The property rights theory of production networks*

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Abstract

This paper investigates the formation of production and trading networks in economies with general interdependencies and complex property rights. We argue that the right to exclude, a core tenet of property, grants asset owners local monopoly power that is amplified by an economy's endogenous production network. Our analysis generalizes the exclusion core, a cooperative solution concept based on the right to exclude, to markets with production. We identify sufficient (and essentially necessary) conditions for the nonemptiness of the exclusion core. Multisourcing and a bias toward shorter supply chains emerge in exclusion-core outcomes. As a methodological contribution, we generalize the top trading cycles algorithm to a production economy and we show that it identifies outcomes in an economy's exclusion core.

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JEL: D51, L14, C71, D23

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1 Introduction

This paper proposes a property rights theory of production networks. We are motivated by the extensive interdependencies in the organization of economic activity. A firm's production process weaves together many inputs. Building an airplane, for example, combines engineering knowledge, parts from many suppliers, and specialized labor. Everything must fit for production, and the product, to get off the ground.

We argue that the distribution of property rights—who owns what and with whom—plays a key role in determining the structure of production and trading networks. Our reasoning traces the economic consequences of a core tenet of property, *the right to exclude*. The United States Supreme Court has called the right to exclude "one of the most essential sticks in the bundle of rights that are commonly characterized as property"[1] and legal scholars have emphasized its significance since at least the eighteenth-century.[2] The right to exclude grants the owner(s) of an asset, including labor and human capital, monopoly power that influences economic interactions at the granular level.[3] By threatening to exclude others from desired goods or critical inputs, an agent can skew outcomes in his favor. Production and trading links further amplify this impact as an agent may upend an entire supply chain if he owns a critical component. Equilibrium production and trading networks must navigate both technical constraints, determining which goods are critical for others' production, and strategic distortions, where agents attempt to extract private gains. Our main contribution is to provide a joint restriction on both the production relationships between firms and the economy's ownership structure that sustains stable outcomes. Absent these conditions, vertical integration, horizontal integration, and input multisourcing are expected compensatory responses over the long run. (The absence of stable outcomes need not be tantamount to economy-wide breakdown; in the short run, asset owners and firms are likely to rely on contracts to reach an acceptable outcome as contracts change the constraints agents face.)

To present our argument, we develop a new theoretical framework to study economies with production interdependencies. Our model is a generalization of the "house exchange" economy of Shapley and Scarf (1974). In this model, agents have unit demand and trade discrete, indivisible goods. This simplicity allows for a detailed

[1] Kaiser Aetna v. United States, 444 U.S. 164 (1979).

[2] William Blackstone described it in his *Commentaries on the Laws of England* (1765). See Merrill (1998), Merrill and Smith (2001a), and Klick and Parchomovsky (2017) for recent discussions.

[3] This monopoly dimension of property is well known. See, e.g., Posner and Weyl (2018).

constructive characterization of any consumption and (in our model) production interdependencies arising in equilibrium through the trading links between agents. As explained below, these links play an especially important role in our theory and would be shrouded in more complex settings.

We enrich the Shapley and Scarf (1974) baseline in two ways. First, we follow Balbuzanov and Kotowski (2019) by assuming both a rich family of property arrangements and by adopting their reinterpretation of "endowments" in an economy. Balbuzanov and Kotowski (2019) argue that, at its most basic level, an endowment can be understood as a distribution of exclusion rights, which may be vested in individual agents, coalitions, or even all members of the economy. This reinterpretation leads Balbuzanov and Kotowski (2019) to define and study an economy's *exclusion core*, which is a cooperative solution tailored for discrete exchange economies with complex property rights, including cases with joint or contested ownership. Roughly, at an exclusion core outcome no coalition has the incentive and the ability to veto or block the allocation by invoking its right to exclude others from the goods that it owns. Crucially, coalitions may use trading interdependencies to extract further concessions from others. Balbuzanov and Kotowski (2019) show that exclusion core assignments are closely related to generalizations of the Top Trading Cycles (TTC) algorithm (attributed to David Gale by Shapley and Scarf, 1974) and have several other desirable properties.

Our second step beyond the Shapley and Scarf (1974) economy—and also beyond the model of Balbuzanov and Kotowski (2019)—is to introduce production.[4] We assume a collection of firms that can transform sets of inputs into sets of outputs. These firms may form a supply chain or, more generally, a production network defined by their mutual input-output relationships. We argue that Balbuzanov and Kotowski's (2019) exclusion core solution extends naturally to this enriched environment and we formally develop that generalization. The generalization is based on the observation that production interdependencies give the owners of critical inputs *de facto* exclusion rights over outputs whose production relies on those inputs. Moreover, this reliance may be indirect and mediated by supply chain relations to "downstream" goods.

Using our model, we investigate when an economy has an outcome in its exclusion core and what economic properties such an outcome may have. An *outcome* consists of a consumption assignment, which describes consumers' choices, and an input assignment, which describes the inputs used by each firm. The latter im-

[4] In a working paper version of Balbuzanov and Kotowski (2019), we explored a prototype production economy. That model of production differed significantly from the analysis here.

plicitly defines the active production network in the economy.

After introducing the model and notation in Section 2, we illustrate our main arguments and conclusions through a series of motivating examples in Section 3. These examples shed new light on several archetypal cases concerning the theory of the firm. In a vertical upstream-downstream relationship between firms, we show that *some* integration of ownership (made precise below) is necessary to ensure a stable outcome when the firms are linked via critical or indispensable inputs. However, when inputs are substitutes or input markets are sufficiently competitive, integration is dispensable. Instead, multisourcing can feature in a stable outcome. Multisourcing cuts the exclusion power of suppliers, thereby reducing hold up threats. Integration, however, reappears as a beneficial characteristic when production requires complementary inputs. Many of these conclusions reinforce classic insights from the organizational economics and industrial organization literatures, a point we elaborate upon in Section 3. Our contribution is to show how these ideas also naturally emerge within a distinct modeling paradigm founded on a very spare interpretation of property rights.

Our primary technical contributions are presented in Section 4. There, we formalize the extension of the exclusion core solution to production economies and we state our main results. Of importance is Theorem 1, which identifies sufficient conditions for the solution to be nonempty. In the class of private ownership economies, those conditions are also necessary, given an unrestricted preference domain (Proposition 2). Roughly, there must be "sufficient integration" in the ownership structure of each good's supply chain. Overlap among owners of critical goods in a supply chain dilutes the veto power of peripheral, but opportunistic, parties.[5] This condition is much weaker than vertical integration and formalizes a minimal criterion neutralizing supplier or hold-up risk in a production and trading network.

We prove Theorem 1 constructively with a novel extension of the TTC algorithm to an economy with production. Our Top Trading Cycles and Supply Chains (TTC-SC) algorithm, outlined in Section 5, also has agents trade iteratively in cycles. However, it additionally identifies supply chains to ensure the production of consumed goods, when necessary. To guarantee feasibility, the algorithm incorporates a cycle trimming procedure that indirectly favors the formation of shorter supply chains. All else equal, shorter supply chains have fewer production interdependencies and hold-up opportunities. A subtle challenge in our construction concerns the presence of production cycles, i.e., situations where good x is (indirectly) an input for

[5]This conclusion is reminiscent of the *tragedy of the anticommons*, which occurs when overly diffuse property rights prevent the production and provision of a good or service (Heller, 1998).

good x' and good x' is (indirectly) an input for good x. We address this complication by applying the TTC-SC algorithm first to a simplified economy (without cycles) and then recovering the implied outcome in the original market. That simplification relies on the graph theoretic notion of a *condensation*. We suggest several economic interpretations of this construction related to firms and joint ventures.

We end in Section 6 by relating our study to the literature on production and trading networks. Of note is the successful literature tackling production networks under the "matching with contracts" paradigm (e.g., Ostrovsky, 2008; Fleiner et al., 2019). As explained below, we examine similar questions, though we build our argument on a distinct set of building blocks—property relations rather than bilateral contracts. Appendix A presents the TTC-SC algorithm in detail. Appendix B contains all omitted proofs.

2 Model

An economy $\mathscr{E} = \langle I, F, X, \succ, \omega\rangle$ consists of agents, firms, goods, a preference profile, and an endowment system. $I \coloneqq \{i_1, \ldots, i_n\}$ and $F \coloneqq \{f_1, \ldots, f_m\}$ are finite sets of agents and production technologies ("firms"), respectively. X is a finite set of goods. It may contain consumption goods (e.g., an apple), production inputs (e.g., a ton of ore), or labor inputs (e.g., an hour of welding). Each good x has unit capacity: x can be consumed or used as an input by only one agent or firm.

Production and Firms Goods are partitioned into a set of primary goods and sets of goods produced by each firm, i.e., $X = X_\varnothing \cup X_{f_1} \cup \cdots \cup X_{f_m}$ where $X_f \cap X_{f'} = \varnothing$ for all $f \neq f'$. Primary goods, the set $X_\varnothing$, are not production outputs. The goods in set X_f are available if and only if they are produced by firm f using the eponymous production function $f : 2^X \to \{\varnothing, X_f\}$. Thus, a firm transforms sets of inputs into a set of (net) outputs.[6] For simplicity, we assume that production has a 0/1 character. Either firm f produces nothing or all goods in X_f are created. We assume each production function is monotone ($Z \subseteq Z' \implies f(Z) \subseteq f(Z')$) and satisfies the "no free lunch" property ($f(\varnothing) = \varnothing$).

We say that the input set Z is *efficient for firm f* if there is no $Z' \subsetneq Z$ that assures f the same output as Z. A firm f with a production function of the form

$$f(Z) = \begin{cases} X_f & \text{if } Z \supseteq W_f; \\ \varnothing & \text{otherwise;} \end{cases} \tag{1}$$

[6]That is, firm f supplies X_f to the market. These goods are not used by f in its own production. However, these goods might be used by other firms in their production.

has a *unique efficient input set* W_f. Such a production process exhibits strong input complementarities, which are important in theories of trade (Grossman et al., 2005) and economic growth (Kremer, 1993; Jones, 2011). Therefore, this class of production functions is of particular interest and plays a key role in our analysis.

When a firm has many efficient input sets that allow it to produce its output, i.e., when there are non-nested sets $W_f \subseteq X$ and $W_f' \subseteq X$ such that $f(Z) = X_f$ whenever $Z \supseteq W_f$ or $Z \supseteq W_f'$, then its production function admits inputs substitution. Anticipating the analysis to follow, such production processes allow for multisourcing.

Agents and Consumers Each agent i has a complete and transitive preference $\succ_i$ defined over $X \cup \{x_0\}$, where $x_0 \notin X$ represents no consumption. We assume each agent has unit demand and his preferences are strict. We write $x \succeq_i x'$ if $x \succ_i x'$ or $x = x'$. In examples, we state an agent's preference by listing goods in his preferred order, e.g., $\succ_i : x, x', \ldots$ Unlisted items are inferior to x_0.

Endowments Property rights are a focus of our analysis. Therefore, we posit a general framework subsuming private and public ownership as special cases. The economy's *endowment system* $\omega : 2^I \to 2^X$ identifies the goods owned by each coalition of agents. Throughout the paper, we assume it satisfies four basic properties.

(A1) Agency: $\omega(\varnothing) = \varnothing$.

(A2) Monotonicity: $C' \subseteq C \implies \omega(C') \subseteq \omega(C)$.

(A3) Exhaustivity: $\omega(I) = X$.

(A4) Weak non-contestability: for each $x \in X$, the set $C^x := \bigcap_{\{C \neq \varnothing : x \in \omega(C)\}} C$ is not empty.

Properties A1–A3 are self-explanatory. Property A4 says that each good x has a set of *principals* C^x and any group that has a good in its endowment includes the good's principals. Property A4 relaxes the non-contestability condition of Balbuzanov and Kotowski (2019).

Many situations satisfy A1–A4. If x is privately owned by i, then $x \in \omega(C)$ if and only if $i \in C$. If x is collectively owned by everyone, C^x is the grand coalition and $x \in \omega(C)$ if and only if $C = I$. An interesting case compatible with our model occurs when $x \notin \omega(C^x)$. In practice this may arise when property rights are imperfectly enforced and a good's principals require others' cooperation to exercise de facto control. If the principal set C^x is the same for all $x \in X_f$, we can interpret C^x as being the set of "co-owners" of firm f. It is also possible that a firm's outputs have different principals, i.e., $x, y \in X_f$ but $C^x \neq C^y$. For example, concert seats might be controlled by the concert promoter but rights to its recording might belong to a record label.

Assignments and Outcomes A *consumption assignment* $\mu\colon I \to X \cup \{x_0\}$ identifies each agent's consumption. For any $C \subseteq I$, let $\mu(C) \coloneqq \bigcup_{i\in C} \mu(i)$. An *input assignment* $\gamma\colon F \to 2^X$ identifies the inputs used by each firm and implicitly defines the economy's input-output network. For any $G \subseteq F$, let $\gamma(G) \coloneqq \bigcup_{f\in G} \gamma(f)$. Abusing notation, let $f(\gamma)$ be the output of firm f at input assignment γ. Analogously, $G(\gamma) \coloneqq \bigcup_{f\in G} f(\gamma)$ is the total output of firms in set $G \subseteq F$. An *outcome* (μ,γ) consists of a consumption and an input assignment. An outcome (μ,γ) is *feasible* if (a) every good that is consumed or used in production is a primary good, a good produced at γ, or the outside option, i.e.,

$$\mu(I) \cup \gamma(F) \subseteq X_\varnothing \cup F(\gamma) \cup \{x_0\}$$

and (b) no more than a single agent or firm is assigned an item, i.e.,

$$|\{i \in I : x = \mu(i)\}| + |\{f \in F : x \in \gamma(f)\}| \leq 1$$

for all $x \in X$. Similarly, an input assignment γ is feasible if (a) $\gamma(F) \subseteq X_\varnothing \cup F(\gamma)$ and (b) $|\{f \in F : x \in \gamma(f)\}| \leq 1$ for all $x \in X$. If (μ,γ) is a feasible outcome, then γ is a feasible input assignment.

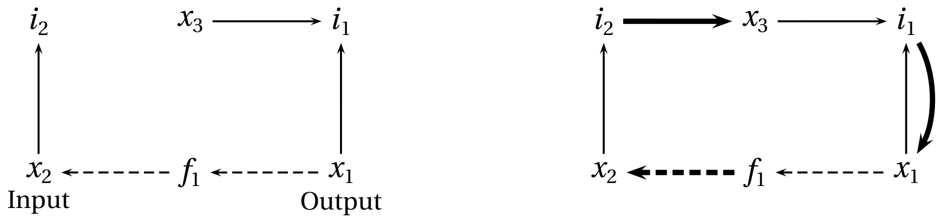
3 Motivating Cases: Integration and Competition

The goal of our analysis is to characterize an economy's plausible outcomes with emphasis on the role of the endowment system and the production relationships between firms. To do so, in Section 4 we formally introduce our solution concept, which extends the exclusion core proposed by Balbuzanov and Kotowski (2019) to economies with production. Working toward that definition, we first consider some examples demonstrating the model's key mechanics and properties.

The high level premise governing resource allocation in our model can be summarized by two informal postulates:

(P1) If a coalition of agents $C \subseteq I$ has a good x in its endowment, i.e., $x \in \omega(C)$, then the coalition can directly exclude other agents or firms from good x by vetoing or blocking its trade or assignment.

(P2) A coalition can exploit trade and production interdependencies to inductively extend its exclusion power beyond the goods in its endowment. Thus, it may



(a) The economy's ownership (solid links) and production (dashed links) relationships.

(b) A production and trading cycle. Links defined by the outcome (μ, γ) are in bold.

Figure 1: The economy of Example 1.

block the consumption of agents and firms that indirectly rely on the coalition's goods at a given outcome.

The inability of any coalition to beneficially leverage P1 and P2 defines the *exclusion core* solution. An outcome (μ, γ) belongs to the economy's exclusion core if no coalition can gain by blocking trade or production directly (P1) or indirectly (P2).

In this section, we explain this idea with a series of motivating examples. These parallel familiar cases from industrial organization and concern a supply chain where a firm sources inputs from suppliers. Example 1 establishes a baseline where there is no conflict of interest among agents. Unsurprisingly, efficient and stable outcomes arise trivially. When agents' interests are in conflict, as in Example 2, instability ensues. Changes to the ownership structure (e.g., vertical integration) or changes in the upstream market (e.g., an increase in supplier competition) become necessary adaptations to support an equilibrium. The former is the focus of Example 3 while the latter is the case described in Examples 4 and 5. All of these market adjustments have received considerable attention in leading theories of firm or market organization, including "transaction cost" (Coase, 1937; Williamson, 1975; Klein et al., 1978) and "property rights" (Grossman and Hart, 1986; Hart and Moore, 1990; Hart, 1995) approaches, and we relate our model to these theories at the end of this section.[7] In the sequel, we extend our analysis to more general economies where the production network among firms need not be linear (as in the following examples) and may, in fact, even contain cycles.

