The winner-take-all dilemma*

Kazuya Kikuchi† Yukio Koriyama‡

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Abstract

We consider collective decision making when society consists of groups endowed with voting weights. Each group chooses an internal rule that specifies the allocation of its weight to alternatives as a function of its members' preferences. Under fairly general conditions, we show that the winner-take-all rule is a dominant strategy, while the equilibrium is Pareto dominated, highlighting the dilemma structure between optimality for each group and for the whole society. We also develop a technique for asymptotic analysis and show Pareto dominance of the proportional rule.

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†Tokyo University of Foreign Studies. E-mail: `kazuya.kikuchi68@gmail.com`.

‡CREST, Ecole Polytechnique, Institut Polytechnique de Paris. E-mail: `yukio.koriyama@polytechnique.edu`.

1 Introduction

In many situations of collective decision making, society consists of distinct groups and decisions are made based on the opinions aggregated within the groups. For example, in the U.S. presidential election, each state allocates its electoral votes to the candidates based on the state-wide popular vote.

Existing institutions use a variety of rules, many of which pertain to how to allocate the weight assigned to each group. The *winner-take-all rule* devotes all the weight to the alternative preferred by the majority of its members. The rule has been used to allocate electoral votes in all but two states in the recent U.S. presidential elections. The *proportional rule* allocates a group's weight in proportion to the number of members who prefer the respective alternatives. The rule corresponds to voting in various parliamentary institutions in which the composition of representatives reflects the preferences of citizens proportionally. The local aggregation rule is often set by each group, as in the case of the U.S. Electoral College, where it is constitutionally mandated that each state decide how electoral votes are allocated (Article II, section 1, clause 2).

However, if groups choose their rules based on local and private motives, the resulting social decisions may make all groups worse off than they could be. Each group may have an incentive to allocate the weight so as to increase the influence of its members' opinions on social decisions, at the expense of the influence of other groups. It is then not clear whether the group-level incentive is consistent with social welfare criteria, such as Pareto efficiency. A society consisting of distinct groups thus faces a dilemma between the local incentive of each group and social objectives. To study the relationship between group incentives and their welfare consequences, we model the choice of rules as a non-cooperative game.

In this paper, we consider a model of collective decision making where a society consists of groups endowed with voting weights. Each group chooses the rule for allocating its weight to binary alternatives, and the winner is the one with the most weight. A *rule* for a group is a function that maps members' preferences to an allocation of the weight to the alternatives. Any monotone function is allowed, including the winner-take-all and proportional rules stated above. Groups independently choose their rules, so as to maximize the expected welfare of their members.

The main result is that the game is an n-player *Prisoner's Dilemma* (Theorem 1). The winner-take-all rule is a *dominant strategy*, i.e., it is an optimal strategy for *each* group, regardless of the rules chosen by the other groups. However, if each group has less than half of the total weight, the winner-take-all profile is *Pareto dominated*, i.e., another profile makes *every* group better off. The dilemma structure exists for any number of groups more than two and with fairly little restriction on the joint distribution of preferences (Assumption 1).

The observation that the winner-take-all rule is an optimal strategy for groups is not new. As we discuss in detail in Sections 1.1 and 3.1, previous studies have already pointed out such incentives for groups in various voting situations. The main contribution of this paper lies in the generality of the circumstances under which a formal proof for the dilemma structure is provided. The fact that the winner-take-all profile is Pareto dominated should be distinguished from the conventional wisdom that the direct popular vote maximizes the *utilitarian* welfare of the society, as it maintains the possibility that *some* groups may be better off under the winner-take-all profile. We show in Example 1 that the winner-take-all profile is not Pareto dominated by either the direct popular vote or the proportional profile. One may then wonder what rule profile Pareto dominates the winner-take-all profile. A full characterization of the Pareto set is provided in Lemma 1.

To further investigate welfare properties, we turn to an asymptotic and normative analysis. We consider situations where the number of groups is sufficiently large, and the preferences are distributed independently across groups and symmetrically with respect to the alternatives. We show that the proportional profile Pareto dominates every other symmetric rule profile (i.e., one in which all groups use the same rule), including the winner-take-all profile.

While the above result suggests that the proportional profile asymptotically performs well in terms of efficiency, it is silent about the equality of welfare distribution. In order to study how rules affect the welfare distribution, we examine an asymmetric rule profile called the *congressional district profile*, inspired by the Congressional District Method currently used by Maine and Nebraska. In these states, two electoral votes are allocated by the winner-take-all rule, and the remainder are awarded to the winner of the popular vote in each district.[1] We show that the rule profile makes groups with a smaller weight better off and achieves a more equal distribution of welfare than any symmetric rule profile.

A technical contribution of this paper is to develop an asymptotic method for analyzing the expected welfare of players in weighted voting games. One of the major challenges in the analysis of these games is their discreteness. Due to the nature of combinatorial problems, obtaining an analytical result often requires a large number of classifications by cases, which may include prohibitively tedious and complex computations in order to obtain general insights. We overcome this difficulty by considering asymptotic properties of games in which there are a sufficiently large number of groups. This technique allows us to obtain an explicit formula that captures the asymptotic behavior of the payoffs, which is valid for a wide class of weight distributions among groups (Lemma A).

1.1 Literature Review

The incentive for groups to use the winner-take-all rule has been studied in several papers (e.g., Hummel (2011)). Gelman (2003) and Eguia (2011a,b) provide theoretical explanations for why voters coordinate their votes. A positive analysis by Beisbart and Bovens (2008) provides a numerical comparison on the basis of *a priori* and *a posteriori* voting power measures, which is complementary to our normative analysis.

Eguia (2011a,b) study endogenous formation of the groups, and De Mouzon et al. (2019) provides a welfare analysis of popular vote interstate compacts. Their findings are coherent with ours: if applied only to a subset of the groups, the winner-take-all rule may be welfare detrimental. The possibility of the National Popular Vote Interstate Compact as a commitment device is discussed also in Cloléry and Koriyama (2020).

[1]The idea of allocating a portion of the votes by the winner-take-all rule and allowing the rest to be awarded to distinct candidates can be seen as a compromise between the winner-take-all and the proportional rules. Symbolically, the two votes allocated by the winner-take-all rule correspond to the number of the Senators from each state, while the remainder is equal to the number of the House representatives. The idea behind such a mixture is in line with the logic supporting bicameralism, which is supposed to provide checks and balances between the state autonomy and federal governance.

Our Theorem 1 translates into an impossibility theorem stating that there is no social choice function that is Bayesian incentive compatible, Pareto efficient and non-dictatorial. This is consistent with the results obtained in Bayesian mechanism design, such as Börgers and Postl (2009), Azrieli and Kim (2014) and Ehlers et al. (2020). See the working-paper version of this article for detailed discussion.[2]

The history, objectives, problems, and reforms of the US Electoral College are summarized, for example, in Edwards (2004), Bugh (2010) and Wegman (2020). The incentive for the candidates to concentrate their campaign resources in swing and decisive states is modeled in Strömberg (2008), which is coherent with the findings of the seminal paper in probabilistic voting by Lindbeck and Weibull (1987). Strömberg (2008) also finds that uneven resource allocation and unfavorable treatment of minority states would be mitigated by implementing a national popular vote, consistent with the classical findings by Brams and Davis (1974). The incentive of voters to turn out is investigated by Kartal (2015), which finds that the winner-take-all rule discourages turnout when the voting cost is heterogeneous.

Constitutional design of weighted voting is studied extensively in the literature. Seminal contributions are found in the context of power measurement: Penrose (1946), Shapley and Shubik (1954), Banzhaf (1968) and Rae (1969). Excellent summaries of theory and applications are given by, above all, Felsenthal and Machover (1998) and Laruelle and Valenciano (2008). The tools and insights obtained in the power measurement literature are often used in the apportionment problem: e.g., Barberà and Jackson (2006), Koriyama et al. (2013), and Kurz et al. (2017).

2 The Model

2.1 Weighted Voting

We consider a society partitioned into n disjoint groups: $i \in \{1, 2, \cdots, n\}$. Each group i is endowed with a voting weight $w_i > 0$. The society makes a decision between two alternatives, denoted -1 and $+1$, through the following two voting stages: (i) each individual votes for his preferred alternative; (ii) each group allocates its

[2] Available at https://arxiv.org/abs/2206.09574.

weight between the alternatives, based on the group-wide voting result. The winner is the alternative that receives the majority of overall weight. Let $\theta_i \in [-1, 1]$ denote the vote margin in group i at the first voting stage. That is, θ_i is the fraction of members of i preferring alternative $+1$ minus the fraction preferring -1.[3] At the second stage, each group's allocation of weight is determined as a function of the group-wide margin.

