A Theory of Fair Random Allocation Under Priorities*

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Abstract

In the allocation of indivisible objects under weak priorities, a common practice is to break the ties using a lottery and randomize over deterministic mechanisms. Such randomizations usually lead to unfairness and inefficiency ex-ante. We propose and study the concept of ex-ante fairness for random allocations, extending some key results in the one-sided and two-sided matching markets. It is shown that the set of ex-ante fair random allocations forms a complete and distributive lattice under first-order stochastic dominance relations, and the agent-optimal ex-ante fair mechanism includes both the deferred acceptance algorithm and the probabilistic serial mechanism as special cases. Instead of randomizing over deterministic mechanisms, our mechanism is constructed using the division method, a new general way of constructing random mechanisms from deterministic mechanisms. As additional applications, we demonstrate that several previous extensions of the probabilistic serial mechanism have their foundations in existing deterministic mechanisms.

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1 Introduction

We study "priority-augmented" allocation of indivisible objects without monetary transfers: while agents have ordinal preferences over heterogeneous objects, each object also has its own priority ranking over the agents. *School choice* (Abdulkadiroğlu and Sönmez, 2003) is one of the most popular applications in practice. The nature of such allocation is closely related to the classical two-sided matching (Gale and Shapley, 1962), and in our context stability is regarded as a key fairness consideration, which requires the differences in priorities be respected. In the case of strict preferences and strict priorities, the *deferred acceptance algorithm* (DA) from Gale and Shapley (1962) is often considered as the best mechanism, since it is agent-optimal stable and strategy-proof. In this study, we consider the more general case with weak priorities, where the appropriate axioms to be imposed and the optimal choice of mechanism are much less obvious. Ties in priorities are commonly observed in practice.[1] Moreover, this framework includes the classical *house allocation* problem (Hylland and Zeckhauser, 1979) as a special case, by treating all the agents as equally ranked by every object.

In the presence of ties in priorities, the additional fairness consideration regarding agents with equal priority requires the use of random allocations. The most common way of constructing a random mechanism, both in theory and in practice, is to randomize over deterministic mechanisms. In particular, we can first break the ties in priorities using a randomly selected ordering of agents, then apply a deterministic mechanism. Some familiar examples include random serial dictatorship in house allocation (Abdulkadiroğlu and Sönmez, 1998), top trading cycles mechanism with random ordering in house allocation with existing tenants (Abdulkadiroğlu and Sönmez, 1999, Sönmez and Ünver, 2005), and DA with random tie-breaking in school choice (Abdulkadiroğlu and Sönmez, 2003). It is well-known that random mechanisms constructed in this way can preserve the strategy-proofness of the deterministic mechanisms, but the outcomes usually suffer from inefficiency and unfairness from the ex-ante perspective.

[1] For instance, in a school choice program students are often prioritized based on only a few criteria (e.g., the walk zone and sibling criteria), and hence many students may have the same priority at a school.

In light of this issue, two notable studies design new random mechanisms that assign probability shares of objects to agents directly. In house allocation, Bogomolnaia and Moulin (2001) introduce the *probabilistic serial mechanism* (PS) that achieves (first-order) stochastic-dominance efficiency and stochastic-dominance envy-freeness. PS determines a random allocation through an "eating" or consumption process, where the agents simultaneously consume their best available objects at the unit rate during the unit time interval. In the general case with weak priorities, Kesten and Ünver (2015) propose a fairness concept from the ex-ante perspective, *strong ex-ante stability*, and construct the *fractional deferred acceptance mechanism*, which is agent-optimal strongly ex-ante stable.

We take a similar approach as Kesten and Ünver (2015), and first propose a new and normatively appealing fairness concept, *ex-ante fairness*, that is generally not satisfied by existing mechanisms (Section 3). It is defined as the combination of two separate axioms: *ex-ante stability* and *ordinal fairness*. Ex-ante stability is studied in Roth et al. (1993) in the context of two-sided matching, and introduced to priority-augmented allocation by Kesten and Ünver (2015). It requires that if agent i is ranked higher than agent j by object a, then j cannot receive a with a positive probability unless i receives an object weakly better than a with probability one. On the other hand, ordinal fairness requires that if i and j are ranked equally by a, and i has a positive probability of receiving it, then her probability of receiving an object weakly better than a cannot be larger than j's probability of receiving an object weakly better than a. This is an adaption of the original ordinal fairness concept defined by Hashimoto et al. (2014) in house allocation, where they use it as a key axiom to characterize PS, and conforms to the idea of *compensation*, a general principle of fairness and social justice (Moulin, 2004).

We then introduce the *deferred consumption mechanism* (DC) to show that an ex-ante fair random allocation always exists (Section 4). DC naturally combines features of DA and PS. In each step, agents propose to consume objects during certain time intervals, and objects tentatively accept some (portions of) proposals based on differences in priorities, as well as the principle of first-come first-serve for equally ranked agents. If an agent's proposal to consume some object during a time interval $[x, y]$ is rejected, then she proposes to consume her next best option during $[x, y]$ in the next step. This propose-and-reject procedure may not be finite, but we show the sequence of tentative assignments converges to the *agent-optimal ex-ante fair random allocation*, i.e., the unique ex-ante fair random allocation that stochastically dominates every other ex-ante fair random allocation for all agents. DC is reduced to DA in the special case of strict priorities, and to PS in the special case of house allocation.

We explore the structural properties of ex-ante fair random allocations further (Section 5). Most notably, the set of ex-ante fair random allocations is a complete and distributive lattice under the stochastic dominance relations of the agents, which extends Conway's lattice theorem of stable matchings (Knuth, 1976). We also establish a generalized and probabilistic version of the rural hospital theorem (McVitie and Wilson, 1970, Roth, 1984, 1986). In some cases, objects are not entirely passive and they have intrinsic preferences that are aligned with their priorities. In this context, agents and objects have opposite interests over ex-ante fair random allocations, and each ex-ante fair random allocation is stochastic-dominance efficient for the two sides of the market.

To better understand our results, we next introduce a new general method of constructing random mechanisms from deterministic mechanisms (Section 6). This method is referred to as *division*, and the basic idea is as follows. Given a positive integer q, we divide (the claim of) each agent into q parts, and each object into q parts as well. Then the *(finitely) divided problem* consists of these agent parts and object parts, where preferences and priorities are extended from the original problem. A deterministic allocation for the divided problem generates a random allocation for the original problem: an agent's probability of being assigned an object is the proportion of her parts that are assigned the object's parts. In general, random allocations generated in this way can have better efficiency and fairness properties from the ex-ante perspective, compared to the usual randomization method. Although the interpretation is different, the idea of the division method in the special case of house allocation first appears in Kesten (2009). We discuss his results in details in Section 7.1.

The finite division framework will be formally used in later applications. To give an alternative perspective on ex-ante fairness, we need to envision a *continuum divided problem*, where each agent and object is divided into a continuum of parts with measure one. Then, every ex-ante fair random allocation is generated by a stable deterministic allocation in the continuum divided problem. This connection with stability helps explain some of the key structural results on ex-ante fairness. Moreover, DC is essentially generated by applying DA to continuum divided problems. While DA is strategy-proof, DC is not, and in general ex-ante fairness is not compatible with strategy-proofness. Compared to the randomization method, one drawback of the division method is that it does not preserve the incentive compatibility of a deterministic mechanism.

In the end, we present additional applications of division (Section 7). In particular, several previous generalizations of PS can be obtained by applying well-known deterministic mechanisms to finitely divided problems. In house allocation under weak preferences, the *extended PS* solution by Katta and Sethuraman (2006) can be obtained from the *serial dictatorships* defined for weak preferences by Svensson (1994). In allocation with private endowments, the generalized PS by Yılmaz (2010) can be obtained from a variation of serial dictatorships that preserves stability,[2] and the generalized PS by Zhang (2017) can be obtained from the *top trading cycles mechanism* (Abdulkadiroğlu and Sönmez, 2003). Then, for house allocation with multi-unit demands, the two generalizations of PS by Kojima (2009) and Heo (2014) can be obtained from two different classes of serial dictatorships under multi-unit demands (Pápai, 2000, Bogolmonaia et al., 2014), respectively. Finally, in the general case with weak priorities, motivated by Harless (2018), we construct a new probabilistic version of *Boston mechanism* using the division method, which restores some key desirable features of the deterministic Boston mechanism under strict priorities.

We discuss closely related studies in details in Section 8, and conclude in Section 9. All the proofs are given in Appendix A.

2 Preliminaries

Let N be a non-empty and finite set of **agents** and A a non-empty and finite set of **objects**. Each agent $i \in N$ has a complete and transitive **preference relation** R_i on $A \cup \{i\}$, with P_i and I_i denoting its asymmetric and symmetric components, respectively. We mainly focus on strict preferences, and assume that R_i is antisymmetric except in the discussions of the division method introduced in Section 6 and an application in Section 7.1. A **preference profile** $R = (R_i)_{i\in N}$ is a list of individual preferences. On the other hand, each object $a \in A$ has a complete and transitive **priority ordering** $\succeq_a$ on N, with $\succ_a$ and $\sim_a$ denoting its asymmetric and symmetric components, respectively. A **priority structure** $\succeq = (\succeq_a)_{a\in A}$ is a profile of priority orderings. Then, a **priority-augmented**

[2] Yılmaz (2009) considers allocation problems with private endowments and weak preferences, and proposes a new solution that extends the mechanisms in Yılmaz (2010) and Katta and Sethuraman (2006). This solution is further extended by Athanassoglou and Sethuraman (2011) to allocation problems with fractional endowments.

allocation problem, or simply a **problem**, is summarized as $p = (N, A, R, \succeq)$.[3]

For a given problem $p = (N, A, R, \succeq)$, a **random allocation**, or simply an **allocation**, is denoted by a $|N| \times |A|$ matrix M such that $M_{ia} \geq 0$, $\sum_{b \in A} M_{ib} \leq 1$, and $\sum_{j \in N} M_{ja} \leq 1$ for all $i \in N$ and $a \in A$, where M_{ia} represents the probability that agent i is assigned object a. For each $i \in N$, let $M_i = (M_{ia})_{a \in A \cup \{i\}}$ denote the lottery obtained by i under the allocation M, where $M_{ii} = 1 - \sum_{a \in A} M_{ia}$ is the probability that i receives her outside option. Similarly, for each $a \in A$, let $M_a = (M_{ia})_{i \in N \cup \{a\}}$, where $M_{aa} = 1 - \sum_{i \in N} M_{ia}$ is the probability that a is unassigned. M is a **deterministic allocation** if $M_{ia} \in \{0, 1\}$ for all $i \in N$ and $a \in A$. For ease of exposition, we also use a one-to-one function $\mu : N \to A \cup N$, where $\mu(i) \in A \cup \{i\}$ for all $i \in N$, to denote a deterministic allocation. By an extension of the Birkhoff-von Neumann theorem (Birkhoff, 1946, von Neumann, 1953, Kojima and Manea, 2010), every random allocation can be represented as a lottery over deterministic allocations.

For $i \in N$ and $a \in A \cup \{i\}$, let $F(R_i, a, M) = \sum_{b \in A \cup \{i\} : b R_i a} M_{ib}$ denote the probability that agent i is assigned an option weakly better than a under the allocation M. Then let $F(P_i, a, M) = \sum_{b \in A \cup \{i\} : b P_i a} M_{ib}$. Similarly, for $a \in A$ and $i \in N$, let $F(\succeq_a, i, M) = \sum_{j \in N : j \succeq_a i} M_{ja}$ and $F(\succ_a, i, M) = \sum_{j \in N : j \succ_a i} M_{ja}$. An allocation M is **individually rational** if $F(R_i, i, M) = 1$ for all $i \in N$. It is **non-wasteful** if $M_{aa} > 0$ implies $F(R_i, a, M) = 1$ for all $i \in N$ and $a \in A$. Given any two allocations M and M', each agent i can compare the lotteries M_i and M'_i using the first-order stochastic dominance relation R_i^{sd}: $M_i R_i^{sd} M'_i$ if $F(R_i, a, M) \geq F(R_i, a, M')$ for all $a \in A \cup \{i\}$. Let $M R_N^{sd} M'$ if $M_i R_i^{sd} M'_i$ for all $i \in N$. M **Pareto dominates** M' if $M R_N^{sd} M'$ and we do not have $M' R_N^{sd} M$. Then, an allocation is stochastic dominance efficient, or **sd-efficient**, if it cannot be Pareto dominated by any allocation. We also define stochastic dominance relations for objects. For each $a \in A$, let $M_a \succeq_a^{sd} M'_a$ if $F(\succeq_a, i, M) \geq F(\succeq_a, i, M')$ for all $i \in N$. Then let $M \succeq_A^{sd} M'$ if $M_a \succeq_a^{sd} M'_a$ for all $a \in A$.

A deterministic allocation μ is **efficient** if it can not be Pareto dominated by any other deterministic allocation. It is **stable** if it is individually rational, non-wasteful, and there is *no justified-envy*, i.e., there do not exist $i, j \in N$ such that $\mu(j) P_i \mu(i)$ and $i \succ_{\mu(j)} j$. A stronger fairness notion ensures that it respects not only the differences but

[3]We have made two simplifying assumptions in the model. First, there is only one copy of each object, i.e., we focus on the *one-to-one* setting. Second, an object does not have an outside option. The main results in the paper can be easily extended to the general *many-to-one* setting where there are multiple copies of each object, and each object $a \in A$ has its priority ordering defined over $N \cup \{a\}$ such that $i \not\succ_a a$ for all $i \in N$.

also the indifferences in priorities: μ is **strongly stable** if it is individually rational, non-wasteful, and there do not exist $i, j \in N$ such that $\mu(j) P_i \mu(i)$ and $i \succeq_{\mu(j)} j$. Applying the *deferred acceptance algorithm* (DA) from Gale and Shapley (1962) after ties in priorities are broken in any way yields a stable deterministic allocation. However, a strongly stable deterministic allocation may not exist. This fact also suggests that in general random allocations are needed to restore fairness regarding equal priority.

A **random mechanism**, or simply a **mechanism**, is a function f that assigns an allocation $f(p)$ to each problem p. If $f(p)$ is deterministic for each p, then f is also called a **deterministic mechanism**. f is said to satisfy a certain property if $f(p)$ satisfies this property for all p. Finally, f is **strategy-proof** if, for each agent, truth-telling yields a lottery that first-order stochastically dominates the lottery obtained from reporting any other preferences: for any $p = (N, A, R, \succeq)$, $i \in N$ and $p' = \big(N, A, (R'_i, R_{-i}), \succeq\big)$, we have $f_i(p) R_i^{sd} f_i(p')$.

3 Ex-Ante Fairness

We propose an axiom of fairness for random allocations from the ex-ante perspective. Consider a problem $p = (N, A, R, \succeq)$. First, we want a random allocation to respect the differences in priorities, such that the assignment of the probability shares of each object always satisfies the demands of higher ranked agents first.

Definition 1. *A random allocation M is **ex-ante stable** if it is individually rational, non-wasteful, and there do not exist $i, j \in N$ and $a \in A$ such that $i \succ_a j, M_{ja} > 0$, and $F(R_i, a, M) < 1$.*

This notion is discussed in Roth et al. (1993) in the context of two-sided matching with strict preferences on both sides of the market, and is introduced to priority-augmented allocation with weak priorities by Kesten and Ünver (2015). It is a direct generalization of the stability concept to random allocations, from the ex-ante perspective. In particular, the last requirement in the definition is a probabilistic version of the no justified-envy condition: if agent i has a higher priority than agent j at object a, then j cannot receive a positive probability share of a, unless i can receive an outcome weakly better than a for sure.

