assume it is weakly increasing, but, for the sake of argument, say it is strictly increasing) looks a lot like the typical single crossing condition that is standard in mechanism design. So, the problem appears to be a variation of Myerson (1981) and Myerson and Satterthwaite (1983). However, unlike what is standard in mechanism design, the fact that the allocation rules depend on exogenous signals makes it so that the probability of receiving each good depends not only on each agent's report but also on his true type (the probability that each agent receives a high-quality object for example might depend on the signal that is realized, whose distribution depends on that agent's true type). As it turns out, this complicates matters considerably, because incentive compatibility no longer implies that types that are closer together receive distributions of goods that are also closer. For example, in Myerson and Satterthwaite (1983), who study bilateral trade, and in much of the literature that followed, larger types are more likely to receive the object that is being traded than lower types in *every* incentive compatible allocation rule. By contrast, in our setting, there are incentive compatible allocation rules where the probability of receiving a high quality object is not monotone with the agent's type.[14]

In order to overcome this difficulty, we first show that we would not have this problem if we were to restrict attention to the class of ordered allocation rules and then show that there are ordered allocation rules which are optimal.

In ordered allocation rules, we can reinterpret the problem by thinking of the two goods as being the two thresholds $\overline{s}$ and $\underline{s}$ (instead of the probability of receiving a high quality object and a low quality object, respectively). The advantage of framing the problem in this manner is that the thresholds $\overline{s}$ and $\underline{s}$ each agent is assigned depend only on his report and not on his true type. Specifically, in any ordered allocation rule, the expected utility of any type θ agent who reports being type θ' is given by

[14] We provide such an example in the Appendix.

$\widehat{U}(\theta, \overline{s}_{\theta'}, \underline{s}_{\theta'})$, where $\widehat{U}(\theta, \overline{s}, \underline{s})$ represents the expected payoff of any agent of type θ when assigned thresholds $\overline{s}$ and $\underline{s}$:

$$\widehat{U}(\theta, \overline{s}, \underline{s}) \equiv u(\theta, h) \int_{\overline{s}}^{1} p(s|\theta)\, ds + u(\theta, l) \int_{\underline{s}}^{\overline{s}} p(s|\theta)\, ds.$$

Notice that $\widehat{U}(\theta, \overline{s}, \underline{s})$ is decreasing with both $\overline{s}$ and $\underline{s}$, so if we were to draw indifference curves of the different types on the space $(\underline{s}, \overline{s})$ they would be downward sloping. Using the MLRP of p, we are able to show that those indifference curves cross at most once as Figure 4 illustrates. The intuition is that larger types are more confident that their signals will be above the upper thresholds. Thus, as we show in the next lemma, if a type is indifferent between two tracks (i.e., two pairs of thresholds) $(\underline{s}', \overline{s}')$ and $(\underline{s}'', \overline{s}'')$, such that $\overline{s}' > \overline{s}'' \geq \underline{s}'' > \underline{s}'$, then all lower types prefer the first pair, while all higher types prefer the second one.

Formally, we have the following lemma:

Lemma 1. *Take any ordered allocation rule x and any two types $\theta' \in \Theta$ and $\theta'' \in \Theta$ such that such that $\overline{s}_{\theta'} > \overline{s}_{\theta''} \geq \underline{s}_{\theta''} > \underline{s}_{\theta'}$. It follows that for all $\theta \in \Theta$,*

$$U(\theta, X(\theta')) \geq U(\theta, X(\theta'')) \Rightarrow U\left(\widehat{\theta}, X(\theta')\right) > U\left(\widehat{\theta}, X(\theta'')\right)$$

for all $\widehat{\theta} < \theta$ and

$$U(\theta, X(\theta'')) \geq U(\theta, X(\theta')) \Rightarrow U\left(\widehat{\theta}, X(\theta'')\right) > U\left(\widehat{\theta}, X(\theta')\right)$$

for all $\widehat{\theta} > \theta$.

The previous lemma makes it much easier to deal with ordered allocation rules (as compared to general allocation rules) because it implies that one only needs to be concerned with "local" incentive constraints. Indeed, the way to find the optimal ordered allocation rule is precisely through a series of arguments of a local nature. Of course, one still has to show that ordered allocation rules are optimal, which we do in the following sections.

The outline for the proof of part 1 of the Theorem is as follows. In Section 5.2, we define a relaxed problem, where we relax some of the incentive constraints. In particular, in the relaxed problem, we maximize W subject to the feasibility constraints (conditions (1) and (2)) and the upper incentives constraints only (which prevent each type from mimicking higher types). Second, we show that, for any solution of the re-

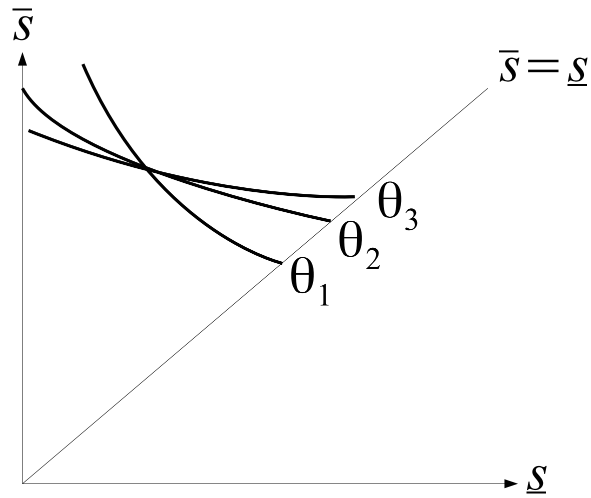

Figure 4: Indifference curves for three types which only cross once. Note that pairs nearer the origin are preferred to pairs away from the origin.

laxed problem, there is an *ordered* allocation rule that is also a solution of the relaxed problem (Lemma 2). Third, we prove that the relaxed problem has a solution (Lemma 3), which, combined with Lemma 2, ensures that there is an ordered allocation rule that is a solution to the relaxed problem. Fourth, in Section 5.3, we show that, for any ordered allocation rule that solves the relaxed problem, upper thresholds are decreasing with the agent's type, lower thresholds are increasing, and any type is indifferent between reporting his type and the next largest type (Lemma 4). Finally, we demonstrate that any ordered allocation rule which solves the relaxed problem satisfies the lower incentive constraints (i.e., is IC), and therefore, is an optimal allocation rule (Lemma 5).

5.2 The relaxed problem

The optimal allocation rule maximizes $W(x)$ subject to the i) feasibility constraints, ii) upper incentive constraints, i.e., for all θ,

$$U(\theta, X(\theta)) \geq U(\theta, X(\theta')) \text{ for all } \theta' > \theta,$$

and iii) lower incentive constraints, i.e., for all θ,

$$U(\theta, X(\theta)) \geq U(\theta, X(\theta')) \text{ for all } \theta' < \theta.$$

We define the relaxed problem as maximizing $W(x)$ subject only to i) and ii). We start by showing that, for any feasible allocation rule which satisfies the incentive constraints of the relaxed problem, there is a feasible ordered allocation rule with the same ex-ante payoff for each agent that also satisfies the same incentive constraints.

Lemma 2. *Let x be any allocation rule that satisfies all the incentive constraints of the relaxed problem. Define $\widehat{x}$ to be an ordered allocation rule with thresholds $\{(\underline{s}_\theta, \overline{s}_\theta)\}_{\theta\in\Theta}$ such that*

$$\int_{\overline{s}_\theta}^{1} p(s|\theta)\, ds = \int_0^1 x_h(\theta, s)\, p(s|\theta)\, ds$$

and

$$\int_{\underline{s}_\theta}^{\overline{s}_\theta} p(s|\theta)\, ds = \int_0^1 x_l(\theta, s)\, p(s|\theta)\, ds$$

for all $\theta \in \Theta$. It follows that $\widehat{x}$ satisfies all the incentive constraints of the relaxed problem.

Allocation rule $\widehat{x}$ in Lemma 2 is such that the probability that each type receives each object is the same as in allocation rule x. The only difference is that the rewards are all "brought up to the top". Therefore, by definition, $W(x) = W(\widehat{x})$. To see why allocation rule $\widehat{x}$ satisfies all upper incentive constraints, it might be convenient to go through the finite steps of transforming allocation rule x into allocation rule $\widehat{x}$. Take allocation rule x and reorder only type θ_1's track as described in Lemma 2; call that allocation rule x^1. It follows that allocation rule x^1 satisfies all incentive constraints because the only incentive constraints considered that involve type θ_1 are the ones that prevent him from mimicking higher types. Seeing as his expected utility is the same under allocation rules x^1 and x, those incentive constraints are satisfied.

Now, do the same reordering with type θ_2 and call the corresponding allocation rule x^2. Once again, by the same reason, type θ_2 does not want to mimic any larger type under allocation rule x^2, so we only need to show that type θ_1 does not want to mimic type θ_2. That is the case because lower types are less likely to draw large signals; therefore, if type θ_2 is made indifferent by bringing all his rewards up, lower types would be made worse off as a result. By continuing with this logic for all the J types, we get to allocation rule $\widehat{x}$.

The following lemma shows that a solution to the relaxed problem exists.

Lemma 3. *There exists an allocation rule that solves the relaxed problem.*

By combining the two previous lemmas, we get that there are ordered allocation rules which solve the relaxed problem. In section 5.3, we are able to characterize them because of the "single crossing property" that ordered allocation rules have; in particular, our arguments are all of a "local" nature as we describe next.

5.3 The optimal ordered allocation rule of the relaxed problem

Lemma 4. *Any ordered allocation rule x that solves the relaxed problem is such that i) $\overline{s}_\theta$ is weakly decreasing; ii) $\underline{s}_\theta$ is weakly increasing; iii) type θ_j is indifferent to reporting to being type θ_{j+1} for all $j < J$.*

To gain some intuition on this result, let us assume that $J = 2$ just like in the example of Section 3. Property iii) states the low type must be indifferent to the high type, i.e., the (only) incentive constraint of the relaxed problem binds. This follows because, if not, it would be possible to manipulate the thresholds of both types in such a way that each agent's ex-ante payoff would increase without violating the incentive constraint or assigning more objects than those available. Property iii) implies that either A) $\overline{s}_1 \geq \overline{s}_2 \geq \underline{s}_2 \geq \underline{s}_1$ or B) $\overline{s}_2 \geq \overline{s}_1 \geq \underline{s}_1 \geq \underline{s}_2$. Properties i) and ii) confirm that A) must be true. The idea is that if one must assign a pair of thresholds to each type with the constraint that the low type must be indifferent, one would rather assign the pair of thresholds with the smaller gap to the high type. That pair of thresholds has the lowest upper threshold, which is more valued by high-type agents, who value going from a low-quality object to a high-quality object more than low-type agents.

In the final step of the proof of part 1, we show that any ordered allocation rule that solves the relaxed problem satisfies the relaxed incentive constraints.

Lemma 5. *Let x be an ordered allocation rule that solves the relaxed problem. Then x is also an optimal allocation rule.*

Lemma 5 directly follows from the previous lemmas as can be seen by considering Figure 1. If type θ_1 is indifferent to mimicking type θ_2, it follows by Lemma 1 that type θ_2, who is more confident that his signal will be high, strictly prefers to report θ_2 over mimicking type θ_1. By the same reasons, type θ_3 strictly prefers to report θ_3 over reporting θ_2 and strictly prefers that over reporting θ_1.

5.4 Uniqueness

Part 2 of the theorem states that ordered allocation rules are essentially the unique optimal allocation rules. The proof is as follows. Consider any optimal allocation rule x and ordered allocation rule $\widehat{x}$ as defined in Lemma 2. By lemmas 4 and 5, it follows that allocation rule $\widehat{x}$ is an optimal allocation rule such that type θ_j is indifferent to reporting to being type θ_{j+1} for all $j \geq 1$. This implies that $X(\theta) =^{a.e.} \widehat{X}(\theta)$ for all $\theta > \theta_1$; if not, there would be some θ_j who would strictly prefer to reporting truthfully over reporting to being type θ_{j+1} under allocation rule $\widehat{x}$, which would be a contradiction (we show this in the proof of Lemma 2). Finally, in regards to type θ_1, by definition of $\widehat{x}$, it follows that:

$$\int_0^1 p(s|\theta_1)\, x_z(\theta_1, s)\, ds = \int_0^1 p(s|\theta_1)\, \widehat{x}_z(\theta_1, s)\, ds$$

for $z = h, l$.

6 Discussion

In this section, we start by discussing non-wasteful allocations and whether these are optimal. We then discuss an extension to the model that allows for multiple quality levels for the objects being assigned (rather than only two). In section 6.3., we discuss how the results extend to the case where there are many finite agents (rather than a continuum). Finally, in section 6.4, we briefly discuss the issue of fairness of the optimal mechanism.