Example 1. Consider the economy presented in panel (a) of Figure 1. This economy has two agents, i_1 and i_2, and one firm, f_1. There are three goods, $X = \{x_1, x_2, x_3\}$. Goods x_2 and x_3 are primary goods while good x_1 is produced by firm f_1, whose

[7]See Gibbons (2005) for a comparative survey of these and other theories.

production function is

$$f_1(Z) = \begin{cases} x_1 & \text{if } x_2 \in Z; \\ \varnothing & \text{otherwise.} \end{cases}$$

In words, f_1 uses x_2 as an input to produce x_1 as an output. This production function has a unique efficient input set, $W_{f_1} = \{x_2\}$, in the notation of (1) above. The implied supply chain relationship is depicted in panel (a) of Figure 1 by the dashed arrows with good x_1 "pointing" to its producer f_1 and firm f_1 "pointing" to the input good x_2 that it consumes for production.[8]

Next we specify the economy's endowment system. As a baseline, we posit a private ownership economy where the endowment system is given by

$$\omega(\varnothing) = \varnothing, \qquad \omega(i_1) = \{x_1, x_3\}, \qquad \omega(i_2) = \{x_2\}, \qquad \omega(i_1, i_2) = \{x_1, x_2, x_3\}.$$

This ownership structure is depicted in panel (a) of Figure 1 by the solid arrows linking each good with its principal. The structure satisfies A1–A4, $C^{x_1} = C^{x_3} = \{i_1\}$, and $C^{x_2} = \{i_2\}$. Thus, agent i_1 owns goods x_1 and x_3, with the latter being a primary good. Agent i_2 starts with good x_2, the primary good needed for the production of x_1. Of course, agent i_1's ability to consume or trade x_1 is contingent on it being produced.

When the agents' preferences are not in conflict, mutually beneficial exchange and production is trivial to arrange. This occurs when, for example, the agents' preferences are

$$\succ_{i_1}: x_1, x_3, x_0, x_2 \qquad \text{and} \qquad \succ_{i_2}: x_3, x_1, x_0, x_2.$$

Both agents consider good x_2 not consumable (it is worse than the outside option x_0), but they disagree which of x_1 or x_3 is better. Given such preferences, the most plausible outcome (and the unique outcome in the exclusion core, which will be defined formally below) is

$$\mu = \begin{pmatrix} i_1 & i_2 \\ x_1 & x_3 \end{pmatrix}, \qquad \gamma = \begin{pmatrix} f_1 \\ x_2 \end{pmatrix}. \tag{2}$$

[8]Throughout the paper we follow the convention that dashed arrows relate to production, e.g., $x_2 \leftarrow\!\!- f_1 \leftarrow\!\!- x_1$ in Figure 1. These arrows point *upstream* or opposite to the direction of the goods' "flow" in a production process. This convention's utility will be apparent in Section 5 where a generalized TTC algorithm will be introduced. Traditionally, such algorithms ask each agent to "point" to the good that he consumes. In our generalization, firms also "point" to the goods that they consume during production.

At (μ,γ), the firm uses good x_2 as an input to produce x_1. Each agent receives his most-preferred good—agent i_1 consumes x_1 and, in exchange for this arrangement, agent i_2 consumes x_3. Efficiency is achieved through an intuitively appealing "cyclic" exchange of goods, as illustrated in panel (b) of Figure 1. It is the same as panel (a) except the agents are now pointing to their assigned consumption good and links representing the consumption assignment μ and the input assignment γ are in boldface for emphasis.

When there is no conflict of interest, as in Example 1, the distribution of property rights is not too important. Agent i_1 could exclude i_2 from x_3, but he has no reason to do so as he would not benefit. Likewise, i_2 could gum up the production of x_1 by blocking access to its critical input. But he too has no reason to do so. The outcome (μ,γ) defined in (2) is therefore consistent with both P1 and P2 and in the market's exclusion core. Matters are different when agents are in conflict as demonstrated in the following extension of the preceding example.

Example 2. Consider again Example 1, but suppose that each agent now desires x_1 the most. That is, each agent's preference is

$$\succ_i: x_1, x_3, x_0, x_2. \tag{3}$$

Given P1 and P2, the likely outcome is now indeterminate. Any outcome where x_1 is not produced is inefficient—given the agent's preferences, it is always better to use x_2 to make x_1 instead of consuming it. Any outcome where i_1 receives x_1 seems unlikely as i_2 can always withhold the critical input x_2 in the hope of securing a more favorable assignment. And, any outcome where i_2 gets x_1 seems incredible as i_1 may renege on the promised exchange once production has occurred. Anticipating hold up, i_2 would hesitate to trade in the first place. The exclusion core is empty in this example.

Trouble arises in Example 2 as postulates P1 and P2 imply that both i_1 and i_2 hold de facto exclusion rights over x_1. Agent i_1 does so *directly* through his outright ownership of that good after it is produced. Agent i_2 does so *indirectly* through his ownership of a critical input. Holding preferences fixed, there are two ways in which to resolve the conflict. Each involves altering a fundamental of the economy—the distribution of property rights or the level of supply competition—to eliminate the ability of one or more agents to exercise his direct or indirect exclusion rights. It is unlikely that such changes can occur quickly in practice, but in the long run we may reasonably expect economies to exhibit fundamentals that reflect this feature.

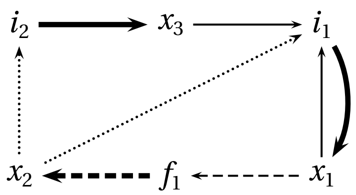


Figure 2: The outcome (μ, γ) in Example 3. Agents $C^{x_2} = \{i_1, i_2\}$ together constitute the principals of critical input good x_2.

In the subsequent examples we describe these changes working toward an outcome where agent i_1 receives good x_1. Parallel arguments can support outcomes where agent i_2 receives good x_1 and we focus on the former case for brevity.

Example 3. The simplest route to address the deficiency in Example 2 is to change the goods' ownership structure through vertical integration. This option is a known method of combatting opportunistic behavior in buyer-supplier relationships with the 1920s buyout of Fisher Body by General Motors being the recurring leading case study (see, e.g., Klein et al., 1978).

Its implementation in our framework is straightforward. In this case, it is sufficient for i_1 to secure joint control of x_2, i.e., the set of principals for x_2 is $C^{x_2} = \{i_1, i_2\}$. In Figure 2, this change is depicted by the dotted links emanating from x_2. The new endowment system is

$$\omega(\varnothing) = \varnothing, \quad \omega(i_1) = \{x_1, x_3\}, \quad \omega(i_2) = \varnothing, \quad \omega(i_1, i_2) = \{x_1, x_2, x_3\}. \tag{4}$$

Now i_2 cannot block the outcome (μ, γ) defined in (2) above. Whereas i_2 prefers x_1, he cannot credibly hold up its production as a bargaining chip. Doing so now requires the agreement of i_1, which will not be forthcoming since he would be made worse off relative to (μ, γ) at any alternative outcome where i_2 gets x_1. Outcome (μ, γ) is also the unique exclusion core outcome. An important feature of this example that reappears in the general model below is that i_1 *does not* need to have outright or independent ownership of x_2. Some overlap in the ownership throughout a supply chain—and not unified control—is the important characteristic.

The next example highlights some of the consequences of postulate P2. It extends Example 2 by adding further layers to the supply chain of good x_1.

Example 4. Consider Example 3 with endowment system ω defined in (4). However, posit that good x_2 is now the output of another firm f_2 using some new primary input x_4. Thus its production function is

$$f_2(Z) = \begin{cases} x_2 & \text{if } x_4 \in Z; \\ \varnothing & \text{otherwise.} \end{cases}$$

Good x_4 belongs to a third agent i_3, i.e., $\omega(i_3) = \{x_4\}$. All agents have the same preferences as in (3), except x_4 is the least-desirable good:

$$\succ_i : x_1, x_3, x_0, x_2, x_4.$$

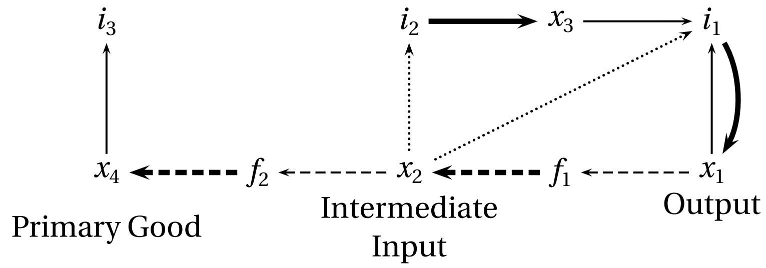
The naive extension of the outcome (μ, γ), which was defined in (2) above, to this situation assigns the input x_4 to firm f_2 to allow for the production of good x_2:

$$\tilde{\mu} = \begin{pmatrix} i_1 & i_2 & i_3 \\ x_1 & x_3 & x_0 \end{pmatrix}, \qquad \tilde{\gamma} = \begin{pmatrix} f_1 & f_2 \\ x_2 & x_4 \end{pmatrix}. \tag{5}$$

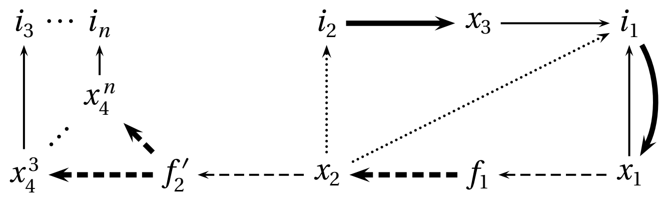
This outcome is depicted in panel (a) of Figure 3. Even though $(\tilde{\mu}, \tilde{\gamma})$ is feasible, it is unlikely to prevail in this market. Just like in Example 2, vertical integration successfully neutralized the ability of agent i_2 to block the outcome. He alone cannot block production of x_1 by withholding x_2. However, agent i_3 can now withhold x_4, making production of x_2 impossible, unless he obtains either x_1 or x_3. The production arrangement unravels just like in Example 2 prior to the change in the ownership structure. Without x_4, x_2 is not made; consequently, neither is x_1. Again, the exclusion core is empty.

One solution to the quandary of Example 4 extends by induction the ideas in Example 3. More vertical integration and a dilution of the ownership structure further up the supply chain will curtail the ability of parties to opportunistically block production. We will not explore this possibility further here; though, it will reappear in the formal analysis to follow. Instead, we will describe how competition may substitute for vertical integration.

Example 5. Amending Example 4, suppose that there now exist many essentially identical copies of good x_4, say $x_4^3, x_4^4, \ldots, x_4^n$, and each copy is initially owned by a distinct agent, $i_3, i_4, \ldots, i_n$, respectively. Call this market's endowment system ω'. Each agent's preferences are the same as in (3) with all x_4^k now being worse than the outside option. Assume each x_4^k is a viable input to make x_2. More concretely,



(a) The unstable outcome $(\tilde{\mu}, \tilde{\gamma})$ in Example 4.



(b) Firm f_2' multisources its inputs at outcome (μ', γ') in Example 5.

Figure 3: Outcomes in Examples 4 and 5.

suppose good x_2 is now made by firm f_2' with production function

$$f_2'(Z) = \begin{cases} x_2 & \text{if } x_4^k \in Z \text{ for some } k = 3, \ldots, n; \\ \varnothing & \text{otherwise.} \end{cases}$$

As before, it is possible for firm f_2' to engage exactly one input—say x_4^k—and produce at capacity. However, at any input assignment where $\gamma(f_2') = x_4^k$, agent i_k would be able to block production, just like i_3 did in Example 4. At that input assignment, input x_4^k is critical.

Matters change when more inputs are engaged and competition takes hold. To take one case, suppose $n \geq 5$ and consider the outcome

$$\mu' = \begin{pmatrix} i_1 & i_2 & i_3 & \cdots & i_n \\ x_1 & x_3 & x_0 & \cdots & x_0 \end{pmatrix}, \qquad \gamma' = \begin{pmatrix} f_1 & f_2' \\ x_2 & \{x_4^3, \ldots, x_4^n\} \end{pmatrix}. \tag{6}$$

At γ', firm f_2' multisources inputs $x_4^3, \ldots, x_4^n$. This outcome is illustrated in panel (b) of Figure 3 with firm f_2' linked to each of $x_4^3, \ldots, x_4^n$. The firm has distorted its input mix by consuming an excess of goods in light of its production technology. However, now no input supplier can credibly block production in the way the monopolist supplier i_3 could in Example 4. At this arrangement, no particular input good is critical for production—x_1 remains in production even if x_4^k is withheld by agent

i_k.[9] Complying with this intuition, outcome (μ', γ') is indeed in the economy's exclusion core.

Taking stock of Examples 4 and 5, we can infer the following principle concerning the extent of vertical integration in a market. In the presence of input specificity (the critical reliance of a producer on an input manufactured by a single upstream supplier), some integration helps stabilize an outcome. In the above examples, we only considered one "layer" of specific intermediate inputs, though the logic extends inductively when goods have longer supply chains. How far should integration extend? Up to any sufficiently competitively supplied goods. These act as a firebreak on the ability of agents to hold-up downstream production. In Example 5, this occurred in the primary sector, through some intermediate goods may too be competitively supplied.[10]

A thread common to the preceding examples is the importance of integration of complementary assets.[11] This is among the main conclusions of "property rights" approach to firm organization pioneered by Grossman and Hart (1986), Hart and Moore (1990), and Hart (1995). That theory also emphasized the distribution of asset ownership, which it argued was instrumental in structuring ex ante investment incentives when contracts are incomplete. Such considerations are absent from our analysis and from property's role or interpretation in our model.[12] Property rights matter for what is, effectively, ex post bargaining in our model. In this regard, our argument is closer in spirit to "rent seeking" or "transaction cost" theories of firm organization where a desire to avoid surplus-destroying conflicts rationalizes ownership structures (Coase, 1937; Williamson, 1975; Klein et al., 1978). A change in the endowment system (i.e., integration) or in the competitive environment ameliorated these conflicts in the examples above. The exclusion rights that agents or coalitions hold can be magnified (or limited) by the entire production or trading network.

[9]In this example we require at least three suppliers of good x_4^k (i.e., $n \geq 5$) for there to be "sufficient competition" in the market. If there are only two (i.e., $n = 4$), then the pair i_3 and i_4 could act as a coalition demanding x_1 and x_3 in exchange for x_4^3 and x_4^4. This would allow both to strictly improve their assignments relative to μ'.

[10]Existing empirical regularities provide suggestive support for these conclusions. For instance, Antràs (2003) shows that, as labor is a competitively supplied primary input, labor-intensive industries see less vertical integration relative to capital-intensive ones, where input specificity is more salient.

[11]Complementarities arose above due to vertical production relationships. Similar examples with horizontal integration are simple to construct.

[12]Notably we do not associate ownership with residual control rights, which is the key interpretation in the property rights approach to the firm. Our model does not preclude this interpretation. Rather, it is simply not part of the theory examined here.

4 The Exclusion Core in Production Economies

This section formalizes the ideas introduced by the motivating examples above and presents our main results.

4.1 Definitions

The exclusion core solution has a familiar structure. It consists of all outcomes that cannot be "blocked" in a particular sense. A coalition will block an outcome if it can benefit from a feasible (for the coalition) rearrangement of the consumption and input assignments. What is feasible was informally stated by postulates P1 and P2 above. We build up to our main definition by first formalizing P1 in Definition 1 and then P2 in Definitions 3 and 4.

P1 says that a coalition can exclude others from goods in its endowment. This suggests the following definition.

Definition 1. Coalition $C \subseteq I$ can *directly exclusion block* the outcome (μ, γ) if there exists a feasible outcome (σ, ψ) such that (a) $\sigma(i) \succ_i \mu(i)$ for all $i \in C$ and (b) $\mu(j) \succ_j \sigma(j) \implies \mu(j) \in \omega(C)$.

Definition 1 has two parts. Part (a) is standard—all coalition members gain at a blocking outcome vis-à-vis the prior outcome. Part (b) reflects exclusion and is the main constraint limiting a coalition's action. If an agent is made worse off by the blocking coalition, it must be the case that he was excluded from something in the blocking coalition's endowment.

P2 relaxes part (b) of Definition 1 by asserting that a coalition has exclusion power over some goods that are not in its endowment. This is because of two channels. (These channels are not mutually exclusive.) The first channel is due to consumption interdependency. If agent i benefits from a coalition's endowment by consuming one of its goods, he is reliant on that coalition. This gives the coalition leverage. It can press agent i to exclude others from goods in his endowment by withholding the good that agent i consumes. The second channel goes through production. If a coalition can exclude a firm from a critical input, it blocks other agents or firms from that firm's production (see, e.g., Example 4 above). A coalition's extended endowment results from the inductive application of these two channels. It is the set of goods over which the coalition holds *de facto* exclusion power at a given outcome. The following definitions operationalize P2 by defining critical inputs (Definition 2) and a coalition's extended endowment (Definition 3).

Definition 2. The set $Z \subseteq X$ is *critical* for the production of good x at input assignment γ if x can be produced by firm f with inputs $\gamma(f)$ and x cannot be produced by f with inputs $\gamma(f) \setminus Z$, i.e., $x \in f(\gamma)$ but $x \notin f(\gamma(f) \setminus Z)$.

Let

$$\alpha_\gamma(Z) := \{x : \exists f \text{ s.t. } x \in f(\gamma) \ \&\ x \notin f(\gamma(f) \setminus Z)\}$$

be the set of goods for which Z is critical at γ.