Second, we also want a random allocation to respect the indifferences in priorities. One fairness notion in this regard is *equal treatment of equals*, which requires that every two agents with the same preferences and the same priorities at all objects should be assigned the same lottery. While it is a more appropriate restriction in the special case of *house allocation*, where all the agents are ranked equally by each object, it is a weak requirement in general priority-augmented allocation, as two agents can differ not only in their preferences but also in their priorities. For instance, if any two agents have the same priority at every object except one, then, regardless of the preferences, equal treatment of equals is satisfied by every deterministic allocation, and fairness considerations regarding equal priorities are clearly ignored. Therefore, in light of the rich priority domain, we define fairness "locally", and impose restrictions on the allocation of each single object among the agents equally ranked by this object.

Definition 2. *A random allocation M is* ***ordinally fair*** *if for any $i, j \in N$ and $a \in A$ with $i \sim_a j$, $M_{ia} > 0$ implies $F(R_j, a, M) \geq F(R_i, a, M)$.*

This concept is first introduced by Hashimoto et al. (2014) for house allocation. They show that, together with individual rationality and non-wastefulness, it characterizes the *probabilistic serial mechanism* (PS) from Bogomolnaia and Moulin (2001). We extend it to the setting with priorities. If $i \sim_a j$, $M_{ia} > 0$ and $F(R_j, a, M) < F(R_i, a, M)$, some probability shares of a can be transferred from i to j to reduce the differences in their probabilities of receiving an object weakly better than a. Therefore, ordinal fairness essentially requires that the allocation of an object among agents in the same priority class should try to equalize their probabilities of receiving a weakly better object (or, equivalently, their chances of receiving a strictly worse outcome). Consequently, an agent with a smaller probability of receiving a strictly better object would generally be assigned a larger share of this object.

Ordinal fairness follows an important general principle of fairness and social justice, *compensation*, which says the allocation of resources should compensate for the differences in those primary individual characteristics, and equalize the shares of the higher-order characteristic or the more essential commodity (Moulin, 2004).[4] In applying this principle to our context, we consider the distribution of the probability shares of an object a among equally ranked agents as an independent resource allocation problem, and choose each agent's probability of receiving an object weakly better than a as the higher-order characteristic, which is a natural choice in our simplistic model of

[4]See Moulin (2004) for detailed and formal discussions regarding four principles of fairness: exogenous rights, compensation, reward, and fitness.

priority-augmented allocation. To achieve equality of the shares of such more essential commodity, we compensate for the agents' primary differences in preferences and priorities for other objects, as well as different supply and demand levels at those objects, which contribute to their different chances of receiving a weakly better object.

The two axioms defined above constitute the central concept in this paper.

Definition 3. *A random allocation M is **ex-ante fair** if it is ex-ante stable and ordinally fair.*

Existing mechanisms generally fail to be ex-ante fair. For example, a widely-used mechanism in school choice is *DA with single tie-breaking* (Abdulkadiroğlu et al., 2009): an ordering of the agents is picked from the uniform distribution to break the ties in priorities before DA is applied. Kesten and Ünver (2015) show that it does not satisfy the probabilistic version of the no justified-envy condition, and is thus not ex-ante stable. Below, we give a simple example in which it is not ordinally fair. We also present all the ex-ante fair allocations in this example.

Example 1. *Suppose that $N = \{1,2,3\}$ and $A = \{a,b,c\}$. The preferences, priorities, and the allocation M selected by DA with single tie-breaking are given as follows:*

R_1	R_2	R_3
a	b	a
b	a	c
c	c	b

$\succsim_a$	$\succsim_b$	$\succsim_c$
2, 3	3	2
1	1, 2	1
		3

$$M = \begin{array}{c|ccc} & a & b & c \\ \hline 1 & 0 & \frac{1}{2} & \frac{1}{2} \\ 2 & \frac{1}{6} & \frac{1}{2} & \frac{1}{3} \\ 3 & \frac{5}{6} & 0 & \frac{1}{6} \end{array}$$

M is ex-ante stable but not ordinally fair: we have $2 \sim_a 3$, $M_{3a} > 0$, and $F(R_2, a, M) = \frac{2}{3} < \frac{5}{6} = F(R_3, a, M)$. In this case, all the probability shares of a are allocated between these two agents, but agent 2 receives too few of a.

There exists a continuum of ex-ante fair allocations for this problem. For every $x \in [\frac{2}{3}, \frac{3}{4}]$, the allocation $M(x)$ below is ex-ante fair.

$$M(x) = \begin{array}{c|ccc} & a & b & c \\ \hline 1 & 0 & 2x-1 & 2-2x \\ 2 & 1-x & 2x-1 & 1-x \\ 3 & x & 3-4x & 3x-2 \end{array}$$

Note that $M(x) R_N^{sd} M(y)$ if $\frac{2}{3} \leq y \leq x \leq \frac{3}{4}$. The ex-ante fair allocation $M(x)$ becomes better for all agents as x increases, with the best one $M(\frac{3}{4})$ and the worst one $M(\frac{2}{3})$.

4 Deferred Consumption

We construct a mechanism to establish the existence of an ex-ante fair allocation. In fact, the mechanism always selects the *best* ex-ante fair allocation. Formally, given a problem $p = (N, A, R, \succeq)$, we say an allocation M is **agent-optimal ex-ante fair** if it is ex-ante fair, and $MR_N^{sd}M'$ for every ex-ante fair allocation M'.

In the special case of strict priorities, ex-ante fairness is reduced to ex-ante stability, which is further equivalent to stability for deterministic allocations. Since every ex-ante stable allocation can be represented as a lottery over some stable deterministic allocations, the outcome of DA Pareto dominates every other ex-ante stable allocation, i.e., it is agent-optimal ex-ante fair. On the other hand, in house allocation ex-ante fairness is reduced to the combination of individual rationality, non-wastefulness and ordinal fairness. Then, by Hashimoto et al. (2014), PS gives the unique ex-ante fair allocation. Therefore, the agent-optimal ex-ante fair mechanism must be reduced to DA and PS in these two special cases respectively.

In each step of DA, agents propose to objects, and every object tentatively accepts a proposer based on priorities. The actual acceptance is *deferred* to the last step. In PS, agents *consume* (or "eat") objects simultaneously at the unit rate during the unit time interval. Our mechanism is defined by a procedure of *deferred consumption* that combines the above features of DA and PS. In each step, agents propose to consume objects during certain time intervals, and every object tentatively accepts some (portions of) proposals based on priorities as well as the "first-come first-serve" principle. We first use the problem in Example 1 to illustrate this procedure.

Example 1 (continued). *In the first step, every agent proposes to consume her favorite object (at the unit rate) during the whole time interval* $[0, 1]$*. Agent* 2*'s proposal is tentatively accepted by object* b*. Given that both agent* 1 *and agent* 3 *want to consume object* a *and* $3 \succ_a 1$*, similar to DA, object* a *tentatively accepts* 3*'s proposal and rejects* 1*'s proposal.*

In the second step, 1 *proposes to consume her second choice,* b*, during* $[0, 1]$*. As* b *tentatively accepted* 2*'s proposal to consume during* $[0, 1]$ *in the previous step and* $1 \sim_b 2$*, similar to PS, its decision is based on the first-come first-serve principle. According to their proposals, the two agents come at the same time* $t = 0$*. If they consume from* $t = 0$*, the object is exhausted at* $t = \frac{1}{2}$*. Therefore,* b *tentatively accepts both agents' proposals to consume during* $[0, \frac{1}{2}]$*, and rejects their proposals to consume during* $[\frac{1}{2}, 1]$*.*

Since they each have a proposal rejected, in the third step, 1 *proposes to consume her* *next choice, object c, during the time interval $[\frac{1}{2}, 1]$, and 2 proposes to consume her next choice a during $[\frac{1}{2}, 1]$. Then 1's proposal is tentatively accepted. On the other hand, object a is now over-demanded: recall that it tentatively accepted 3's proposal to consume during $[0, 1]$ in the first step. Since $2 \sim_a 3$, this is resolved by the first-come first-serve principle as follows. If 2 and 3 consume a according to their proposals, then 2 starts to consume from $t = \frac{1}{2}$, while 3 starts to consume from $t = 0$. a is exhausted at $t = \frac{3}{4}$, and hence it tentatively accepts 2's proposal to consume during $[\frac{1}{2}, \frac{3}{4}]$, and 3's proposal to consume during $[0, \frac{3}{4}]$. In addition, it rejects their proposals to consume during $[\frac{3}{4}, 1]$.*

Finally, in the fourth step, both 2 and 3 propose to consume c during $[\frac{3}{4}, 1]$, which tentatively accepted 1's proposal to consume during $[\frac{1}{2}, 1]$ in the last step. Since the measures of these proposals from the three agents sum to 1, i.e., c can satisfy all the demands, these proposals are accepted, and the deferred consumption procedure terminates. We interpret the final consumption schedule of each agent as a lottery, which gives the agent-optimal ex-ante fair allocation $M(\frac{3}{4})$:

$$M(\tfrac{3}{4}) = \begin{array}{c|ccc} & a & b & c \\ \hline 1 & 0 & \frac{1}{2} & \frac{1}{2} \\ 2 & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ 3 & \frac{3}{4} & 0 & \frac{1}{4} \end{array}$$

As can be seen in this example, the outcome allocation is ex-ante stable since an object accepts proposals in favor of agents with higher priorities. On the other hand, ordinal fairness is guaranteed by the first-come first-serve principle.

We next give the formal definition of the mechanism. Consider any problem $p = (N, A, R, \succeq)$. For an object $a \in A$, a *proposal* from an agent $i \in N$ is represented by an interval $[x, y]$, where $0 \leq x < y \leq 1$, meaning that agent i proposes to consume object a during the time interval $[x, y]$. The object may face proposals from different agents at the same time, and its acceptance or rejection decision is based on its *choice rule* defined as follows.

Let $\{[x_i, y_i]\}_{i \in S}$ be a collection of proposals to consume a from some agents $S \subseteq N$, which is referred to as a *choice set*. If $\sum_{i \in S}(y_i - x_i) \leq 1$, then a can accommodate all these proposals and hence it chooses all of them. If $\sum_{i \in S}(y_i - x_i) > 1$, then a is over-demanded, and it chooses some (portions of) proposals such that their measures sum to one, through multiple consumption processes: the agents in the highest priority class (among S) consume a first according to their proposals; if a is not exhausted after their consumption, then the agents in the second highest priority class consume according to their proposals, and so on; after a is exhausted, for each $i \in S$, her proposal $[x_i, y'_i]$ is chosen if she has consumed a during $[x_i, y'_i]$ in the previous processes.

Therefore, more formally, object a chooses the proposals $\{[x_i, y'_i]\}_{i \in S'}$, where $S' \subseteq S$, that satisfy the following: $\sum_{i \in S'}(y'_i - x_i) = 1$, and there exists $i^* \in S'$ such that y'_{i^*} is the "cut-off", i.e., for any $j \in S$:

- If $j \succ_a i^*$, then $j \in S'$ and $y'_j = y_j$.
- If $j \sim_a i^*$ and $x_j < y'_{i^*}$, then $j \in S'$ and $y'_j = \min\{y_j, y'_{i^*}\}$.
- If $i^* \succ_a j$, or, $j \sim_a i^*$ and $x_j \geq y'_{i^*}$, then $j \notin S'$.

Using these choice rules of objects, we define the **deferred consumption procedure**:

> **Step 1.** Each agent proposes to consume her favorite option (which is an object or her outside option) from $t = 0$ to $t = 1$, i.e., during the whole time interval $[0, 1]$. Each object chooses from the proposals that it receives, tentatively accepts some proposals by its choice rule, and rejects the remaining proposals.
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> **Step $k \geq 2$.** For each agent, if she has a proposal $[x, y]$ rejected by some object a in the last step, then she proposes to consume her next best option to object a during $[x, y]$. Each object considers the proposals received in this step, as well as the proposals tentatively accepted earlier. Then, it tentatively accepts some proposals by its choice rule, and rejects the remaining proposals.
>
> The procedure terminates in some step k if no object rejects any proposal in this step.

In this procedure, it is possible that an agent first has a proposal $[x^2, x^3]$ tentatively accepted by an object a, then in a later step, due to being rejected by a better object, she proposes to consume a again during $[x^1, x^2]$. In this case we combine the proposals: let $[x^1, x^3]$ be the only proposal from this agent in the choice set of object a. We similarly combine proposals to outside options. Then we have the following useful fact that makes the procedure more transparent. It also indicates that the choice rule of each object can always be appropriately applied.

Observation 1. *Consider any step k of the deferred consumption procedure. After agents propose in this step, for each $i \in N$ there exist $x^1 < x^2 <, \dots, < x^\ell$, where $x^1 = 0$ and $x^\ell = 1$, such that:*

- *For any $1 \le \ell_1 \le \ell - 1$, agent i's proposal $[x^{\ell_1}, x^{\ell_1+1}]$ is in the choice set of some $a \in A$, or she has proposed to consume her outside option during $[x^{\ell_1}, x^{\ell_1+1}]$.*
- *If $1 \le \ell_1 < \ell_2 \le \ell - 1$, then her proposal $[x^{\ell_1}, x^{\ell_1+1}]$ is in the choice set of some $a \in A$, and either her proposal $[x^{\ell_2}, x^{\ell_2+1}]$ is in the choice set of some $b \in A$ with $aP_i b$, or she has proposed to consume her outside option during $[x^{\ell_2}, x^{\ell_2+1}]$.*

This partition result can be easily proved by induction. It is worth emphasizing that if an agent has a proposal rejected by an object a, then in the next step she proposes to consume the option next to a in her preference list. Then, in light of Observation 1, generally an agent may propose to multiple objects in a step, and she may also propose to an object that has rejected her before (see Example 2 below).

Given the numbers $x^1, \dots, x^\ell$ in Observation 1, for each $a \in A$, let $M^k_{ia} = x^{\ell_1+1} - x^{\ell_1}$ if there exists $1 \le \ell_1 \le \ell - 1$ such that agent i's proposal $[x^{\ell_1}, x^{\ell_1+1}]$ is in the choice set of a, and $M^k_{ia} = 0$ otherwise. We interpret the $|N| \times |A|$ matrix M^k as the *tentative assignment* after agents propose in step k, which is not necessarily a well-defined random allocation. Let $M^k_{ii} = 1 - \sum_{a \in A} M^k_{ia}$ for each $i \in N$. In general, as in Example 2, the deferred consumption procedure may not be finite, leading to an infinite sequence of tentative assignments $\{M^k\}_{k=1}^\infty$. For simplicity, if the procedure terminates in step $\bar{k}$, then we still construct an infinite sequence $\{M^k\}_{k=1}^\infty$ by setting $M^k = M^{\bar{k}}$ for all $k > \bar{k}$.