Definition 1. A *rule* of group i is a non-decreasing function $\phi_i : [-1, 1] \to [-1, 1]$.

The value $\phi_i(\theta_i)$ is the group-wide weight margin, i.e., the fraction of the weight w_i allocated to alternative $+1$ minus that allocated to -1, given that the vote margin is θ_i. That is, the rule allocates $w_i\phi_i(\theta_i)$ more weight to alternative $+1$ than alternative -1.

Examples of rules. Among all admissible rules, the following examples deserve particular attention.

(i) *Winner-take-all rule*: $\phi_i^{\mathrm{WTA}}(\theta_i) = \operatorname{sgn}\theta_i$.

(ii) *Proportional rule*: $\phi_i^{\mathrm{PR}}(\theta_i) = \theta_i$.

(iii) *Congressional district rule*: $\phi_i^{\mathrm{CD}}(\theta_i) = (c/w_i)\phi_i^{\mathrm{WTA}}(\theta_i) + (1 - c/w_i)\,\phi_i^{\mathrm{PR}}(\theta_i)$ for a constant $c \in (0, \min\{w_1, \cdots, w_n\}]$.

The winner-take-all rule devotes all the weight of a group to the winning alternative in the group. The proportional rule allocates the weight in proportion to the vote shares of the respective alternatives in the group. The congressional district rule allocates the fixed amount of weight c by the winner-take-all rule, and the remaining amount $w_i - c$ by the proportional rule. It is inspired by the congressional district method currently used by Maine and Nebraska for the allocation of presidential electoral votes.

A *rule profile* is a specification of the rule for each group, $\phi = (\phi_i)_{i=1}^n$. We call ϕ *symmetric* if the rules of all groups are the same function; examples include the *winner-take-all profile* ϕ^{WTA} and the *proportional profile* ϕ^{PR}. An example of an

[3] For example, $\theta_i = 0.2$ means that 60% of members of i prefer $+1$ and 40% prefer -1.

asymmetric rule profile is the *congressional district profile* ϕ^{CD} in which the same constant c applies to all groups. The profile is asymmetric since the ratio c/w_i of weight allocated by the winner-take-all rule is larger for groups with smaller weights.

The social decision is the alternative that receives the majority of overall weight. In the case of a tie, we assume that each alternative is chosen with probability 1/2. Thus, given the rules $\phi = (\phi_i)_{i=1}^n$ and the group-wide vote margins $\theta = (\theta_i)_{i=1}^n$, the social decision $d_\phi(\theta)$ is determined as follows:

$$d_\phi(\theta) = \begin{cases} \text{sgn } \sum_{i=1}^n w_i \phi_i(\theta_i) & \text{if } \sum_{i=1}^n w_i \phi_i(\theta_i) \neq 0, \\ \pm 1 \text{ equally likely} & \text{if } \sum_{i=1}^n w_i \phi_i(\theta_i) = 0. \end{cases} \tag{1}$$

The *popular vote* refers to the direct majority voting by all individuals in the society. It is the social decision made according to the sign of $\sum_{i=1}^n m_i \theta_i$, where m_i denotes the population in group i. Equivalently, the popular vote can be represented as the social decision $d_{\phi^{\mathrm{POP}}}(\theta)$ under the rule profile ϕ^{POP} in which the rule of group i is defined by $\phi_i^{\mathrm{POP}}(\theta_i) = k(m_i/w_i)\theta_i$, where $k > 0$ is a sufficiently small constant so that the value $\phi_i^{\mathrm{POP}}(\theta_i)$ lies within $[-1, 1]$.

2.2 The Game

We now define the non-cooperative game Γ in which the groups choose their own rules simultaneously. We assume that after the groups set their rules, all individual members vote sincerely. This assumption might be justified on the grounds that even if individuals can vote against their preferences, truthful voting is a weakly dominant strategy since the rules are non-decreasing.

The game is played under incomplete information about individuals' preferences, and hence about the group-wide vote margins. Each group chooses a rule so as to maximize the expected welfare of its members. Since rules are fixed prior to the realization of preferences, a strategy for a group is a function from the realization of members' preferences to the allocation of weight. Let Θ_i be a random variable that takes values in $[-1, 1]$ and represents the vote margin in group i.[4] Our assumption

[4]Throughout the paper, we use capital Θ_i for the representation of a random variable, and small θ_i for the realization.

on the joint distribution of $(\Theta_i)_{i=1}^n$ will be stated later (Assumption 1).

The ex post payoff for group i is the average payoff for its members from the social decision. For simplicity, we assume that each individual obtains payoff 1 if he prefers the social decision and payoff -1 otherwise.[5] The average payoff of members of group i equals Θ_i or $-\Theta_i$ depending on whether the social decision is $+1$ or -1; more concisely, it is $\Theta_i d_\phi(\Theta)$. The ex ante payoff for group i, denoted $\pi_i(\phi)$, is the expected value of the above expression:

$$\pi_i(\phi) = \mathbb{E}\left[\Theta_i d_\phi(\Theta)\right]. \tag{2}$$

To summarize, the *game* Γ is the one in which: the players are the n groups; the strategy set for each group i is the set of all rules; the payoff function for group i is π_i defined in (2).

In game Γ, a rule (or strategy) ϕ_i for group i *weakly dominates* another rule ψ_i if $\pi_i(\phi_i, \phi_{-i}) \geq \pi_i(\psi_i, \phi_{-i})$ for any ϕ_{-i}, with strict inequality for at least one ϕ_{-i}. A rule ϕ_i is a *weakly dominant strategy* for group i if it weakly dominates every rule not equivalent to ϕ_i, where we call two rules ϕ_i and ψ_i *equivalent* if $\phi_i(\theta_i) = \psi_i(\theta_i)$ for almost every $\theta_i \in [-1, 1]$ (with respect to Lebesgue measure on $[-1, 1]$).

A rule profile ϕ *Pareto dominates* another profile ψ if $\pi_i(\phi) \geq \pi_i(\psi)$ for all i, with strict inequality for at least one i. If ϕ is not Pareto dominated by any rule profile, it is called *Pareto efficient*.

3 The Dilemma

3.1 The Main Result

The main theorem holds under a fairly weak assumption on the joint distribution of preferences (Assumption 1), which allows for arbitrary correlations across groups and group-specific biases. The first two parts of the theorem also refer to the condition that no group has a dictatorial weight (Asssumption 2).

[5]The limitation imposed by the assumption is not essential, because there exists an affine transformation between payoffs with and without the assumption, rendering the strategic incentive equivalent, as we show in Section 3.2.

Assumption 1. The joint distribution of group-wide margins $(\Theta_i)_{i=1}^n$ is absolutely continuous and has full support $[-1,1]^n$.

Assumption 2. No group has more than half the total weight: $w_i \le (1/2)\sum_{j=1}^n w_j$ for all $i = 1, \cdots, n$.

Theorem 1. *Under Assumption 1, the following statements hold:*

(i) The winner-take-all rule ϕ_i^{WTA} is a weakly dominant strategy for each group i if and only if Assumption 2 holds.

(ii) The winner-take-all profile ϕ^{WTA} is Pareto dominated if and only if Assumption 2 holds.

(iii) The proportional profile ϕ^{PR} and the popular vote profile ϕ^{POP} are Pareto efficient.

We use the following lemma to prove the theorem. A rule profile ϕ is called a *generalized proportional profile* if there exists a vector $(\lambda_i)_{i=1}^n \in [0,1]^n \setminus \{\mathbf{0}\}$ such that for each i,

$$\phi_i(\theta_i) = \lambda_i \theta_i \text{ for almost every } \theta_i \in [-1,1].$$

Two rule profiles ϕ and ψ are called *equivalent* if $d_\phi(\theta) = d_\psi(\theta)$ for almost every $\theta \in [-1,1]^n$.

Lemma 1. *(Characterization of the Pareto set) Under Assumption 1, a rule profile $\phi = (\phi_i)_{i=1}^n$ is Pareto efficient if and only if it is equivalent to a generalized proportional profile.*

The proof of Lemma 1 is relegated to Appendix A1.

