EXISTENCE OF MONOTONE EQUILIBRIA IN LARGE DOUBLE AUCTIONS

PAULO BARELLI AND SRIHARI GOVINDAN

Abstract. We provide a shorter proof of the main result in Reny and Perry [7], by establishing a lower semicontinuity property of auctions as the number of traders goes to infinity, leveraging existence of equilibria in the limit "auction". Our proof also eliminates two of the assumptions in their paper.

1. Introduction

Reny and Perry [7] established that double auctions provide strategic foundations for Rational Expectations Equilibria (REE) in a model with a large number of identical buyers and sellers of copies of a single object, where each seller has one unit to sell and each buyer has unitary demand. In particular, they showed that large but finite double auctions, in an environment with quasilinear preferences, symmetric valuations and distributions satisfying the strict Monotone Likelihood Ratio Property (MLRP), have monotone equilibria whose clearing prices converge to the unique REE. Barelli, Govindan, and Wilson [2] showed that the symmetry assumption is crucial: with unrestricted heterogeneity among traders, large but finite auctions cannot even attain allocations close to the REE, let alone have equilibria with outcomes approaching the REE outcome.[1]

We provide an alternative and shorter proof of Reny and Perry's result, using degree theory and asymptotic theory. Instead of circumventing the nonexistence of monotone best replies due to rationing, as in Reny and Perry, we leverage the existence of an equilibrium in the limit auction with a countable number of traders to establish existence of monotone equilibria of auctions with a large but finite number of traders. The sequence

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[1]Specifically, the REE outcome is not in the image of the outcome function of any mechanism, like an auction, that satisfies an "independence of irrelevant messages" kind of property.

of equilibria, in turn, is shown to provide strategic foundations for REE. As our approach is therefore based on a lower semi-continuity property of the equilibrium correspondence, this note also has a methodological purpose of illustrating the power of degree theory in obtaining such a property. We elaborate on this point in the next section.

We work in essentially the set-up of Reny and Perry, with risk-neutral traders with unit demands and supplies for a single commodity. Interdependence of preferences is routed through a one-dimensional hidden state. Traders' private information is also one dimensional, and independently drawn conditional on the state from a distribution that satisfies strict MLRP. The double auction game is the standard uniform-price auction. We do eliminate two of their assumptions. One, we do not assume private values at the boundary (this is their Assumption A.4); and two, our result holds for all sufficiently small bid increments and not just on a residual set of such increments.[2]

2. Degree Theory

Fixed point theorems naturally yield as a corollary the upper semicontinuity of the set of fixed points as we vary the parameters of the problem. But sometimes we are interested in the lower semicontinuity property of the fixed point correspondence. Specifically, if we know that a game G has an isolated equilibrium, do all close-by games have an equilibrium that is close-by? This question is relevant when the games under consideration do not fall into a class where existence of an equilibrium can be ensured by standard fixed-point arguments but we know from direct computation that one such game G has an equilibrium. For instance, the games under consideration have discontinuous payoffs and/or non-compact strategy spaces. For our purposes, we are interested in monotone, non trivial pure-strategy equilibria of double auctions. The usual payoff discontinuities are not a problem because we use bid grids with increments that go to zero as the number

[2]A previous version of [2] shows that the techniques used here allow us to obtain the result allowing for some degree of heterogeneity among traders as well.

of traders goes to infinity.[3] But standard fixed point arguments give us only existence of equilibria in strategies that are not necessarily pure, monotone, or non trivial. An alternative avenue would be to explore the monotone properties of the environment: the literature pioneered by Athey [1] has established that, if the expected payoff of a player satisfies a single-crossing property when the other players employ monotone pure strategies, then the player will have monotone pure-strategy best responses, and hence the existence of a monotone equilibrium can be guaranteed. However, Reny and Perry [7] demonstrated that this single crossing property fails in a double auction. Since it is relatively easy to show that the limit auction (that is, the limiting game with a countable number of traders) has a monotone non trivial equilibrium, if the equilibrium correspondence is lower semi-continuous, then large-but-finite auctions will also have monotone nontrivial equilibrium.