As an agent always proposes to consume her next best option after she has a proposal rejected, for all $i \in N$ and $a \in A$, $\left\{F(R_i, a, M^k)\right\}_{k=1}^\infty$ is a decreasing (and bounded) sequence, and hence it converges.[5] It follows that the sequence $\{M^k_{ia}\}_{k=1}^\infty$ also converges. Therefore, the outcome of the deferred consumption procedure for the current problem p is represented by a $|N| \times |A|$ matrix $f^{\mathrm{DC}}(p)$ such that

$$f^{\mathrm{DC}}_{ia}(p) = \lim_{k \to \infty} M^k_{ia}$$

for all $i \in N$ and $a \in A$.

[5]We abused the notation slightly, since $F(R_i, a, M)$ was only defined for an allocation M. But this will not cause any confusion.

Theorem 1. *For any problem p, $f^{DC}(p)$ is the agent-optimal ex-ante fair random allocation.*

We refer to f^{DC} as the **deferred consumption mechanism** (DC). Due to the specification of the objects' choice rules, the deferred consumption procedure eventually produces a consumption schedule in which, at any point of time, every agent does not envy the object being consumed by another agent, unless the latter agent has a higher priority for the object. In the special case of house allocation, such consumption schedule can be obtained by simply letting the agents simultaneously consume their best available objects. This gives an intuitive explanation on why DC is equivalent to PS in this case. On the other hand, it is obvious from the construction that DC is the same as DA in the special case of strict priorities.

Below we give another example to illustrate the mechanism, where the deferred consumption procedure is infinite.

Example 2. *Suppose that $N = \{1,2,3\}$ and $A = \{a,b,c\}$. The preferences and priorities are given as follows.*

R_1	R_2	R_3
a	b	a
b	a	b
c	c	c

$\succsim_a$	$\succsim_b$	$\succsim_c$
$2,3$	3	2
1	$1,2$	1
		3

In the first two steps, we have $M^1_{1a} = M^1_{2b} = M^1_{3a} = 1$, and $M^2_{1b} = M^2_{2b} = M^2_{3a} = 1$. Then, for every odd number $k \geq 3$, the tentative assignments M^k and M^{k+1} are given by the following matrices:

$$M^k = \begin{array}{c|ccc} & a & b & c \\ \hline 1 & 0 & \frac{1}{2}x_k & 1-\frac{1}{2}x_k \\ 2 & \frac{1}{2}x_k & \frac{1}{2}x_k & 1-x_k \\ 3 & x_k & 1-x_k & 0 \end{array} \qquad M^{k+1} = \begin{array}{c|ccc} & a & b & c \\ \hline 1 & 0 & \frac{1}{2}x_k & 1-\frac{1}{2}x_k \\ 2 & \frac{1}{2}-\frac{1}{4}x_k & \frac{1}{2}x_k & \frac{1}{2}-\frac{1}{4}x_k \\ 3 & \frac{1}{2}+\frac{1}{4}x_k & \frac{1}{2}-\frac{1}{4}x_k & 0 \end{array}$$

where $x_3 = 1$, and if $k \geq 5$,

$$x_k = 1 - \sum_{\ell=1}^{\frac{k-3}{2}} (\frac{1}{4})^{\ell}.$$

For any odd $k \geq 3$, object a is over-assigned in M^k (i.e., $\sum_{i\in N} M^k_{ia} > 1$), and in step k it rejects a proposal with some measure $y > 0$ from both agent 2 and agent 3. Then, 3 *proposes to consume object b in step $k+1$, which leads to both 1 and 2 having a proposal with measure $\frac{1}{2}y$ rejected by b. In step $k+2$, agent 2 proposes to consume object a again, which then rejects a proposal with measure $\frac{1}{4}y$ from both 2 and 3. The procedure continues in this way infinitely, and the tentative assignments converge to the following agent-optimal ex-ante fair allocation:*

	a	b	c
1	0	$\frac{1}{3}$	$\frac{2}{3}$
2	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$
3	$\frac{2}{3}$	$\frac{1}{3}$	0

The infiniteness of the procedure is due to infinite rejection cycles among agents. While each rejection cycle, once detected, can be resolved through a system of linear equations, the procedure is rather complex and a rejection cycle may appear an infinite number of times. The rejection cycles are further attributed to cyclic priority relations that involve ties. Given N and A, we say $\succeq$ is **acyclic** if there do not exist $n \geq 2$ distinct agents $i_1, \ldots, i_n$ and n distinct objects $a_1, \ldots, a_n$ such that $i_k \succeq_{a_k} i_{k+1}$ for each $k \in \{1, \ldots, n\}$, $i_\ell \succ_{a_\ell} i_{\ell+1}$ for some ℓ, and $i_m \sim_{a_m} i_{m+1}$ for some m, where $i_{n+1} = i_1$. Then it can be shown that acyclicity of priorities is sufficient for the deferred consumption procedure to be finite for any preference profile.

Finally, regarding incentive compatibility, DC is not a strategy-proof mechanism. For instance, comparing the agent-optimal ex-ante fair allocations in Examples 1 and 2, agent 3 can manipulate to receive her first choice with a larger probability.

In general, ex-ante fairness is not compatible with strategy-proofness. Recall that a mechanism is defined for every problem $p = (N, A, R, \succeq)$. Consider some N, A and $\succeq$ such that $|N| = |A| \geq 3$ and $i \sim_a j$ for all $i, j \in N$ and $a \in A$. Then for each R the unique ex-ante fair allocation is given by PS. Bogomolnaia and Moulin (2001) show that in such house allocation setting PS is manipulable, i.e., there exist R, $i \in N$ and R'_i such that we do not have $M_i R_i^{sd} M'_i$, where M and M' are the PS outcomes for $p = (N, A, R, \succeq)$ and $p' = \big(N, A, (R'_i, R_{-i}), \succeq\big)$ respectively. Therefore, we obtain the following impossibility.

Theorem 2. *There does not exist a strategy-proof and ex-ante fair mechanism.*

Positive results may arise if we consider a different setup: fix some N, A and $\succeq$, and define a mechanism as a function that maps every preference profile to an allocation. Then a strategy-proof and ex-ante fair mechanism exists when $\succeq$ is strict, as DA satisfies these properties. However, we cannot go much beyond strict priority structures. It can be shown that if $|A| \geq 3$ and there exists a strategy-proof and ex-ante fair mechanism, then there do not exist distinct $i, j, k \in N$ and $a \in A$ such that $i \sim_a j \sim_a k$ (there can only be two-way ties), and either (1) $\left|\left\{a \in A : \succeq_a \text{ is not antisymmetric}\right\}\right| \leq 1$ (at most one object has ties), or (2) $\left|\left\{i \in N : i \sim_a j \text{ for some } j \in N \setminus \{i\} \text{ and some } a \in A\right\}\right| \leq 3$ (at most three agents are involved in ties).

5 Properties of Ex-Ante Fair Allocations

We present additional properties of ex-ante fair allocations. In particular, it will be shown that, as suggested by Example 1, the (potentially infinite) set of ex-ante fair allocations is a complete and distributive lattice under the common preferences of the agents. We will also explicitly construct the least upper bound and the greatest lower bound of any subset of ex-ante fair allocations, which have natural interpretations.

Fix a problem $p = (N, A, R, \succeq)$ in this section. Denote the set of all ex-ante fair allocations as $\mathscr{E}$, and consider the partially ordered set $(\mathscr{E}, R_N^{sd})$. For any non-empty $S \subseteq \mathscr{E}$, we define two $|N| \times |A|$ matrices, $\vee S$ and $\wedge S$, as follows: for every $i \in N$ and $a \in A$,

$$(\vee S)_{ia} = \sup \{F(R_i, a, M) : M \in S\} - \sup \{F(P_i, a, M) : M \in S\}, \text{ and}$$

$$(\wedge S)_{ia} = \inf \{F(R_i, a, M) : M \in S\} - \inf \{F(P_i, a, M) : M \in S\}.$$

Moreover, let $(\vee S)_{ii} = 1 - \sum_{b \in A} (\vee S)_{ib}$ and $(\wedge S)_{ii} = 1 - \sum_{b \in A} (\wedge S)_{ib}$. When $S = \{M, M'\}$, we also write $\vee S$ as $M \vee M'$, and $\wedge S$ as $M \wedge M'$. The following result indicates that if $\vee S$ and $\wedge S$ are ex-ante fair allocations, then they are the least upper bound and the greatest lower bound of S under R_N^{sd}, respectively.

Lemma 1. *For any $i \in N$ and $a \in A \cup \{i\}$, we have $F(R_i, a, \vee S) = sup \{F(R_i, a, M) : M \in S\}$, and $F(R_i, a, \wedge S) = inf \{F(R_i, a, M) : M \in S\}$.*

The next theorem establishes the lattice structure as well as other key properties of ex-ante fair allocations.

Theorem 3. *If $\{M, M'\} \subseteq S \subseteq \mathscr{E}$, then we have the following:*

(i) (Complete lattice) $\vee S \in \mathscr{E}$ and $\wedge S \in \mathscr{E}$. Moreover, for any $i \in N$, either $M_i R_i^{sd} M'_i$ or $M'_i R_i^{sd} M_i$; for any $a \in A$, either $M_a \succeq_a^{sd} M'_a$ or $M'_a \succeq_a^{sd} M_a$.

(ii) *(Distributivity) If $M'' \in \mathscr{E}$, then $M'' \vee (M \wedge M') = (M'' \vee M) \wedge (M'' \vee M')$, and $M'' \wedge (M \vee M') = (M'' \wedge M) \vee (M'' \wedge M')$.*

(iii) *("Rural hospital" theorem) For any $i \in N$ and $a \in A$, $\sum_{b\in A} M_{ib} = \sum_{b\in A} M'_{ib}$, and $\sum_{j\in N} M_{ja} = \sum_{j\in N} M'_{ja}$. Moreover, $M_i = M'_i$ if $\sum_{b\in A} M_{ib} < 1$, and $M_a = M'_a$ if $\sum_{j\in N} M_{ja} < 1$.*

(iv) *(Two-sided efficiency) There does not exist a random allocation $M'' \neq M$ such that $M'' R_N^{sd} M$ and $M'' \succeq_A^{sd} M$.*

(v) *(Conflicting interests) $M R_N^{sd} M'$ if and only if $M' \succeq_A^{sd} M$.*

The above results generalize the familiar properties of stable matchings in the classical two-sided matching market, where both sides of the market have strict preferences, i.e., the *marriage problem* considered in Gale and Shapley (1962).

The first part of statement (i) and statement (ii) extend Conway's lattice theorem of stable matchings (Knuth, 1976). Although $\succeq_A^{sd}$ is generally not antisymmetric on all the random allocations due to the priorities being weak, statement (v) implies that it is a partial order on $\mathscr{E}$, and $(\mathscr{E}, \succeq_A^{sd})$ is also a complete and distributive lattice. The second part of statement (i) indicates that, when S is finite, $\vee S$ (resp. $\wedge S$) can be constructed by letting each agent pick the best (resp. the worst) lottery from the ones under the allocations in S. However, the least upper bound or the greatest lower bound of S under $\succeq_A^{sd}$ cannot be easily obtained by letting objects pick lotteries: for an object $a \in A$, $\succeq_a^{sd}$ may not be antisymmetric on the set $\{M_a : M \in \mathscr{E}\}$.[6]

Besides the lattice structure, the *rural hospital theorem* (McVitie and Wilson, 1970, Roth, 1984, 1986) is another fundamental property of stable matchings in the two-sided matching market. Statement (iii) shows a probabilistic version of this result for ex-ante fair allocations: each object's probability of being assigned to some agent is constant among $\mathscr{E}$, and if this probability is less than one, then the object receives the same lottery among $\mathscr{E}$. An analogous result also holds for each agent.

The last two statements have better economic interpretations if we envision that each object has intrinsic preferences that are aligned with its priority ordering. That is, each object always prefers an agent with a higher priority.[7] In this context, statement (iv) says that every ex-ante fair allocation is "sd-efficient" for the two sides of the market, i.e., when the welfare of every agent and object is taken into account, and statement

[6] For instance, see the lotteries received by object a under ex-ante fair allocations in Example 1.

[7] For example, in school choice, a school's priorities may be determined based on students' academic attainments, and hence largely reflect its preferences over students.

(v) shows that the agents and the objects have conflicting interests regarding ex-ante fair allocations.

6 Division

We introduce a new method of generating random allocations from deterministic allocations, which we refer to as *division*. This offers a new perspective on some of the key results in Theorem 3, as well as the construction of DC.

We start with the simplest example to illustrate the idea. There is only one object, a, to be allocated to two agents, i and j, who have equal claim to it, i.e., $i \sim_a j$. By any standard the only fair random allocation is $M_{ia} = M_{ja} = 0.5$. We take the stance that deterministic allocations are more fundamental than random allocations, and want to understand how the random allocation M can be constructed from deterministic allocations. The first interpretation is that M is generated by the method of *randomization*. That is, we randomize over the two deterministic allocations in which one agent receives a, such that each of the two is picked with probability 0.5.

We next give another possible interpretation. To resolve the conflicting claims of the two agents, we divide the claim of each $k \in \{i, j\}$ into two parts, k^1 and k^2. The object is also divided into two parts, a^1 and a^2, so that each k^x, where $k \in \{i, j\}$ and $x \in \{1, 2\}$, represents a claim to one part of the object. We then prioritize the divided claims: for both parts of the object, i^1 and j^1 have the same priority, i^2 and j^2 have the same priority, and each of i^1 and j^1 has a higher priority than each of i^2 and j^2. This leads to a *divided problem*, in which we essentially allocate two objects, $\{a^1, a^2\}$, to four agents, $\{i^1, i^2, j^1, j^2\}$, where each agent finds the two objects indifferent. This division operation treats the two original agents in the same way, and a fair deterministic allocation can be found in the divided problem due to the more refined priority structure: there exists a (strongly) stable deterministic allocation where i^1 and j^1 each receives one object, which generates the random allocation M for the original problem.

This idea of division can be easily extended to an arbitrary problem. In general, we divide (the claim of) each agent, as well as each object, into some finite number of parts. A part of an agent i has weak preferences over the objects parts, which are simply extended from the preferences of i. Moreover, we grant different priorities to different parts of each agent. Given the priorities in the original problem, the parts of a higher ranked agent are always ranked higher, and, for equally ranked agents, their parts are ranked based on indices.

We provide a formal framework that will also be used throughout Section 7. Consider a problem $p = (N, A, R, \succeq)$. For simplicity, assume that $|N| = |A|$, $aP_i i$ for all $a \in A$ and $i \in N$, and every allocation is a bistochastic matrix.[8] Given an integer $q > 0$, each agent $i \in N$ is divided into q parts, $i^1, \dots, i^q$. Let $N^q = \{i^x : i \in N, x = 1, \dots, q\}$. Each object $a \in A$ is also divided into q parts, $a^1, \dots, a^q$, and let $A^q = \{a^x : a \in A, x = 1, \dots, q\}$. Each $i^x \in N^q$ has a preference relation R_i^q on A^q such that for all $a^y, b^z \in A^q$, $a^y R_i^q b^z$ if and only if $aR_i b$. Then P_i^q and I_i^q denote the asymmetric and symmetric components of R_i^q, respectively. Each $a^x \in A^q$ has a priority ordering $\succeq_a^q$ over N^q such that for all $i^y, j^z \in N^q$, $i^y \succeq_a^q j^z$ if and only if either $i \succ_a j$, or, $i \sim_a j$ and $y \leq z$. Let $\succ_a^q$ and $\sim_a^q$ denote the asymmetric and symmetric components of $\succeq_a^q$, respectively. Then, $p^q = (N^q, A^q, R^q, \succeq^q)$ denotes the ***q*-divided problem** of p.