6.1 On wasteful allocation rules

The fact that agents have the same ordinal preferences over objects implies that the set of ex-post Pareto efficient allocation rules is the set of non-wasteful allocation rules. An allocation rule is wasteful if there are agents who are not assigned any object despite there being objects available. Formally, an allocation rule x is non-wasteful if and only if

$$\sum_{\theta \in \Theta} q(\theta) \int_0^1 p(s|\theta)\, x_h(\theta, s)\, ds = \alpha_h$$

and

$$\sum_{\theta \in \Theta} q(\theta) \int_0^1 p(s|\theta)\, x_l(\theta, s)\, ds = \min\{\alpha_l, 1-\alpha_h\}.$$

Wasteful allocation rules might be considered undesirable as the principal might be unable to commit not to assign all the objects she has available. Indeed, the common assumption in the literature which has studied these assignment problems has been to only consider ex-post efficient allocation rules. Thus, it is natural to wonder what is the optimal non-wasteful allocation rule. It can be shown by following essentially the same steps of the proof of the Theorem of the previous section that the optimal non-wasteful allocation rule is an ordered allocation rule such that properties i), ii) and iii) of part 1 of the Theorem hold. Below, we show that if the measure of low-quality objects is sufficiently large, that allocation rule is not optimal, i.e., we find that any optimal allocation rule is wasteful. In particular, we find that even though all high-quality objects are assigned, there are low-quality objects that are not assigned despite there being agents who receive no object.

Proposition 1. *Assume that $p(0|\theta_1) > p(0|\theta_j) = 0$ for all $j > 1$. Then, for every $\alpha_h \in (0,1)$, there are thresholds $\overline{\alpha}_l \in (0, 1-\alpha_h)$ and $\underline{\alpha}_l \in (0, \overline{\alpha}_l)$ such that:*

i) for all $\alpha_l \leq \underline{\alpha}_l$, every optimal allocation rule is non-wasteful.

ii) for all $\alpha_l > \overline{\alpha}_l$, every optimal allocation rule is wasteful, because, even though every high-quality object is assigned, there are agents who receive no objects despite there being low-quality objects available.

The condition that $p(0|\theta_1) > p(0|\theta_j) = 0$ is rather innocuous (and not necessary). It states that the lowest type is infinitely more likely to generate signals close to 0. Moreover, we show in the Appendix that $\underline{\alpha}_l = \overline{\alpha}_l$ when a unique optimal ordered allocation rule can be guaranteed to exist.[15] In that case, Proposition 1 simply states that the optimal allocation rule is wasteful if and only if α_l is sufficiently high.

The argument builds on what happens when there are enough objects for every agent ($\alpha_h + \alpha_l \geq 1$). In that case, the only incentive compatible ordered allocation rule that is not wasteful is such that $(\underline{s}_\theta, \overline{s}_\theta) = (0, \overline{s}^*)$ for all $\theta \in \Theta$, where $\overline{s}^* \in \mathbb{R}$ is such that

$$\sum_{\theta \in \Theta} q(\theta) \int_{\overline{s}^*}^1 p(s|\theta)\, ds = \alpha_h.$$

This is exactly the allocation rule where objects are assigned exclusively according to

[15]Indeed, it is sufficient that there is a unique optimal ordered allocation rule when $\alpha_h + \alpha_l \geq 1$.

the agents' signals. As we illustrate in the example of Section 3, that allocation rule is not optimal if $p(0|\theta_1) > p(0|\theta_j) = 0$ (which holds in the example). The intuition can be grasped by revisiting Figure 2 and the discussion that follows. With ordered allocation rules, the only way the principal is able to get agents to self-select is by introducing inefficiencies; in Figure 2, high types have a (small) probability of being assigned no object. Assuming that $p(0|\theta_1) > p(0|\theta_j) = 0$ ensures that it is always preferable to have at least an infinitesimally small level of inefficiency (when $s \to 0$), because the probability that a high-type agent generates a signal that is close to 0 is infinitely smaller than the same probability for a mimicking agent with the lowest type.

Let the measure of assigned low-quality objects in the optimal mechanism when $\alpha_h + \alpha_l \geq 1$ be denoted by $\alpha_l^* < 1 - \alpha_h$. The proposition follows directly by setting $\underline{\alpha}_l = \overline{\alpha}_l = \alpha_l^*$: if $\alpha_l \geq \alpha_l^*$, the optimal mechanism is the same as if $\alpha_h + \alpha_l \geq 1$, while if $\alpha_l < \alpha_l^*$, the resource feasibilty constraint associated with low-quality objects binds, which directly implies that there is no waste of low quality objects.

The fact that optimal mechanisms need not be efficient is not only interesting in and of itself, but also because it suggests that decentralized systems for object provision might not be optimal. Consider the discussion over whether college assignment systems should be decentralized. While one could argue that mechanisms that are efficient could be implemented through decentralized systems, where each university decides independently what students to accept, it is much harder to see how a wasteful mechanism like the one we discuss could be implemented in this way (surely, the university with unassigned vacancies would contact the unassigned students to have them attend the university).[16]

6.2 Multiple quality levels

In this section, we extend the model to allow for objects of K different levels of quality, each with measure α_k, where $k = 1, ..., K$ and $\sum_{k=1}^{K} \alpha_k < 1$. As before, each agent's payoff depends on his type θ and on the object k received, and is denoted by $u(\theta, k)$, where $u(\theta, 0)$ represents the payoff of type θ when he receives no object. We extend

[16] Decentralized school choice systems have been studied by Avery and Levin (2010) and Chade, Lewis and Smith (2014). In the latter paper, the low-quality school might end up with vacancies in equilibrium, but that is a product of assuming that schools are able to commit to an acceptance threshold before students choose whether to accept their offers. If schools were unable to commit, there would be no student left unassigned.

the assumptions from before: $u(\theta, k+1) - u(\theta, k)$ is increasing with θ for all $k = 0, 1, ..., K-1$ and $\frac{u(\theta,k+1)-u(\theta,k)}{u(\theta,k)-u(\theta,k-1)}$ is also increasing with θ for all $k = 1, ..., K-1$.

In Section 5, we have characterized the optimal allocation rule when $K = 2$. When $K = 1$, it is straightforward to show that it is optimal to ignore the agents and assign the objects according to the agents' signals: there is a single threshold and every agent whose signal is above the threshold receives the object.[17] The problem becomes more complicated when $K > 2$ and a characterization of optimal allocation rules is not available. The issue is that ordered allocation rules no longer solve the "single crossing problem" described in Section 5.1. The fact that in any ordered allocation rule there are $K > 2$ thresholds associated with each type prevents the use of the methods discussed in Section 5 to determine the optimal ordered allocation rules (essentially, "local" arguments are no longer enough).

Instead of characterizing optimal mechanisms when $K > 2$, we discuss a specific family of mechanisms called *binary* mechanisms. Binary mechanisms are similar to the one presented in the example of Section 3: each agent must choose one of two tracks (A and B). Each track is composed on K thresholds labeled $\left\{s_k^A\right\}_{k=1}^K$ and $\left\{s_k^B\right\}_{k=1}^K$ respectively and each agent who selects track $j = A, B$ is assigned an object of quality k if $s \in \left(s_k^j, s_{k+1}^j\right]$, where $s_{K+1}^j \equiv 1$ for $j = A, B$.

Binary mechanisms are appealing because they are simple. In general, optimal mechanisms require agents to choose many tracks (even when $K = 2$, there might be the need to have as many tracks as there are types), which may make the implementation of the mechanism complicated. We prove in this section that there is a class of binary mechanisms called *ordered binary* mechanisms that perform strictly better (in terms of ex-ante payoffs) than simply assigning agents according exclusively to their signals. Ordered binary mechanisms are such that there are two critical levels $\overline{k}$ and $\underline{k}$ such that $1 \leq \underline{k} < \overline{k} \leq K$, and i) $s_k^A \leq s_k^B$ for all $k \leq \underline{k}$, ii) $s_k^A = s_k^B$ for all $k \in \left(\underline{k}, \overline{k}\right)$ and ii) $s_k^A \geq s_k^B$ for all $k \geq \overline{k}$. Figure 5 illustrates.

The idea is that there are three tiers of objects: a top tier, composed of objects of quality above level $\overline{k}$, a middle tier of objects of quality between levels $\underline{k}$ and $\overline{k}$, and a lower tier, composed of objects of quality level below $\underline{k}$, which includes receiving no object. By choosing track A, an agent increases his chances of receiving a middle tier object but lowers his chances of receiving a top tier object compared to track B. In this way, ordered binary mechanisms are the natural extension of the optimal mechanism

[17] The problem of finding the optimal allocation rule when $K = 1$ is analogous to finding the optimal non-wasteful allocation rule when $K = 2$ and $\alpha_1 + \alpha_2 \geq 1$, which we discuss in the previous section.

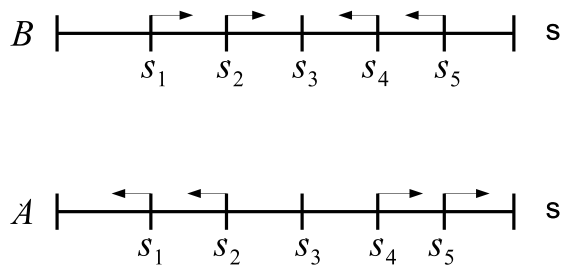

Figure 5: The figure shows the changes from a one-track mechanism, where objects are assigned as a function of the agents' signals, to an ordered binary mechanism when K = 5. The top tier is composed of objects of qualities 5 and 4, the middle tier is composed of objects of qualities 3 and 2 and the lower tier is composed of the lowest quality objects and receiving no object.

characterized in the Theorem but with the restriction that there are only two tracks.

Proposition 2. *In any ordered binary mechanism, there is some type $\widehat{\theta} \in \Theta$ such that the agent chooses track A if $\theta < \widehat{\theta}$ and chooses track B if $\theta > \widehat{\theta}$.*

As expected from the previous sections, the fact that signals have the MLRP leads to agents with higher types preferring track B, because they are more confident that their signal will be high; the downside of not receiving a middle tier object in the event of getting a lower signal is less important to them, because such an event is unlikely.

The following proposition confirms that ordered binary mechanisms are sufficient to improve upon simply assigning objects according to the agents' signals. Let the allocation rules induced by ordered binary mechanisms be called *ordered binary allocation rules.*

Proposition 3. *For any pair of critical levels $\left(\underline{k}, \overline{k}\right)$ such that $1 \leq \underline{k} < \overline{k} \leq K$, there is an ordered binary allocation rule that generates a larger ex-ante payoff than the one generated by assigning objects according exclusively to the agents' signals.*

6.3 Finite number of agents

One of the assumptions of the paper is that there is a continuum of agents and of objects. This assumption is convenient because it implies that there is no aggregate uncertainty due to the proportion of the population with the various types and signals being certain and known by everyone. Each agent is only uncertain about whether his signal will reflect his type, and not about how large his type and/or signal are relative to others. Even though assuming a continuum of agents is relatively standard (see Avery and Levin, 2010, Chade, Lewis and Smith, 2014, and Li 2021, for example), the reader might wonder whether the results presented in Section 5 extend to a model with finitely many objects and agents.

With a finite population, the problem of finding the optimal mechanism becomes more complicated because the number of agents with each type and signal is uncertain. As a result, an allocation for each agent no longer depends exclusively on that agent's type and signal but must also depend on every other agent's type and signal.

Let us rewrite the model for the case when there are N agents. Each agent $i = 1, ..., N$ has an independent private type $\theta_i \in \Theta$ and generates a conditionally independent signal $s_i \in [0,1]$, with the same distributions of the text. There is a total number of τ_H and τ_L high and low-quality objects respectively. An allocation rule is $x = \left(x^1, ..., x^N\right)$, where

$$x^i = \left(x_h^i, x_l^i\right) : \Theta^N \times [0,1]^N \rightarrow [0,1] \times [0,1]$$

such that

$$x_h^i(\theta, s) + x_l^i(\theta, s) \leq 1$$

for all vectors $\theta \in \Theta^N$ and $s \in [0,1]^N$ and for all $i = 1, ..., N$. An allocation rule x is feasible if

$$\sum_{i=1}^{N} x_h^i(\theta, s) \leq \tau_H$$

and

$$\sum_{i=1}^{N} x_l^i(\theta, s) \leq \tau_L$$

for all $\theta \in \Theta^N$ and $s \in [0,1]^N$.