Degree theory is the appropriate tool for the study of the lower semi-continuity properties of the equilibrium correspondence. The degree of a map is an integer that reveals some properties of the map.[4] For instance, letting D and R be subsets of the same finite dimensional space, say that the degree of a map $f : D \to R$ at $y \in R$ is non zero, then there must exist a point $x \in D$ such that $f(x) = y$. One important property of the degree is that it is preserved by local perturbations. If the degree of $f : D \to R$ at $y \in R$ is one, then the degree of $f^n : D \to R$ at y is equal to one for all f^n uniformly close to f. For our purposes, the zero of f identifies an equilibrium of the limit auction and the zeros of f^n identify equilibria of large-but-finite auctions. By establishing that f has degree one at zero, we immediately establish that each f^n has degree one at zero (and hence that these auctions have monotone non trivial equilibria) if we also establish that f^n is uniformly close to f. In general, using f as the displacement map of game (that is, the fixed point map minus the identity map), the reasoning above establishes lower semicontinuity of the equilibrium correspondence of a game.

[3]This is in contrast with the use of bid grids for a fixed population of traders – as the bid increments go to zero, discontinuities may still arise in the form of atoms in the limit. See Bich and Laraki [3], proof of Theorem 4.2, case c1, for a clever way of avoiding such a problem in a two-bidder auction.

[4]See Jezierski and Marzantowicz [6] for a formal treatment.

Another important property of degree that is related to the continuity property mentioned above is that it is invariant under homotopy: the degree of $g : D \to R$ at $y \in R$ is equal to the degree of the homotopy $(1-\lambda)g + \lambda f$ for $\lambda \in [0,1]$ at y as long as the value of the homotopy is not equal to y at the boundary of D. An implication that we explore in our analysis below is that, since the degree of a homeomorphism is one, if we connect a given map f to a homeomorphism g under a homotopy and establish that the homotopy does not have a zero at the boundary, then we necessarily establish that the degree of the zero of f is equal to one (and in particular, we establish the existence of $x \in D$ such that $f(x) = 0$, and, a fortiori, we also establish existence of x^n such that $f^n(x^n) = 0$ for f^n uniformly close to f.)

An application of these ideas appeared in Govindan, Reny, and Robson [4], who used degree theory to obtain a simple proof of Harsanyi's purification theorem. If a game is generic, all of its finitely many equilibria have nonzero degree. Hence, for a perturbed incomplete information game, there exist equilibria close-by. Such equilibria are necessarily in pure strategies, thus showing that all mixed equilibria can be approximately purified.

Apart from the obvious possibility of its use in more general large auctions settings, a potential future application concerns the study of continuous-time games as limits of discrete time games with frequent actions. The limit game is more tractable given that we can use techniques from differential topology; degree theory, hopefully, provides a theoretical justification for its robustness.

3. Model

We start with a base model of a double auction Γ and then we consider a sequence of double auctions obtained by replicating the agents in Γ. The double auction Γ is as follows. The set of buyers is T_{I} with cardinality m_{I} and the set of sellers is T_{II} with cardinality m_{II}, so the set of $m = m_{\mathrm{I}} + m_{\mathrm{II}}$ traders is $T = T_{\mathrm{I}} \cup T_{\mathrm{II}}$. Buyers have unitary demand and each seller has a unit to sell. Let $\mu_1 = m^{-1} m_{\mathrm{II}}$ and $\mu_0 = 1 - \mu_1$ denote the fractions of traders who must end up with one and zero units, respectively.

The set of unobserved states of the world is the interval $\Omega \equiv [0,1]$. For each trader $t \in T$, the interval $X_t \equiv [0,1]$ is his space of signals, with typical element x_t. Let $X \equiv \prod_{t\in T} X_t$, with typical element x. The prior probability distribution over $\Omega \times X$ is Q.

For each $t \in T$ and $\omega \in \Omega$, $P_t(\,\cdot\,|\,\omega)$ is the probability distribution over X_t conditional on the state ω, and $Q(\,\cdot\,|\,\omega)$ is the conditional distribution over X. We assume *symmetry*, so $P_t(\,\cdot\,|\,\omega) \equiv P(\,\cdot\,|\,\omega)$ for all $t \in T$. The marginal distribution on Ω is P_0. The common valuation of a unit is a function $v : \Omega \times X_t \to \mathbb{R}_+$ of the state and signal. That is, interdependence of preferences are rooted through the hidden state ω. We make the following assumptions on Q and v.

Assumption 3.1. The prior distribution Q satisfies the following conditions.

(a) Q has a continuously differentiable and strictly positive density q.