We assume the preferences are strict in the original problem p as before, although p^q is a problem with weak preferences. Alternatively, p^q can be interpreted as a many-to-one problem with strict preferences, as each part of an agent finds all parts of an object indifferent. A deterministic allocation μ for p^q generates a random allocation $M(\mu, p^q)$ for p: for all $i \in N$ and $a \in A$,

$$M_{ia}(\mu, p^q) = \frac{1}{q}\left|\left\{x \in \{1, \dots, q\} : \mu(i^x) = a^y \text{ for some } y \in \{1, \dots, q\}\right\}\right|.$$

Compared with the randomization method, the division method could deliver better efficiency and fairness properties from the ex-ante perspective. To start with, it is straightforward to see that if μ is efficient for p^q, then $M(\mu, p^q)$ is sd-efficient for p. In contrast, a randomization over efficient deterministic allocations gives a random allocation that usually only satisfies the weaker notion of *ex-post efficiency*. Formally, a random allocation is **ex-post efficient** (resp. **ex-post stable**) if it can be represented as a lottery over efficient (resp. stable) deterministic allocations. Regarding fairness, in general, we can choose a deterministic allocation μ that respects the more refined priority structure in the divided problem in some way, such that $M(\mu, p^q)$ satisfies some desirable fairness properties.[9] As one example, while a randomization over stable deterministic allocations only leads to an ex-post stable random allocation, $M(\mu, p^q)$ satisfies

[8]Under these assumptions, we no longer need to consider individual rationality or non-wastefulness. Moreover, for each $i \in N$, we only need to specify her preferences over A, instead of $A \cup \{i\}$.

[9]Depending on the application, we do not necessarily respect the priority structure in the sense of stability. See the random mechanisms discussed in Sections 7.2 and 7.3.

the stronger notion of ex-ante stability if μ is stable. More importantly for the current paper, it can also be easily shown that $M(\mu, p^q)$ is ex-ante fair if μ is strongly stable.

Therefore, strongly stable deterministic allocations can generate ex-ante fair random allocations through the division method. However, this approach to study ex-ante fairness has two limitations. First, it is not clear how to find a proper q such that a strongly stable deterministic allocation for p^q exists. Second, it is impossible to construct the whole set of ex-ante fair allocations by the division method, since there can be a continuum of ex-ante fair allocations. Motivated by these issues, we envision a *continuum divided problem*, where each agent and object $o \in N \cup A$ is divided into a continuum of parts with measure 1, represented by $\{o^x : x \in [0,1]\}$, and preferences and priorities are defined in the same way as in a finitely divided problem.

For every ex-ante fair allocation M, we can find a bijection $\mu : \{i^x : i \in N, x \in [0,1]\} \to \{a^x : a \in A, x \in [0,1]\}$ such that for any $i \in N$, $a \in A$ and $x \in [0,1]$, $\mu(i^x) \in \{a^y : y \in [0,1]\}$ if $F(P_i, a, M) < x \leq F(R_i, a, M)$, or, $F(P_i, a, M) = 0 = x < F(R_i, a, M)$. Then μ is a strongly stable deterministic allocation for the continuum divided problem, which generates M for the original problem. Note that the priorities in the continuum divided problem are almost strict, and hence strong stability is almost equivalent to stability. Therefore, such connection between ex-ante fairness and (strong) stability helps explain why the lattice theorem as well as the rural hospital theorem can be extended to ex-ante fair allocations.[10]

In addition, DC is essentially generated by applying DA to continuum divided problems: when agent i's parts $\{i^x : x \in [y,z]\}$ apply to the parts of object a, this is translated to i's proposal to consume a during the time interval $[y,z]$. However, instead of tracking the atomless parts of agents in a continuum divided problem, in Section 4 we focused on the original problem directly, and gave a simple and intuitive description of the mechanism.

While applying DA to continuum divided problems does not generate a strategy-proof mechanism, DA with single tie-breaking, a randomization over DA, is strategy-proof. In general, there is a clear trade-off between the randomization method and the division method. Although we can achieve better ex-ante properties of fairness and efficiency using the division method, applying a strategy-proof deterministic mechanism to

[10]While we can prove them by extending the proofs of the existing results in the two-sided matching market to the continuum divided problem, several other results in Theorem 3 cannot be fully explained by existing results of stable matchings, including the completeness of the lattice, "for any $i \in N$, either $M_i R_i^{sd} M_i'$ or $M_i' R_i^{sd} M_i$" in statement (i), statements (iv) and (v).

divided problems usually does not generate a strategy-proof random mechanism, which can also be seen in other examples in the next section. In contrast, a randomization over strategy-proof deterministic mechanisms is strategy-proof.

7 Additional Applications of Division

In the end, we present some additional applications of the division method, and mostly focus on allocation problems with simple and special priority structures. For these problems we show that several previous generalizations of PS can be generated by applying well-known deterministic mechanisms to finitely divided problems. For simplicity, in this section we assume that for any problem $p = (N, A, R, \succeq)$ under consideration, $|N| = |A|$, $aP_i i$ for all $a \in A$ and $i \in N$, and every allocation is a bistochastic matrix.

7.1 House Allocation Under Weak Preferences

In a house allocation problem under weak preferences, $p = (N, A, R, \succeq)$, R_i is not necessarily antisymmetric for any $i \in N$. Moreover, $i \sim_a j$ for all $i, j \in N$ and $a \in A$. Let $\mathscr{P}_{\text{HA}}$ denote the collection of all such problems.

Given any problem $p = (N, A, R, \succeq)$ under weak preferences, we say two allocations M and M' are **welfare equivalent** if $MR_N^{sd}M'$ and $M'R_N^{sd}M$. Moreover, for any $i \in N' \subseteq N$ and $A' \subseteq A$, let $B_i(A')$ denote the set of maximal elements in A' according to R_i, and $B_{N'}(A') = \cup_{j \in N'} B_j(A')$.

Katta and Sethuraman (2006) generalize PS to allow for weak preferences by introducing the sd-efficient **extended PS** (EPS) solution. The complete definition through parametric networks is rather involved, and we present a simplified description of the allocation procedure from Heo and Yılmaz (2015). Consider any $p = (N, A, R, \succeq) \in \mathscr{P}_{\text{HA}}$.

> Let $A_0 = A$, $E_0 = \emptyset$, and $d_0(i) = 0$ for all $i \in N$. In each step $k \geq 1$, let $A_k = A_{k-1} \setminus E_{k-1}$, and N_k be the largest set of agents that solves the following problem:
>
> $$\min_{N' \subseteq N, N' \neq \emptyset} \frac{|B_{N'}(A_k)| - \sum_{i \in N'} d_{k-1}(i)}{|N'|}$$
>
> Define
>
> $$\lambda_k = \frac{|B_{N_k}(A_k)| - \sum_{i \in N_k} d_{k-1}(i)}{|N_k|}$$

and $E_k = B_{N_k}(A_k)$. Then the objects E_k are allocated to N_k in step k: each $i \in N_k$ is assigned $\lambda_k + d_{k-1}(i)$ of the objects in $B_i(A_k)$. Set $d_k(i) = 0$ if $i \in N_k$, and $d_k(i) = d_{k-1}(i) + \lambda_k$ otherwise. The procedure terminates in step $\bar{k}$ if $E_{\bar{k}} = A_{\bar{k}}$.

In each step k, the "bottleneck set" N_k is identified. There may be multiple ways of allocating the objects E_k to the agents N_k, but all the outcome allocations are welfare equivalent. Denote the set of these allocations as $f^{\mathrm{EPS}}(p)$. Heo and Yılmaz (2015) show that an allocation M is ordinally fair if and only if $M \in f^{\mathrm{EPS}}(p)$.

Next, we relate EPS to the classical efficient deterministic mechanisms for $\mathscr{P}_{\mathrm{HA}}$: **serial dictatorships** from Svensson (1994). Given any problem $p = (N, A, R, \succeq)$ under weak preferences, choose an ordering σ of the agents, where $\sigma : \{1, \dots, |N|\} \to N$ is a bijection. Let D_0 be the set of all deterministic allocations for p. For $k \geq 1$, define D_k as the set of deterministic allocations most preferred by $\sigma(k)$ among the ones in D_{k-1}. That is,

$$D_k = \left\{\mu \in D_{k-1} : \mu(\sigma(k)) R_{\sigma(k)} \varphi(\sigma(k)) \text{ for all } \varphi \in D_{k-1}\right\}.$$

Then $f^{\mathrm{SD}}(\sigma, p) = D_{|N|}$ is the set of (welfare equivalent) deterministic allocations selected by the serial dictatorship with respect to σ.

Moreover, for any integer $q > 0$, let $\mathscr{O}(p, q)$ be the collection of orderings of N^q such that for any $\sigma \in \mathscr{O}(p, q)$ and $i^x, j^y \in N^q$ with $x < y$, $\sigma^{-1}(i^x) < \sigma^{-1}(j^y)$. We are ready to present the main result in this subsection.

Proposition 1. *Consider any $p = (N, A, R, \succeq) \in \mathscr{P}_{HA}$. Let $q = (n!)^n$, where $n = |N| = |A|$. Then for any $\sigma \in \mathscr{O}(p, q)$ and $\mu \in f^{SD}(\sigma, p^q)$, $M(\mu, p^q) \in f^{EPS}(p)$.*

Thus, applying serial dictatorships to the divided problems generates EPS. The ordering σ has to be chosen from the set $\mathscr{O}(p, q)$ to respect the differences in the priorities in p^q. After dividing each agent and object into the "correct" number of parts, i.e., $q = (n!)^n$, for every $x \in \{1, \dots, q\}$ the ordering among the agents in $\{i^x : i \in N\}$ does not affect the outcome of the serial dictatorship in terms of welfare, and none of them envies another's assignment. Therefore, each $\mu \in f^{\mathrm{SD}}(\sigma, p^q)$ is strongly stable for p^q.

A related result in Kesten (2009) embeds the idea of division. He considers a house allocation problem p under strict preferences, and replicates each object such that there are k copies of it. For each possible ordering of the agents, let the agents choose objects sequentially. They choose one object at a time so that the serial dictatorship is applied

k times. Then a random allocation is computed by taking the average of the $k \cdot (n!)$ possible serial dictatorship outcomes. He shows that this allocation converges to the PS outcome as $k \to \infty$. Technically, this replication method is essentially equivalent to division. In our context, (the proof of) his result implies that for any $\sigma \in \mathcal{O}(p,q)$, as $q \to \infty$, $M(f^{\text{SD}}(\sigma, p^q), p^q)$ converges to the PS outcome. While the focus of Kesten (2009) is on explaining the efficiency loss of random serial dictatorship, we emphasize the idea of generating random mechanisms through finite division in a more general setting with weak preferences.

7.2 House Allocation with Existing Tenants

In *house allocation with existing tenants* (Abdulkadiroğlu and Sönmez, 1999), some objects are private endowments while others are common endowments. This can be modeled via weak priorities such that for each privately owned object the owner has a higher priority than all the other agents. Formally, let $\mathscr{P}_{\text{HET}}$ denote the collection of such problems (under strict preferences). For each $p = (N, A, R, \succeq) \in \mathscr{P}_{\text{HET}}$, there exist non-empty $N(p) \subseteq N$ (the existing tenants), $A(p) \subseteq A$, and a bijection $e_p : N(p) \to A(p)$ such that

- for each $a \in A(p)$, $e_p^{-1}(a) \succ_a i \sim_a j$ for all $i, j \in N \setminus \{e_p^{-1}(a)\}$, and
- for each $a \in A \setminus A(p), i \sim_a j$ for all $i, j \in N$.

In this case, ex-ante stability is equivalent to the key requirement that each existing tenant's lottery first-order stochastically dominates the degenerate lottery of receiving her endowment, which ensures voluntary participation. Both Yılmaz (2010) and Zhang (2017) propose an ex-ante stable and sd-efficient mechanism by generalizing PS.

First, we refer to the mechanism of Yılmaz (2010) as **ex-ante stable PS**. Let $p = (N, A, R, \succeq) \in \mathscr{P}_{\text{HET}}$. In this mechanism, agents still consume objects at the unit rate, but to satisfy the ex-ante stability constraints, for any $N' \subseteq N(p)$, the agents N' are entitled to the objects $U_{N'} = \cup_{i \in N'} \{a \in A : aR_i e_p(i)\}$ in the following sense: if at some point of time, the total remaining fractions of these objects are equal to the total remaining demands of N', then any agent not in N' cannot further consume the objects $U_{N'}$.

One of the simplest stable and efficient deterministic solutions to the problem $p \in \mathscr{P}_{\text{HET}}$ is a **stable serial dictatorship.**[11] It is in the same vein as the serial dictatorships

[11] See Manjunath and Westkamp (2021) and Biró et al. (2022) for discussions of such mechanisms in the context of multi-unit exchange of indivisible objects.

defined in Section 7.1, and the only difference is that the procedure starts with all stable deterministic allocations for p, i.e., the initial set D_0 is the set of deterministic allocations in which no existing tenant receives an object worse than her endowment. Then, we extend this idea of stable serial dictatorships to a divided problem p^q as follows. Let $\sigma \in \mathscr{O}(p,q)$. Define the initial set D_0 as the set of deterministic allocations (for p^q) in which no existing tenant has a part that is assigned a part of an object worse than her endowment. Then, for $k \geq 1$, let D_k be the set of deterministic allocations most preferred by $\sigma(k)$ among the ones in D_{k-1}. If $q = (n!)^{n^2}$, where $n = |N| = |A|$, it can be shown that for any $\mu \in D_{nq}$, $M(\mu, p^q)$ is the outcome of the ex-ante stable PS.[12]

Second, we consider the **eating-trading algorithm** (ETA) (Zhang, 2017, Yu and Zhang, 2021). In each step, if some existing tenants demand each other's endowment and form a cycle, then they exchange probability shares of their endowments. If there is no such cycle, then agents consume their best available objects at possibly different rates: each agent has the same basic consuming rate of 1, and she gets an additional rate equal to the sum of the rates of those agents who are consuming her endowment.

We relate ETA to the original deterministic mechanisms proposed for $\mathscr{P}_{\text{HET}}$ in Abdulkadiroğlu and Sönmez (1999), the **top trading cycles** (TTC) mechanisms.[13] In fact, in order to apply to divided problems we need the more general form of the TTC mechanisms in Abdulkadiroğlu and Sönmez (2003), defined for any priority-augmented allocation problem $p = (N, A, R, \succeq)$. Given an ordering σ of the agents in p, the mechanism is defined through the following procedure.

> In each step, consider the remaining agents and the remaining objects. Let each agent point to her favorite object, and each object point to the agent with the highest priority (if there is a tie, break it using σ). Then there exists at least one cycle. Let each agent in a cycle be assigned the object that she points to. After removing the agents and objects in cycles, we continue to the next step. The procedure terminates when all the agents are assigned.

[12] We omit formal proofs of results in this subsection which use very similar techniques as the proof of Proposition 1.