An allocation rule x is incentive compatible if conditional on θ_i:

$$\begin{aligned}
& \mathbb{E}_{\theta_{-i},s} \ \left(u\left(\theta_i,h\right)x_h^i\left(\left(\theta_i,\theta_{-i}\right),s\right)+u\left(\theta_i,l\right)x_l^i\left(\left(\theta_i,\theta_{-i}\right),s\right)\right) \\
\geq \ & \mathbb{E}_{\theta_{-i},s} \ \left(u\left(\theta_i,h\right)x_h^i\left(\left(\theta_i',\theta_{-i}\right),s\right)+u\left(\theta_i,l\right)x_l^i\left(\left(\theta_i',\theta_{-i}\right),s\right)\right)
\end{aligned}$$

for all $\theta_i' \in \Theta$, $\theta_i \in \Theta$ and $i = 1, ..., N$. Finally, the value for the principal of allocation rule x is given by

$$W\left(x\right)=\frac{1}{N}\sum_{i=1}^{N}\mathbb{E}\left(u\left(\theta_i,h\right)x_h^i\left(\theta,s\right)+u\left(\theta_i,l\right)x_l^i\left(\theta,s\right)\right).$$

Let us momentarily go back to the main model with a continuum of agents. Fix any parameters $\alpha = (\alpha_H, \alpha_L) \in (0,1)^2$ and define the value for the principal of implementing the optimal allocation rule to be W^*. We show that it is possible to construct feasible IC allocation rules in the finite version of the model whose value for the social planner converges to W^* as N goes to infinity when the availability of objects is the same in both versions of the model. In order to do that, consider the floor function $\lfloor\ \rfloor : \mathbb{R} \to \mathbb{Z}$, such that $\lfloor x \rfloor$ denotes the largest integer smaller than or equal to x. Assume that $\tau_H = \lfloor N\alpha_H \rfloor$ and $\tau_L = \lfloor N\alpha_L \rfloor$.

Proposition 4. *For every $\delta > 0$, there is an allocation rule $x = \left(x^1, ..., x^N\right)$ such that*

$$\lim_{N\to\infty} \ W\left(x\right) \geq W^* - \delta.$$

6.4 On fairness

One final concern we wish to address has to do with the preferential treatment given to agents who, for reasons not directly considered by the model, are more predisposed to risking receiving no object. The concern is not so much with risk aversion per se, but with what might correlate with that risk aversion. For example, there is suggestive evidence that low income students tend to be more risk averse (see, for example, Calsamiglia and Guell, 2018; and also Calsamiglia, Martinez-Mora and Miralles, 2021, for a theoretical analysis). As a result, when thinking of our model to assign students to schools or colleges, a reasonable concern is that the optimal mechanism further increases income inequality. However, this problem can be mitigated by adding discriminatory clauses to the mechanism, provided that the students' income is known by the mechanism designer. The idea would be that the set of tracks an agent has available would depend on his socioeconomic status. In the simpler case where all agents are either high or low-income, and all high-income (low-income) agents have the same risk aversion level, that would actually be optimal, as it is easy to see that one could find the optimal mechanism by treating each set of agents independently.

7 Appendix

7.1 Example mentioned in Section 5.1

Consider the following example. Say that θ belongs to $\{\theta_1, \theta_2, \theta_3\}$ with $\theta_i = i$ for $i = 1, 2, 3$, $u(\theta, h) = 2\theta$ and $u(\theta, l) = \theta$, and

$$p(s|\theta) = \begin{cases} 2s & \text{if } \theta = \theta_3 \\ 1 & \text{if } \theta = \theta_2 \\ 2(1-s) & \text{if } \theta = \theta_1 \end{cases}.$$

Notice that p trivially satisfies our assumptions (its support is the interval $[0, 1]$ and the MLRP is satisfied). Consider the following allocation rule, illustrated in Figure 6:

$$x_h(\theta_1, s) = \begin{cases} 1 \text{ if } s \leq 0.4 \\ 0 \text{ if } s > 0.4 \end{cases} \text{ and } x_l(\theta_1, s) = \begin{cases} 1 \text{ if } 0.4 \leq s \leq 0.6 \\ 0 \text{ if } s > 0.6 \end{cases}.$$

$$x_l(\theta_2, s) = 1 \text{ for all } s \in [0, 1].$$

$$x_h(\theta_3, s) = \begin{cases} 1 \text{ if } s \geq 0.6 \\ 0 \text{ if } s < 0.6 \end{cases} \text{ and } x_l(\theta_3, s) = \begin{cases} 1 \text{ if } 0.4 \leq s \leq 0.6 \\ 0 \text{ if } s < 0.4 \end{cases}.$$

It is straightforward to check that, not only is this allocation rule incentive compatible, but also that the distribution of objects that are assigned to type θ_1 is exactly the same as type θ_3 (both types have a 64% probability of receiving a high-quality object and a 20% probability of receiving a low-quality object) but very different than type θ_2 (type θ_2 receives a low-quality object with certainty).

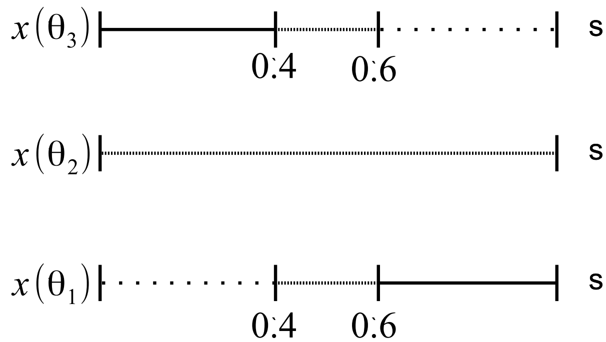

Figure 6: Example of an incentive compatible allocation rules where types θ_1 and θ_3 have the same probability of being assigned the high-quality object and the same probability of being assigned the low-quality object, while type θ_2 is assigned the low-quality object with certainty.

7.2 Proof of part 1 of the Theorem

Part 1 of the Theorem follows by combining lemmas 1-5. Below, we show lemmas 1-4. Lemma 5 directly follows from lemmas 1 and 4 as described in the text.

7.2.1 Proof of Lemma 1

Proof. Take any $\theta \in \Theta$ and notice that

$$U(\theta, X(\theta')) \geq U(\theta, X(\theta'')) \Leftrightarrow \frac{\int\limits_{\underline{s}_{\theta'}}^{\underline{s}_{\theta''}} p(s|\theta)\, ds}{\int\limits_{\overline{s}_{\theta''}}^{\overline{s}_{\theta'}} p(s|\theta)\, ds} \geq \frac{u(\theta, h)}{u(\theta, l)} - 1.$$

The statement of the lemma follows because $\frac{u(\theta,h)}{u(\theta,l)}$ is (weakly) increasing with θ and, as we prove in the following paragraph, the left hand side of the final inequality is strictly decreasing with θ.

Consider any two types θ and $\hat{\theta}$, with $\theta > \hat{\theta}$. We will show that:

$$\frac{\int_{\underline{s}_{\theta'}}^{\underline{s}_{\theta''}} p\left(s|\theta\right) ds}{\int_{\bar{s}_{\theta''}}^{\bar{s}_{\theta'}} p\left(s|\theta\right) ds} < \frac{\int_{\underline{s}_{\theta'}}^{\underline{s}_{\theta''}} p(s|\hat{\theta}) ds}{\int_{\bar{s}_{\theta''}}^{\bar{s}_{\theta'}} p(s|\hat{\theta}) ds}. \tag{3}$$

We know that densities $\{p(\,\cdot\,|\theta) : \theta \in \Theta\}$ have the MLRP. Then, by Proposition 4 in Milgrom (1981), it follows that signal $\{s \in [\bar{s}_{\theta''}, \bar{s}_{\theta'}]\}$ is "more favorable" than signal $\{s \in [\underline{s}_{\theta'}, \underline{s}_{\theta''}]\}$. By definition, this implies that for every nondegenerate prior distribution G for θ, the posterior distribution $G(\,\cdot\,|\{s \in [\bar{s}_{\theta''}, \bar{s}_{\theta'}]\})$ first order stochastic dominates $G(\,\cdot\,|\{s \in [\underline{s}_{\theta'}, \underline{s}_{\theta''}]\})$.

Consider G such that it assigns positive and equal probability only to θ and $\hat{\theta}$. First order stochastic dominance implies:[18]

$$P(\theta|s \in \{s \in [\bar{s}_{\theta''}, \bar{s}_{\theta'}]\}) > P(\theta|s \in \{s \in [\underline{s}_{\theta'}, \underline{s}_{\theta''}]\}), \text{ and}$$

$$P(\hat{\theta}|s \in \{s \in [\bar{s}_{\theta''}, \bar{s}_{\theta'}]\}) < P(\hat{\theta}|s \in \{s \in [\underline{s}_{\theta'}, \underline{s}_{\theta''}]\}).$$

Then,

$$\frac{P(\theta|s \in \{s \in [\bar{s}_{\theta''}, \bar{s}_{\theta'}]\})}{P(\theta|s \in \{s \in [\underline{s}_{\theta'}, \underline{s}_{\theta''}]\})} > \frac{P(\hat{\theta}|s \in \{s \in [\bar{s}_{\theta''}, \bar{s}_{\theta'}]\})}{P(\hat{\theta}|s \in \{s \in [\underline{s}_{\theta'}, \underline{s}_{\theta''}]\})}$$

or equivalently,

$$\frac{P(\theta|s \in \{s \in [\bar{s}_{\theta''}, \bar{s}_{\theta'}]\})}{P(\hat{\theta}|s \in \{s \in [\bar{s}_{\theta''}, \bar{s}_{\theta'}]\})} > \frac{P(\theta|s \in \{s \in [\underline{s}_{\theta'}, \underline{s}_{\theta''}]\})}{P(\hat{\theta}|s \in \{s \in [\underline{s}_{\theta'}, \underline{s}_{\theta''}]\})}$$

By Bayes' theorem:

$$\frac{P(s \in \{s \in [\bar{s}_{\theta''}, \bar{s}_{\theta'}]\}|\theta)}{P(s \in \{s \in [\bar{s}_{\theta''}, \bar{s}_{\theta'}]\}|\hat{\theta})} > \frac{P(s \in \{s \in [\underline{s}_{\theta'}, \underline{s}_{\theta''}]\}|\theta)}{P(s \in \{s \in [\underline{s}_{\theta'}, \underline{s}_{\theta''}]\}|\hat{\theta})}.$$

Then we have:

$$\frac{P(s \in \{s \in [\underline{s}_{\theta'}, \underline{s}_{\theta''}]\}|\hat{\theta})}{P(s \in \{s \in [\bar{s}_{\theta''}, \bar{s}_{\theta'}]\}|\hat{\theta})} > \frac{P(s \in \{s \in [\underline{s}_{\theta'}, \underline{s}_{\theta''}]\}|\theta)}{P(s \in \{s \in [\bar{s}_{\theta''}, \bar{s}_{\theta'}]\}|\theta)}.$$

[18] Given G, there are only two types which have positive probability; as a result, first order stochastic dominance implies that $G(\,\cdot\,|\{s \in [\bar{s}_{\theta''}, \bar{s}_{\theta'}]\})$ should put more probability than $G(\,\cdot\,|\{s \in [\underline{s}_{\theta'}, \underline{s}_{\theta''}]\})$ on the high type, and vice-versa with respect to the low type.

Finally, the last relation implies:

$$\frac{\int_{\underline{s}_{\theta'}}^{\underline{s}_{\theta''}} p(s|\hat{\theta})ds}{\int_{\bar{s}_{\theta''}}^{\bar{s}_{\theta'}} p(s|\hat{\theta})ds} > \frac{\int_{\underline{s}_{\theta'}}^{\underline{s}_{\theta''}} p(s|\theta)ds}{\int_{\bar{s}_{\theta''}}^{\bar{s}_{\theta'}} p(s|\theta)ds}.$$

□

7.2.2 Proof of Lemma 2

Proof. [19] Take any $\theta, \theta' \in \Theta$ such that $\theta' > \theta$. We want to show that:

$$U\left(\theta, \widehat{X}\left(\theta'\right)\right) \leq U\left(\theta, \widehat{X}\left(\theta\right)\right).$$

allocation rule x satisfies the upper incentives constraints, so we know that $U\left(\theta, X\left(\theta'\right)\right) \leq U\left(\theta, X\left(\theta\right)\right)$. Also, by the definition of $\widehat{x}$ we have that $U\left(\theta, \widehat{X}\left(\theta\right)\right) = U\left(\theta, X\left(\theta\right)\right)$. Therefore, it is enough to prove that:

$$U\left(\theta, \widehat{X}\left(\theta'\right)\right) \leq U\left(\theta, X\left(\theta'\right)\right).$$

Define the following two density functions $\widehat{Y}$ and Y:

$$\widehat{Y}\left(s\right) = \frac{1}{\widehat{\varsigma}} p\left(s|\theta'\right)\left(\widehat{x}_h\left(\theta', s\right) u\left(\theta, h\right) + \widehat{x}_l\left(\theta', s\right) u\left(\theta, l\right)\right)$$

and

$$Y\left(s\right) = \frac{1}{\varsigma} p\left(s|\theta'\right)\left(x_h\left(\theta', s\right) u\left(\theta, h\right) + x_l\left(\theta', s\right) u\left(\theta, l\right)\right)$$

for all $s \in [0,1]$ and where

$$\widehat{\varsigma} = \int_0^1 p\left(s|\theta'\right)\left(\widehat{x}_h\left(\theta', s\right) u\left(\theta, h\right) + \widehat{x}_l\left(\theta', s\right) u\left(\theta, l\right)\right) ds$$

and

$$\varsigma = \int_0^1 p\left(s|\theta'\right)\left(x_h\left(\theta', s\right) u\left(\theta, h\right) + x_l\left(\theta', s\right) u\left(\theta, l\right)\right) ds.$$

[19] We are very grateful to an an anonymous referee who made several suggestions that led to a simplification of the original proof.