Definition 3. Given coalition $C \subseteq I$ and outcome (μ, γ), let $Z_0 = \omega(C)$ and for all $k \geq 1$ recursively define

$$Z_k = Z_{k-1} \cup \omega(C \cup \mu^{-1}(Z_{k-1})) \cup \alpha_\gamma(Z_{k-1}). \tag{7}$$

The *extended endowment of coalition C at* (μ, γ) is

$$\Omega_\gamma(C|\omega, \mu) := \bigcup_{k=0}^{\infty} Z_k.$$

The inductive construction of a coalition's extended endowment is apparent in expression (7). Each step k identifies additional goods that a coalition may control, starting from its endowment, $Z_0 = \omega(C)$. Goods in the set "$\omega(C \cup \mu^{-1}(Z_{k-1}))$" enter via the consumption channel—these are goods owned by the coalition or by others who are consuming goods indirectly controlled by the coalition.[13] Goods in the set "$\alpha_\gamma(Z_{k-1})$" enter via the production channel. Production of these goods can be blocked by the coalition as it can withhold critical inputs in Z_{k-1}.

Remark 1. When specialized to an economy without production, Definition 3 differs from the definition of a coalition's extended endowment in Balbuzanov and Kotowski (2019). Lemma 1 in Appendix B demonstrates that Definition 3 is equivalent to the definition in Balbuzanov and Kotowski (2019) given A1–A4.

The next definition takes Definition 1 but replaces the blocking coalition's endowment with its extended endowment in part (b).

Definition 4. Coalition $C \subseteq I$ can *(indirectly) exclusion block* (henceforth, just *exclusion block*) the outcome (μ, γ) if there exists a feasible outcome (σ, ψ) such that (a) $\sigma(i) \succ_i \mu(i)$ for all $i \in C$ and (b) $\mu(j) \succ_j \sigma(j) \implies \mu(j) \in \Omega_\gamma(C|\omega, \mu)$.

[13] Goods that belong only to a coalition of agents are included once all members of that coalition are deemed (in)directly reliant on goods that the coalition (in)directly controls.

As $\omega(C) \subseteq \Omega_\gamma(C|\omega,\mu)$, it follows that if a coalition can directly exclusion block, then it can also exclusion block.[14]

Definition 5. The *exclusion core* is the set of outcomes that cannot be exclusion blocked by any nonempty coalition.

In other words, no coalition can improve upon an exclusion core outcome by invoking its direct or indirect exclusion rights.

An important assumption implicit in Definition 4 concerns the extent to which a blocking coalition can affect the production of firms. Generally, there are many ways that core-like solutions for production economies approach this issue. Per Definition 4, a coalition can seemingly rearrange *all* production in the economy when devising a blocking outcome (σ,ψ). It is not expressly limited to the firms that it "owns." The only constraint on the input assignment in a blocking outcome enters indirectly through part (b) of Definition 4. Satisfying this constraint can limit the production plans pursued by a blocking coalition. All else equal, this formulation makes blocking as easy as possible within our model and makes the exclusion core a stronger solution.[15]

To reinforce the preceding definitions, we revisit the examples from Section 3.

1. Example 1 has one exclusion core outcome that is defined in (2). The agents' preferences are "in alignment" and selecting the only efficient outcome seems natural.

2. In Example 2 all outcomes can be exclusion blocked by some coalition. If agent i_1 is to receive x_1, agent i_2 can block its production as it is in his extended endowment whenever it is produced. If i_2 receives x_1, agent i_1 can exclude him from it directly. As a result, the exclusion core is empty.

3. Example 3 is the same as Example 2 except the endowment system is now ω defined in (4). Now, (μ,γ) defined in (2) is the unique exclusion core outcome.

[14]Exclusion blocking has some superficial similarities but is distinct from setwise blocking (Echenique and Oviedo, 2006) To see that, consider an adaptation of exclusion blocking to many-to-many matching problems under the assumption that an agent can "exclude" any other agent he is matched with from using his services (e.g., a worker unilaterally quitting a firm or a firm unilaterally firing a worker). In this setting, exclusion blocking is stronger than setwise blocking. Specifically, unlike setwise blocking, new links in the blocking matching σ need not be just among members of the blocking coalition C.

[15]The working paper version of this study, Balbuzanov and Kotowski (2022), presented a weaker counterpart to Definition 5 that we called the "ex post exclusion core." It assumed firm's production plans are fixed, i.e., $\psi = \gamma$ in Definition 4. The working paper also considers model extensions to the case of non-rival goods and the consumption of multiple goods. The latter proceeds through an appropriate redefinition of goods and production processes.

The only agent who stands to gain by blocking this outcome is i_2. However, his extended endowment at γ is $\Omega_\gamma(i_2|\omega,\mu)=\varnothing$, so blocking is infeasible. Any outcome where i_2 gets x_1 is exclusion blocked by i_1.

4. The exclusion core is empty in Example 4. The logic parallels Example 2.

5. In Example 5, there are many exclusion core outcomes depending on number of primary good suppliers and the specific sourcing pattern by firm f_2'. However, in every exclusion core outcome, i_1 receives x_1. There are exclusion core outcomes, like (μ',γ') defined in (6) (see also Figure 3b), where i_2 gets x_3; however, there are others where x_3 goes to one of the primary good suppliers instead. At an outcome like (μ',γ'), the extended endowment of a typical primary good supplier i_k, $k\in\{3,\ldots,n\}$, is $\Omega_{\gamma'}(i_k|\omega',\mu')=\{x_4^k\}$. Thus, a lone primary good supplier cannot exclusion block since he cannot prevent the firm f_2' from producing x_2. In contrast, at (μ',γ') a coalition of all primary good suppliers could prevent the production of x_2 by withholding their (collectively) critical inputs. This lets them block production of x_1 and, via the consumption channel, x_3. This coalition's extended endowment is

$$\Omega_{\gamma'}(\{i_3,i_4,\ldots,i_n\}|\omega',\mu')=\{x_4^3,x_4^4,\ldots,x_4^n,x_2,x_1,x_3\}.$$

Despite this endowment, when $n\geq 5$ it is impossible to assure a gain for all coalition members, thus the coalition cannot exclusion block (μ',γ').

4.2 Results and Main Theorem

Our first proposition confirms a basic welfare property of exclusion core outcomes.

Proposition 1. *If (μ,γ) is an exclusion core outcome, then it is Pareto efficient.*

Proof. If (μ,γ) is not Pareto efficient, then there exists a feasible outcome (σ,ψ) such that $\sigma(i)\succeq_i\mu(i)$ for all $i\in I$ and $\sigma(i')\succ_{i'}\mu(i')$ for some $i'\in I$. Thus, (μ,γ) can be exclusion blocked by the coalition $C=\{i'\}$ with (σ,ψ). Part (b) of Definition 4 holds vacuously—there is no agent j for whom $\mu(j)\succ_j\sigma(j)$. □

Note that Pareto efficiency is distinct from the notion of efficient input sets, which was introduced in Section 2. The former concerns the allocation of goods to consumers; the latter concerns the (technically) efficient use of inputs in production and not using more than minimally necessary.

Next we will work toward identifying a large class of economies that have exclusion core outcomes. Examples 2 and 4 above show that the exclusion core may be empty. Thus, some restrictions on the interplay between production technologies, input-output relationships, and the distribution of exclusion rights are necessary. From Examples 3 and 5, we infer that exclusion rights in the economy must be, in some sense, appropriately distributed to neutralize hold up threats throughout the market's active supply chains. Below we will make this intuition precise.

To state our main theorem, we require some further notation. For a given input assignment γ and any set of goods $Z \subseteq X$, let

$$\lambda_\gamma(Z) := \bigcup_{k=0}^{\infty} A_k$$

where $A_0 := Z$ and $A_k := A_{k-1} \cup \alpha_\gamma(A_{k-1})$ for $k \geq 1$. We call $\lambda_\gamma(Z)$ the *(downstream) dependencies of Z at γ*. Starting with the set Z, $\lambda_\gamma(Z)$ includes all goods whose production relies on Z, all goods whose production relies on goods that rely on Z, and so on. Going the other way, for each x let

$$\Lambda_\gamma(x) := \min_{\subseteq} \left\{ Z \in 2^X : x \in \lambda_\gamma(Z) \right\}.$$

$\Lambda_\gamma(x)$ is the set of *critical (upstream) production precursors* of good x at γ. If $Z \in \Lambda_\gamma(x)$, then x is reliant on Z and no proper subset of Z has this property. If a coalition can exclude others from $Z \in \Lambda_\gamma(x)$, it can block the production of x, possibly via indirect dependencies given the input assignment γ. For ease of notation, when $Z = \{z\}$ we may abuse notation by writing $z \in \Lambda_\gamma(x)$ instead (see, e.g., (8′) below).

The set $\Lambda_\gamma(x)$ is never empty. If good x is a primary good, then $\Lambda_\gamma(x) = \{\{x\}\}$.[16] To illustrate cases with non-primary goods, we revisit the production processes in Examples 1–5. In Examples 1–3, good x_1 is produced with x_2 as its critical input. Thus, $\Lambda_\gamma(x_1) = \{\{x_1\}, \{x_2\}\}$ when the input assignment γ is defined in (2). For the remaining goods in these examples, $\Lambda_\gamma(x_k) = \{\{x_k\}\}$ for $k = 2, 3$ as (a set consisting of) good x_k is always in $\Lambda_\gamma(x_k)$. In Example 4, production of good x_2 requires x_4. In this case, $\Lambda_{\tilde{\gamma}}(x_1) = \{\{x_1\}, \{x_2\}, \{x_4\}\}$ where $\tilde{\gamma}$ is defined in (5). Finally, once multisourcing is allowed in Example 5, we have $\Lambda_{\gamma'}(x_1) = \{\{x_1\}, \{x_2\}, \{x_4^3, \ldots, x_4^n\}\}$ instead.

The following theorem provides sufficient conditions for the exclusion core to be nonempty. Its key assumption, B2, links the economy's critical production relationships and the distribution of exclusion rights among coalitions.

Theorem 1. *Let $\mathscr{E} = \langle I, F, X, \succ, \omega \rangle$ be an economy where ω satisfies A1–A4,*

[16] When x is a primary good, then $x \in \lambda_\gamma(Z) \iff x \in Z$. The $(\Leftarrow)$ direction is immediate from the definition of $\lambda_\gamma(Z)$. The $(\Rightarrow)$ direction follows from the fact that $x \notin \alpha_\gamma(Z')$ for all $Z' \not\ni x$. Thus, $x \notin \lambda_\gamma(Z)$ if $Z \not\ni x$, i.e., x has no critical inputs at and γ. Since $x \in \lambda_\gamma(Z)$ if and only if $x \in Z$, the smallest set Z such that $x \in \lambda_\gamma(Z)$ is $Z = \{x\}$.

(B1) each firm has a unique efficient input set,

(B2) there exists a feasible input assignment $\overline{\gamma}$ such that $X = X_{\varnothing} \cup F(\overline{\gamma})$, and

$$C^x_{\overline{\gamma}} := \bigcap_{Z \in \Lambda_{\overline{\gamma}}(x)} \left(\bigcup_{z \in Z} C^z \right) \neq \varnothing \textit{ for all } x \in X. \tag{8}$$

There exists an exclusion core outcome in $\mathscr{E}$.

Assumption B1 refers to a unique efficient input set; this was introduced at (1) above. The first part of assumption B2 ensures that it is feasible to produce all non-primary goods $X \setminus X_{\varnothing}$ under some input assignment $\overline{\gamma}$. This assumption is analogous to Assumption V in Arrow and Debreu (1954, p. 280), which posits a production plan guaranteeing an excess supply of all goods. It *does not* imply that all goods are available for consumption (some goods may be used for producing other goods) or that $\overline{\gamma}$ is the input assignment at an exclusion core outcome.

The second requirement that $\overline{\gamma}$ must satisfy in B2 is (8), which is the theorem's main economic restriction. It alone constrains the relationship between the property rights distribution and production. (In contrast, the rationale for B1 and the first part of B2 is primarily technical.) At a high level, (8) requires there be "sufficient integration" in the ownership structure of the good's critical supply chain.

To unpack condition (8), fix $x \in X$. This good has a nonempty collection $\Lambda_{\overline{\gamma}}(x)$ of critical production precursors at $\overline{\gamma}$. Each set of goods $Z \in \Lambda_{\overline{\gamma}}(x)$ is directly or indirectly crucial for the production of x and x cannot be produced if *all* of Z is unavailable. Therefore, the combined set of principals of Z, $\bigcup_{z \in Z} C^z$, must all agree to credibly withhold Z. Condition (8) then requires that there is a common set of agents who are pivotal in this sense across all critical production precursors. This interpretation is more evident in the special case when condition B1 holds. When B1 holds, Lemma 2(b) in Appendix B shows that all elements of $\Lambda_{\overline{\gamma}}(x)$ are in fact singletons. Thus, with a slight abuse of notation, (8) simplifies to

$$C^x_{\overline{\gamma}} := \bigcap_{z \in \Lambda_{\overline{\gamma}}(x)} C^z \neq \varnothing \text{ for all } x \in X. \tag{8$'$}$$

To further interpret this requirement, we consider a few special cases.[17]

[17]Another alternative to (8) is that the condition "$C^x_{\gamma} \neq \varnothing$ for all $x \in X$" holds for all feasible input assignments γ. However, (8) is weaker and suffices to establish Theorem 1. This is because the input assignment constructed in the theorem's proof is a subgraph of $\overline{\gamma}$ and, therefore, also satisfies (8) in an appropriately adapted sense (Lemma 5 in Appendix B).

First, (8)—or (8′)—is always satisfied for primary goods. As explained above, $\Lambda_\gamma(x) = \{\{x\}\}$ when x is a primary good. Thus, (8) reduces to $C^x_{\overline{\gamma}} = C^x$, which is not empty by A4. More generally, in economies without production (i.e., exchange economies), B1 and B2 hold trivially. And, since A4 relaxes the corresponding condition in Balbuzanov and Kotowski (2019), Theorem 1 implies the corresponding existence result for exchange economies in that paper.

Second, for a good requiring production, condition (8) *does not* imply unified ownership of the good's supply chain. It is weaker, as can be seen by revisiting Example 3. This example satisfies the conditions of Theorem 1. Condition B1 is met since only one input is used for production. To verify B2 it is sufficient to compute (8) for good x_1 at the input assignment $\overline{\gamma} = \gamma$ defined in (2) (the other goods are primary goods):

$$C^{x_1}_{\gamma} = \underbrace{\{i_1\}}_{C^{x_1}} \cap \underbrace{\{i_1, i_2\}}_{C^{x_2}} = \{i_1\}.$$

Thus, the supply chain leading to x_1 is sufficiently integrated to guarantee the existence of an exclusion core outcome. Agent i_1 has sufficient power to neutralize conflict despite not being the sole owner of all the relevant goods.

An economy where (8) does not hold for some good is that of Example 4. In this case, all goods are available at the input assignment $\overline{\gamma} = \tilde{\gamma}$. However, (8) is not satisfied for x_1. For this good $\Lambda_{\tilde{\gamma}}(x_1) = \{\{x_1\}, \{x_2\}, \{x_4\}\}$, which means

$$C^{x_1}_{\tilde{\gamma}} = \underbrace{\{i_1\}}_{C^{x_1}} \cap \underbrace{\{i_1, i_2\}}_{C^{x_2}} \cap \underbrace{\{i_3\}}_{C^{x_4}} = \varnothing.$$

Third, as a sufficient condition, (8) is predicated on the most challenging case. Weaker conditions suffice in most situations. However, in the class of private ownership economies, condition (8) is necessary. Recall that in a private ownership economy for each $x \in X$ there is an $i \in I$ such that $x \in \omega(C)$ if and only if $i \in C$. It is straightforward to verify that a private ownership economy satisfies A1–A4.

Proposition 2. *Consider a tuple $\langle I, F, X, \omega\rangle$, such that each $f \in F$ has a unique efficient input set and for each $x \in X$ there is an agent $i \in I$ such that $x \in \omega(C)$ if and only if $i \in C$. Suppose there exists a feasible input assignment $\overline{\gamma}$ such that $X = X_\varnothing \cup F(\overline{\gamma})$ and*

$$C^x_{\overline{\gamma}} = \bigcap_{Z \in \Lambda_{\overline{\gamma}}(x)} \left(\bigcup_{z \in Z} C^z \right) = \varnothing \text{ for some } x \in X.$$

Then there exists a preference profile $\succ$ such that the exclusion core of the economy $\langle I, F, X, \succ, \omega\rangle$ is empty.

Proof. The unique efficient input set assumption implies that the family $\Lambda_{\overline{\gamma}(x)}$ is comprised only of singleton sets. (It contains x, all goods that are in the efficient input set for manufacturing x, all goods that are in the efficient input sets of those goods and so on.) The emptiness of $C^x_{\overline{\gamma}}$ implies that there exist goods $y, z \in \Lambda_{\overline{\gamma}(x)}$ such that $y \in \omega(i_1)$ and $z \in \omega(i_2)$ for some $i_1 \neq i_2$. Choose a preference profile $\succ$ such that $x_0 \succ_i x'$ for all $x' \in X$ and all $i \in I \setminus \{i_1, i_2\}$, and $x \succ_i x_0 \succ_i x'$ for all $x' \in X \setminus \{x\}$ and all $i \in \{i_1, i_2\}$.