[13] Zhang (2017) shows that ETA can be interpreted as a procedure in which agents trade their fractional endowments, where in each step the remaining fractions of an object that is not owned by any remaining agent are equally distributed as private fractional endowments. Following this study, Yu and Zhang (2021) define a more general parametric class of trading algorithms that incorporate ETA as a special case. We will also interpret ETA as a trading procedure, but from a different perspective. Our focus is on establishing an explicit connection between ETA and the deterministic TTC mechanisms through the method of division.

For TTC to generate ETA, we need to make two slight modifications to the divided problems. Consider $p = (N, A, R, \succeq) \in \mathscr{P}_{\mathrm{HET}}$ and an integer $q > 0$. We first break the ties in preferences: for each $i^x \in N^q$, define $\tilde{R}_i^q$ such that $a^y \tilde{R}_i^q b^z$ if and only if $a P_i b$, or, $a = b$ and $y \leq z$. Then TTC, as defined above, can be applied to $(N^q, A^q, \tilde{R}^q, \succeq^q)$. However, this may cause an existing tenant to lose her basic consuming rate. We give a more favorable treatment to the existing tenants, by reversing the priority ordering of the parts of each privately owned object over the parts of its owner: for each $a^x \in A^q$, define $\tilde{\succeq}_a^q$ such that for all $i^y, j^z \in N^q$, if $a \in A(p)$ and $i = j = e_p^{-1}(a)$, then $i^y \tilde{\succeq}_a^q j^z \iff y \geq z$; otherwise, $i^y \tilde{\succeq}_a^q j^z \iff i^y \succeq_a^q j^z$. Denote the modified q-divided problem as $\tilde{p}^q = (N^q, A^q, \tilde{R}^q, \tilde{\succeq}^q)$. Then, it can be shown that if we choose $q = (n!)^{n+|N(p)|}$, where $n = |N|$, and any ordering of N^q, applying the TTC mechanism to $\tilde{p}^q$ generates the ETA outcome for p.

7.3 A Probabilistic Version of Boston Mechanism

The **Boston mechanism** (Abdulkadiroğlu and Sönmez, 2003) is one of the most popular school choice mechanisms in practice. Under strict priorities, it selects a deterministic allocation through a procedure similar to DA, with the only difference being that in each step the acceptance is final. This mechanism is neither stable nor strategy-proof, and its popularity mainly comes from the property of respecting preference ranks (Kojima and Ünver, 2014), which also implies efficiency, in the sense that if one agent prefers the object received by another agent, then the latter agent must rank the object weakly higher in her preference list.

However, when priorities are weak and ties are broken randomly, the randomized Boston mechanism loses its attractive properties ex-ante. Harless (2018) defines the notion of **respect for rank** for a random allocation M in a problem $p = (N, A, R, \succeq)$, which requires that for any $i, j \in N$ and $a \in A$, if $F(R_i, a, M) < 1$ and $M_{ja} > 0$, then $|\{b \in A : b R_j a\}| \leq |\{b \in A : b R_i a\}|$. Respect for rank implies sd-efficiency, and he shows that the randomized Boston mechanism is not even sd-efficient. Then, instead of randomization, we can apply the Boston mechanism to continuum divided problems to generate a random mechanism similar to DC, with the only difference being that in each step the acceptance of proposals is final. This probabilistic version of the Boston mechanism satisfies respect for rank, and thus sd-efficiency.[14]

[14] In the special case of house allocation, Harless (2018) and Chen et al. (2023) propose and characterize another probabilistic version of the Boston mechanism that also satisfies respect for rank, but their mechanism is different from ours in this domain.

7.4 Multi-Unit Demands

Finally, in house allocation problems with multi-unit demands, there are two classes of serial dictatorships. In the first class, each agent i with a demand of d_i picks the best d_i objects available when it is her turn. Applying such a mechanism to (finitely) divided problems can lead to the generalization of PS by Heo (2014), in which each agent's consuming rate is equal to her demand. In the second class, each agent picks only one object when it is her turn, and hence there are multiple rounds of sequential assignments. Applying such a mechanism to divided problems can lead to the generalization of PS by Kojima (2009), in which each agent i with a demand of d_i consumes objects during the time interval $[0, d_i]$, at the unit rate.[15]

8 Related Studies

We discuss closely related studies in two aspects.

8.1 Random Mechanism Design

DC closely resembles the fractional deferred acceptance mechanism (FDA) of Kesten and Ünver (2015). In each step of FDA, fractions of agents propose to objects. Each object tentatively accepts the fractions of agents with higher priorities first, and for agents with equal priority, it tries to accept an equal fraction of each one. FDA is designed to select the best random allocation that satisfies strong ex-ante stability, which is the combination of ex-ante stability and *no ex-ante discrimination*. The latter axiom differs from ordinal fairness, and essentially requires that the allocation of an object among equally ranked agents should try to equalize their probabilities of receiving this object. Moreover, Kesten and Ünver (2015) also propose a second random mechanism by adding a probability trading stage to improve upon FDA. Its outcome is ex-ante stable, undominated within ex-ante stable allocations, and satisfies equal treatment of equals.

As far as we know, there are two other studies that also unify PS and DA. Afacan (2018) introduces a novel model with random priorities, where a probability distribution over all strict priority structures is given as a component of the allocation problem.

[15] For studies on the first class of serial dictatorships, see, for example, Pápai (2000, 2001), Klaus and Miyagawa (2001) and Ehlers and Klaus (2003). Bogolmonaia et al. (2014) discuss the second class.

He proposes the *constrained probabilistic serial mechanism* that satisfies *claimwise stability* and constrained sd-efficiency. This mechanism is reduced to DA when the priorities are deterministic, and to PS when every two strict priority structures are equally likely.

On the other hand, Aziz and Brandl (2022) extend the eating procedure in PS and introduce the *vigilant eating rule* (VER) that applies to the general many-to-many setting with weak preferences and almost arbitrary constraints. Whenever the set of feasible random allocations is non-empty and compact, it chooses a constrained sd-efficient allocation. In our priority-augmented allocation setting, when constraints are imposed by some stability concept (such as ex-post stability or ex-ante stability), VER is reduced to PS if everyone has the same priority for every object, and to DA if priorities are strict.

Furthermore, when constraints are imposed by ex-ante fairness, VER and DC give equivalent outcomes, although they are defined through very different procedures. While the VER algorithm takes the whole non-empty set of ex-ante fair allocations as inputs, DC shows the existence of an ex-ante fair allocation, as well as the uniqueness of constrained sd-efficient allocation in this case.

8.2 Structural Results on Stable Random Matchings

Roth et al. (1993) establish the lattice structure of *stable fractional matchings* in the classical two-sided market. As they assume strict preferences from both sides and fractional stability is weaker than ex-ante stability, our structural results on ex-ante fair allocations are independent of their study.

Alkan and Gale (2003) take a revealed preference approach to study *schedule matchings* between workers and firms. A schedule matching specifies the amount of time each worker spends with each firm, and can be interpreted as a random allocation when the total available amount of time is one for each worker and firm. Although their general model can potentially be used to handle weak orderings, our Theorem 3 is independent of their similar structural results on stable schedule matchings. There is no systematic way of constructing choice functions based on preferences and priorities, so that we can define an Alkan-Gale problem isomorphic to our problem and establish the equivalence of stable schedule matchings and ex-ante fair allocations.

Although their main focus is on (median) rationalizability of matchings, Echenique et al. (2021) show similar structural results in a two-sided *aggregate matching* model with finite sets of types of men and women. An aggregate matching is a matrix giving the mass of each type of men matched with each type of women. Therefore, when the mass of every type is one, their model corresponds to ours and an aggregate matching can be interpreted as a random allocation. As preferences from both sides are at the type level and strict, their stability notion is equivalent to ex-ante stability as well as ex-ante fairness. Then (i), (ii), the first part of (iii), and (v) in Theorem 3 generalize their lattice result and other structural properties established for stable aggregate matchings.

9 Conclusion

We introduced an appealing fairness concept for the allocation of discrete resources under weak priorities. The main focus of our study is on an ex-ante fair mechanism and the properties of ex-ante fair allocations. The division method underlies our analysis, and also provides a new perspective on various other random mechanisms. In the end, we briefly mention two plausible directions for future research. First, it is interesting to explore the performance of DC in large markets. Given the work of Che and Kojima (2010), it is reasonable to conjecture that DA with single tie-breaking and DC are asymptotically equivalent under some regularity conditions. Second, we hope the division method can be applied to other classes of allocation or matching problems to generate new random mechanisms with good ex-ante properties.

Appendix A

A.1 Proof of Theorem 1

Consider any problem $p = (N,A,R,\succeq)$. First, in the proof we will use the following crucial result regarding the deferred consumption procedure for p and the sequence of tentative assignments $\{M^k\}$, which can be easily shown using Observation 1.

Claim 1. *In any step k of the deferred consumption procedure, an agent $i \in N$ has a proposal $[x,y]$ rejected by an object $a \in A$ if and only if $F(R_i,a,M^k) = y$, $F(R_i,a,M^{k+1}) = x$, and $x < y$.*

Let $f^{\mathrm{DC}}(p) = M^*$. By construction, for all $i \in N$ and $a \in A$, $M^*_{ia} \geq 0$ and $\sum_{b\in A} M^*_{ib} \leq 1$. Suppose that some object $a \in A$ is over-assigned, i.e., $\sum_{i\in N} M^*_{ia} > 1$. Given that $\lim_{k\to\infty} \left\{\sum_{i\in N} M^k_{ia}\right\} = \sum_{i\in N} M^*_{ia} > 1$, there exist K and $\epsilon > 0$ such that for any $k >$

K, $\sum_{i \in N} M_{ia}^k > 1 + \epsilon$. This implies that for any $k > K$, the sum of the measures of the proposals rejected by a is larger than ϵ in step k of the (deferred consumption) procedure. Then, by Claim 1, $\sum_{i \in N} F(R_i, a, M^k) - \sum_{i \in N} F(R_i, a, M^{k+1}) > \epsilon$ for all $k > K$, which is clearly impossible. Therefore, M^* is a well-defined random allocation.

We next show M^* is ex-ante fair. It is individually rational as each agent never proposes to consume an object worse than her outside option. For any object $a \in A$ such that $\sum_{i \in N} M_{ia}^* < 1$, since $\lim_{k \to \infty} \left\{ \sum_{i \in N} M_{ia}^k \right\} < 1$, there exists K such that $\sum_{i \in N} M_{ia}^k < 1$ for all $k > K$. It follows that object a never rejects a proposal in the procedure. Then for any $i \in N$, given that $F(R_i, a, M^1) = 1$, Claim 1 implies that $F(R_i, a, M^k) = 1$ for all k. Hence, $F(R_i, a, M^*) = 1$, and M^* is non-wasteful.

Suppose that M^* is not ex-ante stable. Then there exist $i, j \in N$ and $a \in A$ such that $i \succ_a j$, $F(R_i, a, M^*) < 1$ and $M_{ja}^* > 0$. Since $F(R_i, a, M^1) = 1$ and $\lim_{k \to \infty} F(R_i, a, M^k) < 1$, by Claim 1, agent i has a proposal rejected by a in some step K. Then, by the choice rule of a, it does not tentatively accept any proposal from j in any step $k > K$. Since $M_{ja}^* > 0$, we have $\lim_{k \to \infty} F(P_j, a, M^k) < \lim_{k \to \infty} F(R_j, a, M^k) = F(R_j, a, M^*)$. It follows that for some step $k > K$ of the procedure, $F(P_j, a, M^k) < F(R_j, a, M^*)$. As the sequence $\left\{F(R_j, a, M^\ell)\right\}$ is decreasing, $F(P_j, a, M^k) < F(R_j, a, M^k)$. Therefore, j's proposal $\left[F(P_j, a, M^k), F(R_j, a, M^k)\right]$ is in the choice set of a in step k, and a rejects this proposal. Then, by Claim 1, $F(R_j, a, M^{k+1}) = F(P_j, a, M^k) < F(R_j, a, M^*)$, contradicting to the fact that the decreasing sequence $\left\{F(R_j, a, M^\ell)\right\}$ converges to $F(R_j, a, M^*)$.

Suppose that M^* is not ordinally fair. Then there exist $i, j \in N$ and $a \in A$ such that $i \sim_a j$, $M_{ia}^* > 0$ and $F(R_i, a, M^*) > F(R_j, a, M^*)$. Since $\lim_{k \to \infty} F(R_j, a, M^k) < F(R_i, a, M^*)$ and $F(R_j, a, M^1) = 1$, by Claim 1, in some step K, agent j has a proposal $\left[x, F(R_j, a, M^K)\right]$ rejected by a and $F(R_j, a, M^{K+1}) = x < F(R_i, a, M^*)$. By the choice rule of a, it does not tentatively accept any proposal $[y, z]$ from i such that $z > x$, in any step $k > K$. As in the above proof of ex-ante stability, $M_{ia}^* > 0$ implies that we can find some step $k > K$ of the procedure such that $F(P_i, a, M^k) < F(R_i, a, M^*) \leq F(R_i, a, M^k)$. Hence, i's proposal $\left[F(P_i, a, M^k), F(R_i, a, M^k)\right]$ is in the choice set of a in step k. Since $F(R_i, a, M^k) \geq F(R_i, a, M^*) > x$, i must have a proposal $\left[y, F(R_i, a, M^k)\right]$ rejected by a in this step, and $y \leq \max\left\{x, F(P_i, a, M^k)\right\}$. Then, by Claim 1,

$$F(R_i, a, M^{k+1}) = y \leq \max\left\{x, F(P_i, a, M^k)\right\} < F(R_i, a, M^*).$$

This leads to a contradiction as $\left\{F(R_i, a, M^\ell)\right\}$ is decreasing and converges to $F(R_i, a, M^*)$.

Finally, we show that, for any ex-ante fair allocation M for p, $M^* R_N^{sd} M$. Then it is sufficient to show that $M^k R_N^{sd} M$ for all k, and we prove this by induction. It is obvious that $M^1 R_N^{sd} M$. Suppose that for some $k \geq 1$, $M^k R_N^{sd} M$, but there exists $i \in N$ such that we do not have $M_i^{k+1} R_i^{sd} M_i$. Then for some $a \in A$, $F(R_i, a, M^{k+1}) < F(R_i, a, M) \leq F(R_i, a, M^k)$. By Claim 1, in step k, agent i's proposal $\left[F(P_i, a, M^k), F(R_i, a, M^k)\right]$ is in the choice set of object a and it rejects her proposal $\left[F(R_i, a, M^{k+1}), F(R_i, a, M^k)\right]$. Then we have

$$F(R_i, a, M) > F(R_i, a, M^{k+1}) \geq F(P_i, a, M^k) \geq F(P_i, a, M).$$

It follows that

$$M_{ia} = F(R_i, a, M) - F(P_i, a, M) > F(R_i, a, M^{k+1}) - F(P_i, a, M^k) \geq 0.$$

Consider the proposals tentatively accepted by a in step k. The sum of the measures of these proposals is 1, as i has a proposal rejected. If i has a proposal tentatively accepted by a in this step, then this proposal is $\left[F(P_i, a, M^k), F(R_i, a, M^{k+1})\right]$. The above inequality implies that the measure of this proposal is less than the probability that i receives a under M. Therefore, there must exist some $j \in N$ such that j has a proposal $\left[F(P_j, a, M^k), x\right]$ tentatively accepted by a in step k, and

$$x - F(P_j, a, M^k) > M_{ja} = F(R_j, a, M) - F(P_j, a, M). \tag{1}$$

Since i has a proposal rejected by a in this step, by the choice rule of a, $j \succeq_a i$.