Notice that

$$\int_{0}^{1} p\left(s|\theta'\right) \widehat{x}_z\left(\theta', s\right) ds = \int_{0}^{1} p\left(s|\theta'\right) x_z\left(\theta', s\right) ds$$

for $z = l, h$, which implies that $\varsigma = \widehat{\varsigma}$ and that

$$\int_{y}^{1} \widehat{Y}\left(s\right) ds \geq \int_{y}^{1} Y\left(s\right) ds \tag{4}$$

for every $y \in [0, 1]$. Indeed, unless $\widehat{X}\left(\theta'\right) =^{a.e.} X\left(\theta'\right)$, it will be the case that $\widehat{Y}$ first order stochastically dominates Y.

Finally, notice that

$$U\left(\theta, \widehat{X}\left(\theta'\right)\right) = \int_{0}^{1} \frac{p\left(s|\theta\right)}{p\left(s|\theta'\right)} \widehat{Y}\left(s\right) ds \leq \int_{0}^{1} \frac{p\left(s|\theta\right)}{p\left(s|\theta'\right)} Y\left(s\right) ds = U\left(\theta, X\left(\theta'\right)\right),$$

where the inequality follows by (4) and because $\frac{p(\cdot|\theta)}{p(\cdot|\theta')}$ is strictly decreasing. It holds strictly if $\widehat{Y}$ first order stochastically dominates Y.

□

7.2.3 Proof of Lemma 3

Proof. We first show the existence of an *ordered allocation rule* that solves the relaxed problem. Then, we use Lemma 2 to prove the existence of a solution of the relaxed problem.

The problem of finding a solution of the relaxed problem within the set of ordered allocation rules can be written as:

$$\max_{(\underline{s}_\theta, \bar{s}_\theta)_{\theta \in \Theta}} \sum_{\theta \in \Theta} q(\theta) \left[u(\theta, l) \int_{\underline{s}_\theta}^{\bar{s}_\theta} p(s|\theta) ds + u(\theta, h) \int_{\bar{s}_\theta}^{1} p(s|\theta) ds \right],$$

where for all θ:

$$\underline{s}_\theta \in [0, 1], \bar{s}_\theta \in [0, 1],$$

s.t.

$$\sum_{\theta \in \Theta} q(\theta) \int_{\bar{s}_\theta}^{1} p(s|\theta) ds - \alpha_h \leq 0,$$

$$\sum_{\theta\in\Theta} q(\theta)\int_{\underline{s}_\theta}^{\bar{s}_\theta} p(s|\theta)ds - \alpha_l \leq 0,$$

and, for each pair (θ, θ') with $\theta' > \theta$,

$$u(\theta,l)\left[\int_{\underline{s}_\theta}^{\bar{s}_\theta} p(s|\theta)ds - \int_{\underline{s}_{\theta'}}^{\bar{s}_{\theta'}} p(s|\theta)ds\right] + u(\theta,h)\left[\int_{\bar{s}_\theta}^{1} p(s|\theta)ds - \int_{\bar{s}_{\theta'}}^{1} p(s|\theta)ds\right] \geq 0.$$

Note that all functions are continuous and the domain is the compact set $([0,1]\times[0,1])^J$. Therefore, there is a solution to the problem. Denote the ordered allocation rule that solves the relaxed problem as x^O.

Now, suppose the relaxed problem does not have a solution when we consider the general set of allocation rules. This implies that there is a non-ordered allocation rule x such that $W(x) > W(x^O)$. By Lemma 2, we can construct an ordered allocation rule $\tilde{x}$ such that satisfies all previous constraints and $W(\tilde{x}) = W(x) > W(x^O)$, which is a contradiction.

$\square$

7.2.4 Proof of Lemma 4

Consider any ordered allocation rule $\widehat{x}$ that solves the relaxed problem. Let the associated thresholds be denoted by $\left(\overline{s}_j, \underline{s}_j\right)_{j=1}^{J}$. Consider any type $\theta_j \in \Theta$. We proceed by induction. Assume that, for all $k > 0$,

$$\overline{s}_{j+k} \geq \overline{s}_{j+k+1} \geq \underline{s}_{j+k+1} \geq \underline{s}_{j+k}$$

and

$$U\left(\theta_{j+k}, \widehat{X}\left(\theta_{j+k}\right)\right) = U\left(\theta_{j+k}, \widehat{X}\left(\theta_{j+k+1}\right)\right).$$

We complete the proof by showing that

$$\overline{s}_j \geq \overline{s}_{j+1} \geq \underline{s}_{j+1} \geq \underline{s}_j$$

and

$$U\left(\theta_j, \widehat{X}\left(\theta_j\right)\right) = U\left(\theta_j, \widehat{X}\left(\theta_{j+1}\right)\right).$$ [20]

[20] In doing this, we also prove the induction base; that is:

$$\overline{s}_{J-1} \geq \overline{s}_J \geq \underline{s}_J \geq \underline{s}_{J-1}, \text{ and that } U\left(\theta_{J-1}, \widehat{X}\left(\theta_{J-1}\right)\right) = U\left(\theta_{J-1}, \widehat{X}\left(\theta_J\right)\right).$$

Let $j' \geq j+1$ be such that $\theta_{j'} \in \Theta$ is the largest type such that $\overline{s}_{j+1} = \overline{s}_{j'}$. Then, between θ_{j+1} and $\theta_{j'}$, all upper thresholds are equal, which, in turn, implies that the lower thresholds are also equal (so that the indifference condition for type θ_{j+1} holds).

Let

$$\widehat{q}\left(\theta_{j+1}\right) \equiv \sum_{i=j+1}^{j'} q\left(\theta_i\right)$$

and

$$\widehat{p}\left(s|\theta_{j+1}\right) \equiv \sum_{i=j+1}^{j'} \frac{q\left(\theta_i\right)}{\widehat{q}\left(\theta_{j+1}\right)} p\left(s|\theta_i\right).$$

Notice that for any $s' > s$,

$$\frac{p\left(s'|\theta'\right)}{p\left(s|\theta'\right)} > \frac{p\left(s'|\theta_{j'}\right)}{p\left(s|\theta_{j'}\right)} \geq \frac{\widehat{p}\left(s'|\theta_{j+1}\right)}{\widehat{p}\left(s|\theta_{j+1}\right)} \geq \frac{p\left(s'|\theta_{j+1}\right)}{p\left(s|\theta_{j+1}\right)} > \frac{p\left(s'|\theta''\right)}{p\left(s|\theta''\right)} \tag{5}$$

for any $\theta' > \theta_{j'} \geq \theta_{j+1} > \theta''$, where the second and third inequalities are strict whenever $j' > j+1$.

Indeed,

$$\frac{\widehat{p}\left(s'|\theta_{j+1}\right)}{\widehat{p}\left(s|\theta_{j+1}\right)} = \frac{\sum\limits_{i=j+1}^{j'} q\left(\theta_i\right) p\left(s'|\theta_i\right)}{\sum\limits_{i=j+1}^{j'} q\left(\theta_i\right) p\left(s|\theta_i\right)}.$$

We know that $\frac{p(s'|\theta_i)}{p(s|\theta_i)} \leq \frac{p\left(s'|\theta_{j'}\right)}{p\left(s|\theta_{j'}\right)}$ for every $i = j+1, \ldots, j'$. Then:

$$\frac{\widehat{p}\left(s'|\theta_{j+1}\right)}{\widehat{p}\left(s|\theta_{j+1}\right)} \leq \frac{\sum\limits_{i=j+1}^{j'} q\left(\theta_i\right) \frac{p\left(s'|\theta_{j'}\right)}{p\left(s|\theta_{j'}\right)} p\left(s|\theta_i\right)}{\sum\limits_{i=j+1}^{j'} q\left(\theta_i\right) p\left(s|\theta_i\right)} = \frac{p\left(s'|\theta_{j'}\right)}{p\left(s|\theta_{j'}\right)},$$

where the first inequality is strict whenever $j' > j+1$. By the same reasoning, we can show that $\frac{\widehat{p}(s'|\theta_{j+1})}{\widehat{p}(s|\theta_{j+1})} \geq \frac{p(s'|\theta_{j+1})}{p(s|\theta_{j+1})}$.

The argument is as follows. We first show that allocation rule $\widehat{x}$ is such that there is some type $\theta_{\widetilde{j}} < \theta_{j+1}$ who is indifferent to reporting θ_{j+1} (Claim 1). Then, we show that if $\underline{s}_j > \underline{s}_{j+1}$, there would be no such type $\theta_{\widetilde{j}} < \theta_{j+1}$ who would be indifferent to θ_{j+1} (Claim 2). Claims 1 and 2 imply that $\underline{s}_j \leq \underline{s}_{j+1}$, which, in turn, implies that type θ_j is the one who is indifferent to type θ_{j+1} by Lemma 1. It then directly follows that $\overline{s}_j \geq \overline{s}_{j+1}$ for otherwise type θ_j would not be indifferent.

Claim 1: There is some type $\theta_{\tilde{j}} < \theta_{j+1}$ such that $U\left(\theta_{\tilde{j}}, \widehat{X}\left(\theta_{\tilde{j}}\right)\right) = U\left(\theta_{\tilde{j}}, \widehat{X}\left(\theta_{j+1}\right)\right)$.

Proof. Suppose not so that $U\left(\theta, \widehat{X}\left(\theta\right)\right) > U\left(\theta, \widehat{X}\left(\theta_{j+1}\right)\right)$ for all $\theta < \theta_{j+1}$. Notice that

$$U\left(\theta_{j+1}, \widehat{X}\left(\theta_{j+1}\right)\right) \geq U\left(\theta_{j+1}, \widehat{X}\left(\theta_{j+k}\right)\right) \Rightarrow U\left(\theta, \widehat{X}\left(\theta_{j+1}\right)\right) \geq U\left(\theta, \widehat{X}\left(\theta_{j+k}\right)\right)$$

for all $k > 1$ and $\theta < \theta_{j+1}$ by the induction hypothesis and Lemma 1. As a result, it follows that no type $\theta < \theta_{j+1}$ is indifferent to reporting any type larger than θ_{j+1}. There are three cases to consider.

The **first case** is when $\overline{s}_j < 1$ and $\overline{s}_{j+1} > \underline{s}_{j+1}$. Consider the following alternative ordered allocation rule x', where x' is equal to $\widehat{x}$ except that $\overline{s}'_j = \overline{s}_j + \varepsilon$ and $\overline{s}'_{j+1} = \overline{s}_{j+1} - \delta\left(\varepsilon\right)$, where $\varepsilon > 0$ and $\delta\left(\varepsilon\right)$ is such that the measure of each object assigned is the same as with allocation rule $\widehat{x}$, i.e.,

$$q\left(\theta_j\right) \int_{\overline{s}_j}^{\overline{s}_j+\varepsilon} p\left(s|\theta_j\right) ds = q\left(\theta_{j+1}\right) \int_{\overline{s}_{j+1}-\delta(\varepsilon)}^{\overline{s}_{j+1}} p\left(s|\theta_{j+1}\right) ds.$$

If ε is sufficiently small, allocation rule x' is feasible and satisfies all the considered incentive constraints, because one is slightly improving the payoff of reporting θ_{j+1} and reducing the payoff of reporting θ_j. Provided ε is sufficiently small, low types will not mimic type θ_{j+1} (because they were not indifferent in $\widehat{x}$) nor will type θ_j mimic some type $\theta > \theta_{j+1}$ (because he was not indifferent in $\widehat{x}$). Finally, $W\left(x'\right) > W\left(\widehat{x}\right)$ because

$$u\left(\theta_{j+1}, h\right) - u\left(\theta_{j+1}, l\right) > u\left(\theta_j, h\right) - u\left(\theta_j, l\right),$$

which means that $\widehat{x}$ does not solve the relaxed problem; a contradiction.