No outcome (μ, γ) in which $[\mu(i_1) \neq x \;\&\; \mu(i_2) \neq x]$ is Pareto efficient and so it is not in the exclusion core. Instead, consider an outcome (μ, γ) such that $\mu(i_1) = x$. We claim that this outcome can be blocked by i_2 with the outcome (σ, γ) where σ is the consumption assignment for which $\sigma(i_2) = x$ and $\sigma(i) = x_0$ for all $i \in I \setminus \{i_2\}$. It is clear that $\sigma(i_2) \succ_{i_2} \mu(i_2)$. To verify condition (b) of Definition 4, note that $\{i \in I : \mu(i) \succ_i \sigma(i)\} = \{i_1\}$. However, as $z \in \omega(i_2)$ and z is critical for the production of x at all input assignments, it follows that $x = \mu(i_1) \in \Omega_\gamma(i_2|\omega, \mu)$. The argument that i_1 can exclusion block any outcome (μ, γ), in which $\mu(i_2) = x$, is identical. □

Finally, we can relax some of the assumptions concerning production in Theorem 1 if we strengthen the assumptions we place on the endowment system.

Proposition 3. *Consider an economy, in which all goods are part of the social endowment, i.e., $C^x = I$ for all $x \in X$. This economy's exclusion core equals the set of Pareto efficient outcomes and is,* a fortiori*, non-empty.*

Proof. By Proposition 1, the exclusion core is contained in the Pareto efficient set. To show the opposite inclusion, assume that (μ, γ) is not in the exclusion core. There exists a coalition C that can exclusion block (μ, γ) with (σ, ψ) such that $\sigma(i) \succ_i \mu(i)$ for all $i \in C$. If $C = I$, it follows that $\{i \in I : \mu(i) \succ_i \sigma(i)\} = \varnothing$. If $C \neq I$, then $\Omega_\gamma(C|\omega, \mu) = \varnothing$ because every good in the economy is part of the social endowment. In order for C to be able to block, this implies $\{i \in I : \mu(i) \succ_i \sigma(i)\} = \varnothing$.

Either way, we see that $\{i \in I : \mu(i) \succ_i \sigma(i)\} = \varnothing$. This means that (σ, ψ) is a Pareto improvement over (μ, γ), and the latter is not Pareto efficient. This completes our proof. □

Note that Proposition 3 does not require that each firm has a unique efficient input set (assumption B1 in Theorem 1) or the existence of an input assignment that allows the production of all goods (assumption B2). Although not invoked in the proposition, condition (8) is nevertheless satisfied at all possible input assignments γ since $C^x_\gamma = I$ for all $x \in X$ and all γ. An immediate corollary to Proposition 3 is that the assumptions B1 and B2 pertaining to production in Theorem 1 are dispensable in the "Robinson Crusoe" version of our economy where there is a single agent.

4.3 Comparative Statics

We conclude this section by examining two comparative statics on the set of exclusion core outcomes. Both are implied by the monotonicity of $\Omega_\gamma(\cdot|\omega,\mu)$ (in the sense of set inclusion) in ω and γ. First, suppose (μ,γ) is an exclusion core outcome when the endowment system is ω. If the endowment system changes to ω' and $\omega'(C) \subseteq \omega(C)$ for all $C \subseteq I$, then (μ,γ) remains in the exclusion core. If instead $\omega(C) \subsetneq \omega'(C)$, this may no longer be the case. Thus, an expansion of exclusion rights tends to contract the exclusion core. This is due to the increased opportunities for rent-seeking behavior.

Second, consider an exclusion core outcome (μ,γ) and suppose the input assignment changes to γ'. If each firm's output is constant but more production links are formed (i.e., $\gamma(f) \subseteq \gamma'(f)$ for all $f \in F$), then (μ,γ') is also in the exclusion core. If instead $\gamma'(f) \subsetneq \gamma(f)$, this may no longer be the case. We can interpret the latter situation as one where forming or maintaining production relationships becomes more difficult, possibly due to not-modeled transaction costs. Even if the remaining production links keep potential output constant, each input's importance is amplified since firms pursue fewer substitute inputs. This change in strategic balance increases suppliers' (indirect) exclusion power, thereby shrinking the exclusion core.

5 Proof of Theorem 1

There are several challenges in establishing Theorem 1. Foremost, Definition 4 assumes that coalitions are unusually powerful. A coalition's blocking ability is determined by its extended endowment, which is large due to trading or production interdependencies. Moreover, the mutability of *all* production plans allows a coalition to craft a preferred assignment easily. An input assignment in an exclusion core outcome must be both constrained (to curtail coalitions' power) and expansive (to supply desired goods). This is a delicate balance.

The proof of Theorem 1 has two parts due to a technical subtlety caused by production cycles. These occur if a good is (indirectly) needed for the production of another good and vice versa. In Part I, we construct an exclusion core outcome in an acyclic economy (defined below), which precludes such cases. This construction proceeds via a generalized TTC algorithm. In Part II, we extend the analysis to a general economy with production cycles.

5.1 Top Trading Cycles and Supply Chains in Acyclic Economies

Consider an economy $\mathscr{E} = \langle I, F, X, \succ, \omega \rangle$ satisfying the hypotheses of Theorem 1. The economy's *input network* is a directed graph where the set of nodes is F and there is a directed edge from $f \in F$ to $f' \in F$ if and only if f uses an output of f' as an input, i.e., if $W_f \cap X_{f'} \neq \emptyset$.[18] The input network is *acyclic* if whenever there is a path in the input network from f to f', there is no path from f' back to f. An economy is acyclic if its input network is acyclic. Acyclicity is a common assumption in studies of production or trading networks (e.g., Ostrovsky, 2008; Manea, 2018). A linear supply chain is an example of an acyclic input network.

A generalized TTC algorithm constructs an exclusion core outcome in an acyclic economy (Lemmas 8 and 9 in Appendix B). We call this generalization the Top Trading Cycles and Supply Chains (TTC-SC) algorithm and present it as Algorithm 1 in Appendix A.[19] Here we explain its intuition and main features.

The TTC algorithm was introduced in Shapley and Scarf's (1974) "house exchange" economy where each good has one owner and each agent owns one item. The TTC algorithm proceeds as follows. Each good "points" to its owner and each agent "points" to his favorite item, thus forming a directed graph with at least one cycle of alternating goods and agents. Each agent in the cycle leaves the market with the good to which he is pointing. The process then repeats with each remaining agent pointing to his favorite remaining item, until all goods are assigned. Our algorithm proceeds similarly; however, we must also ensure that production, if needed, is feasible by simultaneously defining necessary supply chains. This addition sets TTC-SC apart from other TTC generalizations.

The TTC-SC algorithm initializes with each agent pointing to his most-preferred good and each good $x \in X$ pointing to its principal $i_k \in C^x$ with the lowest index number.[20] Each non-primary good also points to its producer (i.e., if $x \in X_f$, then $x \dashrightarrow f$) and each firm points to each of its required inputs (i.e., if $x \in W_f$, then $f \dashrightarrow x$). Necessarily, the graph has at least one cycle of alternating agents and goods.[21] This cycle may be one of three nontrivial types, of increasing complexity.[22] These types are (a) a cycle without production, (b) a cycle with production, and (c)

[18]Recall from (1) that $X_{f'}$ is the set of outputs of firm f' while W_f is the unique efficient input set for firm f.

[19]Despite similar nomenclature, TTC-SC differs from the Trading Cycles and Chains algorithm of Roth et al. (2004), which concerns transplant organ allocation and has no provision for production.

[20]The lowest index principal is chosen for expositional ease. Other selection rules are possible, but are unnecessary for the purpose at hand.

[21]Cycles involving only agents and goods are disjoint. If there are many, we can select any of them.

[22]A trivial fourth case occurs when an agent prefers the outside option. In this case, the agent can be assigned x_0 and removed from the market. The algorithm then moves to its next step.

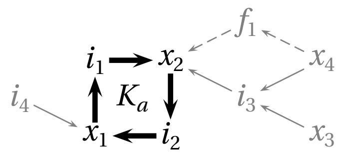
a cycle with an intersecting supply chain. Next, we explain how assignments are defined in each case using Figure 4, which presents independent instances of each case.

(a) A *cycle without production* (Figure 4a) involves only goods that do not require simultaneous production. TTC-SC assigns to each agent in the cycle the good that he is pointing to, like in the classic TTC algorithm. In Figure 4a, i_1 gets x_2 and i_2 gets x_1.

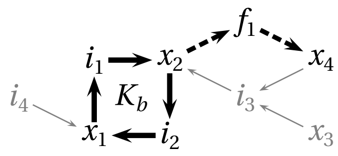
(b) In a *cycle with production* (Figure 4b), at least one of the assigned goods is produced by a firm. Accordingly, its producer must simultaneously get its required inputs to ensure the consumer-demanded good is available. The TTC-SC algorithm clears the cycle by assigning each agent in the cycle the good that he is pointing to *and* it assigns any producing firms their inputs. In Figure 4b, x_2 is made by f_1 using x_4 as an input. This input is a primary good and is not consumed by an agent. Thus, i_1 and i_2 get their desired goods and f_1 is assigned its required input. If, additionally, x_4 was produced by another firm, that firm would be assigned its inputs, and so on.

It is interesting to observe that i_3 cannot exclusion block the resulting outcome even though Figure 4b seems to imply that he wants x_2 and he "gave up" x_4 without receiving anything in return. He is unable to block because, by the assumption of "sufficient integration" in the ownership structure of each good's critical supply chain (B2 in Theorem 1), there must be a principal of x_4 (not in the figure) who is distinct from i_3 and who is also a principal of x_2. The proof of Theorem 1 shows that this co-principal receives a better good than anything a blocking action with i_3 could secure.

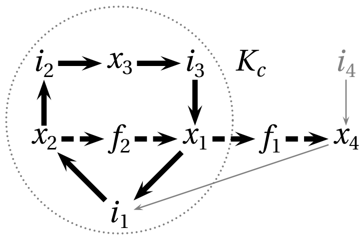
(c) The most complex case is that of a *cycle with an intersecting supply chain* (Figure 4c). Such a cycle involves a produced good, however its production is infeasible because an input is itself assigned as a consumption good in the cycle. In Figure 4c, good x_1 is part of supply chain for x_2 (i.e., $x_2 \dashrightarrow f_2 \dashrightarrow x_1 \dashrightarrow \cdots$) *and* it is assigned to i_3 (i.e., $i_3 \rightarrow x_1$). This "double assignment" is infeasible. The TTC-SC introduces a novel "cycle trimming procedure" to resolve this case. This procedure works by asking the (over-demanded) *upstream* good to point to the same agent as the *downstream* good. In this case, this process amounts to good x_1 pointing to agent i_2 instead of i_1, thus leading to a shorter cycle K_d as shown in Figure 4d. The result is a cycle with production like in case (b) and



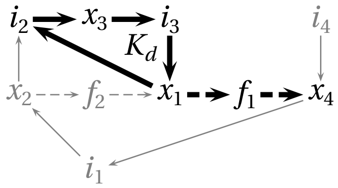
(a) K_a is a cycle without production.



(b) K_b is a cycle where good x_2 is made by f_1 using x_4 as an input.



(c) The cycle K_c has an intersecting supply chain.



(d) The cycle K_d is formed by "trimming" K_c.

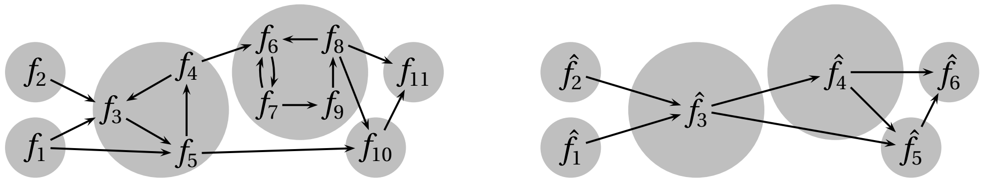
Figure 4: Representative cases encountered by the TTC-SC algorithm.

the assignment proceeds as described above. Condition (8) ensures that the change does not create blocking opportunities.

After an assignment is made, agents and firms receiving goods are removed from the market along with their assignment. Goods whose production is no longer feasible are also removed. The process then iterates: Each remaining agent points to his most-preferred still-available good and each good points to its principal with the lowest index number who remains in the market. (If all of the good's principals have left the market, it points to the remaining agent with the lowest index.) Each remaining produced good points to its producer and each remaining firm points to its inputs. The process stops when no agents remain. Termination occurs as there are finitely many agents and at least one is removed each iteration.

5.2 Acyclic Condensations of Cyclic Economies

To go beyond the acyclic case, we rely on the graph-theoretic concept of a condensation (Bondy and Murty, 2008, pp. 91–92). Firms f and f' are *strongly connected* in an input network if there is a path from f to f' and from f' back to f. A *strongly connected component* is a set of firms F_k such that each pair $f, f' \in F_k$ are strongly connected and F_k is not a proper subset of any other set of strongly connected firms. A firm that is not strongly connected to any other firm forms a strongly connected component by itself. Figure 5a illustrates an input network with six strongly connected components. Within each component, there is either a single firm or an input-output cycle. Figure 5b presents the network's *condensation*, which is formed by contracting the nodes in each strongly connected component into a single node



(a) The original input network. (b) The input network's condensation.

Figure 5: An input network with 11 firms and 6 strongly connected components.

while preserving any external links. A directed graph's condensation is a directed acyclic graph. An acyclic graph's condensation is the graph itself.

We adapt the idea of a condensation of a directed graph to define the *condensation of an economy* $\mathscr{E}$, denoted as $\hat{\mathscr{E}} = \langle \hat{I}, \hat{F}, \hat{X}, \hat{\succ}, \hat{\omega} \rangle$ (formally presented as Definition 6 in Appendix B). Intuitively, the concept involves "merging" the firms in each strongly connected component of the economy's input network.[23] For example, in Figure 5 $f_3, f_4, f_5 \in F$ become $\hat{f}_3 \in \hat{F}$. The economy $\hat{\mathscr{E}}$ is acyclic and has an exclusion core outcome identifiable by the TTC-SC algorithm. The proof of Theorem 1 shows that this implies existence of an exclusion core outcome in the original economy $\mathscr{E}$.

Depending on the modeling context, the concept of a condensation may have an economic interpretation beyond its narrow technical role in our model. One possibility views each "firm" $f \in F$ as a production task, plant, or division within some larger entity, which is the "condensed firm" $\hat{f}_k \in \hat{F}$. More general interpretations in the same spirit include patent pools, joint ventures, or other situations with dense production interdependencies.

6 Related Literature and Concluding Remarks

In this study we have analyzed the interplay between property relations and the economy's network structure. By identifying agents' endowments with a distribution of exclusion rights, our framework has let us determine key contributors to a production and trading network's durability. Ownership—in the narrow sense of exclusion rights—must be sufficiently integrated to ensure that opportunistic peripheral parties do not block production. Our model's generality and its grounding in a basic characterization of property sets it apart from prior studies of these questions.

The economy's network structure has been examined from many complementary perspectives. Related macroeconomic studies date to at least the input-output

[23]There are some further technicalities. Endowments and primary goods must also be defined in $\hat{\mathscr{E}}$.

models of Leontief (1941). Our model's specifics distinguish it from the macro- and trade-oriented literature, which is surveyed by Carvalho and Tahbaz-Salehi (2019). We share its premise that input-output relationships propagate shocks between firms. Elliott et al. (2022) study a model of production network formation where random shocks compromise production links. In their model, firms multisource and invest in link strength as a hedge. By abstracting from risk, our model isolates multisourcing's twin role as a limit on suppliers' relationship-specific monopoly power. That is, it also helps absorb "shocks" arising from firms' strategic actions.

Related to our analysis are studies of trading networks and intermediation (Kranton and Minehart, 2001; Gale and Kariv, 2007; Elliott, 2015; Condorelli et al., 2017; Manea, 2018). A common finding in this literature is that an agent's market power is tied to his position in the network. Our model reinforces this intuition. If a good is a critical input for many firms, possibly indirectly via supply chains, its owners can block many unfavorable outcomes.

Finally, our analysis complements research by Ostrovsky (2008), Hatfield et al. (2013), and Fleiner et al. (2019). These authors extend Hatfield and Milgrom's (2005) "matching with contracts" model to the case of supply chains and trading networks. Our study shares this literature's motivation, but differs on technical and conceptual grounds. The technical distinction concerns the solution concept. Generalizations of "stability" are the preferred solutions in contract-based matching models. Roughly, a set of contracts is stable if no coalition can profitably recontract. In contrast, the exclusion core allows agents to veto others' assignments by invoking their exclusion rights. An exclusion core outcome cannot be upset by any agent profitably exercising such claims.

A conceptual contrast is also noteworthy. At a high level, the matching with contracts framework builds upon Gale and Shapley's (1962) "marriage market" model. Our model's roots are in Shapley and Scarf's (1974) "house exchange" economy. This difference is intriguing given the former's emphasis on (bilateral) *contracts* and the latter's connection to *property* (as argued above). The contracts-property dichotomy is a recognized, though fluid, distinction in legal analysis.[24] It is interesting, therefore, that two seminal models of markets have distinct legal institutions in their foundations. Two related observations follow from this. First, the matching with contracts literature focuses almost exclusively on private ownership, while our approach captures a wide variety of property-rights arrangements. These include joint, disputed, and conditional ownership as well as extensions accommodating

[24] Merrill and Smith (2001b, p. 774) write that "[p]roperty and contracts are bedrock institutions of the legal system, but it is often difficult to say where the one starts and the other leaves off."

government interventions, such as trade sanctions. For discussion and further examples see Balbuzanov and Kotowski (2019, 2021), and Section 6.2 in Balbuzanov and Kotowski (2022).