Proof of Theorem 1. It is useful to introduce the notation $\pi_i(x_i, \phi_{-i}|\theta_i)$ for group i's interim payoff given $\Theta_i = \theta_i$ when the group chooses the weight margin $x_i \in [-1,1]$. By conditioning on whether the social decision is $+1$ or -1, we have:

$$\begin{aligned}\pi_i(x_i, \phi_{-i}|\theta_i) =& \theta_i \mathbb{P}\left\{ w_i x_i + \textstyle\sum_{j\neq i} w_j \phi_j(\Theta_j) > 0 \middle| \Theta_i = \theta_i \right\} \\ &- \theta_i \mathbb{P}\left\{ w_i x_i + \textstyle\sum_{j\neq i} w_j \phi_j(\Theta_j) < 0 \middle| \Theta_i = \theta_i \right\}.\end{aligned} \tag{3}$$

We first check that without any assumption,

$$\pi_i(\phi_i^{\mathrm{WTA}}, \phi_{-i}) \geq \pi_i(\phi_i, \phi_{-i}) \tag{4}$$

for any (ϕ_i, ϕ_{-i}). By (3), if $\theta_i > 0$ (resp. $\theta_i < 0$), then the interim payoff $\pi_i(x_i, \phi_{-i}|\theta_i)$ is non-decreasing (resp. non-increasing) in $x_i \in [-1, 1]$. We thus have $\pi_i(\phi_i^{\mathrm{WTA}}(\theta_i), \phi_{-i}|\theta_i) \geq \pi_i(\phi_i(\theta_i), \phi_{-i}|\theta_i)$ for any (ϕ_i, ϕ_{-i}) and $\theta_i \neq 0$. Since $\Theta_i = 0$ occurs with probability 0, this implies (4).

"If" part of (i). We show that if no group has a dictatorial weight, then for any rule profile ϕ_{-i} in which each $\phi_j(\Theta_i)$ $(j \neq i)$ has full support $[-1, 1]$ (e.g., ϕ_j^{PR}), the strict inequality

$$\pi_i(\phi_i^{\mathrm{WTA}}, \phi_{-i}) > \pi_i(\phi_i, \phi_{-i}) \tag{5}$$

holds for any rule ϕ_i that differs from ϕ_i^{WTA} on a set $A \subset [-1, 1]$ of positive measure; combined with (4), this establishes that ϕ_i^{WTA} is weakly dominant. To show (5), note that for such ϕ_{-i} and any θ_i, the conditional distribution of $\sum_{j \neq i} w_j \phi_j(\Theta_j)$ given $\Theta_i = \theta_i$ has support $I := \left[-\sum_{j \neq i} w_j, \sum_{j \neq i} w_j\right]$. Since $w_i \leq \sum_{j \neq i} w_j$ by Assumption 2, as x_i moves in $[-1, 1]$, $w_i x_i$ moves in the interval I. Formula (3) thus implies that if $\theta_i > 0$ (resp. $\theta_i < 0$), then $\pi_i(x_i, \phi_{-i}|\theta_i)$ is strictly increasing (resp. decreasing) in $x_i \in [-1, 1]$. Hence $\pi_i(\phi_i^{\mathrm{WTA}}(\theta_i), \phi_{-i}|\theta_i) > \pi_i(\phi_i(\theta_i), \phi_{-i}|\theta_i)$ at any $\theta_i \in A$. Since Θ_i has full support, result (5) follows.

"Only if" part of (i). Suppose that group i has a dictatorial weight. Consider the mixed rule $\phi_i^{a_i} := a_i \phi_i^{\mathrm{WTA}} + (1 - a_i)\phi_i^{\mathrm{PR}}$ for $a_i \in (0, 1)$. If a_i is sufficiently close to 1, this strategy gives group i dictatorial power, i.e., the social decision always coincides with the alternative preferred by the majority of its members, whatever rules the other groups choose. Thus, for any ϕ_{-i}, the strategy $\phi_i^{a_i}$ with a_i close to 1 always gives group i the same payoff as ϕ^{WTA} does; in particular, ϕ_i^{WTA} does not weakly dominate $\phi_i^{a_i}$.

"If" part of (ii). By the characterization of the Pareto set (Lemma 1), it suffices to check that ϕ^{WTA} is not equivalent to any generalized proportional profile. Suppose, on the contrary, that ϕ^{WTA} is equivalent to a generalized proportional profile with

coefficients $\lambda \in [0,1]^n \setminus \{\mathbf{0}\}$. Then, since $(\Theta_i)_{i=1}^n$ has full support,

$$d_{\phi^{\text{WTA}}}(\theta) = \operatorname{sgn} \sum_{i=1}^{n} w_i \lambda_i \theta_i \text{ at almost every } \theta \in [-1,1]^n. \tag{6}$$

Since no group dictates the social decision, the coefficients λ_i are positive for at least two groups. Without loss of generality, assume $\lambda_1 > 0$ and $\lambda_2 > 0$. Now, fix θ_i for $i \neq 1, 2$ so that they are sufficiently small in absolute value. Then, according to (6), there exists $\bar{\varepsilon} > 0$ such that for (almost any) $\varepsilon \in [0, \bar{\varepsilon}]$, $d_{\phi^{\text{WTA}}}(\theta) = +1$ if $\theta_1 = 1 - \varepsilon$ and $\theta_2 = -\varepsilon$, while $d_{\phi^{\text{WTA}}}(\theta) = -1$ if $\theta_1 = \varepsilon$ and $\theta_2 = -1 + \varepsilon$. This contradicts the fact that $d_{\phi^{\text{WTA}}}(\theta)$ depends only on the signs of $(\theta_i)_{i=1}^n$.

"Only if" part of (ii). This is immediate from Lemma 1: if group i has a dictatorial weight, ϕ^{WTA} is equivalent to the generalized proportional profile in which the coefficient is positive only for group i, and therefore is Pareto efficient.

(iii). This is also immediate from Lemma 1: ϕ^{PR} and ϕ^{POP} are generalized proportional profiles with the coefficients defined by $\lambda_i = 1$ and $\lambda_i = k(m_i/w_i)$ for a constant $k > 0$, respectively. □

Theorem 1 shows that, while the dominant strategy for each group is the winner-take-all rule, the dominant-strategy equilibrium is Pareto dominated by a generalized proportional profile. This typical Social Dilemma (or, n-player Prisoner's Dilemma) situation suggests that a Pareto efficient outcome is not expected to be achieved under decentralized decision making, and a commitment device can be effective in order to attain a Pareto improvement.[6] Our characterization lemma tells us that, at the *first-best*, the society should use rules that are proportional in nature, so that the cardinal information of the group-wide preferences is aggregated without distortion.

The observation that groups have an incentive to use the winner-take-all rule is not new. Beisbart and Bovens (2008) consider Colorado's deviation from the winner-take-all rule to the proportional rule, following the state's attempt in 2004 to amend the state constitution, and show that the citizens in Colorado are worse

[6] See the working paper version for concrete examples of the device. `https://arxiv.org/abs/2206.09574`

off under both *a priori* and *a posteriori* measures. Hummel (2011) shows that a majority of the voters in a state is worse off by unilaterally switching to the proportional rule from the winner-take-all profile.

Our results are also consistent with the findings in the literature of the coalition formation games in which individuals may have incentive to raise their voices by forming a coalition and aligning their votes. Gelman (2003) illustrates that individuals are better off by forming a coalition and assign all their weights to one alternative. Eguia (2011a) considers a game in which the members in an assembly decide whether to accept the party discipline to align their votes, and shows that the voting blocs form in equilibrium if preferences are sufficiently polarized. Eguia (2011b) considers a dynamic model and shows the conditions under which voters form two polarized voting blocs in a stationary equilibrium.

A novelty of Theorem 1 lies in its generality. Earlier studies have introduced a specific structure either on the distribution of the preferences and/or of the weights, or on the set of the rules that groups can use. In contrast, we only impose fairly mild conditions on the preference distribution (in particular, Assumption 1 imposes no restriction on across-group correlation), on the weight distribution (Assumption 2 imposes no specific weight structure such as one big group and several smaller ones, or equally sized groups), and on the set of the available rules (Definition 1 admits all non-decreasing rules, not just the winner-take-all and the proportional rules).

3.2 Discussion

3.2.1 An illustrative example

It is worth emphasizing that our main result does not imply merely utilitarian (i.e., benthamite) inefficiency of the equilibrium profile. The profile is Pareto dominated, implying that it is in *every* group's interest to move from the winner-take-all equilibrium to another profile. From the utilitarian perspective, it is straightforward to see that the social optimum is obtained by the *popular vote*, i.e., direct majority voting by all individuals. However, this observation is not sufficient to establish that the winner-take-all profile is Pareto dominated. After all, the utilitarian optimum is merely one point in the Pareto set.

The following example illustrates that the winner-take-all profile is not necessarily Pareto dominated by either the popular vote or the proportional profile.

Example 1. Consider a society which consists of two large groups with an equal weight and one small group. For an illustrative purpose, let us consider three American states: Florida, New York and Wyoming. Their populations and weights are summarized in Table 1.