The **second case** applies to two sets of conditions; when $\overline{s}_j = 1$ and when both $1 > \overline{s}_j > \underline{s}_j$ and $\overline{s}_{j+1} = \underline{s}_{j+1}$. In that case, the alternative ordered allocation rule is x'', where x' is equal to $\widehat{x}$ except that $\underline{s}''_j = \underline{s}_j + \varepsilon$ and $\underline{s}''_{j+1} = \underline{s}_{j+1} - \delta\left(\varepsilon\right)$ and where $\varepsilon > 0$ and $\delta\left(\varepsilon\right)$ is such that the measure of each object assigned is unchanged. By the same logic as before, allocation rule x'' is feasible, satisfies the considered incentive constraints and is such that $W\left(x''\right) > W\left(\widehat{x}\right)$, which is a contradiction.

The **third case** is when $\overline{s}_j = \underline{s}_j < 1$ and $\overline{s}_{j+1} = \underline{s}_{j+1}$. In this case, the alternative ordered allocation rule is x''', whose only difference to $\widehat{x}$ is that $\overline{s}''_j = \underline{s}''_j = \underline{s}_j + \varepsilon$ and $\overline{s}''_{j+1} = \underline{s}''_{j+1} = \underline{s}_{j+1} - \delta\left(\varepsilon\right)$, where $\varepsilon > 0$ and $\delta\left(\varepsilon\right)$ is such that the measure of each object assigned is unchanged. As before, by letting ε be sufficiently small, we get a contradiction. $\square$

Claim 2: $\underline{s}_j > \underline{s}_{j+1} \Rightarrow U\left(\theta, \widehat{X}(\theta)\right) > U\left(\theta, \widehat{X}(\theta_{j+1})\right)$ for all $\theta < \theta_{j+1}$.

Proof. Suppose the statement is false and let $\theta_{\widetilde{j}}$ be the largest type smaller than θ_{j+1} such that $U\left(\theta_{\widetilde{j}}, \widehat{X}\left(\theta_{\widetilde{j}}\right)\right) = U\left(\theta_{\widetilde{j}}, \widehat{X}(\theta_{j+1})\right)$. Consider the following ordered allocation rule x', where x' is equal to x except that $\overline{s}'_j = \overline{s}_j + \varepsilon$, $\underline{s}'_j = \underline{s}_j - \beta(\varepsilon)$, $\overline{s}'_i = \overline{s}_{j+1} - \delta(\varepsilon)$ and $\underline{s}'_i = \underline{s}_{j+1} + \gamma(\varepsilon)$ for all i such that $j+1 \leq i \leq j'$, where $\varepsilon > 0$ and $\delta(\varepsilon)$,$\beta(\varepsilon)$ and $\gamma(\varepsilon)$ are such that the measure of each object assigned is the same and that type $\theta_{\widetilde{j}}$ is kept indifferent. Formally, we have that

$$\widehat{q}(\theta_{j+1}) \int_{\overline{s}_{j+1}-\delta(\varepsilon)}^{\overline{s}_{j+1}} \widehat{p}(s|\theta_{j+1})\, ds = q(\theta_j) \int_{\overline{s}_j}^{\overline{s}_j+\varepsilon} p(s|\theta_j)\, ds,$$

$$\widehat{q}(\theta_{j+1}) \int_{\underline{s}_{j+1}}^{\underline{s}_{j+1}+\gamma(\varepsilon)} \widehat{p}(s|\theta_{j+1})\, ds = q(\theta_j) \int_{\underline{s}_j-\beta(\varepsilon)}^{\underline{s}_j} p(s|\theta_j)\, ds$$

and

$$\left(u\left(\theta_{\widetilde{j}}, h\right) - u\left(\theta_{\widetilde{j}}, l\right)\right) \int_{\overline{s}_{j+1}-\delta(\varepsilon)}^{\overline{s}_{j+1}} p\left(s|\theta_{\widetilde{j}}\right) ds = u\left(\theta_{\widetilde{j}}, l\right) \int_{\underline{s}_{j+1}}^{\underline{s}_{j+1}+\gamma(\varepsilon)} p\left(s|\theta_{\widetilde{j}}\right) ds.$$

Functions δ,β and γ are all differentiable and all converge to 0 when $\varepsilon \to 0$. Using the three previous equations, we have that

$$\delta'(0) = \frac{q(\theta_j)}{\widehat{q}(\theta_{j+1})} \frac{p(\overline{s}_j|\theta_j)}{\widehat{p}(\overline{s}_{j+1}|\theta_{j+1})}, \tag{6}$$

$$\beta'(0) = \frac{\left(u\left(\theta_{\widetilde{j}}, h\right) - u\left(\theta_{\widetilde{j}}, l\right)\right)}{u\left(\theta_{\widetilde{j}}, l\right)} \frac{\widehat{p}(\underline{s}_{j+1}|\theta_{j+1})}{\widehat{p}(\overline{s}_{j+1}|\theta_{j+1})} \frac{p\left(\overline{s}_{j+1}|\theta_{\widetilde{j}}\right)}{p\left(\underline{s}_{j+1}|\theta_{\widetilde{j}}\right)} \frac{p(\overline{s}_j|\theta_j)}{p(\underline{s}_j|\theta_j)} \tag{7}$$

and

$$\gamma'(0) = \frac{q(\theta_j)}{\widehat{q}(\theta_{j+1})} \frac{\left(u\left(\theta_{\widetilde{j}}, h\right) - u\left(\theta_{\widetilde{j}}, l\right)\right)}{u\left(\theta_{\widetilde{j}}, l\right)} \frac{p\left(\overline{s}_{j+1}|\theta_{\widetilde{j}}\right)}{p\left(\underline{s}_{j+1}|\theta_{\widetilde{j}}\right)} \frac{p(\overline{s}_j|\theta_j)}{\widehat{p}(\overline{s}_{j+1}|\theta_{j+1})}. \tag{8}$$

We complete the proof by showing that if $\varepsilon \to 0$, then $W(x') > W(\widehat{x})$, x' is feasible and satisfies all the considered incentive constraints, which is a contradiction.

That x' is feasible if $\varepsilon \to 0$ follows by construction. Let $V(\varepsilon) \equiv W(x'(\varepsilon))$ for any given $\varepsilon > 0$ and notice that

$$V(\varepsilon) = \left\{ \begin{array}{c} \sum_{i=j+1}^{j'} q(\theta_i) \left(u(\theta_i, h) \int\limits_{\overline{s}_{j+1}-\delta(\varepsilon)}^{1} p(s|\theta_i)\, ds + u(\theta_i, l) \int\limits_{\underline{s}_{j+1}+\gamma(\varepsilon)}^{\overline{s}_{j+1}-\delta(\varepsilon)} p(s|\theta_i)\, ds \right) + \\ q(\theta_j) \left(u(\theta_j, h) \int\limits_{\overline{s}_j+\varepsilon}^{1} p(s|\theta_j)\, ds + u(\theta_j, l) \int\limits_{\underline{s}_j-\beta(\varepsilon)}^{\overline{s}_j+\varepsilon} p(s|\theta_j)\, ds - \right) \end{array} \right\}.$$

Let

$$\widehat{V}(\varepsilon) = \left\{ \begin{array}{c} \widehat{q}(\theta_{j+1}) \left(u(\theta_{j+1}, h) \int\limits_{\overline{s}_{j+1}-\delta(\varepsilon)}^{1} \widehat{p}(s|\theta_{j+1})\, ds + u(\theta_{j+1}, l) \int\limits_{\underline{s}_{j+1}+\gamma(\varepsilon)}^{\overline{s}_{j+1}-\delta(\varepsilon)} \widehat{p}(s|\theta_{j+1})\, ds \right) + \\ q(\theta_j) \left(u(\theta_j, h) \int\limits_{\overline{s}_j+\varepsilon}^{1} p(s|\theta_j)\, ds + u(\theta_j, l) \int\limits_{\underline{s}_j-\beta(\varepsilon)}^{\overline{s}_j+\varepsilon} p(s|\theta_j)\, ds - \right) \end{array} \right\}$$

and notice that $V(\varepsilon) \geq \widehat{V}(\varepsilon)$ because both $u(\theta, h)$ and $u(\theta, l)$ are increasing with θ. Using (6), (7) and (8) we get that $\widehat{V}'(0) > 0$ if and only if

$$\begin{aligned} & u(\theta_{j+1}, h) - u(\theta_{j+1}, l) - u(\theta_j, h) + u(\theta_j, l) \\ > \; & (u(\theta_{j+1}, l) - u(\theta_j, l)) \frac{\left(u\left(\theta_{\tilde{j}}, h\right) - u\left(\theta_{\tilde{j}}, l\right)\right)}{u\left(\theta_{\tilde{j}}, l\right)} \frac{\widehat{p}\left(\underline{s}_{j+1}|\theta_{j+1}\right)}{\widehat{p}\left(\overline{s}_{j+1}|\theta_{j+1}\right)} \frac{p\left(\overline{s}_{j+1}|\theta_{\tilde{j}}\right)}{p\left(\underline{s}_{j+1}|\theta_{\tilde{j}}\right)}. \end{aligned}$$

Notice that

$$\frac{p\left(\overline{s}_{j+1}|\theta_{\tilde{j}}\right)}{p\left(\underline{s}_{j+1}|\theta_{\tilde{j}}\right)} < \frac{\widehat{p}\left(\overline{s}_{j+1}|\theta_{j+1}\right)}{\widehat{p}\left(\underline{s}_{j+1}|\theta_{j+1}\right)}.$$

Therefore, in order to show that $\widehat{V}'(0) > 0$ it is enough to show that

$$u(\theta_{j+1}, h) - u(\theta_{j+1}, l) - u(\theta_j, h) + u(\theta_j, l) \geq (u(\theta_{j+1}, l) - u(\theta_j, l)) \frac{\left(u\left(\theta_{\tilde{j}}, h\right) - u\left(\theta_{\tilde{j}}, l\right)\right)}{u\left(\theta_{\tilde{j}}, l\right)}$$

which is equivalent to

$$\frac{u(\theta_{j+1}, l)}{u(\theta_j, l)} \left(\frac{u(\theta_{j+1}, h)}{u(\theta_{j+1}, l)} - \frac{u\left(\theta_{\tilde{j}}, h\right)}{u\left(\theta_{\tilde{j}}, l\right)} \right) \geq \frac{u(\theta_j, h)}{u(\theta_j, l)} - \frac{u(\theta_{j'}, h)}{u(\theta_{j'}, l)}$$

which is true, because $\frac{u(\theta_{j+1},l)}{u(\theta_j,l)} > 1$ and $\frac{u(\theta,h)}{u(\theta,l)}$ is increasing with θ. This proves that, if ε is sufficiently small, $W(x') > W(\widehat{x})$.

Finally, we turn to the incentive constraints. There are only three type of deviations that one has to rule out; deviations to reporting some type $\theta > \theta_{j'}$, deviations to reporting θ_{j+1} and deviations to reporting θ_j.

Let us start with the deviations to reporting types larger than $\theta_{j'}$. Let $B(\varepsilon) \equiv U(\theta_{j+1}, X'(\theta_{j+1})) - U\left(\theta_{j+1}, \widehat{X}(\theta_{j+1})\right)$ denote the payoff increase for type θ_{j+1} as a function of ε, i.e.,

$$B(\varepsilon) = (u(\theta_{j+1}, h) - u(\theta_{j+1}, l)) \int_{\overline{s}_{j+1}-\delta(\varepsilon)}^{\overline{s}_{j+1}} p(s|\theta_{j+1})\, ds - u(\theta_{j+1}, l) \int_{\underline{s}_{j+1}}^{\underline{s}_{j+1}+\gamma(\varepsilon)} p(s|\theta_{j+1})\, ds.$$

It follows that $B'(0) > 0$, because

$$\frac{(u(\theta_{j+1}, h) - u(\theta_{j+1}, l))}{u(\theta_{j+1}, l)} \frac{p(\overline{s}_{j+1}|\theta_{j+1})}{p\left(\underline{s}_{j+1}|\theta_{j+1}\right)} > \frac{\left(u\left(\theta_{\widetilde{j}}, h\right) - u\left(\theta_{\widetilde{j}}, l\right)\right)}{u\left(\theta_{\widetilde{j}}, l\right)} \frac{p\left(\overline{s}_{j+1}|\theta_{\widetilde{j}}\right)}{p\left(\underline{s}_{j+1}|\theta_{\widetilde{j}}\right)}$$

(which follows because $\frac{u(\theta,h)-u(\theta,l)}{u(\theta,l)}$ is weakly increasing with θ and because $\frac{p(\overline{s}_{j+1}|\theta)}{p\left(\underline{s}_{j+1}|\theta\right)}$ is strictly increasing with θ). Therefore, if ε is small enough, type θ_{j+1} gets better and, as a result, does not want to deviate. By Lemma 1, it also follows that $U(\theta_i, X'(\theta_i)) > U\left(\theta_i, \widehat{X}(\theta_i)\right)$ for all i such that $j+1 \leq i \leq j'$ when ε is sufficiently small, so neither of those types deviate either. Finally, notice that $U\left(\theta_j, \widehat{X}(\theta_j)\right) > U\left(\theta_j, \widehat{X}(\theta_i)\right)$ for all $i > j'$ by Lemma 1 and by the induction hypothesis. As a result, type θ_j will not want to deviate to any type larger than $\theta_{j'}$ provided ε is sufficiently small.