Second, our model dispenses with the assumption of bilateral transactions that is central in the literature on matching with contracts. Within that framework, multilateral transactions, such as a firm receiving a good owned by multiple agents, would require separate contracts among each pair of participating agents. Our approach is more parsimonious (though, see Hatfield and Kominers 2014, Bando and Hirai 2021 and references therein). The further requirement that contracts satisfy some form of substitutability may preclude such outcomes from being stable at all as, for example, a firm's contracts with the two joint owners of a critical input are necessarily complements.

A Appendix: Top Trading Cycles and Supply Chains

The Top Trading Cycles and Supply Chains (TTC-SC) algorithm requires several preliminary definitions. Fix an economy and let Γ be the set of all feasible input assignments. The input assignment $\gamma \in \Gamma' \subseteq \Gamma$ is *maximal in* Γ' if $\nexists\, \gamma' \in \Gamma'$ such that $\gamma' \neq \gamma$ and $\gamma'(f) \supseteq \gamma(f)$ for every f. For any $X' \subseteq X$ and $F' \subseteq F$, the input assignment $\gamma\colon F' \to 2^X$ is (X', F')*-feasible* if (a) $\gamma(F') \subseteq X' \cup F'(\gamma)$ and (b) $|\{f \in F' : x \in \gamma(f)\}| \leq 1$ for all $x \in X' \cup F'(\gamma)$. If $\gamma(f)$ is efficient for each firm, then $\gamma \in \Gamma$ is *efficient*. When γ is efficient, every element of $\Lambda_\gamma(x)$ is a singleton (the proof is analogous to that of parts (a) and (b) of Lemma 2 in Appendix B). A *maximal efficient* input assignment is maximal among efficient input assignments.

Algorithm 1 (TTC-SC)**.** Given $\mathscr{E} = \langle I, F, X, \succ, \omega\rangle$, construct the outcome (μ, γ) in a series of steps. In step $t \geq 1$, the algorithm proceeds as follows with inputs $(I^t, F^t, X^t_\varnothing)$. I^t is the set of unassigned agents, F^t is the set of unassigned firms, and $X^t_\varnothing$ is the set of primary and already produced goods that are unassigned. Initialize $I^1 := I$, $F^1 := F$, and $X^1_\varnothing := X_\varnothing$.

Step t Let $\gamma^t\colon F^t \to 2^X$ be the maximal efficient $(X^t_\varnothing, F^t)$-feasible input assignment.[25] Let $X^t := X^t_\varnothing \cup F^t(\gamma^t)$. Construct a directed graph as follows. Let $I^t \cup X^t \cup \{x_0\}$ be the set of nodes. Draw an arc from each $i \in I^t$ to the $\succ_i$-maximal element in $X^t \cup \{x_0\}$. If $x \in X^t$ and $C(x) := C^x_{\gamma^t} \cap I^t \neq \varnothing$, draw an arc from x to the lowest-index agent in $C(x)$. Else if $C(x) = \varnothing$, draw an arc from x to the lowest-index agent in I^t.

If there is a link from agent i to x_0, then define $\mu(i) = x_0$, $\tilde{I}^t = \{i\}$, and $\tilde{X}^t = \varnothing$. Update the state variables—$I^{t+1} := I^t \setminus \tilde{I}^t$, $F^{t+1} := F^t$, $X^{t+1}_\varnothing := X^t_\varnothing$—and go to step

[25] Lemma 4 in Appendix B shows that γ^t exists and is uniquely defined.

$t+1$.

Otherwise, from each agent i there is a link to some $x \in X^t$. Since each agent is pointing to a good and each good is pointing to an agent, there exists a cycle of alternating agents and goods. (A cycle may involve one agent and one good.) If there are multiple cycles, they are disjoint and we may focus on any of them.

Select a cycle K and pick two distinct goods $x, y \in K \cap X^t$ such that $x \in \Lambda_{\gamma^t}(y)$. If there are no such goods, continue to Step-t Assignment below. Otherwise, iterate the following operation to define a new cycle until it has no distinct goods x and y such that $x \in \Lambda_{\gamma^t}(y)$.

> *Cycle Trimming.* Since x and y belong to the same cycle, these goods are pointing to different agents. Say, $x \to i$ and $y \to j$. Delete the arc from x to i and draw a new arc from x to j, thus defining the new cycle $K' \subseteq K$. The new cycle K' does not contain good y or agent i.

Step-t Assignment. Given the identified cycle K, perform the following assignments.

(a) If $i \to x$ in the cycle, set $\mu(i) = x$. Let $\tilde{I}^t$ be the set of agents whose assignment has just been defined. Let $I^{t+1} := I^t \setminus \tilde{I}^t$ be the set of agents for whom $\mu(\cdot)$ is yet undefined.

(b) Each $x \in (\bigcup_{z \in K \cap X^t} \Lambda_{\gamma^t}(z)) \setminus X^t_\varnothing$ is either a produced good that is assigned to an agent in (a) or a produced good that is an (indirect) input for a good that is assigned to an agent in (a). This good's producer, say f, belongs to F^t. For each such firm define $\gamma(f) = \gamma^t(f)$. Let $\tilde{F}^t$ be the set of firms whose input assignment has just been defined.

(c) Let $\tilde{X}^t \subseteq \mu(\tilde{I}^t) \cup \gamma(\tilde{F}^t)$ be the set of goods assigned to agents or firms in parts (a) and (b). Let $X^{t+1}_\varnothing := (X^t_\varnothing \cup \bar{F}^t(\gamma^t)) \setminus \tilde{X}^t$ be the set of primary goods or goods produced up to step t that are unassigned.

(d) If $F^t \setminus \tilde{F}^t = \varnothing$, set $F^{t+1} = \varnothing$. Otherwise, define $\check{\gamma}^t$ as the maximal efficient $(X^{t+1}_\varnothing, F^t \setminus \tilde{F}^t)$-feasible input assignment. Let $\check{F}^t := \{f \in F^t \setminus \tilde{F}^t : \check{\gamma}^t(f) = \varnothing\}$ be the set of remaining firms that are assigned no inputs at $\check{\gamma}^t$. For each $f \in \check{F}^t$, assign $\gamma(f) = \varnothing$ and denote the set of these firms' (henceforth, not produced) outputs by $\check{X}^t$. Let $F^{t+1} := F^t \setminus (\tilde{F}^t \cup \check{F}^t)$ be the set of firms for which $\gamma(\cdot)$ is still undefined.

Given the newly defined parameters, $(I^{t+1}, F^{t+1}, X^{t+1}_\varnothing)$, proceed to step $t+1$.

Stop when $I^t = \varnothing$. At this point, $\mu(\cdot)$ has been defined for all $i \in I$ and $\gamma(\cdot)$ has been defined for all $f \in F \setminus F^t$. Set $\gamma(f) = \varnothing$ for any remaining $f \in F^t$. ■

The number of agents is finite and at least one is assigned in each step. Thus, Algorithm 1 terminates in finitely many steps.

B Appendix: Proofs

This section adopts the convention that $Z_{-1} = \varnothing$. We start by showing the equivalence of the definition of an exchange economy's extended endowment with the definition in Balbuzanov and Kotowski (2019), as claimed in Remark 1.

Lemma 1. *In an exchange economy where the endowment system satisfies A1–A4,* $\Omega(C|\omega,\mu) = \omega\left(\bigcup_{k=0}^{\infty} C_k\right)$*, where* $C_0 = C$ *and* $C_k = C_{k-1} \cup (\mu^{-1} \circ \omega)(C_{k-1})$ *for each* $k \geq 1$.

Proof of Lemma 1. Fix C and define $C_0 = C$ and $C_k = C_{k-1} \cup (\mu^{-1} \circ \omega)(C_{k-1})$ for each $k \geq 1$. We make two preliminary observations. First, since $C_k \subseteq C_{k+1}$ and $\omega(\cdot)$ is monotone, it follows that $\omega(\bigcup_{k=0}^{\infty} C_k) = \bigcup_{k=0}^{\infty} \omega(C_k)$. And second, $C_k = C \cup (\mu^{-1} \circ \omega)(C_{k-1})$ for each $k \geq 1$. We can prove this fact by induction. The base case is true since $C_1 = C_0 \cup (\mu^{-1} \circ \omega)(C_0)$ and $C_0 = C$. Let $k \geq 2$ and suppose $C_{k-1} = C \cup (\mu^{-1} \circ \omega)(C_{k-2})$. By definition, $C_k = C_{k-1} \cup (\mu^{-1} \circ \omega)(C_{k-1}) = C \cup (\mu^{-1} \circ \omega)(C_{k-2}) \cup (\mu^{-1} \circ \omega)(C_{k-1})$. Since $C_{k-2} \subseteq C_{k-1}$, $\mu^{-1}(\omega(C_{k-2})) \subseteq \mu^{-1}(\omega(C_{k-1}))$. Hence, $C_k = C \cup (\mu^{-1} \circ \omega)(C_{k-1})$.

To prove the lemma it suffices to show $Z_k = \omega(C_k)$ for all k. If $k = 0$, then $Z_0 = \omega(C) = \omega(C_0)$. Proceeding by induction, let $k \geq 1$. If $Z_{k-1} = \omega(C_{k-1})$, then $Z_k = Z_{k-1} \cup \omega(C \cup \mu^{-1}(Z_{k-1})) = \omega(C_{k-1}) \cup \omega(C \cup \mu^{-1}(\omega(C_{k-1}))) = \omega(C_{k-1}) \cup \omega(C_k) = \omega(C_k)$. □

The proof of Theorem 1 relies on Lemmas 2–10. Lemmas 2–7 are technical preliminaries. Given an acyclic economy, Lemma 8 shows that the TTC-SC algorithm identifies a feasible outcome and Lemma 9 shows that it belongs to the exclusion core. Definition 6 introduces the condensation of an economy. Lemma 10 shows that a condensed economy has an exclusion core outcome. Theorem 1's proof invokes Lemma 10 to show that any economy satisfying the theorem's hypotheses has an exclusion core outcome.

Lemma 2. *Let* $\mathscr{E} = \langle I, X, F, \succ, \omega\rangle$ *be an economy satisfying A1–A4 and B1. Fix a feasible input assignment* γ *and for any* $Z \subseteq X$ *let* $\lambda_\gamma(Z) = \bigcup_{k=0}^{\infty} A_k$ *where* $A_0 = Z$ *and* $A_k = A_{k-1} \cup \alpha_\gamma(A_{k-1})$ *for each* $k \geq 1$.

(a) If $a_K \in A_K$ *and* $a_K \notin A_{K-1}$*, then there exists a sequence* $(a_K, a_{K-1}, \ldots, a_0)$ *such that* $a_k \in A_k$ *for each* $k \geq 0$ *and* $a_k \in \alpha_\gamma(a_{k-1})$ *for each* $k \geq 1$.

(b) For all $x \in X$*, every* $Z \in \Lambda_\gamma(x)$ *consists of a single element.*

(c) *If $z \in \Lambda_\gamma(y)$ and $y \in \Lambda_\gamma(x)$, then $z \in \Lambda_\gamma(x)$.*

(d) *If $y \in \Lambda_\gamma(x)$, then $\bigcap_{z\in\Lambda_\gamma(x)} C^z \subseteq \bigcap_{z\in\Lambda_\gamma(y)} C^z$.*

(e) *If the economy's input network is acyclic, $[x \neq y \ \&\ y \in \Lambda_\gamma(x)] \implies x \notin \Lambda_\gamma(y)$.*

(f) *Suppose F_k is a strongly connected component of the economy's input network and $f, f' \in F_k$ are distinct firms. If $x \in X_f$ is produced at γ and $y \in W_{f'}$,[26] then $y \in \Lambda_\gamma(x)$.*

Proof of Lemma 2. (a) Suppose $a_K \in A_K$ and $a_K \notin A_{K-1}$. Thus, $a_K \in \alpha_\gamma(A_{K-1})$ and A_{K-1} is critical for a_K at γ. Since each firm has a unique efficient production plan, $a_K \in \alpha_\gamma(a_{K-1})$ for some $a_{K-1} \in A_{K-1}$. Moreover, $a_{K-1} \notin A_{K-2}$ (else, $a_K \in A_{K-1}$, which is assumed not true). Repeating this same construction, we can define a sequence $(a_K, a_{K-1}, \dots, a_1, a_0)$ such that $a_k \in A_k$ for each k and $a_k \in \alpha_\gamma(a_{k-1})$.

(b) Suppose $Z \in \Lambda_\gamma(x)$. Thus, $x \in \lambda_\gamma(Z)$. It suffices to show that $x \in \lambda_\gamma(z)$ for some $z \in Z$. This is trivial if $x \in Z$. Thus, suppose $x \notin Z$. Since $x \in \lambda_\gamma(Z) = \bigcup_{k=0}^\infty A_k$, $x \in A_K$ and $x \notin A_{K-1}$ for some some $K \geq 1$. By part (a), there is a sequence $x = a_K, a_{K-1}, \dots, a_1, a_0 = z$ such that $a_k \in \alpha_\gamma(a_{k-1})$ for each $k \geq 1$ and $z \in Z$. Given z, consider the sequence $A_0^z = \{z\}$ and $A_k^z = A_{k-1}^z \cup \alpha_\gamma(A_{k-1}^z)$ for each $k \geq 1$. Clearly, $a_k \in A_k^z$ for each $k = 0, \dots, K$. And so, $x \in \bigcup_{k=0}^\infty A_k^z = \lambda_\gamma(z)$.

(c) If $z \in \Lambda_\gamma(y)$, then $y \in \lambda_\gamma(z) = \bigcup_{k=0}^\infty A_k^z$ where $A_0^z = \{z\}$ and $A_k^z = A_{k-1}^z \cup \alpha_\gamma(A_{k-1}^z)$. Likewise, if $y \in \Lambda_\gamma(x)$, then $x \in \lambda_\gamma(y) = \bigcup_{k=0}^\infty A_k^y$ where $A_0^y = \{y\}$ and $A_k^y = A_{k-1}^y \cup \alpha_\gamma(A_{k-1}^y)$. Let K be the smallest value for which $y \in \bigcup_{k=0}^K A_k^z$. Thus, for all $k \geq K$, $A_{k-K}^y \subseteq A_k^z$. Hence, $\bigcup_{k=0}^\infty A_k^y \subseteq \bigcup_{k=0}^\infty A_k^z$. Therefore, $x \in \lambda_\gamma(z)$ and $z \in \Lambda_\gamma(x)$.

(d) By part (c), $[z \in \Lambda_\gamma(y) \ \&\ y \in \Lambda_\gamma(x)] \implies z \in \Lambda_\gamma(x)$. Thus, $\Lambda_\gamma(y) \subseteq \Lambda_\gamma(x)$. Hence, $\bigcap_{z\in\Lambda_\gamma(x)} C^z \subseteq \bigcap_{z\in\Lambda_\gamma(y)} C^z$.

(e) Suppose $x \neq y$ and $y \in \Lambda_\gamma(x)$. Thus, $x \in \lambda_\gamma(y)$. Given part (a), there exists a sequence of goods $(a_K, \dots, a_0)$ such that $x = a_K$, $y = a_0$ and $a_k \in \alpha_\gamma(a_{k-1})$ for all $k \geq 1$. Since each firm has a unique efficient production plan, there is a link from the firm producing a_k to the firm producing a_{k-1} in the input network. Thus, there is a path from the producer of x to the producer of y. If $x \in \Lambda_\gamma(y)$, the same reasoning implies a path from the producer of y to the producer of x. As the input network is acyclic, this is impossible. Thus, $x \notin \Lambda_\gamma(y)$.

(f) Because $f, f' \in F_k$, there is a path in the input network such that $f = f^1 \to \cdots \to f^L = f'$. If $y \in W_{f'}$, then it is critical for the production of all $x' \in X_{f^L}$. Since $f^{L-1} \to f^L$, there exist $y^L \in X_{f^L} \cap W_{f^{L-1}}$, which is critical for the production of all $x' \in X_{f^{L-1}}$. Continuing in this way, we can construct a sequence $y^\ell \in X_{f^\ell} \cap W_{f^{\ell-1}}$

[26] Recall that $W_{f'}$ is the minimal set of inputs f' requires for production. See (1).

where y^ℓ is a critical input for firm $f^{\ell-1}$'s production. Thus, $x = y^1 \in \alpha_\gamma(y^2)$, $y^2 \in \alpha_\gamma(y^3), \ldots, y^L \in \alpha_\gamma(y)$. Thus, $x \in \lambda_\gamma(y)$ and, therefore, $y \in \Lambda_\gamma(x)$. $\square$

Lemma 3. *Suppose $\mathscr{E} = \langle I, X, F, \succ, \omega\rangle$ satisfies A1–A4 and B1. Let (μ, γ) be a feasible outcome and $C \subseteq I$. Let $Z_k = Z_{k-1} \cup \omega(C \cup \mu^{-1}(Z_{k-1})) \cup \alpha_\gamma(Z_{k-1})$ for each $k \geq 0$. If $x \in Z_k$, then there exists $y \in \omega(C \cup \mu^{-1}(Z_{k-1}))$ such that $y \in \Lambda_\gamma(x)$.*

Proof of Lemma 3. If $x \in Z_0$, then $x \in \omega(C)$ and $x \in \Lambda_\gamma(x)$. Proceeding by induction, let $k \geq 1$ and suppose $x \in Z_{k'} \implies [\exists\ y \in \omega(C \cup \mu^{-1}(Z_{k'-1}))$ s.t. $y \in \Lambda_\gamma(x)]$ is true for all $k' \leq k-1$. Let $x \in Z_k$. There are two cases.