Table 1: Comparison of the expected payoffs in an example of the society which consists of three states: Florida, New York and Wyoming. Weights are the electoral votes assigned in the Electoral College in 2020. Population is an estimation of the voting-age population in 2018 (in thousands). Source: US Census Bureau.

State	Weight	Population	$\pi_i\left(\phi^{\mathrm{WTA}}\right)$	$\pi_i\left(\phi^{\mathrm{PR}}\right)$	$\pi_i\left(\phi^{\mathrm{POP}}\right)$	$\pi_i\left(\hat{\phi}\right)$
Florida	29	15,047	0.250	0.332	0.343	0.271
New York	29	13,684	0.250	0.332	0.323	0.271
Wyoming	3	422	0.250	0.034	0.008	0.271
Per capita average			0.250	0.328	0.329	0.271

The vote margins $(\Theta_i)_{i=1,2,3}$ are drawn from the uniform distribution on $[-1, 1]$ independently across the states.

Since there is no dictator state (i.e., Assumpion 2 is satisfied), any pair of two states is a minimal winning coalition under the winner-take-all profile, implying that the expected payoffs are exactly the same across states under ϕ^{WTA}. The two larger states are better off under the proportional profile ϕ^{PR}, while the smaller state is worse off. This is because the social decision is more likely to coincide with the alternative preferred by the majority of the large states under ϕ^{PR}.

The payoff of the small state is larger under ϕ^{PR} than the popular vote ϕ^{POP}. This is due to the advantage to the small state induced by degressive proportionality of the apportionment.[7] We can also verify that the utilitarian welfare is maximized under the popular vote ϕ^{POP} by comparing the per capita average payoffs.

[7]The degressive proportionality is a consequence of the rule specified in the US Constitution. The number of electoral votes of each state is the sum of the numbers of Senate members (constant) and of the House (proportional to population in principle). Under such a rule, per capita weight is decreasing in population.

Finally, let $\hat{\phi}$ be the generalized proportional profile with coefficients $\lambda_i = 1/w_i$. We observe that it Pareto dominates ϕ^{WTA}. Remember that our characterization lemma tells us that a profile is Pareto efficient if and only if it is equivalent to a generalized proportional profile. We can show that among the profiles which Pareto dominate the equilibrium profile ϕ^{WTA}, one is obtained by letting $\lambda_i = 1/w_i$, because the payoffs are equal across the states in this example, and we can obtain the particular point in the Pareto set with the equal Pareto coefficients by setting $\lambda_i = 1/w_i$.

This example illustrates that the winner-take-all, proportional profiles, and the popular vote may be all Pareto imcomparable. Even though Theorem 1 shows that the winner-take-all profile is Pareto dominated, it may not be dominated by either the proportional profile or the popular vote. This may happen when the number of groups is small. For the cases in which there are sufficiently many groups, we provide clear-cut insights in Section 4 by using an asymptotic model.

3.2.2 Heterogeneous preference intensities

We have assumed that all individuals have the same preference intensities (i.e., each individual receives a unit payoff whenever she prefers the social decision), and that each group's objective is to maximize the ex ante average payoff of its members. However, our formal definition (2) can be generalized to the case with heterogeneous preference intensities. It only suffices for the group-wide payoff from the social decision to be more generally defined, not necessarily as the average of members' payoffs with identical preference intensities.

To be more precise, suppose that the the ex post payoff of each group i is U_i^+ or U_i^- depending on whether the social decision is $+1$ or -1, where U_i^+ and U_i^- are random variables that can take *any* values in $[0, 1]$. Redefine the variable Θ_i as the difference: $\Theta_i := U_i^+ - U_i^-$. Then the group's (ex ante) payoff from the social decision under rule profile ϕ is

$$\begin{aligned} u_i(\phi) &= \mathbb{E}\left[U_i^+ \frac{1+d_\phi(\Theta)}{2} + U_i^- \frac{1-d_\phi(\Theta)}{2}\right] \\ &= \frac{1}{2}\mathbb{E}\left[\Theta_i d_\phi(\Theta)\right] + \frac{1}{2}\mathbb{E}\left[U_i^+ + U_i^-\right] \\ &= \frac{1}{2}\pi_i(\phi) + \text{constant}. \end{aligned}$$

Since this is a positive affine transformation of $\pi_i(\phi)$, our model captures the general case where each group maximizes the group-wide payoff u_i. In particular, the group-wide ex post payoffs U_i^+ and U_i^- can be any functions of members' ex post payoffs, including heterogeneous preference intensities.

4 Asymptotic Results

4.1 Asymptotic Analysis

In this section, we provide asymptotic and normative analysis. More precisely, we focus on the situation in which: (i) the number of groups is sufficiently large, and (ii) preferences of the members are symmetrically distributed.

In order to study the case with a sufficiently large number of groups, let us consider a sequence of weights $(w_i)_{i=1}^\infty$, exogenously given as a fixed parameter.

Assumption 3. The sequence of weights $(w_i)_{i=1}^\infty$ satisfies the following properties.

(i) $w_1, w_2, \cdots$ are in a bounded interval $[\underline{w}, \bar{w}]$ for some $0 \leq \underline{w} < \bar{w}$.

(ii) Let G_n be the statistical distribution of weights in $(w_i)_{i=1}^n$, defined by $G_n(x) = \#\{i \leq n | w_i \leq x\}/n$ for each x. As $n \to \infty$, G_n weakly converges to a distribution G with support $[\underline{w}, \bar{w}]$.

Assumption 3 guarantees that for large n, the statistical distribution of the weights G_n is sufficiently close to a well-behaved distribution G, on which our asymptotic analysis is based.

The difficulty of analyzing weighted voting often arises from the discrete nature of the problem. Since the social decision d_ϕ is determined by the sum of the weights allocated to the alternatives across groups, computation of expected payoffs may require classification of a large number of success configurations, which increases exponentially as the number of groups increases, rendering the analysis prohibitively costly. The asymptotic technique developed here allows us to overcome this difficulty.

Additionally, we impose the following assumption for the sake of our normative analysis:

Assumption 4. The variables $(\Theta_i)_{i=1}^{\infty}$ are drawn independently from a common distribution F which is symmetric and absolutely continuous, and has a full support $[-1, 1]$.

Assumption 4 abstracts away from the specific structure of the biases and correlations in group preferences. Such an assumption allows our normative analysis to be independent of the distributional details, which is in line with Felsenthal and Machover (1998).

Following the symmetry of the preference distribution, our analysis also focuses on *symmetric* rule profiles, in which all groups use the same rule: $\phi_i = \phi$ for all i. With a slight abuse of notation, we write ϕ both for a *single rule* ϕ and for the *symmetric rule profile* $(\phi, \phi, \cdots)$, which should not create confusion as long as we refer to symmetric rule profiles. As for the alternatives, it is natural to consider that the label should not matter when the group-wide vote margin is translated into a weight allocation, given the symmetry of the preference distribution.

Assumption 5. We assume that the rule is neutral, i.e., ϕ is an odd function: $\phi(\theta_i) = -\phi(-\theta_i)$.

Let $\pi_i(\phi; n)$ denote the expected payoff for group $i (\leq n)$ under the profile ϕ when the set of groups is $\{1, \cdots, n\}$ and each group j's weight is w_j, the jth component of the weight sequence. The definition of $\pi_i(\phi; n)$ is the same as that of $\pi_i(\phi)$ in the preceding sections; the new notation simply clarifies its dependence on the number of groups n.

The main welfare criterion employed in this section is asymptotic Pareto dominance.

Definition 2. For two symmetric rule profiles ϕ and ψ, we say that ϕ *asymptotically Pareto dominates* ψ if there exists N such that for all $n > N$ and all $i = 1, \cdots, n$,

$$\pi_i(\phi; n) > \pi_i(\psi; n).$$

4.2 Pareto Dominance

The following is the main result in our asymptotic analysis.

Theorem 2. *Under Assumptions 3-5, the proportional profile asymptotically Pareto dominates all other symmetric rule profiles. In particular, it asymptotically Pareto dominates the dominant-strategy equilibrium of the game, i.e., the symmetric winner-take-all profile.*

Proof. The heart of the proof is in the correlation result shown in part (iii) of Lemma A in Appendix A3. It follows that if correlation of $\phi(\Theta)$ with Θ is higher than that of $\psi(\Theta)$, then for each group i, there exists N_i such that if the number of groups (n) is greater than N_i, group i ($\leq n$) will be better off under ϕ than ψ.