Now we turn to deviations to reporting θ_j. Because there is some type $\theta_{\widetilde{j}}$ who is indifferent to reporting θ_{j+1}, it must be that $\overline{s}_{j+1} > \overline{s}_j \geq \underline{s}_j > \underline{s}_{j+1}$. This plus Lemma 1 implies that any type $\theta_i < \theta_{\widetilde{j}}$ is such that

$$U\left(\theta_i, \widehat{X}(\theta_i)\right) \geq U\left(\theta_i, \widehat{X}(\theta_{j+1})\right) > U\left(\theta_i, \widehat{X}(\theta_j)\right).$$

As a result, they will not deviate provided ε is sufficiently small. Next, consider type $\theta_{\widetilde{j}}$. Let $C(\varepsilon)$ represents the payoff difference when type $\theta_{\widetilde{j}}$ reports θ_j between allocation rules x' and $\widehat{x}$:

$$C(\varepsilon) \equiv u\left(\theta_{\widetilde{j}}, l\right) \int\limits_{\underline{s}_j - \beta(\varepsilon)}^{\underline{s}_j} p\left(s|\theta_{\widetilde{j}}\right) ds - \left(u\left(\theta_{\widetilde{j}}, h\right) - u\left(\theta_{\widetilde{j}}, l\right)\right) \int\limits_{\overline{s}_j}^{\overline{s}_j + \varepsilon} p\left(s|\theta_{\widetilde{j}}\right) ds.$$

It follows that $C'(0) < 0$ if and only if

$$\frac{\frac{p(\overline{s}_j|\theta_j)}{p\left(\underline{s}_j|\theta_j\right)}}{\frac{p\left(\overline{s}_j|\theta_{\widetilde{j}}\right)}{p\left(\underline{s}_j|\theta_{\widetilde{j}}\right)}} < \frac{\frac{\widehat{p}(\overline{s}_{j+1}|\theta_{j+1})}{\widehat{p}\left(\underline{s}_{j+1}|\theta_{j+1}\right)}}{\frac{p\left(\overline{s}_{j+1}|\theta_{\widetilde{j}}\right)}{p\left(\underline{s}_{j+1}|\theta_{\widetilde{j}}\right)}},$$

which is indeed true because

$$\begin{aligned}
\frac{\frac{\widehat{p}(\overline{s}_{j+1}|\theta_{j+1})}{\widehat{p}\left(\underline{s}_{j+1}|\theta_{j+1}\right)}}{\frac{p\left(\overline{s}_{j+1}|\theta_{\widetilde{j}}\right)}{p\left(\underline{s}_{j+1}|\theta_{\widetilde{j}}\right)}} &= \frac{\frac{\widehat{p}(\overline{s}_j|\theta_{j+1})}{\widehat{p}\left(\underline{s}_j|\theta_{j+1}\right)}}{\frac{p\left(\overline{s}_j|\theta_{\widetilde{j}}\right)}{p\left(\underline{s}_j|\theta_{\widetilde{j}}\right)}} \frac{\frac{\widehat{p}(\overline{s}_{j+1}|\theta_{j+1})}{\widehat{p}(\overline{s}_j|\theta_{j+1})}}{\frac{p\left(\overline{s}_{j+1}|\theta_{\widetilde{j}}\right)}{p\left(\overline{s}_j|\theta_{\widetilde{j}}\right)}} \frac{\frac{\widehat{p}\left(\underline{s}_j|\theta_{j+1}\right)}{\widehat{p}\left(\underline{s}_{j+1}|\theta_{j+1}\right)}}{\frac{p\left(\underline{s}_j|\theta_{\widetilde{j}}\right)}{p\left(\underline{s}_{j+1}|\theta_{\widetilde{j}}\right)}} \\
&> \frac{\frac{\widehat{p}(\overline{s}_j|\theta_{j+1})}{\widehat{p}\left(\underline{s}_j|\theta_{j+1}\right)}}{\frac{p\left(\overline{s}_j|\theta_{\widetilde{j}}\right)}{p\left(\underline{s}_j|\theta_{\widetilde{j}}\right)}} > \frac{\frac{p(\overline{s}_j|\theta_j)}{p\left(\underline{s}_j|\theta_j\right)}}{\frac{p\left(\overline{s}_j|\theta_{\widetilde{j}}\right)}{p\left(\underline{s}_j|\theta_{\widetilde{j}}\right)}}.
\end{aligned}$$

This means that if ε is small enough, type $\theta_{\widetilde{j}}$ does not want to deviate. Finally, Lemma 1 also implies that $U(\theta_i, X'(\theta_j)) < U\left(\theta_i, \widehat{X}(\theta_j)\right)$ for all i such that $\widetilde{j} < i < j$, so that none of those types will want to deviate either.

Finally, we consider deviations to reporting θ_{j+1}. By construction, type $\theta_{\widetilde{j}}$ does not want to deviate (because he is indifferent). As for any type θ_i such that $\widetilde{j} < i < j+1$, notice that $U\left(\theta_i, \widehat{X}(\theta_i)\right) > U\left(\theta_i, \widehat{X}(\theta_{j+1})\right)$, which implies that none of them will deviate provided ε is sufficiently small. Finally, notice that, because $C'(0) < 0$, $U\left(\theta_{\widetilde{j}}, X'(\theta_{j+1})\right) \leq U\left(\theta_{\widetilde{j}}, \widehat{X}(\theta_{j+1})\right)$, which implies that $U(\theta_i, X'(\theta_{j+1})) < U\left(\theta_i, \widehat{X}(\theta_{j+1})\right)$ for any type $\theta_i < \theta_{\widetilde{j}}$, which means those types will not deviate either. □

7.3 Proofs of Section 6

7.3.1 Proof of Proposition 1

As we discuss in the text, the proof builds on a characterization of optimal allocation rules when $\alpha_h + \alpha_l \geq 1$. We refer to this problem as the *canonical* problem. Let allocation rule x^* be such that objects are assigned to agents according exclusively to their signals. Formally, x^* is an ordered allocation rule such that $(\underline{s}_\theta, \overline{s}_\theta) = (0, \overline{s}^*)$ for all $\theta \in \Theta$, where $\overline{s}^* \in \mathbb{R}$ is such that

$$\sum_{\theta \in \Theta} q(\theta) \int_{\overline{s}^*}^{1} p(s|\theta)\, ds = \alpha_h.$$

In the following two lemmas, we prove that allocation rule x^* is not an optimal allocation rule of the canonical problem whenever $p(0|\theta_1) > p(0|\theta_j) = 0$ for all $j > 1$ (Lemma 6) despite being an optimal non-wasteful allocation rule (Lemma 7).

Lemma 6. *If $p(0|\theta_1) > p(0|\theta_j) = 0$ for all $j > 1$, then allocation rule x^* is not an optimal allocation rule of the canonical problem.*

Proof. Consider the following ordered allocation rule x', where $\overline{s}'_1 = \overline{s}^* + \varepsilon$, $\overline{s}'_j = \overline{s}^* - \delta(\varepsilon)$ for all $j > 1$ and $\underline{s}'_j = \gamma(\varepsilon)$ for all $j > 1$, where

$$q(\theta_1) \int_{\overline{s}^*}^{\overline{s}^*+\varepsilon} p(s|\theta_1)\, ds = \widehat{q}(\theta_2) \int_{\overline{s}^*-\delta(\varepsilon)}^{\overline{s}^*} \widehat{p}(s|\theta_2)\, ds,$$

and

$$\begin{aligned} & -\left(u(\theta_1, h) - u(\theta_1, l)\right) \int_{\overline{s}^*}^{\overline{s}^*+\varepsilon} p(s|\theta_1)\, ds \\ = \;& \left(u(\theta_1, h) - u(\theta_1, l)\right) \int_{\overline{s}^*-\delta(\varepsilon)}^{\overline{s}^*} p(s|\theta_1)\, ds - u(\theta_1, l) \int_{0}^{\gamma(\varepsilon)} p(s|\theta_1)\, ds. \end{aligned}$$

In words, we are perturbing allocation rule x^* by shifting some of the high-quality objects from type θ_1 to the higher types, while keeping constant the measure of high-quality objects being assigned (but reducing the measure of low-quality objects assigned) and keeping type θ_1 indifferent.

Notice that

$$\delta'(0) = \frac{q(\theta_1)}{\widehat{q}(\theta_2)} \frac{p(\overline{s}^*|\theta_1)}{\widehat{p}(\overline{s}^*|\theta_2)}$$

and

$$\gamma'(0) = \frac{\left(u(\theta_1, h) - u(\theta_1, l)\right)}{u(\theta_1, l)} \frac{p(\overline{s}^*|\theta_1)}{p(0|\theta_1)} \left(\frac{q(\theta_1)}{\widehat{q}(\theta_2)} \frac{p(\overline{s}^*|\theta_1)}{\widehat{p}(\overline{s}^*|\theta_2)} + 1 \right).$$

Once again, let $V(\varepsilon) \equiv W(x'(\varepsilon))$ for any given $\varepsilon > 0$ and notice that

$$V(\varepsilon) = \left\{ \begin{array}{c} \sum_{i=2}^{J} q(\theta_i)\left(u(\theta_i, h)\int_{\overline{s}^*-\delta(\varepsilon)}^{1} p(s|\theta_i)\,ds + u(\theta_i, l)\int_{\gamma(\varepsilon)}^{\overline{s}^*-\delta(\varepsilon)} p(s|\theta_i)\,ds\right) + \\ q(\theta_1)\left(u(\theta_1, h)\int_{\overline{s}^*+\varepsilon}^{1} p(s|\theta_1)\,ds + u(\theta_1, l)\int_{0}^{\overline{s}^*+\varepsilon} p(s|\theta_1)\,ds - \right) \end{array} \right\}.$$

Moreover, notice that $V(\varepsilon) \geq \widehat{V}(\varepsilon)$ for all $\varepsilon > 0$, where

$$\widehat{V}(\varepsilon) = \left\{ \begin{array}{c} \widehat{q}(\theta_2)\left(u(\theta_2, h)\int_{\overline{s}^*-\delta(\varepsilon)}^{1} \widehat{p}(s|\theta_2)\,ds + u(\theta_2, l)\int_{\gamma(\varepsilon)}^{\overline{s}^*-\delta(\varepsilon)} \widehat{p}(s|\theta_2)\,ds\right) + \\ q(\theta_1)\left(u(\theta_1, h)\int_{\overline{s}^*+\varepsilon}^{1} p(s|\theta_1)\,ds + u(\theta_1, l)\int_{0}^{\overline{s}^*+\varepsilon} p(s|\theta_1)\,ds - \right) \end{array} \right\}.$$

Finally, notice that $\widehat{V}'(0) > 0$ if and only if

$$\begin{aligned} & u(\theta_2, h) - u(\theta_2, l) - u(\theta_1, h) + u(\theta_1, l) \\ > \; & \frac{\widehat{q}(\theta_2)}{q(\theta_1)} u(\theta_2, l) \frac{(u(\theta_1, h) - u(\theta_1, l))}{u(\theta_1, l)} \frac{\widehat{p}(0|\theta_2)}{p(0|\theta_1)} \left(\frac{q(\theta_1)}{\widehat{q}(\theta_2)} \frac{p(\overline{s}^*|\theta_1)}{\widehat{p}(\overline{s}^*|\theta_2)} + 1 \right), \end{aligned}$$

which holds whenever $\frac{\widehat{p}(0|\theta_2)}{p(0|\theta_1)} = 0$.