If $j \succ_a i$, we have $F(R_j, a, M) = 1$, since M is ex-ante stable and $M_{ia} > 0$. Then, given $x \leq 1$, (1) implies that $F(P_j, a, M^k) < F(P_j, a, M)$, contradicting to $M^k R_N^{sd} M$. Next, consider the case that $j \sim_a i$. Since i's proposal $\left[F(R_i, a, M^{k+1}), F(R_i, a, M^k)\right]$ is rejected by a in step k, by the choice rule of a, $x \leq F(R_i, a, M^{k+1})$. Hence,

$$x \leq F(R_i, a, M^{k+1}) < F(R_i, a, M) \leq F(R_j, a, M), \tag{2}$$

where the last inequality follows from $M_{ia} > 0$ and the ordinal fairness of M. Then, (1) and (2) imply that $F(P_j, a, M^k) < F(P_j, a, M)$, and a contradiction is reached.

A.2 Proof of Lemma 1

Consider any $i \in N$ and $a \in A \cup \{i\}$. First, since every $M \in S$ is individually rational, for each $b \in A$ such that $iP_i b$, $(\vee S)_{ib} = 0$. Therefore, if $iR_i a$, then $F(R_i, a, \vee S) = (\vee S)_{ii} + \sum_{b \in A} (\vee S)_{ib} = 1 = \sup\{F(R_i, a, M) : M \in S\}$. Next, consider the case that $aP_i i$. Let $A' = \{b \in A : bR_i a\}$ and $|A'| = k$. If $k = 1$, then $F(R_i, a, \vee S) = \sup\{F(R_i, a, M) : M \in S\}$. Suppose that $k > 1$. We list the objects in A' in the order of i's preferences: let $A' = \{a_1, \ldots, a_k\}$ such that $a_1 P_i a_2, \ldots, a_{k-1} P_i a_k$, where $a_k = a$. Then,

$$\begin{aligned}
F(R_i, a_k, \vee S) &= \sum_{\ell=1}^{k} (\vee S)_{ia_\ell} \\
&= \sum_{\ell=1}^{k} \Big\{ \sup\{F(R_i, a_\ell, M) : M \in S\} - \sup\{F(P_i, a_\ell, M) : M \in S\} \Big\} \\
&= \sup\{F(R_i, a_1, M) : M \in S\} + \\
&\quad \sum_{\ell=2}^{k} \Big\{ \sup\{F(R_i, a_\ell, M) : M \in S\} - \sup\{F(R_i, a_{\ell-1}, M) : M \in S\} \Big\} \\
&= \sup\{F(R_i, a_k, M) : M \in S\}.
\end{aligned}$$

By similar arguments, it can be shown that $F(R_i, a, \wedge S) = \inf\{F(R_i, a, M) : M \in S\}$.

A.3 Proof of Theorem 3

We prove the results in Theorem 3 in a particular order. The proof consists of nine parts. We first show (iv) and (v), in Part 1 and Part 2, by similar techniques. In Part 3, we show $M \vee M' \in \mathscr{E}$. The first statement in the rural hospital theorem is also proved along the way. Part 4 deals with the second statement in (i). Building on these results, in Part 5, we show $M \wedge M' \in \mathscr{E}$, and hence $(\mathscr{E}, R_N^{sd})$ is a lattice. Part 6 shows the lattice is distributive, and Part 7 and Part 8 establish completeness. Finally, we show the second statement in the rural hospital theorem in Part 9.

Part 1: two-sided efficiency.

Assume to the contrary, there exists some allocation $M'' \neq M$ such that $M'' R_N^{sd} M$ and $M'' \succeq_A^{sd} M$. Let $N' = \{i \in N : M_i \neq M_i''\}$. Clearly $N' \neq \emptyset$. For each $i \in N'$, since $M_i'' R_i^{sd} M_i$ and M is individually rational, $\{a \in A : aP_i i, M_{ia}'' > M_{ia}\} \neq \emptyset$. Define a_i as the maximal element in this set under R_i. Then $M_{ia}'' = M_{ia}$ for all $a \in A$ such that $aP_i a_i$.

Now, consider any $i \in N'$. Since $F(R_i, a_i, M) < F(R_i, a_i, M'') \leq 1$ and M is ex-ante stable, we have $F(\succeq_{a_i}, i, M) = 1$. Then $M''_{a_i} \succeq^{sd}_{a_i} M_{a_i}$ implies $F(\succeq_{a_i}, i, M'') = 1$ and $F(\succ_{a_i}, i, M'') \geq F(\succ_{a_i}, i, M)$. Hence $\sum_{j \in N: j \sim_{a_i} i} M''_{ja_i} \leq \sum_{j \in N: j \sim_{a_i} i} M_{ja_i}$. Since $M''_{ia_i} > M_{ia_i}$, there exists some $j \in N \setminus \{i\}$ such that $j \sim_{a_i} i$ and $M''_{ja_i} < M_{ja_i}$. Then $j \in N'$ and $a_j P_j a_i$. As M is ordinally fair and $M_{ja_i} > 0$, we have

$$F(R_i, a_i, M) \geq F(R_j, a_i, M) > F(R_j, a_j, M).$$

In sum, it has been shown that for any $i \in N'$, there exists $j \in N'$ such that $F(R_i, a_i, M) > F(R_j, a_j, M)$. This leads to a contradiction as N' is finite.

Part 2: conflicting interests.

Suppose that $M R^{sd}_N M'$, but for some $a \in A$ and $i \in N$, $F(\succeq_a, i, M) > F(\succeq_a, i, M')$. Then there exists $j \in N$ such that $j \succeq_a i$ and $M_{ja} > M'_{ja}$. Since $F(\succeq_a, j, M') \leq F(\succeq_a, i, M') < 1$ and M' is ex-ante stable, $F(R_j, a, M') = 1$. Then

$$F(P_j, a, M') = 1 - M'_{ja} > 1 - M_{ja} \geq 1 - \sum_{b \in A \cup \{j\}: a R_j b} M_{jb} = F(P_j, a, M).$$

This contradicts to $M_j R^{sd}_j M'_j$.

To show the other direction, suppose that $M' \succeq^{sd}_A M$, but we do not have $M R^{sd}_N M'$. Let $N' = \{i \in N : F(R_i, a, M') > F(R_i, a, M)$ for some $a \in A$ such that $a P_i i\}$. Then, given that M is individually rational, $N' \neq \emptyset$. For each $i \in N'$, define a_i as the maximal object in the set $\{a \in A : a P_i i, F(R_i, a, M') > F(R_i, a, M)\}$ under R_i.

Consider any $i \in N'$. By the definition of a_i, we have $M'_{ia_i} > M_{ia_i}$. Since $F(R_i, a_i, M) < 1$ and M is ex-ante stable, $F(\succeq_{a_i}, i, M) = 1$. It follows from $M' \succeq^{sd}_A M$ that $F(\succeq_{a_i}, i, M') = 1$ and $\sum_{j \in N: j \sim_{a_i} i} M'_{ja_i} \leq \sum_{j \in N: j \sim_{a_i} i} M_{ja_i}$. Therefore, $M'_{ia_i} > M_{ia_i}$ implies the existence of some $j \in N \setminus \{i\}$ such that $j \sim_{a_i} i$ and $M'_{ja_i} < M_{ja_i}$. Since M' is ordinally fair and $M'_{ia_i} > 0$,

$$F(R_j, a_i, M') \geq F(R_i, a_i, M') > F(R_i, a_i, M). \tag{3}$$

Since M is ordinally fair and $M_{ja_i} > 0$,

$$F(R_i, a_i, M) \geq F(R_j, a_i, M). \tag{4}$$

Hence (3) and (4) imply $F(R_j, a_i, M') > F(R_j, a_i, M)$. So $j \in N'$. Since $M'_{ja_i} < M_{ja_i}$,

$a_j P_j a_i$. Then by (4) and $M_{ja_i} > 0$,

$$F(R_i, a_i, M) \geq F(R_j, a_i, M) > F(R_j, a_j, M).$$

In sum, we have shown that for any $i \in N'$, there exists $j \in N'$ such that $F(R_i, a_i, M) > F(R_j, a_j, M)$, which leads to a contradiction since N' is finite.

Part 3: $M \vee M' \in \mathscr{E}$, and the first part of rural hospital theorem.

Denote $\bar{M} = M \vee M'$. By the construction, $\bar{M}_{ia} \geq 0$ for all $i \in N$ and $a \in A$. Then by Lemma 1, for any $i \in N$, $\sum_{a \in A} \bar{M}_{ia} = F(P_i, i, \bar{M}) = \max\{F(P_i, i, M), F(P_i, i, M')\} \leq 1$. Therefore, for $\bar{M}$ to be a well-defined random allocation, it remains to show that the probability shares of any object are not over-assigned.

Lemma 2. *For any $M^1, M^2 \in \mathscr{E}$, $i \in N$ and $a \in A$, if $F(R_i, a, M^1) \geq F(R_i, a, M^2)$, then $M^1_{ia} \geq \left(M^1 \vee M^2\right)_{ia}$.*

Proof of Lemma 2.

$$\begin{aligned} M^1_{ia} &= F(R_i, a, M^1) - F(P_i, a, M^1) \\ &= \max\left\{F(R_i, a, M^1), F(R_i, a, M^2)\right\} - F(P_i, a, M^1) \\ &\geq \max\left\{F(R_i, a, M^1), F(R_i, a, M^2)\right\} - \max\left\{F(P_i, a, M^1), F(P_i, a, M^2)\right\} \\ &= \left(M^1 \vee M^2\right)_{ia} \end{aligned}$$

■

Claim 2. *For any $a \in A$, either $\bar{M}_{ia} \leq M_{ia}$ for all $i \in N$, or $\bar{M}_{ia} \leq M'_{ia}$ for all $i \in N$.*

Proof of Claim 2. Consider any $a \in A$. Let $N(a) = \{i \in N : \bar{M}_{ia} > 0\}$. If $F(R_i, a, M) = F(R_i, a, M')$ for all $i \in N(a)$, then Claim 2 follows directly from Lemma 2. Now, suppose that there exists $i \in N(a)$ such that $F(R_i, a, M) \neq F(R_i, a, M')$. Without loss of generality, let $F(R_i, a, M) > F(R_i, a, M')$. Then by Lemma 2, $M_{ia} \geq \bar{M}_{ia} > 0$. Next, we want to show that $F(R_j, a, M) \geq F(R_j, a, M')$ for all $j \in N(a) \setminus \{i\}$. Assume to the contrary, for some $j \in N(a)$, $F(R_j, a, M) < F(R_j, a, M')$. Then by Lemma 2, $M'_{ja} \geq \bar{M}_{ja} > 0$. Since M is ex-ante stable, $M_{ia} > 0$ and $F(R_j, a, M) < F(R_j, a, M') \leq 1$ imply $i \succeq_a j$. Similarly, given that M' is ex-ante stable, $M'_{ja} > 0$ and $F(R_i, a, M') < F(R_i, a, M) \leq 1$ imply $j \succeq_a i$. Hence, $i \sim_a j$. Then we have

$$F(R_i, a, M') \geq F(R_j, a, M') > F(R_j, a, M) \geq F(R_i, a, M),$$

where the first inequality follows from $M'_{ja} > 0$ and the ordinal fairness of M', and the last inequality follows from $M_{ia} > 0$ and the ordinal fairness of M. This contradicts to the assumption that $F(R_i, a, M) > F(R_i, a, M')$. Therefore, $F(R_k, a, M) \geq F(R_k, a, M')$ for all $k \in N(a)$. By Lemma 2, $\bar{M}_{ka} \leq M_{ka}$ for all $k \in N$. ■

It follows immediately from Claim 2 that for every $a \in A$, $\sum_{i \in N} \bar{M}_{ia} \leq 1$. So $\bar{M}$ is a well-defined random allocation. Next, we show that $\bar{M}$ is ex-ante fair. Lemma 1 implies that $\bar{M}$ is individually rational. The non-wastefulness of $\bar{M}$ can be deduced from the following critical result by Erdil (2014).

Lemma 3 (Reshuffling Lemma). *Let M^1 and M^2 be two random allocations. If M^1 is individually rational, non-wasteful and $M^2 R_N^{sd} M^1$, then $\sum_{b \in A} M^1_{ib} = \sum_{b \in A} M^2_{ib}$ for every $i \in N$, and $\sum_{j \in N} M^1_{ja} = \sum_{j \in N} M^2_{ja}$ for every $a \in A$.*

If $\bar{M}$ is wasteful, then there exist $i \in N$ and $a \in A$ such that $F(R_i, a, \bar{M}) < 1$ and $\sum_{j \in N} \bar{M}_{ja} < 1$. As $\bar{M} R_N^{sd} M$, $F(R_i, a, M) < 1$. Then, since M is non-wasteful, $\sum_{j \in N} M_{ja} = 1$, contradicting to Lemma 3.

We pause the proof of the ex-ante fairness of $\bar{M}$ and present two other important implications of the Reshuffling Lemma.

- First, since $\bar{M} R_N^{sd} M$, $\bar{M} R_N^{sd} M'$, and both M and M' are individually rational and non-wasteful, Lemma 3 and Claim 2 together imply the following result.

  Claim 3. *For any $a \in A$, either $\bar{M}_a = M_a$ or $\bar{M}_a = M'_a$.*

- Second, for any $i \in N$ and $a \in A$, by Lemma 3 we have

$$\sum_{b \in A} M_{ib} = \sum_{b \in A} \bar{M}_{ib} = \sum_{b \in A} M'_{ib},$$

$$\sum_{j \in N} M_{ja} = \sum_{j \in N} \bar{M}_{ja} = \sum_{j \in N} M'_{ja}.$$

  That is, we have shown the first part of the rural hospital theorem.

Now, suppose that $\bar{M}$ is not ex-ante stable, then there exist $i, j \in N$ and $a \in A$ such that $i \succ_a j$, $\bar{M}_{ja} > 0$ and $F(R_i, a, \bar{M}) < 1$. Clearly, $M''_{ja} > 0$ for some $M'' \in \{M, M'\}$. But $\bar{M} R_N^{sd} M''$ implies $F(R_i, a, M'') \leq F(R_i, a, \bar{M}) < 1$, contradicting to the ex-ante stability of M''. Finally, suppose that $\bar{M}$ is not ordinally fair, then there exist $i, j \in N$ and $a \in A$ such that $i \sim_a j$, $\bar{M}_{ia} > 0$ and $F(R_i, a, \bar{M}) > F(R_j, a, \bar{M})$. By Lemma 1, $F(R_i, a, \bar{M}) =$

$\max\{F(R_i,a,M),F(R_i,a,M')\}$. Without loss of generality, let $F(R_i,a,\bar{M})=F(R_i,a,M)\geq F(R_i,a,M')$. Then by Lemma 2, $M_{ia}\geq \bar{M}_{ia}>0$. But we have

$$F(R_j,a,M)\leq F(R_j,a,\bar{M})<F(R_i,a,\bar{M})=F(R_i,a,M),$$

contradicting to the ordinal fairness of M. In sum, $\bar{M}$ is ex-ante fair.

Part 4: for any $i\in N$, either $M_i R_i^{sd} M_i'$ or $M_i' R_i^{sd} M_i$; for any $a\in A$, either $M_a \succeq_a^{sd} M_a'$ or $M_a' \succeq_a^{sd} M_a$.