Note that the convergence in part (iii) of Lemma A is uniform in $w_i \in [\underline{w}, \bar{w}]$. This implies that the convergence is uniform in $i = 1, 2, \cdots$.[8] Thus there is N with the above property, without subscript i, which applies to all groups $i = 1, 2, \cdots$. Therefore, if correlation of $\phi(\Theta)$ with Θ is higher than that of $\psi(\Theta)$, then ϕ asymptotically Pareto dominates ψ.

Since the perfect correlation $\mathrm{Corr}[\Theta, \phi^{\mathrm{PR}}(\Theta)] = 1$ is attained by the proportional rule, Theorem 2 follows. □

[8] A more detailed explanation of this step is the following. By Lemma A (i), $\sqrt{2\pi n}\pi_i(\phi; n)$ asymptotically behaves as $2\sqrt{2\pi n}\int_0^1 \theta \mathbb{P}\{-w_i\phi(\theta) < \sum_{j\leq n} w_j\phi(\Theta_j) \leq w_i\phi(\theta)\}dF(\theta)$, where whether the sum $\sum_{j\leq n} w_j\phi(\Theta_j)$ includes the ith term or not is immaterial in the limit. The estimate of $\sqrt{2\pi n}\pi_i(\phi; n)$ therefore has the form $f_n(w_i)$, where $f_n(x) := 2\sqrt{2\pi n}\int_0^1 \theta \mathbb{P}\{-x\phi(\theta) < \sum_{j\leq n} w_j\phi(\Theta_j) \leq x\phi(\theta)\}dF(\theta)$. Lemma A (iii) implies that $f_n(x)$ converges uniformly in $x \in [\underline{w}, \bar{w}]$, which in turn implies that the convergence of $\sqrt{2\pi n}\pi_i(\phi; n) \approx f_n(w_i)$ is uniform in $i = 1, 2, \cdots$.

The above results show that while the winner-take-all rule is characterized by its strategic dominance, the proportional rule is characterized by its asymptotic Pareto dominance in a symmetric environment.

4.3 Congressional District Method

The analysis in the preceding subsection shows that the proportional profile is optimal in terms of Pareto efficiency. However, our model also implies that this profile produces an unequal distribution of welfare; in fact, this unequal nature pertains to all symmetric rule profiles. Lemma A (iii) shows that for these profiles, the payoff for a group is asymptotically proportional to its weight, providing high payoffs to groups with a large weight.

In this subsection, we examine whether such inequality can be alleviated by the congressional district profile ϕ^{CD}, introduced in Section 2.1. Recall that this profile is *asymmetric* across the groups since the ratio of weight allocated by the winner-take-all rule, c/w_i, is larger for groups with smaller weights. Therefore, we cannot apply Theorem 2 in order to obtain a Pareto dominance relationship. However, we can obtain a small-group advantage result (Theorem 3) and a Lorenz dominance result (Theorem 4). To ensure that the profile is well-defined, we impose that the lower bound of weights $\underline{w}$ is strictly positive and $c \in (0, \underline{w}]$.

Theorem 3. *Under Assumptions 3-5, let us consider the congressional district profile ϕ^{CD} with parameter $c \leq \underline{w}$. For any symmetric rule profile ϕ, there exists $w^* \in [\underline{w}, \bar{w}]$ with the following property: for any $\varepsilon > 0$, there is N such that for all $n > N$ and $i = 1, \cdots, n$,*

$$w_i < w^* - \varepsilon \Rightarrow \pi_i(\phi^{\mathrm{CD}}; n) > \pi_i(\phi; n),$$
$$w_i > w^* + \varepsilon \Rightarrow \pi_i(\phi^{\mathrm{CD}}; n) < \pi_i(\phi; n).$$

Proof. By Lemma A (iii), the payoff for group i under a symmetric rule profile ϕ

tends to a linear function of w_i. Let A^ϕ be the coefficient:

$$\begin{aligned}\lim_{n\to\infty}\sqrt{2\pi n}\pi_i(\phi;n) &= \frac{2w_i\mathbb{E}[\Theta\phi(\Theta)]}{\sqrt{\mathbb{E}[\phi(\Theta)^2]\int_{\underline{w}}^{\bar{w}}w^2dG(w)}} \\ &=: A^\phi w_i.\end{aligned}\tag{7}$$

We denote the congressional district rule for group i by $\phi^{\mathrm{CD}}(\theta_i, w_i)$, clarifying its dependence on the weight w_i. Remember the definition:

$$\begin{aligned}w_j\phi^{\mathrm{CD}}(\theta_j,w_j) &= c\phi^{\mathrm{WTA}}(\theta_j)+(w_j-c)\,\phi^{\mathrm{PR}}(\theta_j) \\ &= c\,\mathrm{sgn}(\theta_j)+(w_j-c)\,\theta_j.\end{aligned}$$

We claim that the limit function is affine in w_i:

$$\lim_{n\to\infty}\sqrt{2\pi n}\pi_i(\phi^{\mathrm{CD}};n) = Bw_i + C.\tag{8}$$

To see that, let us apply Lemma A (ii):

$$\begin{aligned}\lim_{n\to\infty}\sqrt{2\pi n}\pi_i(\phi^{\mathrm{CD}};n) &= 2\cdot\frac{w_i\mathbb{E}\left[\Theta\phi^{\mathrm{CD}}(\Theta,w_i)\right]}{\sqrt{\int_{\underline{w}}^{\bar{w}}w^2\mathbb{E}\left[\phi^{\mathrm{CD}}(\Theta,w)^2\right]dG(w)}} \\ &= 2\cdot\frac{c\mathbb{E}\left[|\Theta|\right]+(w_i-c)\,\mathbb{E}\left[\Theta^2\right]}{\sqrt{\int_{\underline{w}}^{\bar{w}}w^2\mathbb{E}\left[\phi^{\mathrm{CD}}(\Theta,w)^2\right]dG(w)}}.\end{aligned}$$

Since $|\theta| \geq \theta^2$ with a strict inequality for $0 < |\theta| < 1$, the full support condition for Θ implies $\mathbb{E}\left[|\Theta|\right] > \mathbb{E}\left[\Theta^2\right]$, and thus the intercept C is positive. The coefficient of w_i is:

$$B = \frac{2\mathbb{E}\left[\Theta^2\right]}{\sqrt{\int_{\underline{w}}^{\bar{w}}w^2\mathbb{E}\left[\phi^{\mathrm{CD}}(\Theta,w)^2\right]dG(w)}}.$$

If $A^\phi < B$, combined with $C > 0$, the right-hand side of (8) is above that of (7). Then, set $w^* = \bar{w}$. If $A^\phi > B$, again combined with $C > 0$, the two limit functions (7) and (8) intersect only once at a positive value $\hat{w}$. Let $w^* = \max\left\{\underline{w}, \min\{\hat{w}, \bar{w}\}\right\}$.

Since the convergences (7) and (8) are uniform in w_i, for any $\varepsilon > 0$ there is N with the property stated in Theorem 3. □

Theorem 3 shows that the congressional district profile makes the members of groups with small weights better off, compared with *any* symmetric rule profile. If the weight is an increasing function of the group size, it means that the congressional district profile is favorable for the members of small groups.[9]

The intuitive reason why the congressional district profile is advantageous for small groups is as follows. Under this profile, the ratio of weights cast by the winner-take-all rule (i.e., c/w_i) is higher for small groups than for large groups. Therefore, the rules used by the smaller groups are relatively close to the dominant strategy, inducing a relative advantage for the small groups.

In addition to Theorem 3, we can also show that the congressional district profile distributes payoffs more equally than any symmetric rule profile does, in the sense of Lorenz dominance. A profile of group payoffs, $\pi = (\pi_1, \cdots, \pi_n)$, is said to *Lorenz dominate* another profile, $\pi' = (\pi'_1, \cdots, \pi'_n)$, if the share of payoffs acquired by any bottom fraction of groups is larger in π than in π'.

Lorenz dominance, whenever it occurs, agrees with equality comparisons by various inequality indices including the coefficient of variation, the Gini coefficient, the Atkinson index, and the Theil index (see Fields and Fei (1978) and Atkinson (1970)). To see why the congressional district profile is more equal than any symmetric rule profile, recall equations (7) and (8) in the proof of Theorem 3, which assert that when the number of groups is large, the payoff for group i is approximately $A^\phi w_i$ for the symmetric profile, and it is approximately $Bw_i + C$ for the congressional district profile. The constant term $C > 0$ for the congressional district profile assures equal additions to all groups' payoffs, which results in a more equal distribution than when there is no such term. More precisely, we can prove the following statement. The proof is relegated to the Appendix A4.