(a) If $x \in \omega(C \cup \mu^{-1}(Z_{k'}))$ for some $k' < k$, then monotonicity of ω and μ^{-1} imply that $\omega(C \cup \mu^{-1}(Z_{k'})) \subseteq \omega(C \cup \mu^{-1}(Z_{k-1}))$. The conclusion follows.

(b) If $x \in \alpha_\gamma(Z_{k'})$ for some $k' < k$, then $Z_{k'}$ is a critical set of inputs for x. By B1, there exists some $x' \in Z_{k'}$ that is a critical input for x, i.e., $x' \in \Lambda_\gamma(x)$. By the induction hypothesis, there exists $y \in \omega(C \cup \mu^{-1}(Z_{k'-1})) \subseteq \omega(C \cup \mu^{-1}(Z_{k-1}))$ such that $y \in \Lambda_\gamma(x')$. Since, $x' \in \Lambda_\gamma(x)$, $y \in \Lambda_\gamma(x)$ by Lemma 2(c). $\square$

Lemma 4. *Suppose $\mathscr{E} = \langle I, X, F, \succ, \omega\rangle$ satisfies B1 and B2. Let $X' \subseteq X$ and $F' \subseteq F$. A maximal efficient (X', F')-feasible input assignment exists and is unique.*

Proof of Lemma 4. The input assignment γ where $\gamma(f) = \varnothing$ for all $f \in F'$ is efficient and (X', F')-feasible. Thus, a maximal efficient (X', F')-feasible input assignment exists. To show uniqueness, let $\gamma \neq \gamma'$ be maximal efficient (X', F')-feasible input assignments. Let $\tilde{\gamma}(f) := \gamma(f) \cup \gamma'(f)$ for all $f \in F'$. As γ and γ' are efficient, so is $\tilde{\gamma}$. (A producing firm must be assigned a unique set of inputs; a non-producing firm must be assigned no inputs.) Due to their maximality, γ and γ' are both nonempty. Therefore, $\tilde{\gamma} \supseteq \gamma$ and $\tilde{\gamma} \neq \gamma$. Then $\tilde{\gamma}$ must not be (X', F')-feasible. Since γ and γ' are (X', F')-feasible, $\tilde{\gamma}(F') \subseteq X' \cup F'(\tilde{\gamma})$. Therefore, there exists a good x such that $|\{f \in F' : x \in \tilde{\gamma}(f)\}| > 1$. Since each firm has a unique efficient production plan, there is more than one firm in F' requiring x as an input. At least one of these firms cannot produce at $\overline{\gamma}$, contradicting B2. $\square$

Lemma 5. *Suppose A1–A4 and B1–B2 hold in economy $\mathscr{E} = \langle I, X, F, \succ, \omega\rangle$. Let $X' \subseteq X$ and $F' \subseteq F$. If γ is (X', F')-feasible, then $C^x_\gamma = \bigcap_{z \in \Lambda_\gamma(x)} C^z \neq \varnothing \forall\ x \in X' \cup F'(\gamma)$.*

Proof of Lemma 5. By Lemma 2(b), $C^x_\gamma = \bigcap_{z \in \Lambda_\gamma(x)} C^z$ for all γ. By (8), $C^x_{\overline{\gamma}} = \bigcap_{z \in \Lambda_{\overline{\gamma}}(x)} C^z \neq \varnothing$. Thus, to prove the lemma it suffices to show that $\Lambda_\gamma(x) \subseteq \Lambda_{\overline{\gamma}}(x)$ for all $x \in X' \cup F'(\gamma)$. Suppose γ is (X', F')-feasible and let $x \in X' \cup F'(\gamma)$. By Lemma 2(b), $\Lambda_\gamma(x)$ contains only singletons. Let $z \in \Lambda_\gamma(x)$. Therefore, $x \in \lambda_\gamma(z) = \bigcup_{k=0}^{\infty} A^\gamma_k$ where $A^\gamma_0 = \{z\}$

and $A_k^\gamma = A_{k-1}^\gamma \cup \alpha_\gamma(A_{k-1}^\gamma)$. (We include the γ superscript on A_k^γ for clarity.) By Lemma 2(a), there is a sequence $(a_K, a_{K-1}, \ldots, a_0)$ such that $x = a_K$, $a_0 = z$, and $a_k = \alpha_\gamma(a_{k-1})$ for each $k \geq 1$. Since each firm has a unique efficient production plan, if a_{k-1} is critical for a_k at γ, it is critical for a_k at $\overline{\gamma}$. Thus, $a_k = \alpha_{\overline{\gamma}}(a_{k-1})$ for each $k \geq 1$. Hence, $a_0 \in A_0^{\overline{\gamma}}$ and $a_k \in A_k^{\overline{\gamma}} = A_{k-1}^{\overline{\gamma}} \cup \alpha_{\overline{\gamma}}(A_{k-1}^{\overline{\gamma}})$ for each $k \geq 1$. And so, $x \in \bigcup_{k=0}^{\infty} A_k^{\overline{\gamma}} = \lambda_{\overline{\gamma}}(z)$, which implies $z \in \Lambda_{\overline{\gamma}}(x)$. Thus, $\Lambda_\gamma(x) \subseteq \Lambda_{\overline{\gamma}}(x)$. $\square$

Lemma 6. *Let $\mathscr{E} = \langle I, X, F, \succ, \omega \rangle$ be an economy satisfying A1–A4 and B1–B2. Consider step t of Algorithm 1 where γ^t is the maximal efficient $(X_\varnothing^t, F^t)$-feasible input assignment and $C(x) = C_{\gamma^t}^x \cap I^t$. Let $K_1 \supseteq \cdots \supseteq K_L$ be a sequence of cycles identified by iterating the "cycle trimming" procedure within this step of the algorithm. Let $x, y \in K_\ell \cap X$, $x \in \Lambda_{\gamma^t}(y)$, and suppose $x \to i$ and $y \to j$ within cycle K_ℓ. Suppose the procedure reassigns x to point to j at iteration ℓ.*

(a) *If $j \in C(y)$, then $j \in C(x)$.*

(b) *If $j \notin C(y)$, then some $i' \in \bigcap_{z \in \Lambda_{\gamma^t}(y)} C^z$ was assigned by Algorithm 1 in step $t' < t$.*

Proof of Lemma 6. Recall that $C(x) = C_{\gamma^t}^x \cap I^t = (\bigcap_{z \in \Lambda_{\gamma^t}(x)} C^z) \cap I^t$. The term in parenthesis is not empty by Lemma 5. The proof is by induction. Consider cycle K_1. Let $x, y \in K_1 \cap X$, $x \in \Lambda_{\gamma^t}(y)$, and suppose $x \to i$ and $y \to j$ within cycle K_1. Suppose the procedure reassigns x to point to j. Since $x \in \Lambda_{\gamma^t}(y)$, $\bigcap_{z \in \Lambda_{\gamma^t}(y)} C^z \subseteq \bigcap_{z \in \Lambda_{\gamma^t}(x)} C^z$ by Lemma 2(d). Thus, if $j \in C(y)$, then $j \in C(x)$. Otherwise, if $j \notin C(y)$, then $C(y) = \varnothing$. Thus, all $i' \in \bigcap_{z \in \Lambda_{\gamma^t}(y)} C^z$ must have been assigned prior to step t.

Proceeding by induction, suppose statements (a) and (b) of the lemma are true for all cycles $K_1, \ldots, K_{\ell-1}$. Let $x, y \in K_\ell \cap X$, $x \in \Lambda_{\gamma^t}(y)$, and suppose $x \to i$ and $y \to j$ within cycle K_ℓ. Suppose the procedure reassigns x to point to j. If $j \in C(y)$, then $j \in C(x)$ as above. Otherwise, $j \notin C(y)$ and there are two cases.

(i) $C(y) = \varnothing$. Thus, all $i' \in \bigcap_{z \in \Lambda_{\gamma^t}(y)} C^z$ have been assigned prior to step t.

(ii) $C(y) \neq \varnothing$. Since y is not pointing to an element of $C(y)$, it must have been reassigned to point to j in some prior iteration $\ell' < \ell$ of the cycle trimming procedure. Suppose at iteration ℓ', $x' \to j$ and $y \in \Lambda_{\gamma^t}(x')$. Invoking the induction hypothesis, if $j \in C(x')$, then $j \in C(y)$, a contradiction. Thus, $j \notin C(x')$ and there exists an $i' \in \bigcap_{z \in \Lambda_{\gamma^t}(x')} C^z \subseteq \bigcap_{z \in \Lambda_{\gamma^t}(y)} C^z$ who was assigned by Algorithm 1 in step $t' < t$. $\square$

Lemma 7. *Let $\mathscr{E} = \langle I, X, F, \succ, \omega \rangle$ be an economy satisfying A1–A4 and B1–B2. Let (μ, γ) be a feasible outcome identified by Algorithm 1. Suppose agent j was assigned* *good y at step t of Algorithm 1. For every $x \in \Lambda_\gamma(y)$, there exists $i \in C^x$ who was assigned his consumption in step t or earlier.*

Proof of Lemma 7. Let $x \in \Lambda_\gamma(y)$. As good y was assigned in step t, good x must be produced in step $s \le t$. (If x is a primary good, $s = 1$.) Let K be the final cycle identified by Algorithm 1 in step s that determines the agents' consumption assignments. There exists a good $x' \in K \cap X$ such that $x \in \Lambda_{\gamma^s}(x')$. (If $x \in K$, then $x = x'$.) Let $i' \in K \cap I^s$ be the agent in the cycle such that $x' \to i'$. Agent i' was assigned $\mu(i')$ in this step of the algorithm. Recalling that $C(x) = C^x_{\gamma^t} \cap I^t = (\bigcap_{z \in \Lambda_{\gamma^t}(x)} C^z) \cap I^t$, there are three cases.

(a) $i' \in C(x')$. Because $x \in \Lambda_{\gamma^s}(x')$, $i' \in C(x') = (\bigcap_{z \in \Lambda_{\gamma^s}(x')} C^z) \cap I^s \subseteq C^x$. Thus, agent $i' \in C^x$ was assigned $\mu(i')$ in step $s \le t$.

(b) $i' \notin C(x')$ and $C(x') = \varnothing$. This implies that $(\bigcap_{z \in \Lambda_{\gamma^s}(x')} C^z) \cap I^s = \varnothing$. By Lemma 5, $\bigcap_{z \in \Lambda_{\gamma^s}(x')} C^z \neq \varnothing$. Thus, every $j' \in \bigcap_{z \in \Lambda_{\gamma^s}(x')} C^z \subseteq C^x$ must have been assigned $\mu(j')$ before step $s \le t$.

(c) $i' \notin C(x')$ and $C(x') \neq \varnothing$. If x' was pointing to an agent not in $C(x')$, it is because during the trimming procedure in step s it was reassigned to point to agent i'. By Lemma 6(b), $\exists\ j' \in \bigcap_{z \in \Lambda_{\gamma^s}(x')} C^z \subseteq C^x$ who was assigned $\mu(j')$ in step $s' < s \le t$ of Algorithm 1.

In each case, there is a member of C^x assigned in or before step t. □

Lemma 8. *Let $\mathcal{E} = \langle I, X, F, \succ, \omega \rangle$ be an acyclic economy satisfying A1–A4 and B1–B2. The outcome (μ, γ) identified by Algorithm 1 is feasible.*

Proof of Lemma 8. It suffices to show that if a good is assigned to agent i in step t of Algorithm 1, then it is not also assigned to any firm. (The other possibilities are ruled out by the algorithm's definition and the feasibly of γ^t.) Assume the contrary. Suppose y is assigned to an agent and to a firm in step t. If y is a produced good, then it cannot be assigned to the firm that produces y (γ^t is efficient and y is not an input for itself). Thus, there exists $x \neq y$ such that $y \in \Lambda_{\gamma^t}(x)$. If x is assigned to an agent at step t, then both x and y belong to the cycle determining the consumption assignment at step t. But, this violates the trimming procedure's stopping criterion. Therefore, x is not assigned to another agent at step t. But this implies $x \in \Lambda_{\gamma^t}(x')$ for some x' that is assigned to an agent at step t. By parts (b) and (c) of Lemma 2, $y \in \Lambda_{\gamma^t}(x')$. Thus, if $x' \neq y$, we arrive at a contradiction, as above. If $x' = y$, then we contradict Lemma 2(e) because $x \neq y$, $y \in \Lambda_{\gamma^t}(x)$, and $x \in \Lambda_{\gamma^t}(y)$. Each case leads to a contradiction. Thus, no good is assigned to an agent and to a firm. □

Lemma 9. *Let $\mathscr{E} = \langle I, X, F, \succ, \omega \rangle$ be an acyclic economy satisfying A1–A4 and B1–B2. The outcome constructed by Algorithm 1 is in the economy's exclusion core.*

Proof of Lemma 9. Let (μ, γ) be an outcome identified by Algorithm 1 in $\mathscr{E}$. Before presenting the proof, we define some useful terminology and notation. Algorithm 1 constructed (μ, γ) sequentially by removing sets of agents $(\tilde{I}^1, \tilde{I}^2, \dots)$ and goods $(\tilde{X}^1 \cup \check{X}^1, \tilde{X}^2 \cup \check{X}^2, \dots)$. Each $i \in \tilde{I}^t$ was *assigned (his consumption) in step* t and each $x \in \tilde{X}^t \cup \check{X}^t$ was *removed from the market in step* t. The latter can occur for two reasons: (a) each $x \in \tilde{X}^t$ was assigned to an agent or to a firm in step t; or, (b) the production of each $x \in \check{X}^t$ became impossible given the assignments in step t. Colloquially, the firm (potentially) producing x was "shut down."

To derive a contradiction, assume (μ, γ) is not in the exclusion core. Thus, there exists a feasible outcome (σ, ψ) and coalition $C \subseteq I$ such that $\sigma(i) \succ_i \mu(i)$ for all $i \in C$ and

$$\mu(j) \succ_j \sigma(j) \Longrightarrow \mu(j) \in \Omega_\gamma(C|\omega, \mu). \tag{9}$$

Without loss of generality, we may assume that $C = \{i \in I : \sigma(i) \succ_i \mu(i)\}$. Algorithm 1 ensures that $\sigma(i) \succ_i \mu(i) \succeq_i x_0$ for all $i \in C$. For every $i \in C$ assigned in step t, $\sigma(i)$ must have been removed from the market in step $t' < t$. Otherwise, i would not have been pointing to his most preferred still available good at the start of step t.

Claim 1. *There exists an agent j such that $\mu(j) \succ_j \sigma(j)$. Moreover, this agent was assigned his consumption before any member of coalition C.*

Proof of Claim 1. Consider the good $x \in \sigma(C)$ removed from the market earliest by Algorithm 1. (If there are multiple such goods, pick any of them.) Suppose this occurs in step t. Since x was removed from the market, $x \notin \mu(C)$. (Otherwise, there would be some $x' \in \sigma(C)$ removed from the market strictly earlier than x.) Thus, $C \cap \tilde{I}^{t'} = \varnothing$ for all $t' \leq t$.

As the assignment of x is different at (μ, γ) than at (σ, ψ), three cases are possible.

(a) There exists $j \in I$ such that $\mu(j) = x$. Thus, j is assigned x in step t of Algorithm 1. As preferences are strict, $j \notin C$ and $\sigma(j) \neq \mu(j)$ imply that $\mu(j) \succ_j \sigma(j)$.

(b) There exists $f \in F$ such that $x \in \gamma(f)$. Thus, there is some produced good x' such that (i) x' is assigned to some agent j in step t, and (ii) $x \in \Lambda_{\gamma^t}(x')$. If $x \in \sigma(C)$ then it is unavailable as an input at (σ, ψ). Because each firm has a unique efficient production plan, if x is unavailable as an input for f, then f cannot produce. (The input assignment at γ was efficient.) Therefore, good x' also cannot be produced at (σ, ψ). Thus, agent j's consumption must be different at (σ, ψ), i.e., $x' = \mu(j) \neq \sigma(j)$. Since $j \notin C$, it follows that $\mu(j) \succ_j \sigma(j)$.

(c) The firm producing good x is "shut down" at step t of Algorithm 1. This occurs only if an (indirect) input x' for the production of x becomes unavailable at step t. An (indirect) input becomes unavailable only if it is assigned as a consumption good to some agent j (i.e., $\mu(j) = x'$) in step t. (This is because x is available at the beginning of step t and the input assignment γ^t is feasible.) But, if x is consumed at (σ, ψ), then x' cannot be consumed by j at σ. Hence, $\sigma(j) \neq \mu(j)$. Since no member of C is assigned in step t or earlier, $j \notin C$ and $\mu(j) \succ_j \sigma(j)$.

In each case, some agent j is assigned in step t of Algorithm 1 and $\mu(j) \succ_j \sigma(j)$. $\diamond$

Given Claim 1, let j be the agent assigned earliest by Algorithm 1 and for whom $\mu(j) \succ_j \sigma(j)$. (If there are multiple such agents, choose any of them.) Suppose j's assignment was set in step t^*. By Claim 1, each $i \in C$ was assigned strictly after step t^*.