We can then conclude that if $\varepsilon > 0$ is sufficiently small, allocation rule x' will generate a larger ex-ante payoff than allocation rule x^*, while being feasible and incentive compatible (type θ_1 is indifferent by construction, while higher types do not deviate by Lemma). □

Lemma 7. *Allocation rule x^* is an optimal non-wasteful allocation rule of the canonical problem.*

Proof. Consider the same relaxed problem as in the proof of part 1 of the Theorem, where the only incentive constraints considered are the upward ones, and add to it an additional constraint that states that every agent must be assigned an object. By Lemma 2, it follows that there is an ordered allocation rule x^1 that solves the relaxed problem with thresholds $\{\underline{s}_\theta, \overline{s}_\theta\}$ (it is straightforward to see that a solution exists by following the same steps as in Lemma 3). It also directly follows that $\underline{s}_\theta = 0$ (because only non-wasteful allocation rules are considered) and that $\overline{s}_\theta$ is weakly increasing with θ (for the incentive constraints that are considered to hold). The proof is completed by showing that $\overline{s}_\theta$ must be constant with θ.

Suppose not, so that there is some j such that $\overline{s}_{\theta_j} < \overline{s}_{\theta_{j+1}}$. Consider alternative allocation rule x' that is equal to allocation rule x^1 except that

$$\overline{s}'_{\theta_j} = \overline{s}_{\theta_j} + \varepsilon \text{ and } \overline{s}'_{\theta_{j+1}} = \overline{s}_{\theta_j} - \delta(\varepsilon)$$

for some small enough $\varepsilon > 0$, where $\delta(\varepsilon)$ is such that the total proportion of high-quality objects assigned is the same as under allocation rule x^1. It follows that allocation rule x' would generate a strictly larger ex-ante payoff because

$$u(\theta_{j+1}, h) - u(\theta_{j+1}, l) > u(\theta_j, h) - u(\theta_j, l).$$

Furthermore, provided $\varepsilon > 0$ is small enough, no type $\widehat{j} < j+1$ would like to mimic type $j+1$, because

$$U\left(\theta_{\widehat{j}}, X^1(\theta_j)\right) > U\left(\theta_{\widehat{j}}, X^1(\theta_{j+1})\right)$$

for all $\widehat{j} < j+1$, which is a contradiction. $\square$

We can now use lemmas 6 and 7 to complete the proof of Proposition 1.

Proof of Proposition 1. Consider the problem of finding the optimal allocation rule for some (α_l, α_h). In that linear problem, in addition to the incentive constraints we have two feasibility constraints; the h-feasibility constraint and the l-feasibility constraint. Let us call that problem P. Consider a relaxed version of problem P (called R_h), where the h-feasibility constraint is not considered. The unique solution of that problem is not feasible, as it would assign a high-quality object to every agent. Because any solution of R_h violates the h-feasibility constraint (and because both P and R_h are linear problems), any solution of P is such that the h-feasibility constraint holds with equality, so that there is no waste of h-quality objects.

Now, let us do the same for the l-feasibility constraint. Consider a relaxed version of problem P (called R_l) where the l-feasibility constraint is not considered. Let x be any ordered allocation rule that solves R_l and let its corresponding thresholds be denoted by $\{(\underline{s}_\theta, \overline{s}_\theta)\}_{\theta\in\Theta}$. For each such x, let

$$\rho(x) \equiv \sum_{\theta\in\Theta} q(\theta) \int_{\underline{s}_\theta}^{\underline{s}_\theta} p(s|\theta)\, ds$$

denote the measure of l-quality objects assigned to agents. Define

$$\overline{\alpha}_l = \max\{\rho(x) \in [0,1] : x \text{ solves } R_l\}$$

and

$$\underline{\alpha}_l = \min\left\{\rho\left(x\right) \in [0,1] : x \text{ solves } R_l\right\}.$$

Notice that problem R_l does not depend on α_l, so any solution x of R_l must also be the solution P whenever $\alpha_l \geq 1-\alpha_h$, which would imply that x would be wasteful (by lemmas 6 and 7). Hence, it follows that $\overline{\alpha}_l < 1-\alpha_h$. Moreover, notice that if the solution to P when $\alpha_l \geq 1-\alpha_h$ is unique, then $\overline{\alpha}_l = \underline{\alpha}_l$.

By construction, whenever $\alpha_l \geq \underline{\alpha}_l$, the set of solutions of P is equal to the set of solutions x of R_l such that $\rho\left(x\right) \geq \alpha_l$. As a result, if $\alpha_l > \overline{\alpha}_l$, then all solutions of P waste l-quality objects. By constrast, if $\alpha_l < \underline{\alpha}_l$, none of the solutions of R_l solves P. Therefore, because both P and R_l are linear problems, it follows that any solution of P is such that the l-feasibility constraint holds with equality, which implies that there is no waste.

□

7.3.2 Proof of Proposition 2

Proof. Take any ordered binary mechanism. Denote as $\overline{S}$ the set of all thresholds of the corresponding ordered binary allocation rule. That is:

$$\overline{S} \equiv \left\{s \in [0,1] : \exists k = 1,...,K, t = A, B : s = s_k^t\right\},$$

and let $i = 1,...,\left|\overline{S}\right|$ denote i^{th} smallest element of $\overline{S}$. The element which corresponds to $s^B_{\overline{k}}$ is denoted by $\overline{i}$ and the element which corresponds to $s^B_{\underline{k}}$ is denoted by $\underline{i} < \overline{i}$. Let

$$F\left(i|\theta\right) \equiv \int_0^i p\left(s|\theta\right) ds$$

for all $i \in [0,1]$.

Let the difference between reporting B and A as a function of θ be denoted by $\Lambda\left(\theta\right)$. Notice that one can write

$$\Lambda\left(\theta\right) = \Lambda_+\left(\theta\right) - \Lambda_-\left(\theta\right)$$

for some $\Lambda_+\left(\theta\right) > 0$ and $\Lambda_-\left(\theta\right) > 0$.

It follows that

$$\Lambda_+(\theta) = \sum_{i=\bar{i}+1}^{|\overline{S}|} \Lambda_+^i(\theta)\left(F(i|\theta) - F(i-1|\theta)\right),$$

where, for all $i \geq \bar{i}+1$,

$$\Lambda_+^i(\theta) = \left(u(\theta, k'(i)) - u(\theta, k(i))\right)$$

for some $(k(i), k'(i))$ such that $\overline{k} \leq k(i) \leq k'(i) \leq K$. Likewise, it also follows that

$$\Lambda_-(\theta) = \sum_{i=1}^{\underline{i}} \Lambda_-^i(\theta)\left(F(i|\theta) - F(i-1|\theta)\right),$$

where, for all $i \leq \underline{i}$,

$$\Lambda_-^i(\theta) = \left(u(\theta, k'(i)) - u(\theta, k(i))\right)$$

for some $(k(i), k'(i))$ such that $1 \leq k(i) \leq k'(i) \leq \underline{k}$.

Notice that $\frac{\Lambda_+^i(\theta)}{\Lambda_-^{i'}(\theta)}$ is weakly increasing with θ whenever $i \geq \bar{i}+1$ and $i' \leq \underline{i}$ because

$$\frac{u(\theta, k) - u(\theta, k-1)}{u(\theta, k') - u(\theta, k'-1)}$$

is weakly increasing with θ for all $k \geq k'$. Notice also that

$$\frac{F(i|\theta) - F(i-1|\theta)}{F(i'|\theta) - F(i'-1|\theta)}$$

is strictly increasing with θ whenever $i > i'$ because of the MLRP of $p(s|\theta)$ (see (3) in the proof of Lemma 1). Therefore, it follows that

$$\theta' > \theta \Rightarrow \frac{\Lambda_+(\theta')}{\Lambda_-(\theta')} > \frac{\Lambda_+(\theta)}{\Lambda_-(\theta)}$$

for any $\theta, \theta' \in \Theta$. The result follows by defining $\widehat{\theta}$ as follows: if $\frac{\Lambda_+(\theta_J)}{\Lambda_-(\theta_J)} < 1$, then $\widehat{\theta} = \theta_J$; if not, then

$$\widehat{\theta} \in \arg\min_{\theta\in\Theta} \frac{\Lambda_+(\theta)}{\Lambda_-(\theta)} \text{ s.t. } \frac{\Lambda_+(\theta)}{\Lambda_-(\theta)} \geq 1.$$

□

7.3.3 Proof of Proposition 3

Proof. Notice that, for any $s' > s$,

$$\frac{p(s'|\theta_J)}{p(s|\theta_J)} > \frac{\widehat{p}(s'|\theta_2))}{\widehat{p}(s|\theta_2)} > \frac{p(s'|\theta_2)}{p(s|\theta_2)} > \frac{p(s'|\theta_1)}{p(s|\theta_1)}.$$

Let $\{s_k\}_{k=1}^{K+1}$ denote the allocation rule where objects are assigned according exclusively to the agents' signals (with $s_{K+1} = 1$) and consider the following alternative ordered binary allocation rule $\left\{s_k^A, s_k^B\right\}_{k=1}^{K+1}$, where $s_{\overline{k}}^A = s_{\overline{k}} + \varepsilon$, $s_{\overline{k}}^B = s_{\overline{k}} - \delta(\varepsilon)$, $s_{\underline{k}}^A = s_{\overline{k}} - \beta(\varepsilon)$, $s_{\underline{k}}^B = s_k + \gamma(\varepsilon)$ and $s_k^t = s_k$ for all other k and $t \in \{A, B\}$ and where $\varepsilon > 0$ is small and $(\delta(\varepsilon), \beta(\varepsilon), \gamma(\varepsilon)) \in (0,1)^3$ are such that

$$q(\theta_1) \int_{s_{\overline{k}}}^{s_{\overline{k}}+\varepsilon} p(s|\theta_1)\, ds = \widehat{q}(\theta_2) \int_{s_{\overline{k}}-\delta(\varepsilon)}^{s_{\overline{k}}} \widehat{p}(s|\theta_2)\, ds, \tag{9}$$

$$q(\theta_1) \int_{s_{\underline{k}}^*-\beta(\varepsilon)}^{s_{\underline{k}}^*} p(s|\theta_1)\, ds = \widehat{q}(\theta_2) \int_{s_{\underline{k}}^*}^{s_{\underline{k}}^*+\gamma(\varepsilon)} \widehat{p}(s|\theta_2)\, ds \tag{10}$$

and

$$(u(\theta_1, \underline{k}) - u(\theta_1, \underline{k}-1)) \int_{s_{\underline{k}}^*-\beta(\varepsilon)}^{s_{\underline{k}}^*+\gamma(\varepsilon)} p(s|\theta_1)\, ds = \left(u\left(\theta_1, \overline{k}\right) - u\left(\theta_1, \overline{k}-1\right)\right) \int_{s_{\overline{k}}-\delta(\varepsilon)}^{s_{\overline{k}}+\varepsilon} p(s|\theta_1)\, ds. \tag{11}$$

In words, we are perturbing the original allocation rule slightly as indicated in Figure 7 while keeping the lowest type indifferent and the measure of assigned objects constant for each quality; there is merely a change in the composition of the types of agents who receive objects of levels $\overline{k}$, $\overline{k}-1$, $\underline{k}$ and $\underline{k}-1$. Provided ε is sufficiently small, the new allocation rule is feasible and such that the object assigned to type θ_1 depends on thresholds $\left\{s_k^A\right\}_{k=1}^{K+1}$, while the objects assigned to every other type depend on thresholds $\left\{s_k^B\right\}_{k=1}^{K+1}$.