Theorem 4. *Under Assumptions 3-5, let us consider the payoff profile under the congressional district profile:* $\pi\left(\phi^{\mathrm{CD}}; n\right) = \left(\pi_i\left(\phi^{\mathrm{CD}}; n\right)\right)_{i=1}^n$. *Let* ϕ *be any symmetric*

[9] As a special case, we cannot rule out the possibility where w^* is equal to $\underline{w}$ so that ϕ^{CD} is Pareto dominated by ϕ. However, this can only happen when A^ϕ is greater than B, which implies that the ratio $\pi_i(\phi^{\mathrm{CD}})/\pi_i(\phi)$ is decreasing with respect to w_i (see (7) and (8)). Thus, even in such a case, the congressional district profile favors groups with small weights in terms of relative comparison of payoffs.

rule profile and $\pi(\phi; n) = (\pi_i(\phi; n))_{i=1}^n$ *the payoff profile under* ϕ. *For sufficiently large* n, $\pi\left(\phi^{\mathrm{CD}}; n\right)$ *Lorenz dominates* $\pi(\phi; n)$.

To sum up, under the asymptotic and symmetric assumptions, we show that the proportional profile asymptotically Pareto dominates all other symmetric rule profiles (Theorem 2), while the congressional district profile is advantageous to small-weight groups (Theorem 3) and reduce inequality (Theorem 4). Our asymptotic model highlights the efficiency-equity trade-off between the two profiles. A Monte Carlo simulation using the parameters from the US Presidential Election is provided in the working paper version in order to verify the relevance of our asymptotic analysis.

Appendix

A1 Proof of Lemma 1

Preliminaries. Let Φ be the set of all rule profiles $\phi = (\phi_i)_{i=1}^n$. Given any two rule profiles $\phi, \phi' \in \Phi$ and a number $\alpha \in [0, 1]$, we define the randomization $\alpha * \phi + (1 - \alpha) * \phi'$ to be the random choice of ϕ with probability α and ϕ' with probability $1-\alpha$. More precisely, for any θ, the social decision $d_{\alpha*\phi+(1-\alpha)*\phi'}(\theta)$ is the $\{-1, +1\}$-valued random variable that equals $d_\phi(\theta)$ with probability α and $d_{\phi'}(\theta)$ with probability $1 - \alpha$. Any mixture of rule profiles obtained in this way is called a *random rule profile*, and denoted generically as $\tilde{\phi}$. Let $\tilde{\Phi}$ be the set of all random rule profiles.

For any subset $A \subset \tilde{\Phi}$ of random rule profiles, let $\pi(A) := \{(\pi_i(\tilde{\phi}))_{i=1}^n | \tilde{\phi} \in A\}$ be the set of payoff profiles induced by random rule profiles in A. Then $\pi(\tilde{\Phi})$ is the convex hull of $\pi(\Phi)$.

For any subset $U \subset \pi(\tilde{\Phi})$ of payoff profiles (i.e., vectors of n real numbers), let $\mathrm{Pa}(U)$ be the Pareto frontier of U.

We divide the proof of the lemma into the following two claims.

Claim A1.1. *Let $q \in \mathbb{R}^n_+ \setminus \{\mathbf{0}\}$, and let ϕ^* be the generalized proportional profile with coefficients $\lambda^* := (q_i/w_i)_{i=1}^n$. Consider the following maximization problem:*

$$\max_{u \in \pi(\tilde{\Phi})} \sum_{i=1}^{n} q_i u_i. \tag{9}$$

The following statements hold:

(i) The unique solution to (9) is the payoff profile $u^* := (\pi_i(\phi^*))_{i=1}^n$.

(ii) A random rule profile $\tilde{\phi} \in \tilde{\Phi}$ satisfies $\left(\pi_i(\tilde{\phi})\right)_{i=1}^n = u^*$ if and only if $\tilde{\phi}$ is equivalent to ϕ^* (i.e., they induce the same social decision almost surely).

Proof of Claim A1.1. Let $\tilde{\phi}$ be any random rule profile. Then

$$\sum_{i=1}^{n} q_i \pi_i(\tilde{\phi}) = \sum_{i=1}^{n} q_i \mathbb{E}\left[\Theta_i d_{\tilde{\phi}}(\Theta)\right] = \mathbb{E}\left[d_{\tilde{\phi}}(\Theta) \sum_{i=1}^{n} q_i \Theta_i\right]. \tag{10}$$

Since Θ is absolutely continuous, and so $\sum_{i=1}^n q_i\Theta_i \neq 0$ almost surely, the $\{-1, +1\}$-valued variable $d_{\tilde{\phi}}(\Theta)$ maximizes (10) if and only if $d_{\tilde{\phi}}(\Theta) = \operatorname{sgn} \sum_{i=1}^n q_i\Theta_i$ almost surely. That is,

$$\tilde{\phi} \text{ maximizes (10)} \Leftrightarrow \tilde{\phi} \text{ is equivalent to } \phi^*. \tag{11}$$

This implies statement (i) in the claim. Result (11) also implies that if $\tilde{\phi}$ is not equivalent to ϕ^*, then $\pi_i(\tilde{\phi}) \neq \pi_i(\phi^*)$ for some i, which proves the "only if" part of statement (ii) in the claim. The "if" part is trivial. □

Claim A1.2. *A payoff profile $u \in \pi(\tilde{\Phi})$ induced by a random rule profile is in the Pareto frontier $\mathrm{Pa}\,(\pi(\tilde{\Phi}))$ if and only if there exists $\lambda \in \mathbb{R}^n_+ \setminus \{\mathbf{0}\}$ such that $u = (\pi_i(\phi))_{i=1}^n$, where ϕ is the generalized proportional profile with coefficients λ. Combined with part (ii) of Claim A1.A1.1, this completes the proof of Lemma 1.*

Proof of Claim A1.2. Since $\pi(\tilde{\Phi})$ is convex, we can apply Mas-Colell et al. (1995, Proposition 16.E.2) to show the "only if" part of the claim.

To show the "if" part, suppose on the contrary that there exists $\lambda \in \mathbb{R}^n_+ \setminus \{\mathbf{0}\}$ such that the payoff profile $u := (\pi_i(\phi))_{i=1}^n$ induced by the generalized proportional profile ϕ with coefficients λ does not belong to the Pareto set $\mathrm{Pa}\,(\pi(\tilde{\Phi}))$. Then there exists $\bar{u} \in \pi(\tilde{\Phi})$ such that $\bar{u} \neq u$ and $\bar{u}_i \geq u_i$ for all i. Letting $q_i := w_i \lambda_i$ for $i = 1, \cdots, n$, we have $\sum_{i=1}^n q_i \bar{u}_i \geq \sum_{i=1}^n q_i u_i$. However, part (i) of Claim A1.1 implies that u is the only solution to the problem (9), a contradiction. □

A2 Local Limit Theorem

We quote a version of the Local Limit Theorem (LLT) shown in Mineka and Silverman (1970, Theorem 1). We will use it in the proof of part (ii) of Lemma 1.

LLT. *(Mineka and Silverman, 1970) Let (X_i) be a sequence of independent random variables with mean 0 and variances $0 < \sigma_i^2 < \infty$. Write F_i for the distribution of X_i. Write also $S_n = \sum_{i=1}^n X_i$ and $s_n^2 = \sum_{i=1}^n \sigma_i^2$. Suppose the sequence (X_i) satisfies the following conditions:*

(α) There exists $\bar{x} > 0$ and $c > 0$ such that for all i,

$$\frac{1}{\sigma_i^2} \int_{|x|<\bar{x}} x^2 dF_i(x) > c.$$

(β) Define the set

$$A(t, \varepsilon) = \{x |\ |x| < \bar{x} \text{ and } |xt - \pi m| > \varepsilon \text{ for all integers } m \text{ with } |m| < \bar{x}\}.$$

Then, for some bounded sequence (a_i) such that $\inf_i \mathbb{P}\{|X_i - a_i| < \delta\} > 0$ for all $\delta > 0$, and for any $t \neq 0$, there exists $\varepsilon > 0$ such that

$$\frac{1}{\log s_n} \sum_{i=1}^n \mathbb{P}\{X_i - a_i \in A(t, \varepsilon)\} \to \infty.$$

(γ) (Lindeberg's condition.) For any $\varepsilon > 0$,

$$\frac{1}{s_n^2} \sum_{i=1}^n \int_{|x|/s_n > \varepsilon} x^2 dF_i(x) \to 0.$$

Under conditions (α)-(γ), if $s_n^2 \to \infty$, we have

$$\sqrt{2\pi s_n^2}\,\mathbb{P}\{S_n \in (a,b]\} \to b-a.^{10} \tag{12}$$

A3 Asymptotic Formula of Payoffs

Lemma A below shows an asymptotic formula of payoffs for a class of rule profiles such that the weight allocation rules have the following specific form of separability.