Next, we show that $\mu(j) \notin \Omega_\gamma(C|\omega, \mu)$, which will contradict (9) and thus prove the theorem. Define $Z_\ell = \varnothing$ for all $\ell \leq -1$ and $Z_\ell = Z_{\ell-1} \cup \omega(C \cup \mu^{-1}(Z_{\ell-1})) \cup \alpha_\gamma(Z_{\ell-1})$ for each $\ell \geq 0$. Suppose $\mu(j) \in Z_0 = \omega(C)$. By Lemma 7, there exists $i \in C^{\mu(j)} \subseteq C$ who was assigned at step t^* or earlier. However, from above we know that no member of C was assigned in step t^*, or earlier, of Algorithm 1—a contradiction.

Proceeding by induction, let $k \geq 1$ and assume that for $k' = k-1$, (a) no agent in $C \cup \mu^{-1}(Z_{k'-1})$ was assigned at any step $t \leq t^*$ by Algorithm 1, and (b) $\mu(j) \notin \bigcup_{\ell=0}^{k'} Z_\ell$. We will verify that (a) and (b) are true for $k' = k$.

Verification of (a). Suppose $i \in C \cup \mu^{-1}(Z_{k-1})$ was assigned at step $t \leq t^*$ by Algorithm 1. Since $Z_0 \subseteq \cdots \subseteq Z_{k-1}$ and no member of $C \cup \mu^{-1}(Z_{k-2})$ was assigned at any step $t \leq t^*$, $i \in \mu^{-1}(Z_{k-1} \setminus Z_{k-2})$. Because $Z_{k-1} = Z_{k-2} \cup \omega(C \cup \mu^{-1}(Z_{k-2})) \cup \alpha_\gamma(Z_{k-2})$, it must be the case that $\mu(i) \in \omega(C \cup \mu^{-1}(Z_{k-2})) \cup \alpha_\gamma(Z_{k-2})$. There are two cases.

(i) $\mu(i) \in \omega(C \cup \mu^{-1}(Z_{k-2}))$. By Lemma 7, there exists $i' \in C^{\mu(i)} \subseteq C \cup \mu^{-1}(Z_{k-2})$ who was assigned at step $t \leq t^*$ of Algorithm 1—a contradiction with (a) above.

(ii) $\mu(i) \in \alpha_\gamma(Z_{k-2})$. Since Z_{k-2} is a critical input set for $\mu(i)$, there exists $x \in Z_{k-2}$ such that $x \in \Lambda_\gamma(\mu(i))$. By Lemma 7, there exists $i' \in C^x$ who was assigned his consumption in step $t \leq t^*$ of Algorithm 1. Since i' is assigned before step t^*, by the induction hypothesis $i' \notin C \cup \mu^{-1}(Z_{k-2})$. Therefore, $x \notin \omega(C \cup \mu^{-1}(Z_{k-2}))$. By Lemma 3, there exists $y \in \omega(C \cup \mu^{-1}(Z_{k-3}))$ such that $y \in \Lambda_\gamma(x)$. By Lemma 2(c), $y \in \Lambda_\gamma(\mu(i))$. By Lemma 7, $\exists i'' \in C^y \subseteq C \cup \mu^{-1}(Z_{k-3})$ who was assigned his consumption in step $t \leq t^*$ of Algorithm 1. However, this is a contradiction as no member of $C \cup \mu^{-1}(Z_{k-3})$ can be assigned in step t^* or earlier.

As each case leads to a contradiction, no agent in $i \in C \cup \mu^{-1}(Z_{k-1})$ is assigned at step $t \leq t^*$ by Algorithm 1.

Verification of (b). Toward a contradiction, suppose $\mu(j) \in \bigcup_{\ell=0}^{k} Z_\ell$. Necessarily, this implies $\mu(j) \in Z_k \setminus Z_{k-1}$ and, in particular, $\mu(j) \in \omega(C \cup \mu^{-1}(Z_{k-1})) \cup \alpha_\gamma(Z_{k-1})$. Applying the same arguments (with all indices shifted up by one) from the verification of (a) above, together with the induction conclusion of (a), we reach a contradiction and establish that $\mu(j) \notin \bigcup_{\ell=0}^{k} Z_\ell$.

As the number of goods is finite, $\bigcup_{\ell=0}^{\infty} Z_\ell = \bigcup_{\ell=0}^{L} Z_\ell$ for some $L \in \mathbb{N}$. Thus, the preceding induction argument confirms that $\mu(j) \notin \bigcup_{\ell=0}^{\infty} Z_\ell$. $\square$

Definition 6. Let $\mathscr{E} = \langle I, X, F, \succ, \omega \rangle$ be an economy satisfying A1–A4 and B1–B2. Let $\{F_1, \dots, F_L\}$ denote the strongly connected components of $\mathscr{E}$'s input network. For each $k = 1, \dots, L$, let $W_{\hat{f}_k} := (\bigcup_{f \in F_k} W_f) \setminus (\bigcup_{f \in F_k} X_f)$ and $X_{\hat{f}_k} := (\bigcup_{f \in F_k} X_f) \cap \{x \in X : \nexists f \in F_k \text{ s.t. } x \in W_f\}$. The *condensation* of $\mathscr{E}$ is the economy $\hat{\mathscr{E}} = \langle \hat{I}, \hat{F}, \hat{X}, \hat{\succ}, \hat{\omega} \rangle$ where:

- The set of agents is $\hat{I} := I$.
- The set of firms is $\hat{F} := \{\hat{f}_1, \dots, \hat{f}_K\}$ where, relabeling if necessary, $\hat{F}$ includes all $\hat{f}_k$ such that $W_{\hat{f}_k} \neq \varnothing$. Each firm's production function is $\hat{f}_k(Z) = X_{\hat{f}_k}$ if $Z \supseteq W_{\hat{f}_k}$ and $\hat{f}_k(Z) = \varnothing$ otherwise. Let $\tilde{F} := \{\hat{f}_{K+1}, \dots, \hat{f}_L\}$ be the other $\hat{f}_k$ for whom $W_{\hat{f}_k} = \varnothing$.
- The set of goods is $\hat{X} := \hat{X}_\varnothing \cup (\bigcup_{\hat{f} \in \hat{F}} X_{\hat{f}})$ where $\hat{X}_\varnothing := X_\varnothing \cup (\bigcup_{\hat{f} \in \tilde{F}} X_{\hat{f}})$ are primary.
- The preference of each $i \in \hat{I}$ equals $\succ_i$ restricted to $\hat{X}$, i.e., $\hat{\succ}_i = \succ_i|_{\hat{X}}$.
- The endowment of each $C \subseteq \hat{I}$ is defined as $x \in \hat{\omega}(C) \iff [x \in \hat{X} \;\&\; \hat{C}^x \subseteq C]$ and each $\hat{C}^x$ is defined as follows. If $x \in X_\varnothing$, then $\hat{C}^x := C^x$. Otherwise, if $x \in X_{\hat{f}_k}$ for some $\hat{f}_k \in \hat{F} \cup \tilde{F}$, then

  $$\hat{C}^x := C^x \cap \left(\bigcap_{y \in Y_{\hat{f}_k}} C^y \right) \tag{10}$$

  where $Y_{\hat{f}_k} = (\bigcup_{f \in F_k} W_f) \cap (\bigcup_{f \in F_k} X_f)$.

Lemma 10. *Let $\mathscr{E} = \langle I, X, F, \succ, \omega \rangle$ be an economy satisfying A1–A4, B1, and B2. There exists an exclusion core outcome in the condensation of $\mathscr{E}$.*

Proof of Lemma 10. It suffices to show that the condensed economy $\hat{\mathscr{E}} = \langle \hat{I}, \hat{F}, \hat{X}, \hat{\succ}, \hat{\omega} \rangle$ satisfies the hypotheses of Lemma 9. First, $\hat{\mathscr{E}}$ is acyclic. This is because the input network of $\hat{\mathscr{E}}$ is a subgraph of the condensation of the input network of $\mathscr{E}$. A directed graph's condensation is acyclic (Bondy and Murty, 2008, pp. 91–92). Second, by definition, each $\hat{f}_k \in \hat{F}$ has unique efficient production plan, is monotone, and satisfies the no free lunch property. We verify the remaining three requirements as separate claims.

Claim 1. *The endowment system $\hat{\omega}$ satisfies A1–A4.*

Proof of Claim 1. It suffices to show that $\hat{C}^x \neq \varnothing$ for all $x \in \hat{X}$. When this is true, properties A1–A4 follow from the definition of $\hat{\omega}$. If $x \in X_{\varnothing}$, then $\hat{C}^x = C^x \neq \varnothing$. If $x \in X_{\hat{f}_k}$ for some $\hat{f}_k \in \hat{F} \cup \tilde{F}$, then $\hat{C}^x = C^x \cap (\bigcap_{y \in Y_{\hat{f}_k}} C^y)$ where $Y_{\hat{f}_k} = (\bigcup_{f \in F_k} W_f) \cap (\bigcup_{f \in F_k} X_f)$. If $y \in Y_{\hat{f}_k}$, then $y \in \Lambda_{\overline{\gamma}}(x)$ by Lemma 2(f). By Lemma 2(b), $\bigcap_{Z \in \Lambda_{\overline{\gamma}(x)}}(\bigcup_{z \in Z} C^z) \neq \varnothing$ reduces to $\bigcap_{z \in \Lambda_{\overline{\gamma}(x)}} C^z \neq \varnothing$. Thus,

$$\hat{C}^x = C^x \cap \left(\bigcap\nolimits_{y \in Y_{\hat{f}_k}} C^y\right) \supseteq C^x \cap \left(\bigcap\nolimits_{y \in \Lambda_{\overline{\gamma}}(x)} C^y\right) = \bigcap\nolimits_{z \in \Lambda_{\overline{\gamma}}(x)} C^z \neq \varnothing \tag{11}$$

where the second equality follows from the fact that $x \in \Lambda_{\overline{\gamma}}(x)$. ◇

Claim 2. *There exists a feasible input assignment $\gamma' : \hat{F} \to 2^{\hat{X}}$ such that $\hat{X} = \hat{X}_{\varnothing} \cup \hat{f}_{\hat{F}}(\gamma')$.*

Proof of Claim 2. Define $\gamma' : \hat{F} \to 2^{\hat{X}}$ as follows. Let $\gamma'(\hat{f}_k) = (\bigcup_{f \in F_k} W_f) \setminus (\bigcup_{f \in F_k} X_f)$ for each $\hat{f}_k \in \hat{F}$. At γ', the output of $\hat{f} \in \hat{F}$ is $X_{\hat{f}}$. Thus, $\hat{X}_{\varnothing} \cup \hat{f}_{\hat{F}}(\gamma') = \hat{X}_{\varnothing} \cup (\bigcup_{\hat{f} \in \hat{F}} X_{\hat{f}}) = \hat{X}$.

Next we verify that γ' is feasible. Suppose $x \in \gamma'(\hat{F})$. Thus, $x \in \gamma'(\hat{f}_k) = (\bigcup_{f \in F_k} W_f) \setminus (\bigcup_{f \in F_k} X_f)$ for some $\hat{f}_k \in \hat{F}$. In particular, $x \in W_f$ for some $f \in F_k$. There are two possibilities. If x is a primary good, then $x \in X_{\varnothing} \subseteq \hat{X}_{\varnothing}$. Otherwise, $x \in X_{f'}$ for some $f' \in F_\ell \neq F_k$. We know that $X_{\hat{f}_\ell} = (\bigcup_{f \in F_\ell} X_f) \cap \{x \in X : \nexists f \in F_\ell \text{ s.t. } x \in W_f\}$. Clearly, $x \in X_{\hat{f}_\ell}$ if and only if no other firms in F_ℓ employ x as an input. If $f'' \in F_\ell$ uses x as an input, then f and f'' cannot both produce in $\mathscr{E}$ at $\overline{\gamma}$, contradicting B2. Thus, $\gamma'(\hat{F}) \subseteq \hat{X}_{\varnothing} \cup \hat{f}_{\hat{F}}(\gamma')$.

Finally, suppose $1 < |\{\hat{f} \in \hat{F} : x \in \gamma'(\hat{f})\}|$ for some $x \in \hat{X}$. This implies there exist $f, f' \in F$ such that $W_f \cap W_{f'} \neq \varnothing$. Thus, both f and f' cannot produce at $\overline{\gamma}$, contradicting B2. ◇

Claim 3. $\hat{C}^x_{\gamma'} := \bigcap_{Z \in \Lambda_{\gamma'}(x)}(\bigcup_{z \in Z} \hat{C}^z) \neq \varnothing$ *for all $x \in \hat{X}$.*

Proof of Claim 3. By Lemma 2(b), $\hat{C}^x_{\gamma'} = \bigcap_{Z \in \Lambda_{\gamma'}(x)}(\bigcup_{z \in Z} \hat{C}^z) = \bigcap_{z \in \Lambda_{\gamma'}(x)} \hat{C}^z$. If x is a primary good or is not produced at γ', then $\Lambda_{\gamma'}(x) = \{x\}$ and $\hat{C}^x_{\gamma'} = \hat{C}^x \neq \varnothing$. Otherwise, $x \in X_{\hat{f}_k}$ for some $\hat{f}_k \in \hat{F}$. If z is an (indirect) critical input for x at γ', it must also be an (indirect) critical input for x at $\overline{\gamma}$ in $\mathscr{E}$. This is because each firm has a unique efficient production plan. Therefore, $\Lambda_{\gamma'}(x) \subseteq \Lambda_{\overline{\gamma}}(x)$. Thus, $\bigcap_{z \in \Lambda_{\gamma'}(x)} \hat{C}^z \supseteq$

$\bigcap_{z\in\Lambda_{\gamma'}(x)}(\bigcap_{y\in\Lambda_{\overline{\gamma}}(z)} C^y) \supseteq \bigcap_{y\in\Lambda_{\overline{\gamma}}(x)} C^y \neq \varnothing$. The first set inclusion is by (11). The second is because if z is an indirect critical input for x at γ' and y is an indirect critical input for z at $\overline{\gamma}$, then y is an indirect critical input for x at $\overline{\gamma}$. The inequality is because (8) holds in $\mathscr{E}$ and Lemma 2(b). $\diamond$

$\square$

Proof of Theorem 1. Consider the economy $\mathscr{E} = \langle I, F, X, \succ, \omega\rangle$. Let $\{F_1, \ldots, F_L\}$ be the strongly connected components of its input network. By Lemma 10, its condensation $\hat{\mathscr{E}} = \langle \hat{I}, \hat{F}, \hat{X}, \hat{\succ}, \hat{\omega}\rangle$ has an exclusion core outcome $(\hat{\mu}, \hat{\gamma})$. As in Definition 6, each $\hat{f}_k \in \hat{F} \cup \tilde{F}$ is defined with respect to the corresponding strongly connected component F_k. If $\hat{f}_k \in \hat{F}$, then $W_{\hat{f}_k} \neq \varnothing$. If $\hat{f}_k \in \tilde{F}$, then $W_{\hat{f}_k} = \varnothing$ and $X_{\hat{f}_k} \subseteq \hat{X}_\varnothing$ in $\hat{\mathscr{E}}$. Define the outcome (μ, γ) in $\mathscr{E}$ as follows. For each $i \in I$, let $\mu(i) = \hat{\mu}(i)$. For each $f \in F_k \subseteq F$, let

$$\gamma(f) = \begin{cases} W_f & \text{if } \hat{f}_k \in \hat{F} \text{ produces its output } X_{\hat{f}_k} \text{ at } (\hat{\mu}, \hat{\gamma}) \text{ in } \hat{\mathscr{E}} \text{ (i.e., } \hat{\gamma}(\hat{f}_k) \neq \varnothing\text{);} \\ W_f & \text{if } \hat{f}_k \in \tilde{F}; \\ \varnothing & \text{otherwise.} \end{cases}$$

We will verify that (μ, γ) is an exclusion core outcome in $\mathscr{E}$.

Claim 1. *The outcome (μ, γ) is feasible in $\mathscr{E}$.*

Proof of Claim 1. We first show that $\mu(I) \cup \gamma(F) \subseteq X_\varnothing \cup \{x_0\} \cup F(\gamma)$. It suffices to show that $[x \in \mu(I) \cup \gamma(F)\ \&\ x \notin X_\varnothing \cup \{x_0\}] \implies x \in F(\gamma)$. There are two cases.

(a) If $x \in \mu(I)$, then x is available at $(\hat{\mu}, \hat{\gamma})$ in $\hat{\mathscr{E}}$. Since $x \notin X_\varnothing \cup \{x_0\}$, there exists some $\hat{f}_k \in \hat{F} \cup \tilde{F}$ such that $x \in X_{\hat{f}_k}$ and some $f \in F_k$ such that $x \in X_f$. By definition of γ, $\gamma(f) = W_f$. Thus, x is produced at (μ, γ) in $\mathscr{E}$, i.e., $x \in F(\gamma)$.

(b) Suppose $x \in \gamma(F)$. Thus, there exists $f \in F_k \subseteq F$ such that $x \in W_f = \gamma(f)$. Since $\gamma(f) \neq \varnothing$, $X_{\hat{f}_k}$ is available at $(\hat{\mu}, \hat{\gamma})$ in $\hat{\mathscr{E}}$. There are two subcases.