(b) The conditional distributions $P(\,\cdot\,|\,\omega)$ over the x_t's given ω are independent, i.e. $q(x\,|\,\omega)$ is the product of the densities $p(x_t\,|\,\omega)$ of $P(x_t\,|\,\omega)$ for $t \in T$.

(c) The common density $p(x_t\,|\,\omega)$ satisfies the strict monotone likelihood ratio property (MLRP).

Recall that strict MLRP means that $\frac{p(x'_t\,|\,\omega')}{p(x_t\,|\,\omega')} > \frac{p(x'_t\,|\,\omega)}{p(x_t\,|\,\omega)}$ for $x'_t > x_t$ and $\omega' > \omega$.

Given the above assumption, we view $Q(\,\cdot\,|\,\omega)$ as the conditional CDF and thus write $Q(x\,|\,\omega)$ for $Q(\prod_t [0, x_t]\,|\,\omega)$, and $P(x_t\,|\,\omega)$ for $P([0, x_t]\,|\,\omega)$.

Assumption 3.2. The valuation v satisfies the following conditions:

(a) v is positive and twice-continuously differentiable;

(b) $\frac{\partial v(\omega, x_t)}{\partial \omega} \geqslant 0$ and $\frac{\partial v(\omega, x_t)}{\partial x_t} > 0$.

For each $n = 1, 2, \ldots$, Γ^n is the n-fold replica of Γ. Specifically, the set of traders is T^n, which has nm_{I} buyers and nm_{II} sellers, for a total of nm traders and $\mu_1 nm$ objects for sale.[5] The set of states of nature remains Ω but the signal space is X^n, the n-fold product

[5]We keep the fraction μ_1 independent of n for simplicity of notation. We could allow for a ratio $\mu_1(n)$ that depends on n, so long as there is a well-defined limit that is strictly between 0 and 1.

of X. The distribution Q^n over $\Omega \times X^n$ is generated by the distribution P_0 on Ω, as in Γ, and the conditionally independent distributions $P(x_t \,|\, \omega)$, for $t \in T^n$.

Turn now to the rules of the double auctions. Traders observe signals and submit bids in $\mathbb{R}_+$. The bids in a profile b^n are ordered $b_{(1)} \geqslant \cdots \geqslant b_{(nm)}$, where $b_{(k)}$ is the k-th highest bid. If $b_{(nm_1)} > b_{(nm_1+1)}$, each buyer (resp. seller) bidding at least $b_{(nm_1)}$ gets to buy a copy of the object (resp. does not get to sell his copy) and each seller (resp. buyer) bidding below $b_{(nm_1)}$ gets to sell a copy of the object (resp. does not get to buy a copy of the object). In the event of a tie, i.e. $b_{(nm_1)} = b_{(nm_1+1)}$, those bidding above $b_{(nm_1)}$ end up with a copy of the object and allocations are made randomly among those tied. The price at which trade occurs is $\varrho^n(b^n) = \alpha b_{(nm_1)} + (1-\alpha) b_{(nm_1+1)}$, where $0 \leqslant \alpha \leqslant 1$.

A pure strategy for trader $t \in T^n$ is a measurable map $\sigma_t^n : X_t \to \mathbb{R}_+$. Given a profile σ^n of pure strategies, the payoff to a buyer t in Γ^n from bid b can be written as

$$\pi_t^n(b, \sigma^n; x_t) = \int_\Omega \tau_t^n(b, \omega, \sigma^n)[v(\omega, x_t) - \varrho_t^n(b, \omega, \sigma^n)] dP(\omega \,|\, x_t)$$

where $\tau_t^n(b, \omega, \sigma^n)$ is the probability that trader t trades in ω if he bids b and others play according to σ^n, with $\varrho_t^n(\cdot)$ being the expected clearing price for this event. The payoff for a seller is defined analogously. Hence σ^n is an equilibrium if for every t and for a.e. x_t under the marginal of Q^n on X_t,

$$\pi_t^n(\sigma_i(x_t), \sigma^n; x_t) \geqslant \pi_t^n(b, \sigma^n; x_t)$$

for each $b \in \mathbb{R}_+$.