Let $V(\varepsilon)$ denote the welfare of this new allocation rule as a function of ε. We complete the proof by showing that $\lim_{\varepsilon\to 0} V(\varepsilon) > 0$. By combining (9), (10) and (11), we get that

$$\delta'(0) = \frac{q(\theta_1)}{\widehat{q}(\theta_2)} \frac{p(s_{\overline{k}}|\theta_1)}{\widehat{p}(s_{\overline{k}}|\theta_2)},$$

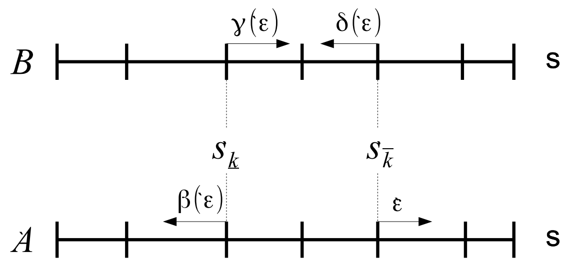

Figure 7: Perturbing the allocation rule.

$$\gamma'(0) = \frac{q(\theta_1)}{\widehat{q}(\theta_2)} \frac{p(s_{\underline{k}}|\theta_1)}{\widehat{p}(s_{\underline{k}}|\theta_2)} \frac{\left(u\left(\theta_1, \overline{k}\right) - u\left(\theta_1, \overline{k}-1\right)\right)}{\left(u\left(\theta_1, \underline{k}\right) - u\left(\theta_1, \underline{k}-1\right)\right)} \frac{p(s_{\overline{k}}|\theta_1)}{p(s_{\underline{k}}|\theta_1)} \frac{\left(1 + \frac{q(\theta_1)}{\widehat{q}(\theta_2)} \frac{p\left(s_{\overline{k}}|\theta_1\right)}{\widehat{p}\left(s_{\overline{k}}|\theta_2\right)}\right)}{\left(1 + \frac{q(\theta_1)}{\widehat{q}(\theta_2)} \frac{p\left(s_{\underline{k}}|\theta_1\right)}{\widehat{p}\left(s_{\underline{k}}|\theta_2\right)}\right)}$$

and

$$\beta'(0) = \frac{\left(u\left(\theta_1, \overline{k}\right) - u\left(\theta_1, \overline{k}-1\right)\right)}{\left(u\left(\theta_1, \underline{k}\right) - u\left(\theta_1, \underline{k}-1\right)\right)} \frac{p(s_{\overline{k}}|\theta_1)}{p(s_{\underline{k}}|\theta_1)} \frac{\left(1 + \frac{q(\theta_1)}{\widehat{q}(\theta_2)} \frac{p\left(s_{\overline{k}}|\theta_1\right)}{\widehat{p}\left(s_{\overline{k}}|\theta_2\right)}\right)}{\left(1 + \frac{q(\theta_1)}{\widehat{q}(\theta_2)} \frac{p\left(s_{\underline{k}}|\theta_1\right)}{\widehat{p}\left(s_{\underline{k}}|\theta_2\right)}\right)}.$$

We also have that

$$\begin{aligned} V(\varepsilon) &= q(\theta_1) \left[\begin{array}{c} u\left(\theta_1, \overline{k}\right) \int\limits_{s_{\overline{k}}+\varepsilon}^{s_{\overline{k}+1}} p(s|\theta_1)\, ds + u\left(\theta_1, \overline{k}-1\right) \int\limits_{s_{\overline{k}-1}}^{s_{\overline{k}}+\varepsilon} p(s|\theta_1)\, ds + \\ u\left(\theta_1, \underline{k}\right) \int\limits_{s_{\underline{k}}-\beta(\varepsilon)}^{s_{\underline{k}+1}} p(s|\theta_1)\, ds + u\left(\theta_1, \underline{k}-1\right) \int\limits_{s_{\underline{k}-1}}^{s_{\underline{k}}-\beta(\varepsilon)} p(s|\theta_1)\, ds \end{array} \right] + \\ & \sum_{j=2}^{J} q(\theta_j) \left[\begin{array}{c} u\left(\theta_j, \overline{k}\right) \int\limits_{s_{\overline{k}}-\delta(\varepsilon)}^{s_{\overline{k}+1}} p(s|\theta_j)\, ds + u\left(\theta_j, \overline{k}-1\right) \int\limits_{s_{\overline{k}-1}}^{s_{\overline{k}}-\delta(\varepsilon)} p(s|\theta_j)\, ds + \\ u\left(\theta_j, \underline{k}\right) \int\limits_{s_{\overline{k}}+\gamma(\varepsilon)}^{s_{\underline{k}+1}} p(s|\theta_j)\, ds + u\left(\theta_j, \underline{k}-1\right) \int\limits_{s_{\overline{k}-1}}^{s_{\overline{k}}+\gamma(\varepsilon)} p(s|\theta_j)\, ds \end{array} \right] + c \end{aligned}$$

where c is some expression that does not depend on ε. Define

$$\begin{aligned}\widehat{V}(\varepsilon) &= q(\theta_1)\left[\begin{array}{c} u\left(\theta_1,\overline{k}\right)\int_{s_{\overline{k}}+\varepsilon}^{s_{\overline{k}+1}} p(s|\theta_1)\,ds + u\left(\theta_1,\overline{k}-1\right)\int_{s_{\overline{k}-1}}^{s_{\overline{k}}+\varepsilon} p(s|\theta_1)\,ds+ \\ u(\theta_1,\underline{k})\int_{s_{\underline{k}}-\beta(\varepsilon)}^{s_{\underline{k}+1}} p(s|\theta_1)\,ds + u(\theta_1,\underline{k}-1)\int_{s_{\underline{k}-1}}^{s_{\underline{k}}-\beta(\varepsilon)} p(s|\theta_1)\,ds \end{array}\right]+ \\ &\quad \widehat{q}(\theta_2)\left[\begin{array}{c} u\left(\theta_2,\overline{k}\right)\int_{s_{\overline{k}}-\delta(\varepsilon)}^{s_{\overline{k}+1}} \widehat{p}(s|\theta_2)\,ds + u\left(\theta_2,\overline{k}-1\right)\int_{s_{\overline{k}-1}}^{s_{\overline{k}}-\delta(\varepsilon)} \widehat{p}(s|\theta_2)\,ds+ \\ u(\theta_2,\underline{k})\int_{s_{\overline{k}}+\gamma(\varepsilon)}^{s_{\overline{k}+1}} \widehat{p}(s|\theta_2)\,ds + u(\theta_2,\underline{k}-1)\int_{s_{\overline{k}-1}}^{s_{\overline{k}}+\gamma(\varepsilon)} \widehat{p}(s|\theta_2)\,ds \end{array}\right]+c\end{aligned}$$

and notice that $V(\varepsilon) \geq \widehat{V}(\varepsilon)$ for all $\varepsilon > 0$ because $u(\theta_j, k) > u(\theta_2, k)$ for all $j > 2$. Notice also that $\widehat{V}'(0) > 0$ if and only if

$$\frac{\overline{\Delta}_2-\overline{\Delta}_1}{\overline{\Delta}_1} > \frac{\underline{\Delta}_2-\underline{\Delta}_1}{\underline{\Delta}_1}\frac{\left(1+\frac{q(\theta_1)}{\widehat{q}(\theta_2)}\frac{p\left(s_{\overline{k}}|\theta_1\right)}{\widehat{p}\left(s_{\overline{k}}|\theta_2\right)}\right)}{\left(1+\frac{q(\theta_1)}{\widehat{q}(\theta_2)}\frac{p\left(s_{\underline{k}}|\theta_1\right)}{\widehat{p}\left(s_{\underline{k}}|\theta_2\right)}\right)}$$

where

$$\overline{\Delta}_j \equiv u\left(\theta_j,\overline{k}\right)-u\left(\theta_j,\overline{k}-1\right)$$

and

$$\underline{\Delta}_j \equiv u(\theta_j,\underline{k})-u(\theta_j,\underline{k}-1)$$

for $j = 1, 2$, which is true because

$$\frac{\left(1+\frac{q(\theta_1)}{\widehat{q}(\theta_2)}\frac{p\left(s_{\overline{k}}|\theta_1\right)}{\widehat{p}\left(s_{\overline{k}}|\theta_2\right)}\right)}{\left(1+\frac{q(\theta_1)}{\widehat{q}(\theta_2)}\frac{p\left(s_{\underline{k}}|\theta_1\right)}{\widehat{p}\left(s_{\underline{k}}|\theta_2\right)}\right)} < 1$$

and

$$\frac{\overline{\Delta}_2-\overline{\Delta}_1}{\overline{\Delta}_1} > \frac{\underline{\Delta}_2-\underline{\Delta}_1}{\underline{\Delta}_1}.$$

This means that

$$\lim_{\varepsilon\to 0} V(\varepsilon) \geq \lim_{\varepsilon\to 0} \widehat{V}(\varepsilon) > 0.$$

$\square$

7.3.4 Proof of Proposition 4

Proof. In order to construct allocation rule x, let us first go back to the model with a continuum of agents and consider the optimal ordered allocation rule when the parameters of the model are $\alpha = (\alpha_H - \varepsilon, \alpha_L - \varepsilon)$ for some $\varepsilon \in (0, \min\{\alpha_L, \alpha_H\})$. Let the thresholds of that allocation rule be denoted by $\{\underline{s}_\theta(\varepsilon), \overline{s}_\theta(\varepsilon)\}_{\theta\in\Theta}$ and denote the value for the principal of implementing it by W_ε. Notice that by continuity of the objective function of the principal and of all the incentive constraints considered, it follows that $\lim_{\varepsilon\to 0} W_\varepsilon = W^*$. As a result, in order to prove the proposition, it is enough to construct an allocation rule x such that $\lim_{N\to\infty} W(x) = W_\varepsilon$ and then, for each $\delta > 0$, select a small enough ε.

In order to construct said allocation rule, let us start by defining

$$z_i^h = \begin{cases} 1 \text{ if } s_i \in [\overline{s}_{\theta_i}(\varepsilon), 1] \\ 0 \text{ if } s_i \notin [\overline{s}_{\theta_i}(\varepsilon), 1] \end{cases}$$

and

$$z_i^l = \begin{cases} 1 \text{ if } s_i \in \left(\underline{s}_{\theta_i}(\varepsilon), \overline{s}_{\theta_i}(\varepsilon)\right) \\ 0 \text{ if } s_i \notin (\underline{s}_{\theta}(\varepsilon), \overline{s}_{\theta}(\varepsilon)) \end{cases}$$

for all $i = 1, ..., N$.

Construct each $x^i : \Theta^N \times [0,1]^N \to [0,1] \times [0,1]$ of allocation rule $x = \left(x^1, ..., x^N\right)$ as follows: i) for all $\theta \in \Theta^N$ and $s \in [0,1]^N$ such that either

$$\sum_{j\neq i} z_j^h \geq \lfloor N\alpha_H \rfloor$$

or

$$\sum_{j\neq i} z_j^l \geq \lfloor N\alpha_L \rfloor$$

then $x_h^i(\theta, s) = x_l^i(\theta, s) = 0$; ii) if not, then $x_h^i(\theta, s) = z_i^h$ and $x_l^i(\theta, s) = z_i^l$.

The logic of the mechanism is to assign the objects as they would have been assigned if there was a continuum of agents (part ii) of the definition of x) unless not enough objects are available. In that case (part i of the definition of x), which is increasingly rare as the number of agents grows as we show below, one has to select an assignment that is both feasible and does not alter the agents' incentives. The simplest way of doing this (but not the only one) is by not assigning any object to any agent; basically it is as if the system breaks down in those circumstances. Alternatively, when possible, the principal might simply break the feasibility constraint and simply provide those extra objects even if that requires a larger financial effort (something that is routinely done in the college assignment problem for example and is known as over-enrollment - see Avery, Fairbanks, and Zeckhauser, 2009).

In order to complete the proof, we note that, by construction, allocation rule x is feasible and incentive compatible for any $N \geq 1$, and that the probability that condition i) is realized converges to 0 as $N \to \infty$. To see why that is, notice that condition i) can be written as

$$\frac{\sum_{j \neq i} z_j^h}{N} \geq \frac{\lfloor N\alpha_H \rfloor}{N}$$

and

$$\frac{\sum_{j \neq i} z_j^l}{N} \geq \frac{\lfloor N\alpha_L \rfloor}{N}.$$

Notice that both z_j^h and z_j^l are independent across $j = 1, ..., N$, so, by the weak law of large numbers, we have that

$$\frac{\sum_{j \neq i} z_j^h}{N} \leq \frac{\sum_{j=1}^{N} z_j^h}{N} \to^p \sum_{\theta \in \Theta} q(\theta) \int_{\overline{s}_\theta(\varepsilon)}^{1} p(s|\theta)\, ds \leq \alpha_H - \varepsilon < \frac{\lfloor N\alpha_H \rfloor}{N}.$$

and

$$\frac{\sum_{j \neq i} z_j^l}{N} \leq \frac{\sum_{j=1}^{N} z_j^l}{N} \to^p \sum_{\theta \in \Theta} q(\theta) \int_{\underline{s}_\theta(\varepsilon)}^{\overline{s}_\theta(\varepsilon)} p(s|\theta)\, ds \leq \alpha_L - \varepsilon < \frac{\lfloor N\alpha_L \rfloor}{N},$$

where both final inequalities hold if N is sufficiently large.

This means that each agent is assigned the same lottery of rewards as in the case of the ordered allocation rule $\{\underline{s}_\theta(\varepsilon), \overline{s}_\theta(\varepsilon)\}_{\theta \in \Theta}$ of the model with a continuum of agents with probability 1, which implies that $\lim_{N \to \infty} W(x) = W_\varepsilon$.

$\square$

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