First, consider any $a\in A$. By Claim 3, $\bar{M}_a=M_a$ or $\bar{M}_a=M_a'$. Since $\bar{M}$ is ex-ante fair, $\bar{M}R_N^{sd}M$, and $\bar{M}R_N^{sd}M'$, Part 2 (conflicting interests) implies that $M\succeq_A^{sd}\bar{M}$ and $M'\succeq_A^{sd}\bar{M}$. Therefore, $M_a'\succeq_a^{sd}M_a$ if $\bar{M}_a=M_a$, and $M_a\succeq_a^{sd}M_a'$ if $\bar{M}_a=M_a'$.

Second, assume to the contrary, for some $i\in N$, M_i and M_i' are not comparable using the first-order stochastic dominance relation. Since M and M' are individually rational, there exist $a,b\in A$ such that $aP_i i$, $bP_i i$, $F(R_i,a,M)>F(R_i,a,M')$ and $F(R_i,b,M)<F(R_i,b,M')$. Without loss of generality, assume $aP_i b$. Let c be the worst object in $\{a'\in A: a'P_i b, F(R_i,a',M)>F(R_i,a',M')\}$, and d the object next to c (and worse than c) on i's preference list. Then we have $aR_i cP_i dR_i b$, $F(R_i,c,M)>F(R_i,c,M')$ and $F(R_i,d,M)\leq F(R_i,d,M')$. Moreover,

$$\begin{aligned}
M_{id}' &= F(R_i,d,M')-F(R_i,c,M')\\
&> F(R_i,d,M')-F(R_i,c,M)\\
&= \max\left\{F(R_i,d,M'),F(R_i,d,M)\right\}-\max\left\{F(R_i,c,M'),F(R_i,c,M)\right\}\\
&= \bar{M}_{id}.
\end{aligned}$$

By Claim 3, $M_{id}'>\bar{M}_{id}$ implies $\bar{M}_d=M_d$. It follows from Part 2 (conflicting interests) that $M_d'\succeq_d^{sd}\bar{M}_d=M_d$. Since $F(R_i,d,M)\leq F(R_i,b,M)<F(R_i,b,M')\leq 1$ and M is ex-ante stable, $F(\succeq_d,i,M)=1$. Therefore, $F(\succeq_d,i,M')=1$ and $\sum_{j\in N:j\sim_d i}M_{jd}'\leq \sum_{j\in N:j\sim_d i}M_{jd}$. Then $M_{id}'>\bar{M}_{id}=M_{id}$ implies that there exists $j\in N$ such that $j\sim_d i$ and $M_{jd}'<M_{jd}$, and we have

$$F(R_j,d,M')\geq F(R_i,d,M')\geq F(R_i,d,M)\geq F(R_j,d,M),$$

where the first inequality follows from $M_{id}'>0$ and the ordinal fairness of M', and the last inequality follows from $M_{jd}>0$ and the ordinal fairness of M. Since $F(R_j,d,M')\geq$

$F(R_j, d, M)$, by Lemma 2, $\bar{M}_{jd} \le M'_{jd} < M_{jd}$. This is a contradiction since $\bar{M}_d = M_d$.

Part 5: $M \wedge M' \in \mathscr{E}$.

Denote $\hat{M} = M \wedge M'$. Let $N' = \{i \in N : M'_i R^{sd}_i M_i\}$. If $i \in N'$, then $\bar{M}_i = M'_i$ and $\hat{M}_i = M_i$. By Part 4, if $i \in N \setminus N'$, then $\bar{M}_i = M_i$ and $\hat{M}_i = M'_i$. It follows that $\hat{M}_{ia} \ge 0$ and $\sum_{b \in A} \hat{M}_{ib} \le 1$ for all $i \in N$ and $a \in A$. For any $a \in A$, by the first part of the rural hospital theorem,

$$\sum_{i \in N'} M_{ia} + \sum_{i \in N \setminus N'} M_{ia} = \sum_{i \in N} M_{ia} = \sum_{i \in N} \bar{M}_{ia} = \sum_{i \in N'} M'_{ia} + \sum_{i \in N \setminus N'} M_{ia}.$$

Then

$$\sum_{i \in N'} M_{ia} = \sum_{i \in N'} M'_{ia}.$$

This implies that

$$\sum_{i \in N} \hat{M}_{ia} = \sum_{i \in N'} M_{ia} + \sum_{i \in N \setminus N'} M'_{ia} = \sum_{i \in N'} M'_{ia} + \sum_{i \in N \setminus N'} M'_{ia} = \sum_{i \in N} M'_{ia} \le 1.$$

That is, the probability shares of a are not over-assigned. So $\hat{M}$ is a well-defined random allocation. We show that it is ex-ante fair. First, $\hat{M}$ is individually rational by Lemma 1. Second, suppose that for some $a \in A$, $\sum_{i \in N} \hat{M}_{ia} < 1$. It was shown above that $\sum_{i \in N} M'_{ia} = \sum_{i \in N} \hat{M}_{ia}$. The first part of the rural hospital theorem further implies that $\sum_{i \in N} M_{ia} = \sum_{i \in N} M'_{ia} = \sum_{i \in N} \hat{M}_{ia} < 1$. Since M and M' are non-wasteful, for all $i \in N$, we have $F(R_i, a, M) = F(R_i, a, M') = 1$. It follows that $F(R_i, a, \hat{M}) = 1$ for all $i \in N$, and hence $\hat{M}$ is non-wasteful.

Suppose that $\hat{M}$ is not ex-ante stable. Then there exist $i, j \in N$ and $a \in A$ such that $i \succ_a j$, $F(R_i, a, \hat{M}) < 1$ and $\hat{M}_{ja} > 0$. Without loss of generality, assume $\hat{M}_i = M_i$. Since M is ex-ante stable, $F(\succ_a, j, M) = 1$. In particular, $M_{ja} = 0$. Then $M'_{ja} = \hat{M}_{ja} > 0$ and $M_j R^{sd}_j M'_j$. It follows that $\bar{M}_{ja} = M_{ja} \neq M'_{ja}$. By Claim 3, $\bar{M}_a = M_a$. Then, as $\bar{M} R^{sd}_N M'$, Part 2 (conflicting interests) implies $M'_a \succeq^{sd}_a M_a$. However, this contradicts to the facts that $M'_{ja} > 0$ and $F(\succ_a, j, M) = 1$.

Finally, suppose that $\hat{M}$ is not ordinally fair. Then there exist $i, j \in N$ and $a \in A$ such that $i \sim_a j$, $\hat{M}_{ia} > 0$ and $F(R_i, a, \hat{M}) > F(R_j, a, \hat{M})$. Without loss of generality, let $\hat{M}_i = M_i$. Then by the ordinal fairness of M, $\hat{M}_j = M'_j$. Since $M'_i R^{sd}_i M_i$,

$$F(R_i, a, M') \ge F(R_i, a, M) = F(R_i, a, \hat{M}) > F(R_j, a, \hat{M}) = F(R_j, a, M').$$

By the ordinal fairness of M', $M'_{ia} = 0$. Given that $M'_{ia} \neq M_{ia}$ and $\bar{M}_{ia} = M'_{ia}$, by Claim 3 we have $\bar{M}_a = M'_a$. Then Part 2 (conflicting interests) implies $M_a \succeq_a^{sd} M'_a$.

Since $F(R_j, a, M') < F(R_i, a, M) \leq 1$, the ex-ante stability of M' implies $F(\succeq_a, j, M') = 1$. Then $F(\succeq_a, j, M) = 1$ and $\sum_{k \in N : k \sim_a j} M_{ka} \leq \sum_{k \in N : k \sim_a j} M'_{ka}$. As $M_{ia} > M'_{ia}$ and $i \sim_a j$, there exists $k \in N$ such that $k \sim_a j$ and $M_{ka} < M'_{ka}$. Since $\bar{M}_a = M'_a$, we have $\bar{M}_k = M'_k$. Hence $M'_k R_k^{sd} M_k$. Then

$$F(R_k, a, M') \geq F(R_k, a, M) \geq F(R_i, a, M) > F(R_j, a, M'),$$

where the second inequality follows from the ordinal fairness of M and $M_{ia} > 0$. However, given that $M'_{ka} > 0$, this contradicts to the ordinal fairness of M'.

Part 6: distributivity.

Consider any $M'' \in \mathscr{E}$, and any $i \in N$. Given Part 4, without loss of generality, assume $M_i R_i^{sd} M'_i$. Then $(M'' \wedge (M \vee M'))_i = (M'' \wedge M)_i$. It also follows from $M_i R_i^{sd} M'_i$ that $(M'' \wedge M)_i R_i^{sd} (M'' \wedge M')_i$. Therefore

$$\big((M'' \wedge M) \vee (M'' \wedge M')\big)_i = \big(M'' \wedge M\big)_i = \big(M'' \wedge (M \vee M')\big)_i .$$

That is, we have $M'' \wedge (M \vee M') = (M'' \wedge M) \vee (M'' \wedge M')$. By similar arguments, it can be shown that $M'' \vee (M \wedge M') = (M'' \vee M) \wedge (M'' \vee M')$.

Part 7: $\vee S \in \mathscr{E}$.

First, given that any agent can compare the lotteries obtained under two ex-ante fair allocations using the first-order stochastic dominance relation (Part 4), it is straightforward to see that for any $M'' \in \mathscr{E}$, and any non-empty and finite set $S' \subseteq \mathscr{E}$ such that $\vee S' \in \mathscr{E}$, we have $\vee \{S' \cup \{M''\}\} = \vee \{\vee S', M''\}$. Therefore, by Part 3 and an induction argument, for any non-empty and finite $S'' \subseteq \mathscr{E}$, $\vee S'' \in \mathscr{E}$.

Now, consider the set $S \subseteq \mathscr{E}$, which can be infinite. As in the case of $\bar{M}$, it can be easily shown that for all $i \in N$ and $a \in A$, $(\vee S)_{ia} \geq 0$ and $\sum_{b \in A} (\vee S)_{ib} \leq 1$. For $\vee S$ to be a well-defined random allocation, it remains to show that $\sum_{i \in N} (\vee S)_{ia} \leq 1$ for every $a \in A$. Assume to the contrary, $\sum_{i \in N} (\vee S)_{ia} > 1$ for some $a \in A$. For each $i \in N$, we can find $M(i) \in S$ such that

$$F(R_i, a, M(i)) > \sup \left\{F(R_i, a, M'') : M'' \in S\right\} - \frac{1}{|N|} \left\{\sum_{j \in N} (\vee S)_{ja} - 1\right\}.$$

Let $S' = \{M(i) : i \in N\} \subseteq S$. Then for each $i \in N$,

$$\begin{aligned}
(\vee S')_{ia} &= \max\left\{F(R_i, a, M'') : M'' \in S'\right\} - \max\left\{F(P_i, a, M'') : M'' \in S'\right\} \\
&\geq F(R_i, a, M(i)) - \sup\left\{F(P_i, a, M'') : M'' \in S\right\} \\
&> \sup\left\{F(R_i, a, M'') : M'' \in S\right\} - \frac{1}{|N|}\left\{\sum_{j\in N}(\vee S)_{ja} - 1\right\} - \sup\left\{F(P_i, a, M'') : M'' \in S\right\} \\
&= (\vee S)_{ia} - \frac{1}{|N|}\left\{\sum_{j\in N}(\vee S)_{ja} - 1\right\}.
\end{aligned}$$

Summing over N, we have

$$\sum_{i\in N}(\vee S')_{ia} > 1.$$

A contradiction is reached, since S' is finite and $\vee S' \in \mathscr{E}$.

Next, we show that the allocation $\vee S$ is ex-ante fair. First, the individual rationality of $\vee S$ follows from Lemma 1. As in the case of $\bar{M}$, the non-wastefulness of $\vee S$ can be deduced from the Reshuffling Lemma, since $(\vee S) R_N^{sd} M$, and M is individually rational and non-wasteful. Second, suppose that $\vee S$ is not ex-ante stable, then there exist $i, j \in N$ and $a \in A$ such that $i \succ_a j, (\vee S)_{ja} > 0$ and $F(R_i, a, \vee S) < 1$. Clearly $M''_{ja} > 0$ for some $M'' \in S$. But $F(R_i, a, M'') \leq F(R_i, a, \vee S) < 1$, contradicting to the ex-ante stability of M''. Finally, suppose that $\vee S$ is not ordinally fair. Then there exist $i, j \in N$ and $a \in A$ such that $i \sim_a j, (\vee S)_{ia} > 0$, and $F(R_i, a, \vee S) > F(R_j, a, \vee S)$. Pick some number x such that

$$\max\left\{F(R_j, a, \vee S), F(P_i, a, \vee S)\right\} < x < F(R_i, a, \vee S).$$

Since $F(R_i, a, \vee S) = \sup\{F(R_i, a, M'') : M'' \in S\}$, there exists $M'' \in S$ such that $F(R_i, a, M'') > x$. Then $F(R_i, a, M'') > F(R_j, a, \vee S) \geq F(R_j, a, M'')$. However, $M''_{ia} = F(R_i, a, M'') - F(P_i, a, M'') > x - F(P_i, a, \vee S) > 0$, contradicting to the ordinal fairness of M''.

Part 8: $\wedge S \in \mathscr{E}$.

First, as in Part 7, it is easy to show that for any non-empty and finite $S' \subseteq \mathscr{E}$, $\wedge S' \in \mathscr{E}$. It is also straightforward to see that for all $i \in N$ and $a \in A$, $(\wedge S)_{ia} \geq 0$ and $\sum_{b\in A}(\wedge S)_{ib} \leq 1$. Below, we show that the probability shares of any object are not over-assigned in $\wedge S$, using arguments similar to those in Part 7. Assume to the contrary, for some $a \in A$, $\sum_{i\in N}(\wedge S)_{ia} > 1$. For each $i \in N$, there exists $M^1(i) \in S$ such that

$$F(P_i, a, M^1(i)) < \inf\left\{F(P_i, a, M'') : M'' \in S\right\} + \frac{1}{|N|}\left\{\sum_{j\in N}(\wedge S)_{ja} - 1\right\}.$$

Define $S^1 = \left\{M^1(i) : i \in N\right\} \subseteq S$. Then for each $i \in N$,

$$\begin{aligned}
(\wedge S^1)_{ia} &= \min\left\{F(R_i, a, M'') : M'' \in S^1\right\} - \min\left\{F(P_i, a, M'') : M'' \in S^1\right\} \\
&\geq \inf\left\{F(R_i, a, M'') : M'' \in S\right\} - F(P_i, a, M^1(i)) \\
&> \inf\left\{F(R_i, a, M'') : M'' \in S\right\} - \left(\inf\left\{F(P_i, a, M'') : M'' \in S\right\} + \frac{1}{|N|}\left\{\sum_{j\in N}(\wedge S)_{ja} - 1\right\}\right) \\
&= (\wedge S)_{ia} - \frac{1}{|N|}\left\{\sum_{j\in N}(\wedge S)_{ja} - 1\right\}.
\end{aligned}$$

Summing over N, we have

$$\sum_{i\in N}(\wedge S^1)_{ia} > 1.$$

This contradicts to the fact that S^1 is finite and $\wedge S^1 \in \mathscr{E}$. Hence, $\wedge S$ is a well-defined random allocation. We show that it is ex-ante fair. First, the individual rationality follows from Lemma 1. Second, to see non-wastefulness, consider any $a \in A$ such that $\sum_{i\in N}(\wedge S)_{ia} < 1$. For each $i \in N$, there exists $M^2(i) \in S$ such that

$$F(R_i, a, M^2(i)) < \inf\left\{F(R_i, a, M'') : M'' \in S\right\} + \frac{1}{|N|}\left\{1 - \sum_{j\in N}(\wedge S)_{ja}\right\}.$$

Let $S^2 = \left\{M^2(i) : i \in N\right\} \subseteq S$. By similar arguments as above, it can be shown that $\sum_{i\in N}(\wedge S^2)_{ia} < 1$. Since S^2 is finite, $\wedge S^2 \in \mathscr{E}$. By the first part of the rural hospital theorem, for every $M'' \in \mathscr{E}$, $\sum_{i\in N} M''_{ia} = \sum_{i\in N}(\wedge S^2)_{ia} < 1$. Then for every $M'' \in \mathscr{E}$, by non-wastefulness, $F(R_i, a, M'') = 1$ for all $i \in N$. It follows that $F(R_i, a, \wedge S) = 1$ for all $i \in N$, and hence $\wedge S$ is non-wasteful.