We focus here on monotone and symmetric pure strategies. A pure strategy profile σ^n is monotone if for each $t \in T^n$, σ_t^n is monotone in x_t. It is symmetric if for each pair $t, t' \in T^n$ such that either both of them are buyers or sellers, $\sigma_t^n(s) = \sigma_{t'}^n(s)$ for all $s \in [0,1]$. A symmetric pure strategy profile can thus be represented by two functions $\sigma_{\mathrm{I}}^n, \sigma_{\mathrm{II}}^n : [0,1] \to \mathbb{R}_+$, where σ_{I}^n (resp. σ_{II}^n) is the strategy employed by all buyers (resp. sellers). We shall still use σ^n to denote a symmetric pure strategy profile.

For each n and $\zeta > 0$, let $\Gamma^{n,\zeta}$ be the game where the set of bids is restricted to be $\{0, \zeta, 2\zeta, \dots\}$. The game $\Gamma^{n,\zeta}$ obviously has an equilibrium in behavioral strategies.

Reny and Perry [7] showed, as will we, that for a fixed grid size ζ, $\Gamma^{n,\zeta}$ has a symmetric monotone equilibrium in pure strategies if n is large.

3.1. **The Limit Economy E^∞.** A limit competitive economy E^∞ is obtained as follows. Let $X^\infty = X \times X \times \cdots$, where $X = \prod_{t\in T} X_t$ is the space of signals in Γ. The denumerable set of traders is $T^\infty \equiv \lim_{n\uparrow\infty} T^n$. Each seller has one unit to sell and each buyer wants one unit. The valuation of each trader is the function v from Γ. Let $\mathcal{O}$ be the Borel σ-algebra on Ω; and let $\mathcal{X}^\infty$ be the product σ-algebra on X^∞, using the Borel σ-algebra on each factor. Q^* is the probability distribution over $(\Omega \times X^\infty, \mathcal{O} \otimes \mathcal{X}^\infty)$ for which P_0 is the marginal on Ω and for each ω, conditional on ω, the distribution over X^∞ is a product distribution with the distribution over X_t being the same for all traders and equal to the one from Γ.

A state of E^∞ is given by (ω, x^∞). A price map is a random variable $\phi : \Omega \to \mathbb{R}_+$. Given ϕ, the valuation of the object for a trader with signal s is a random variable $\mathbb{E}(v(\cdot, s) \,|\, \phi) : \Omega \to \mathbb{R}_+$, where the expectation is w.r.t. $P(\,\cdot\,|\,s)$ and is conditional on the σ-algebra generated by the price function ϕ. Observe that this expectation is strictly monotone in s, since v is strictly increasing in s and $P(\,\cdot\,|\,s)$ satisfies strict MLRP; moreover, it is continuously differentiable in s. If t is a buyer (resp. seller) with signal s, his demand $\mathcal{D}_t(\omega, s, \phi)$ is 1 or 0 (resp. 0 or -1) depending on whether $\mathbb{E}(v(\cdot, s) \,|\, \phi)(\omega)$ is greater or smaller than $\phi(\omega)$.

Given ϕ, excess demand is a function $Z(\omega, x^\infty, \phi)$ defined for each state of the economy (ω, x^∞) by

$$Z(\omega, x^\infty, \phi) = \lim_{n\to\infty} \frac{1}{n} \sum_{t\in T^n} \mathcal{D}_t(\omega, x_t, \phi).$$

This function is well-defined by the Strong Law of Large Numbers. A price function ϕ is a *Rational Expectations Equilibrium* (REE) if $Z(\omega, x^\infty, \phi) = 0$ for Q^*-a.e. (ω, x^∞). An REE ϕ is fully revealing if it is strictly monotone. We now describe an REE of this economy. For each $\omega \in \Omega$, let $s^*(\omega)$ be the unique $s \in [0, 1]$ such that $P(s \,|\, \omega) = \mu_0$. Because the density is continuously differentiable and satisfies strict MRLP, $s^*(\omega)$ is well-defined and differentiable in ω. Define $\phi^* : \Omega \to \mathbb{R}_+$ by $\phi^*(\omega) = v(\omega, s^*(\omega))$. At the price $\phi^*(\omega)$, all buyers with signals higher than $s^*(\omega)$ will demand one unit and all sellers with signals less than $s^*(\omega)$ want to sell one unit and the markets clear. By the SLLN, the excess demand function is a.e. equal to $\mu_0 - P(s^*(\omega) \mid \omega)$. Thus ϕ^* is an REE, and it is also fully revealing.[6]