Assumption 6. Let $\phi = (\phi_i)_{i=1}^{\infty}$ be a rule profile. There exist functions h_1, h_2, h_3 such that

$$w_i\phi_i(\theta_i, w_i) = h_1(w_i)h_2(\theta_i) + h_3(w_i)\operatorname{sgn}\theta_i, \text{ for all } i$$

where (i) h_1 is bounded, (ii) h_2 is an odd function such that the support of the distribution of $h_2(\Theta_i)$ contains 0, and (iii) h_3 is continuous but not constant.[11]

It is straightforward to show that Assumption 6 is satisfied for any symmetric rule profile as well as the congressional district profile. For a symmetric rule profile ϕ, let $h_1(w_i) = w_i$, $h_2(\theta_i) = \phi(\theta_i) - r\operatorname{sgn}\theta_i$, and $h_3(w_i) = w_i r$ where $r > 0$ is any positive number in the support of the distribution of $\phi(\Theta)$.[12] For the congressional district profile ϕ^{CD}, let $h_1(w_i) = w_i - c$, $h_2(\theta_i) = \theta_i - \operatorname{sgn}\theta_i$, and $h_3(w_i) = w_i$.

Lemma A. *Under Assumptions 3-5, let ϕ be a rule profile which satisfies Assumption 6. Then the following statements hold.*

[10] The original conclusion of Theorem 1 in Mineka and Silverman (1970) is stated in terms of the open interval (a, b). Applying the theorem to $(a, b+c)$ and $(b, b+c)$ and then taking the difference gives the result for $(a, b]$. In addition, the original statement allows for cases where s_n^2 does not go to infinity, and also mentions uniform convergence. These considerations are not necessary for our purpose, so we omit them.

[11] Under this form, $\phi_i(\cdot,\cdot)$ is the same for all i so that we can omit subscript i whenever there is no confusion.

[12] This is possible since $\phi(\Theta)$ is symmetrically distributed, and since we exclude the trivial case in which $\phi(\Theta) = 0$ almost surely.

(i) For any n,

$$\pi_i(\phi; n)$$
$$= 2\int_0^1 \theta_i \mathbb{P}\left\{ -w_i\phi(\theta_i, w_i) < \sum_{j\le n,\, j\ne i} w_j\phi(\Theta_j, w_j) \le w_i\phi(\theta_i, w_i)\right\} dF(\theta_i).$$

(ii) As $n \to \infty$,

$$\sqrt{2\pi n}\,\pi_i(\phi; n) \to \frac{2w_i\mathbb{E}[\Theta\phi(\Theta, w_i)]}{\sqrt{\int_{\underline{w}}^{\bar{w}} w^2\mathbb{E}[\phi(\Theta, w)^2]dG(w)}},$$

uniformly in $w_i \in [\underline{w}, \bar{w}]$, where Θ is a random variable having the same distribution F as Θ_i.

(iii) If ϕ is a symmetric rule profile, then, as $n \to \infty$,

$$\sqrt{2\pi n}\,\pi_i(\phi; n) \to 2w_i\sqrt{\frac{\mathbb{E}[\Theta^2]}{\int_{\underline{w}}^{\bar{w}} w^2 dG(w)}}\,\mathrm{Corr}\,[\Theta, \phi(\Theta)],$$ [13]

uniformly in $w_i \in [\underline{w}, \bar{w}]$, where Θ is a random variable having the same distribution F as Θ_i. The limit depends on the rule profile ϕ only through the factor $\mathrm{Corr}[\Theta, \phi(\Theta)]$.

Proof of Lemma A (i)

We prove the statement for group 1. Let $\pi_1(\phi; n|\theta_1)$ be the interim payoff for group 1 given that the group-wide margin is $\Theta_1 = \theta_1$. As in (3), we have:

$$\pi_1(\phi; n|\theta_1) = \theta_1(\mathbb{P}\{w_1\phi(\theta_1, w_1) + S_{\phi_{-1}} > 0\} - \mathbb{P}\{w_1\phi(\theta_1, w_1) + S_{\phi_{-1}} < 0\}),$$

where $S_{\phi_{-1}} := \sum_{j\ne 1} w_j\phi(\Theta_j, w_j)$. The probabilities on the right-hand side are unconditional on θ_1, since Θ_i's are independent. Since $S_{\phi_{-1}}$ is symmetrically distributed, the second probability can be written as $\mathbb{P}\{-w_1\phi(\theta_1, w_1) + S_{\phi_{-1}} > 0\}$.

[13]Since Θ and $\phi(\Theta)$ are symmetrically distributed, the correlation is given by $\mathrm{Corr}\,[\Theta, \phi(\Theta)] = \mathbb{E}[\Theta\phi(\Theta)]/\sqrt{\mathbb{E}[\Theta^2]\mathbb{E}[\phi(\Theta)^2]}$ unless $\phi(\Theta)$ is almost surely zero. If $\phi(\Theta)$ is almost surely zero, then the correlation is zero.

Thus, for $\theta_1 \in [0,1]$, the above expression equals

$$\pi_1(\phi; n|\theta_1) = \theta_1 \mathbb{P}\{-w_1\phi(\theta_1, w_1) < S_{\phi_{-1}} \leq w_1\phi(\theta_1, w_1)\}.$$

By symmetry, twice the integral of this expression over $\theta_1 \in [0,1]$ (instead of $[-1,1]$) equals the unconditional expected payoff $\pi_1(\phi; n)$, which proves Lemma A (i). □

Proof of Lemma A (ii)

Preliminaries. We prove the statement for group 1. The proof uses the notation of the Local Limit Theorem (LLT). Let

$$X_i := w_i\phi(\Theta_i, w_i),\ i = 1, 2, \cdots,$$

and $S_n := \sum_{i=1}^n X_i$. Then X_i has mean 0 and variance $\sigma_i^2 := w_i^2\mathbb{E}[\phi(\Theta, w_i)^2]$, and so the partial sum of variances is $s_n^2 := \sum_{i=1}^n w_i^2\mathbb{E}[\phi(\Theta, w_i)^2]$, where Θ represents a random variable that has the same distribution F as Θ_i.

Define the event

$$\Omega_n(\theta_1, w_1) = \left\{-w_1\phi(\theta_1, w_1) < \sum_{i=2}^n X_i \leq w_1\phi(\theta_1, w_1)\right\}.$$

We divide the proof into four claims. Claims A3.1-A3.3 show that the sequence (X_i) defined above satisfies the conditions of LLT. Claim A3.4 applies LLT to complete the proof of Lemma A (ii).

Claim A3.1. $s_n^2/n \to \int_{\underline{w}}^{\bar{w}} w^2\mathbb{E}[\phi(\Theta, w)^2]dG(w)$.

Proof of Claim A3.1. This holds since sequence (σ_i^2) is bounded and the statistical distribution G_n induced by $(w_i)_{i=1}^n$ converges weakly to G. □

Claim A3.2. *Conditions (α) and (γ) in LLT hold.*

Proof of Claim A3.2. This immediately follows from the fact that sequence (X_i) is bounded and $s_n^2 \to \infty$. In particular, it is enough to define $\bar{x}$ to be any finite number greater than $\bar{w}$. □

Claim A3.3. *Condition (β) in LLT holds.*

Proof of Claim A3.3. Recall that ϕ has the form

$$w_i\phi(\theta_i, w_i) = h_1(w_i)h_2(\theta_i) + h_3(w_i)\operatorname{sgn}\theta_i.$$

Let $a_i = h_3(w_i)$. We first check that the sequence (a_i) satisfies the requirements in condition (β). First, (a_i) is bounded since h_3 is bounded. Now, for any i and any $\delta > 0$,

$$\begin{aligned}\mathbb{P}\{|X_i - a_i| < \delta\} &\geq \mathbb{P}\{|X_i - a_i| < \delta \text{ and } \Theta_i > 0\} \\ &\geq \mathbb{P}\left\{|w_i\phi(\Theta_i, w_i) - h_3(w_i)\operatorname{sgn}\Theta_i| < \delta \text{ and } \Theta_i > 0\right\} \\ &= \mathbb{P}\{|h_1(w_i)h_2(\Theta_i)| < \delta \text{ and } \Theta_i > 0\}.\end{aligned}$$

Letting $\bar{h}_1 > 0$ be an upper bound of $|h_1|$ and Θ a random variable distributed as Θ_i, the last expression has the following lower bound independent of i:

$$\mathbb{P}\{|h_2(\Theta)| < \delta/\bar{h}_1 \text{ and } \Theta > 0\} > 0,$$

which is positive by the assumptions on h_2 and on the distribution of Θ.