Third, suppose that $\wedge S$ is not ex-ante stable. Then there exist $i, j \in N$ and $a \in A$ such that $i \succ_a j, (\wedge S)_{ja} > 0$ and $F(R_i, a, \wedge S) < 1$. Given that

$$\inf\,\{F(R_i, a, M'') : M'' \in S\} < 1, \text{ and}$$

$$\inf\{F(P_j,a,M''): M''\in S\} < \inf\{F(R_j,a,M''): M''\in S\},$$

we can find some $M^1, M^2 \in S$ such that

$$F(R_i,a,M^1) < 1, \text{ and } F(P_j,a,M^2) < \inf\{F(R_j,a,M''): M''\in S\}.$$

This implies that $F(R_i,a,M^1 \wedge M^2) < 1$ and $(M^1 \wedge M^2)_{ja} > 0$, contradicting to the ex-ante stability of $M^1 \wedge M^2$.

Finally, suppose that $\wedge S$ is not ordinally fair. Then there exist $i,j \in N$ and $a \in A$ such that $i \sim_a j, (\wedge S)_{ia} > 0$, and $F(R_i,a,\wedge S) > F(R_j,a,\wedge S)$. Since

$$\inf\{F(P_i,a,M''): M''\in S\} < \inf\{F(R_i,a,M''): M''\in S\}, \text{ and}$$

$$\inf\{F(R_j,a,M''): M''\in S\} < \inf\{F(R_i,a,M''): M''\in S\},$$

we can find $M^3, M^4 \in S$ such that

$$F(P_i,a,M^3) < \inf\{F(R_i,a,M''): M''\in S\}, \text{ and}$$

$$F(R_j,a,M^4) < \inf\{F(R_i,a,M''): M''\in S\}.$$

It follows that $(M^3 \wedge M^4)_{ia} > 0$ and $F(R_j,a,M^3 \wedge M^4) < F(R_i,a,M^3 \wedge M^4)$, contradicting to the ordinal fairness of $M^3 \wedge M^4$.

Part 9: the second part of rural hospital theorem.

We want to show that for all $i \in N$ and $a \in A$, $M_i = M'_i$ if $\sum_{b\in A} M_{ib} < 1$, and $M_a = M'_a$ if $\sum_{j\in N} M_{ja} < 1$.

First, consider any $a \in A$ such that $\sum_{i\in N} M_{ia} < 1$. By Part 7, $\vee\mathscr{E} \in \mathscr{E}$. To prove that $M_a = M'_a$, it is sufficient to show that for any $M'' \in \mathscr{E}$, $M''_a = (\vee\mathscr{E})_a$. Assume to the contrary, $M'' \in \mathscr{E}$ and $M''_a \neq (\vee\mathscr{E})_a$. By the first part of the rural hospital theorem, $\sum_{i\in N} M''_{ia} = \sum_{i\in N}(\vee\mathscr{E})_{ia} < 1$. So there exists $i \in N$ such that $M''_{ia} < (\vee\mathscr{E})_{ia}$. Moreover, as M'' and $\vee\mathscr{E}$ are non-wasteful, $F(R_i,a,M'') = F(R_i,a,\vee\mathscr{E}) = 1$. Therefore, $F(P_i,a,M'') = 1 - M''_{ia} > 1 - (\vee\mathscr{E})_{ia} = F(P_i,a,\vee\mathscr{E})$, contradicting to the fact that $(\vee\mathscr{E})_i R^{sd}_i M''_i$.

Second, consider any $i \in N$ such that $\sum_{a\in A} M_{ia} < 1$. We show that for any $M'' \in \mathscr{E}$, $M''_i = (\wedge\mathscr{E})_i$, where $\wedge\mathscr{E} \in \mathscr{E}$ by Part 8. Suppose that $M'' \in \mathscr{E}$ and $M''_i \neq (\wedge\mathscr{E})_i$. Let a be the worst object in the set $\{b \in A : M''_{ib} \neq (\wedge\mathscr{E})_{ib}\}$, according to R_i. Clearly, $aP_i i$.

By the first part of the rural hospital theorem, $\sum_{b\in A} M''_{ib} = \sum_{b\in A}(\wedge\mathscr{E})_{ib} < 1$. It follows that $F(R_i, a, M'') = F(R_i, a, \wedge\mathscr{E})$, since $M''_{ib} = (\wedge\mathscr{E})_{ib}$ for all $b \in A$ such that $aP_i b$. Then $M''_i R^{sd}_i (\wedge\mathscr{E})_i$ implies that $F(P_i, a, M'') \geq F(P_i, a, \wedge\mathscr{E})$, and hence $M''_{ia} < (\wedge\mathscr{E})_{ia}$.

Since $F(R_i, a, M'') = F(R_i, a, \wedge\mathscr{E}) \leq \sum_{b\in A}(\wedge\mathscr{E})_{ib} < 1$, by ex-ante stability we have $F(\succeq_a, i, M'') = F(\succeq_a, i, \wedge\mathscr{E}) = 1$. By Part 2 (conflicting interests), $(\wedge\mathscr{E})_a \succeq^{sd}_a M''_a$. Then $M''_{ia} < (\wedge\mathscr{E})_{ia}$ implies that there exists some $j \in N$ such that $i \sim_a j$ and $M''_{ja} > (\wedge\mathscr{E})_{ja}$. Since $M''_j R^{sd}_j (\wedge\mathscr{E})_j$, $F(P_j, a, M'') \geq F(P_j, a, \wedge\mathscr{E})$. Thus, $F(R_j, a, M'') > F(R_j, a, \wedge\mathscr{E})$, and we have

$$F(R_i, a, M'') \geq F(R_j, a, M'') > F(R_j, a, \wedge\mathscr{E}) \geq F(R_i, a, \wedge\mathscr{E}),$$

where the first inequality follows from the ordinal fairness of M'' and $M''_{ja} > 0$, and the last inequality follows from the ordinal fairness of $\wedge\mathscr{E}$ and $(\wedge\mathscr{E})_{ia} > 0$. This is a contradiction since it was already shown that $F(R_i, a, M'') = F(R_i, a, \wedge\mathscr{E})$.

A.4 Proof of Proposition 1

Let $p = (N, A, R, \succeq) \in \mathscr{P}_{\mathrm{HA}}$ with $|N| = |A| = n$. Consider the allocation procedure in the definition of EPS, which terminates in some step $\bar{k}$. By construction, we have $\bar{k} \leq n$, and $\{\lambda_k\}_{k=1}^{\bar{k}}$ is a sequence of positive numbers with $\sum_{k=1}^{\bar{k}} \lambda_k = 1$. For each $k \in \{1, \dots, \bar{k}\}$ and $i \in N_k$, define $k(i) \in \{0, \dots, k-1\}$ such that $i \in N_{k(i)}$, and $i \notin N_\ell$ if $k(i) < \ell < k$, where we set $N_0 = N$. Then $d_{k-1}(i) = \lambda_{k(i)+1} + \ldots + \lambda_{k-1}$ if $d_{k-1}(i) > 0$.

Let $q = (n!)^n$. We first use induction to show that $\lambda_k q$ is an integer for every $k \in \{1, \dots, \bar{k}\}$. Since $\lambda_1 = |E_1|/|N_1|$, $\lambda_1 \cdot n!$ is an integer. Assume that $\lambda_{k-1} \cdot (n!)^{k-1}$ is an integer, where $1 < k \leq \bar{k}$. Then

$$\begin{aligned}
\lambda_k \cdot (n!)^k &= \frac{|E_k| - \sum_{i\in N_k} d_{k-1}(i)}{|N_k|} \cdot (n!)^k \\
&= \frac{n!}{|N_k|} \cdot \Big\{|E_k| \cdot (n!)^{k-1} - \sum_{i\in N_k} d_{k-1}(i) \cdot (n!)^{k-1}\Big\},
\end{aligned}$$

which is an integer since for each $i \in N_k$ with $d_{k-1}(i) > 0$,

$$d_{k-1}(i) \cdot (n!)^{k-1} = \lambda_{k(i)+1} \cdot (n!)^{k-1} + \ldots + \lambda_{k-1} \cdot (n!)^{k-1}$$

is an integer. Given that $\bar{k} \leq n$ and $q = (n!)^n$, $\lambda_k q$ is an integer for all $k \in \{1, \dots, \bar{k}\}$.

Next, we construct a deterministic allocation for p^q based on EPS. Consider any

$k \in \{1, \ldots, \bar{k}\}$. Define

$$E_k^q = \{a^x \in A^q : a \in E_k, x = 1, \ldots, q\},$$

and

$$N_k^q = \Big\{i^x \in N^q : i \in N_k, \sum_{\ell=0}^{k(i)} \lambda_\ell q + 1 \leq x \leq \sum_{\ell=0}^{k} \lambda_\ell q\Big\},$$

where $\lambda_0 = 0$. Note that, since every $\lambda_\ell q$ is an integer, for each $i \in N_k$ we have

$$\left|\{i^x : i^x \in N_k^q\}\right| = \sum_{\ell=k(i)+1}^{k} \lambda_\ell q = (d_{k-1}(i) + \lambda_k)q. \tag{5}$$

We argue that the objects E_k^q can be assigned to the agents N_k^q such that each $i^x \in N_k^q$ receives some $a^y \in E_k^q$ with $a \in B_i(A_k)$. If this is not true, then by Hall's theorem, there exists $\tilde{N}_k^q \subseteq N_k^q$ such that $|\tilde{N}_k^q| > |B_{N'}(A_k)|q$, where $N' = \{i \in N_k : i^x \in \tilde{N}_k^q \text{ for some } x\}$. It follows that

$$\left|\{i^x : i \in N', i^x \in N_k^q\}\right| \geq |\tilde{N}_k^q| > \left|B_{N'}(A_k)\right|q.$$

Then by Equation 5,

$$\begin{aligned} &\sum_{i \in N'} (d_{k-1}(i) + \lambda_k)q > \left|B_{N'}(A_k)\right|q \\ \Longrightarrow\ &\lambda_k > \frac{\left|B_{N'}(A_k)\right| - \sum_{i \in N'} d_{k-1}(i)}{|N'|}, \end{aligned}$$

which contradicts to the definition of λ_k.

Therefore, we can construct a deterministic allocation φ such that for each $k \in \{1, \ldots, \bar{k}\}$ and $i^x \in N_k^q$, $\varphi(i^x) = a^y \in E_k^q$ for some $a \in B_i(A_k)$ and y. Then by Equation 5, under the allocation $M(\varphi, p^q)$, each $i \in N_k$ is assigned the objects in $B_i(A_k)$ with a probability of $d_{k-1}(i) + \lambda_k$. Therefore, $M(\varphi, p^q) \in f^{\text{EPS}}(p)$.

To finish the proof, it remains to show that for any $\sigma \in \mathcal{O}(p, q)$ and $\mu \in f^{\text{SD}}(\sigma, p^q)$, φ and μ are welfare equivalent.[16] Assume to the contrary, φ and μ are not welfare equivalent. Let i^x be the first agent in N^q who is not indifferent between φ and μ. That is, we do not have $\mu(i^x)\ I_i^q\, \varphi(i^x)$. Moreover, $\mu(j^y)\ I_j^q\, \varphi(j^y)$ for any $j^y \in N^q$ with $\sigma^{-1}(j^y) <$

[16] Specifically, it is easy to see that when φ and μ are welfare equivalent, $M(\varphi, p^q)$ and $M(\mu, p^q)$ are welfare equivalent. Then it can be shown that the ordinal fairness of $M(\varphi, p^q)$ implies $M(\mu, p^q)$ is also ordinally fair, and hence $M(\mu, p^q) \in f^{\text{EPS}}(p)$.

$\sigma^{-1}(i^x)$. It follows from the definition of serial dictatorships that $\mu(i^x)\, P_i^q\, \varphi(i^x)$. Suppose that $\mu(i^x) = a^y$ and $i^x \in N_k^q$. Then $aP_i b$ for any $b \in B_i(A_k)$. Note that, if $k(i) < \ell < k$, then $i \notin N_\ell$, and we have $B_i(A_\ell) \setminus E_\ell \neq \emptyset$, since otherwise

$$\frac{|B_{N_\ell \cup \{i\}}(A_\ell)| - \sum_{j \in N_\ell \cup \{i\}} d_{\ell-1}(j)}{|N_\ell \cup \{i\}|} = \frac{|E_\ell| - \sum_{j \in N_\ell \cup \{i\}} d_{\ell-1}(j)}{|N_\ell \cup \{i\}|} < \lambda_\ell.$$

Therefore, $B_i(A_{\ell+1}) \subseteq B_i(A_\ell)$. It follows that for any ℓ with $k(i) < \ell \leq k$, we have $aP_i b$ for any $b \in B_i(A_\ell)$, and hence $a \notin A_\ell$. This implies $a \in E_{k'}$ for some k' such that $1 \leq k' \leq k(i)$.

By the construction, all the objects $\cup_{\ell=1}^{k(i)} E_\ell^q$ are assigned to the agents $\cup_{\ell=1}^{k(i)} N_\ell^q$ under φ. In addition, $x > z$ for any $j^z \in \cup_{\ell=1}^{k(i)} N_\ell^q$, and $i^x \notin \cup_{\ell=1}^{k(i)} N_\ell^q$. Then, $\mu(i^x) = a^y \in E_{k'}^q \subseteq \cup_{\ell=1}^{k(i)} E_\ell^q$ implies the existence of some $j^z \in \cup_{\ell=1}^{k(i)} N_\ell^q$ such that $\mu(j^z) \notin \cup_{\ell=1}^{k(i)} E_\ell^q$. Let $j^z \in N_{k''}^q$, where $1 \leq k'' \leq k(i)$, and $\mu(j^z) = b^w$. Then $b \notin E_\ell$ for any $\ell \in \{1, \ldots, k(i)\}$. This first implies $b \in A_{k''}$. However, since $x > z$ and $\sigma^{-1}(j^z) < \sigma^{-1}(i^x)$, we have $\mu(j^z)\, I_j^q\, \varphi(j^z)$, and hence $b \in B_j(A_{k''}) \subseteq E_{k''}$, which leads to a contradiction.

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