3.2. **The Limit Auction Γ^∞.** As we are focusing on symmetric pure strategy profiles, there is a well-defined auction game Γ^∞ with the player set T^∞. The probability model is the same as for E^∞. As in Γ, a symmetric pure strategy profile is again given by a pair $(\sigma_{\mathrm{I}}, \sigma_{\mathrm{II}})$ with σ_{I} (resp. σ_{II}) being the common strategy used by all buyers (resp. sellers). Let us still denote it by σ. Given such a profile, we define

$$\varrho^\infty(\omega, \sigma) \equiv \sup\{\, b \mid \mu_0 P(\sigma_{\mathrm{I}}^{-1}([0,b] \mid \omega) + \mu_1 P(\sigma_{\mathrm{II}}^{-1}([0,b] \mid \omega) \leqslant \mu_0 \,\}$$

with the convention that the supremum of the empty set is 0. As for allocations, a trader gets a copy of the object when his bid is above $\varrho^\infty(\omega, \sigma)$ and does not get a copy when his bid is below $\varrho^\infty(\omega, \sigma)$. For a generic buyer, identified by the subscript I, the probability $\tau_{\mathrm{I}}^\infty(b, \omega, \sigma)$ that he gets a copy of the object bidding $b = \varrho^\infty(\omega, \sigma)$ is given by the equation $[\mu_0 P(\sigma_{\mathrm{I}}^{-1}(\{\, b \,\}) \mid \omega) + \mu_1 P(\sigma_{\mathrm{II}}^{-1}(\{\, b \,\}) \mid \omega)]\tau_{\mathrm{I}}^\infty(b, \omega, \sigma) = \mu_1 - [\mu_0 P(\sigma_{\mathrm{I}}^{-1}((b, \infty) \mid \omega) + \mu_1 P(\sigma_{\mathrm{II}}^{-1}((b, \infty) \mid \omega)]$. For a generic seller, identified by the subscript II, the probability is given by the equation $[\mu_0 P(\sigma_{\mathrm{I}}^{-1}(\{\, b \,\}) \mid \omega) + \mu_1 P(\sigma_{\mathrm{II}}^{-1}(\{\, b \,\}) \mid \omega)]\tau_{\mathrm{II}}^\infty(b, \omega, \sigma) = \mu_0 - [\mu_0 P(\sigma_{\mathrm{I}}^{-1}((0, b) \mid \omega) + \mu_1 P(\sigma_{\mathrm{II}}^{-1}((0, b) \mid \omega)]$. Observe that if b is not an atom of σ then the choice of $\tau_i^\infty(b, \omega, \sigma)$ for $b = \varrho^\infty(\omega, \sigma)$ and $i = \mathrm{I}, \mathrm{II}$ is immaterial.

The payoff to a buyer I and signal x_{I} from a bid b against a symmetric profile σ is

$$\pi_{\mathrm{I}}^\infty(b, \sigma; x_{\mathrm{I}}) = \int_\Omega \tau_{\mathrm{I}}^\infty(b, \omega, \sigma)[v(\omega, x_{\mathrm{I}}) - \varrho^\infty(\omega, \sigma)] dP(\omega \mid x_{\mathrm{I}}) \,,$$

and similarly for a seller II. An equilibrium of Γ^∞ is defined as for Γ^n.

[6]Moreover, as an REE must necessarily solve the system of equations just described, ϕ^* is the unique REE of E^∞.

Define a strategy profile $\sigma^* = (\sigma^*_{\mathrm{I}}, \sigma^*_{\mathrm{II}})$ as follows. For $i = \mathrm{I}, \mathrm{II}$, and $\omega \in \Omega$, let $\sigma^*_i(s^*(\omega)) = \phi^*(\omega)$; for $s < s^*(0)$, $\sigma^*_i(s) = v(0, s)$; for $s > s^*(1)$, $\sigma^*_i(s) = v(1, s)$. It is easily verified that σ^* implements the REE.

For each grid size $\zeta > 0$, we can define a limit game $\Gamma^{\infty,\zeta}$ with bids restricted to $\{0, \zeta, 2\zeta, \dots\}$ as we did for Γ^n.