   (i) If $x \in W_{\hat{f}_k}$, then $x \in X_{\hat{f}_\ell}$ for some $\hat{f}_\ell \in \hat{F} \cup \tilde{F}$ and $\hat{f}_\ell \neq \hat{f}_k$. This implies there exists some $f' \in F_\ell \subseteq F$ such that $x \in X_{f'}$ and, by definition of γ, $\gamma(f') = W_{f'}$. Therefore, x is produced at (μ, γ) in $\mathscr{E}$, i.e., $x \in F(\gamma)$.

   (ii) If $x \notin W_{\hat{f}_k}$, then $\hat{f}_k \in \tilde{F}$ and the producer of x, say f', must belong to the same strongly connected component F_k. In this case, the definition of γ implies that $\gamma(f') = W_{f'}$. Thus, good x is produced by f' at (μ, γ) in $\mathscr{E}$, i.e., $x \in F(\gamma)$.

The following three points together imply that $|\{i \in I : \mu(i) = x\}| + |\{f \in F : x \in \gamma(f)\}| \leq 1$ for all $x \in X$ and prove the claim.

(a) At most one agent can be assigned x at (μ,γ) in $\mathscr{E}$. This is because $\mu(i)=\hat{\mu}(i)$ for each i and $(\hat{\mu},\hat{\gamma})$ is feasible in $\hat{\mathscr{E}}$.

(b) At most one firm can be assigned x at (μ,γ) in $\mathscr{E}$. To see this, suppose $f \neq f'$ are both assigned x as an input. Because x has unit capacity, firms f and f' cannot both produce at $\overline{\gamma}$, a contradiction since all goods are produced at $\overline{\gamma}$.

(c) It is impossible for an agent and a firm to be both assigned x at (μ,γ) in $\mathscr{E}$. Suppose this was not true, i.e., $\mu(i)=x$ and $x\in\gamma(f)=W_f$. By definition, $\hat{\mu}(i)=x$ in $\hat{\mathscr{E}}$. Since firm $f\in F_k\subseteq F$ is assigned an input at (μ,γ), the set of goods $X_{\hat{f}_k}$ must be available at $(\hat{\mu},\hat{\gamma})$ in $\hat{\mathscr{E}}$. If $x\in W_{\hat{f}_k}=(\bigcup_{f\in F_k} W_f)\setminus(\bigcup_{f\in F_k} X_f)$ then we have a contradiction as x would be assigned to i and $\hat{f}_k$ in the (feasible) outcome $(\hat{\mu},\hat{\gamma})$ in $\hat{\mathscr{E}}$. Thus, $x\in\bigcup_{f\in F_k} X_f$ and, because x is assigned to agent i at $(\hat{\mu},\hat{\gamma})$, $x\in X_{\hat{f}_k}=(\bigcup_{f\in F_k} X_f)\cap\{x\in X:\nexists f\in F_k \text{ s.t. } x\in W_f\}$. But then $\nexists f\in F_k$ s.t. $x\in W_f$, which is a contradiction since the set F_k contains at least one firm that uses x as an input. $\diamond$

Next we show that (μ,γ) cannot be exclusion blocked in $\mathscr{E}$. Suppose the contrary. Thus, for some $C\subseteq I$ and a feasible outcome (σ,ψ) in $\mathscr{E}$, $\sigma(i)\succ_i\mu(i)$ for all $i\in C$ and $\mu(j)\succ_j\sigma(j)\implies\mu(j)\in\Omega_\gamma(C|\omega,\mu)$. Without loss of generality, assume that C contains all agents i for whom $\sigma(i)\succ_i\mu(i)$ and that ψ is efficient.

Consider the outcome $(\hat{\sigma},\hat{\psi})$ in $\hat{\mathscr{E}}$ defined as follows. For each $i\in\hat{I}$, $\hat{\sigma}(i)=\sigma(i)$. For each $\hat{f}_k\in\hat{F}$, $\hat{\psi}(\hat{f}_k)=(\bigcup_{f\in F_k}\psi(f))\setminus(\bigcup_{f\in F_k} X_f)$. We will show that coalition C can exclusion block $(\hat{\mu},\hat{\gamma})$ in $\hat{\mathscr{E}}$ with $(\hat{\sigma},\hat{\psi})$. This will contradict $(\hat{\mu},\hat{\gamma})$ being in the exclusion core of $\hat{\mathscr{E}}$. Together, Claims 2–4 imply this conclusion and complete the proof.

Claim 2. *The outcome $(\hat{\sigma},\hat{\psi})$ is feasible in $\hat{\mathscr{E}}$.*

Proof of Claim 2. Let $x\in\hat{\sigma}(\hat{I})\cup\hat{\psi}(\hat{F})$. Suppose $x\notin X_\varnothing\cup\{x_0\}$. There are two cases.

(a) If $x\in\hat{\sigma}(\hat{I})$, then $\hat{\sigma}(i)=x$ for some $i\in\hat{I}$. Since $\hat{\sigma}(i)=\sigma(i)$, good x must be available at (σ,ψ) in $\mathscr{E}$. As $x\notin X_\varnothing\cup\{x_0\}$, $x\in X_f$ for some $f\in F_k\subseteq F$ and $\psi(f)=W_f$. (Since ψ is efficient, $f(\psi)\neq\varnothing\implies\psi(f)=W_f$.) There are two subcases.

(i) If $\hat{f}_k\in\tilde{F}$, then $x\in X_{\hat{f}_k}\subseteq\hat{X}_\varnothing$. This is because x was assigned to an agent. Hence, it must not be assigned to any other $f'\in F_k$.

(ii) Otherwise, $\hat{f}_k\in\hat{F}$. We know that $\psi(f')=W_{f'}$ for all $f'\in F_k$; else, $f\in F_k$ would not be able to produce its output. Therefore, $\hat{\psi}(\hat{f}_k)=(\bigcup_{f\in F_k} W_f)\setminus$

$(\bigcup_{f\in F_k} X_f)$ and $X_{\hat{f}_k}$ is available at $(\hat{\sigma},\hat{\psi})$. If $x \notin X_{\hat{f}_k}$, then $x \in W_{f'}$ for some $f' \in F_k$. But, this implies that at (σ,ψ), $\sigma(i) = x$ and $x \in \psi(f')$—a contradiction. Thus, $x \in X_{\hat{f}_k}$.

(b) If $x \in \hat{\psi}(\hat{F})$, then $x \in \hat{\psi}(\hat{f}_k)$ for some $\hat{f}_k \in \hat{F}$. Hence, $x \in \psi(f)$ for some $f \in F_k$. Since ψ is efficient, $x \in W_f$. Because (σ,ψ) is feasible, the firm producing x, say f', must produce at (σ,ψ). There are two subcases.

(i) Suppose $f' \in F'_k$ and $\hat{f}'_k \in \tilde{F}$. Since x was assigned to firm f at ψ, x was not assigned as an input to any other firm in F'_k. But then, we know that $x \in X_{\hat{f}'_k} \subseteq \hat{X}_\varnothing$.

(ii) Otherwise, suppose $f' \in F'_k$ and $\hat{f}'_k \in \hat{F}$. All firms in the strongly connected component F'_k must also produce at (σ,ψ). If $F'_k = F_k$, then $x \in \hat{\psi}(\hat{f}_k)$, which is not possible. Therefore, $F'_k \neq F_k$. If all $f \in F'_k$ produce at (σ,ψ), then $\psi(f) = W_f$ for all $f \in F'_k$. Thus, $\hat{\psi}(\hat{f}'_k) = (\bigcup_{f\in F'_k} W_f) \setminus (\bigcup_{f\in F'_k} X_f)$ and $\hat{f}'_k$ produces $X_{\hat{f}'_k} = (\bigcup_{f\in F'_k} X_f) \cap \{x \in X : \nexists f \in F'_k \text{ s.t. } x \in W_f\}$, at $(\hat{\sigma},\hat{\psi})$. If $x \notin X_{\hat{f}'_k}$, then there exists some firm $f'' \in F'_k$ such that $x \in W_{f''}$. However, above we saw that $x \in W_f$ and $f \notin F'_k$. Thus there exist firms $f \neq f'$ that require the same input good for production. However, this contradicts condition B2 which holds in $\mathscr{E}$.

Cases (a) and (b) imply that if $x \notin X_\varnothing \cup \{x_0\}$, then $x \in \hat{F}(\hat{\psi})$. Hence, $\hat{\sigma}(\hat{I}) \cup \hat{\psi}(\hat{F}) \subseteq \hat{X}_\varnothing \cup \hat{F}(\hat{\psi}) \cup \{x_0\}$. The following three points together imply that $|\{i \in \hat{I} : \hat{\sigma}(i) = x\}| + |\{\hat{f} \in \hat{F} : x \in \hat{\psi}(\hat{f})\}| \leq 1$ for all $x \in \hat{X}_\varnothing \cup \hat{F}(\hat{\psi}) \cup \{x_0\}$ and prove the claim.

(a) At most one agent can be assigned x at $(\hat{\sigma},\hat{\psi})$ since (σ,ψ) is feasible and $\sigma = \hat{\sigma}$.

(b) At most one firm can be assigned x at $(\hat{\sigma},\hat{\psi})$. To see this, suppose $\hat{f}_k$ and $\hat{f}_\ell$ are both assigned x at $(\hat{\sigma},\hat{\psi})$. This implies there exist two distinct firms $f_k \in F_k$ and $f_\ell \in F_\ell$ such that $x \in \psi(f_k)$ and $x \in \psi(f_\ell)$. But this means that both firms would not be able to produce at $\overline{\gamma}$, contradicting B2.

(c) It is impossible for an agent and a firm to both be assigned x at $(\hat{\sigma},\hat{\psi})$ in $\hat{\mathscr{E}}$. Suppose the contrary. If $\hat{\sigma}(i) = x$ and $x \in \hat{\psi}(\hat{f}_k)$, then there exists a firm $f \in F_k \subseteq F$ such that $x \in \psi(f)$. Thus, agent i and firm f are both assigned x at (σ,ψ), a contradiction. ⋄

Claim 3. *$\hat{\sigma}(i) \succ_i \hat{\mu}(i)$ for all $i \in C$.*

Proof of Claim 3. Since $\hat{\sigma} = \sigma$ and $\hat{\mu} = \mu$, it follows that $\hat{\sigma}(i) \succ_i \hat{\mu}(i)$ for all $i \in C$. ⋄

Claim 4. $\hat{\mu}(j) \succ_j \hat{\sigma}(j) \implies \hat{\mu}(j) \in \Omega_{\hat{\gamma}}(C|\hat{\omega},\hat{\mu})$.

Proof of Claim 4. Recall that $\Omega_\gamma(C|\omega,\mu) = \bigcup_{k=0}^{\infty} Z_k$ where $Z_0 = \omega(C)$ and $Z_k = Z_{k-1} \cup \omega(C \cup \mu^{-1}(Z_{k-1})) \cup \alpha_\gamma(Z_{k-1})$ for all $k \geq 1$. Likewise, $\Omega_{\hat{\gamma}}(C|\hat{\omega},\hat{\mu}) = \bigcup_{k=0}^{\infty} \hat{Z}_k$ where $\hat{Z}_0 = \hat{\omega}(C)$ and $\hat{Z}_k = \hat{Z}_{k-1} \cup \hat{\omega}(C \cup \hat{\mu}^{-1}(\hat{Z}_{k-1})) \cup \alpha_{\hat{\gamma}}(\hat{Z}_{k-1})$ for all $k \geq 1$. Since $\mu(j) \succ_j \sigma(j) \implies \mu(j) \in \Omega_\gamma(C|\omega,\mu)$, $\sigma(i) = \hat{\sigma}(i) \in \hat{X} \cup \{x_0\}$ for all i, and $\mu(j) = \hat{\mu}(j) \in \hat{X} \cup \{x_0\}$ for all j, to prove the claim it suffices to show that $\Omega_\gamma(C|\omega,\mu) \cap \hat{X} \subseteq \Omega_{\hat{\gamma}}(C|\hat{\omega},\hat{\mu})$. Thus, it suffices to show that $Z_k \cap \hat{X} \subseteq \hat{Z}_k$ for all $k \geq 0$.

Let $k = 0$. If $x \in Z_0 \cap \hat{X} = \omega(C) \cap \hat{X}$, then $C^x \subseteq C$. Thus, $\hat{C}^x \subseteq C^x \subseteq C$. Which implies, $x \in \hat{\omega}(C) = \hat{Z}_0$. Proceeding by induction, suppose $Z_{k'} \cap \hat{X} \subseteq \hat{Z}_{k'}$ for all $k' \leq k-1$. Let $x \in Z_k \cap \hat{X}$. If $x \in Z_{k-1} \cap \hat{X}$, then the induction hypothesis implies that $x \in \hat{Z}_{k-1} \subseteq \hat{Z}_k$. Instead, suppose $x \notin Z_{k-1}$ and $x \in (\omega(C \cup \mu^{-1}(Z_{k-1})) \cup \alpha_\gamma(Z_{k-1})) \cap \hat{X}$. There are two cases.

(a) Suppose $x \in \omega(C \cup \mu^{-1}(Z_{k-1}))$. Since $\mu = \hat{\mu}$, the range of $\mu(\cdot)$ is contained in $\hat{X} \cup \{x_0\}$. Thus, $\mu^{-1}(Z_{k-1}) = \mu^{-1}(Z_{k-1} \cap \hat{X}) = \hat{\mu}^{-1}(Z_{k-1} \cap \hat{X}) \subseteq \hat{\mu}^{-1}(\hat{Z}_{k-1})$. Therefore, $\omega(C \cup \mu^{-1}(Z_{k-1})) \subseteq \omega(C \cup \hat{\mu}^{-1}(\hat{Z}_{k-1}))$. And so, $C^x \subseteq C \cup \hat{\mu}^{-1}(\hat{Z}_{k-1})$, which implies $\hat{C}^x \subseteq C \cup \hat{\mu}^{-1}(\hat{Z}_{k-1})$. Therefore, $x \in \hat{\omega}(C \cup \hat{\mu}^{-1}(\hat{Z}_{k-1}))$ and $x \in \hat{Z}_k$.

(b) Suppose $x \in \alpha_\gamma(Z_{k-1})$. Because each firm has a unique efficient production plan there exists $y^1 \in Z_{k-1}$ such that $x \in \alpha_\gamma(y^1)$. There are two subcases.

(i) $y^1 \in \hat{X}$. Thus, $y^1 \in Z_{k-1} \cap \hat{X} \subseteq \hat{Z}_{k-1}$. As y^1 is critical for x at γ, it remains so at $\hat{\gamma}$ since each firm has a unique efficient production plan. Thus, $x \in \alpha_{\hat{\gamma}}(\hat{Z}_{k-1})$, which implies $x \in \hat{Z}_k$.

(ii) $y^1 \notin \hat{X}$. Thus, there exists some firm $f^1 \in F_k \subseteq F$ such that $y^1 \in X_{f^1}$ and there must exist some other firm $f^0 \in F_k$ that uses y^1 as an input. Since y^1 is a critical input for x, it follows that the firm producing x must be f^0, i.e., $x \in X_{f^0}$ and $y^1 \in W_{f^0}$

We know that $y^1 \in Z_{k-1} = Z_{k-2} \cup \omega(C \cup \mu^{-1}(Z_{k-2})) \cup \alpha_\gamma(Z_{k-2})$. If $y^1 \in Z_{k-2}$, then $x \in Z_{k-1}$, which is a contradiction. Three cases remain.

Case 1. $y^1 \in \omega(C \cup \mu^{-1}(Z_{k-2}))$. In this case, $C^y \subseteq C \cup \mu^{-1}(Z_{k-2})$. However, $f^0, f^1 \in F_k$. Thus, $y^1 \in Y_{\hat{f}_k}$. Noting (10), this implies that $\hat{C}^x \subseteq C^y$. Hence, $\hat{C}^x \subseteq C \cup \mu^{-1}(Z_{k-2})$ which (by reasoning analogous to case (a) above) implies that $\hat{C}^x \subseteq C \cup \hat{\mu}^{-1}(\hat{Z}_{k-2})$. Therefore, $x \in \hat{\omega}(C \cup \hat{\mu}^{-1}(\hat{Z}_{k-2}))$. Thus, we can conclude that $x \in \hat{Z}_{k-1} \subseteq \hat{Z}_k$.

Case 2. $y^1 \in \alpha_\gamma(y^2)$ where $y^2 \in Z_{k-2} \cap \hat{X}$. This implies that $x \in \alpha_{\hat{\gamma}}(y^2)$. This is because y^2 is an input for $\hat{f}_k$ and $x \in X_{\hat{f}_k}$. However, $y^2 \in Z_{k-2} \cap \hat{X}$ implies $y^2 \in \hat{Z}_{k-2}$. Thus, $x \in \alpha_{\hat{\gamma}}(\hat{Z}_{k-2})$, which implies $x \in \hat{Z}_{k-1} \subseteq \hat{Z}_k$.

Case 3. $y^1 \in \alpha_\gamma(y^2)$ where $y^2 \in Z_{k-2}$ and $y^2 \notin \hat{X}$. In this case, we can repeat the preceding argument starting at (ii) either establishing that $x \in \hat{Z}_k$, as in cases 1 and 2, or identifying a new good y^3 such that $y^3 \notin \hat{X}$, as in this case. As there is a finite number of goods, this argument must eventually stop and it can only stop after showing $x \in \hat{Z}_k$. $\diamond$

$\square$

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