Next we check the limit condition in (β). Recall that $A(t, \varepsilon)$ is the union of intervals

$$\left(\frac{\pi m + \varepsilon}{|t|}, \frac{\pi(m+1) - \varepsilon}{|t|}\right), \ m = 0, \pm 1, \pm 2, \cdots,$$

restricted to $(-\bar{x}, \bar{x})$, where we can choose $\bar{x}$ to be any number greater than $\bar{w}$. To prove the limit condition in (β), it therefore suffices to verify that one such interval contains $X_i - a_i$ with probability bounded away from zero, for all groups i in some sufficiently large subset of groups. To do this, note that if $\Theta_i < 0$, then $X_i - a_i = h_1(w_i)h_2(\Theta_i) - 2h_3(w_i)$. The assumptions on h_2 and on the distribution of Θ imply that for any $\eta > 0$, there exists a set $O_\eta \subset [-1, 0]$ with $\mathbb{P}\{\Theta \in O_\eta\} > 0$ such that if $\Theta \in O_\eta$ then $|h_2(\Theta)| \leq \eta$. Therefore,

$$\Theta_i \in O_\eta \Rightarrow X_i - a_i \in T_{w_i,\eta},$$

where

$$T_{w_i,\eta} := [-2h_3(w_i) - \eta h_1(w_i),\ -2h_3(w_i) + \eta h_1(w_i)].$$

Since h_1 is bounded, we can make $T_{w_i,\eta}$ an arbitrarily small interval around $-2h_3(w_i)$ by letting $\eta > 0$ be sufficiently small. Moreover, since h_3 is continuous and not a constant, we can find a sufficiently small interval $[\underline{v}, \bar{v}] \subset [\underline{w}, \bar{w}]$ with $\underline{v} < \bar{v}$ such that if $w_i \in [\underline{v}, \bar{v}]$, then $-2h_3(w_i)$ is between, and bounded away from, $(\pi m)/|t|$ and $(\pi(m+1))/|t|$ for some integer m. Fix such an interval $[\underline{v}, \bar{v}]$ and define

$$I := \{i | w_i \in [\underline{v}, \bar{v}]\}.$$

Then, for sufficiently small $\eta > 0$ and $\varepsilon > 0$, we have $T_{w_i,\eta} \subset A(t,\varepsilon)$ for all $i \in I$. Fixing such $\eta > 0$ and $\varepsilon > 0$, it follows that

$$\Theta_i \in O_\eta \text{ and } i \in I \Rightarrow X_i - a_i \in A(t,\varepsilon).$$

This implies that

$$\mathbb{P}\{X_i - a_i \in A(t,\varepsilon)\} \geq \mathbb{P}\{\Theta \in O_\eta\} =: p > 0 \text{ for all } i \in I,$$

and hence

$$\frac{1}{\log s_n} \sum_{i=1}^{n} \mathbb{P}\{X_i - a_i \in A(t,\varepsilon)\} \geq \frac{n}{\log s_n} \cdot \frac{\sharp\{i \in I | i \leq n\}}{n} \cdot p.$$

As $n \to \infty$, the first factor on the right-hand side tends to ∞ since s_n has an asymptotic order of $\sqrt{n}$. The second factor tends to $G(\bar{v}) - G(\underline{v}) > 0$, which is positive since G has full support on $[\underline{w}, \bar{w}]$. Therefore the left-hand side tends to ∞. □

Claim A3.4. *As $n \to \infty$, uniformly in $w_1 \in [\underline{w}, \bar{w}]$,*

$$2\int_0^1 \theta_1 \sqrt{2\pi n}\,\mathbb{P}\{\Omega_n(\theta_1, w_1)\} dF(\theta_1) \to \frac{2w_1 \mathbb{E}[\Theta\phi(\Theta, w_1)]}{\sqrt{\int_{\underline{w}}^{\bar{w}} w^2 \mathbb{E}[\phi(\Theta, w)^2] dG(w)}}. \tag{13}$$

By Lemma A (i), the left-hand side of (13) is $\sqrt{2\pi n}\pi_i(\phi; n)$, and therefore Lemma

A (ii) holds.

Proof of Claim A3.4. By Claims A3.2 and A3.3, we may apply LLT to obtain

$$\sqrt{2\pi s_n^2}\mathbb{P}\{\Omega_n(\theta_1, w_1)\} \to 2w_1\phi(\theta_1, w_1).$$

By Claim A3.1, this means that

$$\sqrt{2\pi n}\theta_1\mathbb{P}\{\Omega_n(\theta_1, w_1)\} \to \frac{2w_1\theta_1\phi(\theta_1, w_1)}{\sqrt{\int_{\underline{w}}^{\bar{w}} w^2\mathbb{E}[\phi(\Theta, w)^2]dG(w)}}. \tag{14}$$

Letting $\theta_1 = 1$ maximizes the left-hand side of (14) with the maximum value $\sqrt{2\pi n}\mathbb{P}\{\Omega_n(1, w_1)\}$. This maximum value itself converges to a finite limit. Hence the expression $\sqrt{2\pi n}\theta_1\mathbb{P}\{\Omega_n(\theta_1, w_1)\}$ is uniformly bounded for all n and $\theta_1 \in [0, 1]$. By the Bounded Convergence Theorem,

$$2\int_0^1 \theta_1\sqrt{2\pi n}\mathbb{P}\{\Omega_n(\theta_1, w_1)\}dF(\theta_1) \to 2 \cdot \frac{2w_1\int_0^1 \theta_1\phi(\theta_1, w_1)dF(\theta_1)}{\sqrt{\int_{\underline{w}}^{\bar{w}} w^2\mathbb{E}[\phi(\Theta, w)^2]dG(w)}}.$$

Since F is symmetric and ϕ is odd, this limit is exactly the one in (13).

To check the uniform convergence, note that for each n, the integral on the left-hand side of (13) is non-decreasing in w_1, since event $\Omega_n(\theta_1, w_1)$ weakly expands as w_1 increases.[14] We have shown that this integral converges pointwise to a limit that is proportional to the factor $w_1\mathbb{E}[\Theta\phi(\Theta, w_1)]$, which is continuous in w_1.[15] Therefore, the convergence in (13) is uniform in $w_1 \in [\underline{w}, \bar{w}]$.[16] □

[14] Let $\theta_1 \in [0, 1]$. If ϕ is a symmetric rule profile, i.e., if $\phi(\theta_1, w_1) = \phi(\theta_1)$, then $w_1\phi(\theta_1)$ is non-decreasing in w_1. If $\phi = \phi^{\mathrm{CD}}$, then $w_1\phi^{\mathrm{CD}}(\theta_1, w_1) = c\,\mathrm{sgn}(\theta_1) + (w_1 - c)\theta_1$, which is non-decreasing in w_1 again. Thus event $\Omega_n(\theta_1, w_1)$ weakly expands as w_1 increases.

[15] If ϕ is a symmetric rule profile, this factor is linear in w_i. If $\phi = \phi^{\mathrm{CD}}$, the factor equals $c\mathbb{E}(|\Theta|) + (w_i - c)\mathbb{E}(\Theta^2)$, which is affine in w_i.

[16] It is known that if (f_n) is a sequence of non-decreasing functions on a fixed finite interval and f_n converges pointwise to a continuous function, then the convergence is uniform. See Buchanan and Hildebrandt (1908).

Proof of Lemma A (iii)

This follows immediately from Lemma A (ii), by noting that if ϕ is a symmetric rule profile, each group's rule can be written as $\phi(\theta_j, w_j) = \phi(\theta_j)$. $\square$

A4 Proof of Theorem 4

Clearly, Lorenz dominance is invariant under linear transformations of payoffs. Thus, it suffices to prove that for large enough n, the payoff profile $\sqrt{2\pi n}\pi(\phi^{\text{CD}}; n)$ Lorenz dominates the payoff profile $\sqrt{2\pi n}\pi(\phi; n)$. By equations (7) and (8) in the proof of Theorem 3, as $n \to \infty$ these amounts converge to $Bw_i + C$ and $A^\phi w_i$, respectively. A result by Moyes (1994, Proposition 2.3) implies that if f and g are continuous, nondecreasing, and positive-valued functions such that $f(w_i)/g(w_i)$ is decreasing in w_i, then the distribution of $f(w_i)$ Lorenz dominates that of $g(w_i)$. The ratio $(Bw_i + C)/(A^\phi w_i)$ is decreasing in w_i, and so the claimed Lorenz dominance holds in the limit as $n \to \infty$. Recalling that the convergences are uniform, the dominance holds for sufficiently large n. $\square$

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