3.3. **The Main Result.** For each finite n, grid size ζ, a monotone pure strategy profile $\sigma^{n,\zeta} = (\sigma^{n,\zeta}_{\mathrm{I}}, \sigma^{n,\zeta}_{\mathrm{II}})$, and $x^n \in X^n$, let $\varrho^{n,\zeta}(\omega, x^n, \sigma^{n,\zeta})$ be the clearing price under $\sigma^{n,\zeta}$; for each trader $t \in T^n$, let $\tau^{n,\zeta}_t(\omega, x^n, \sigma^{n,\zeta})$ be the probability of getting the object, and let

$$\bar{\tau}^{n,\zeta}(\omega, x^n, \sigma^{n,\zeta}) = \frac{1}{nm} \sum_{t \in T^n} \tau^{n,\zeta}_t(\omega, x^n, \sigma^{n,\zeta}) \mathbb{1}(x^n_t \geqslant s^*(\omega)).$$

We now state Reny and Perry [7]'s result in our framework.

Theorem 3.3. *Fix $\varepsilon > 0$. For each sufficiently small $\zeta > 0$, there exists $n_0(\zeta)$ such that for each $n \geqslant n_0(\zeta)$, there is a monotone pure strategy equilibrium $\sigma^{n,\zeta}$ of $\Gamma^{n,\zeta}$ with the following properties:*

(a) $\|\sigma^{n,\zeta} - \sigma^*\| < \varepsilon$;

(b) *for each $\omega \in \Omega$, there exists a subset A^n of X^n such that $\mathcal{P}(A^n \mid \omega) < \varepsilon$ and for each $x^n \in X^n \setminus A^n$:*
   (i) $|\varrho^n(\omega, x^n, \sigma^{n,\zeta}) - \phi^*(\omega)| < \varepsilon$;
   (ii) $|\bar{\tau}^{n,\zeta}(\omega, x^n, \sigma^{n,\zeta}) - \mu_1| < \varepsilon$.

Remark 3.4. If ζ is small, Properties (a) and (b) hold for all equilibria when $n = \infty$, i.e., in the limit auction; as limits of monotone equilibria of $\Gamma^{n,\zeta}$ (as n goes to ∞) are equilibria of $\Gamma^{\infty,\zeta}$, Properties (a) and (b) can be easily shown to hold for any monotone pure strategy equilibrium of $\Gamma^{n,\zeta}$ when ζ is small and n is large. The real task is to prove existence of one such equilibrium.

Remark 3.5. One could strengthen Property (b) of the theorem by making A^n independent of n, as follows. By viewing ϱ^n and $\bar{\tau}^n$ as functions defined on $\Omega \times X^\infty$, we can obtain, for each ω, a subset A^∞ of X^∞ with measure less than ε and such that items (i) and (ii) of Property (b) hold outside A^∞. Thus, one obtains an almost uniform convergence result.

4. Proof of Theorem 3.3

We proceed in steps. In Step 1, for each fixed grid size ζ, we construct a finite-dimensional compact set Θ^ζ parametrizing a set of monotone pure strategies that will be used in our search for an equilibrium. It is anchored by the strategy profile σ^* defined above, so our search for an equilibrium is localized around σ^*. Θ^ζ is defined in such a way that the homotopy constructed in Step 4 has no zeros on the boundary of Θ^ζ. In Step 2, for $i = \mathrm{I}, \mathrm{II}$ and all n, we construct a function $\bar{\pi}_i^{n,\zeta,k}$, defined on $\Theta^\zeta \times X_i$, which we will use to identify equilibria of $\Gamma^{n,\zeta}$, and use large deviations arguments to show that $\bar{\pi}_i^{n,\zeta,k}$ and its derivative converge uniformly to $\bar{\pi}_i^{\infty,\zeta,k}$ and its derivative (Lemma 4.1). In Step 3, we use Lemma 4.1 and an appropriate approximation of $\bar{\pi}_i^{\infty,\zeta,k}$ to verify that $\bar{\pi}_i^{n,\zeta,k} = 0$ indeed identifies an equilibrium of $\Gamma^{n,\zeta}$. Step 4 is where we apply degree theory to establish the desired lower semicontinuity result. We first associate each point in Θ^ζ to the corresponding image of the functions $\bar{\pi}_i^{n,\zeta,k}$. For each n, that's a mapping $\Upsilon^{n,\zeta}$ between finite-dimensional spaces. We show that, for $n = \infty$, the mapping has no zeros on the boundary of Θ^ζ and that the degree of its zero over Θ^ζ is one. To show this, we construct a homotopy deforming another such mapping $\Upsilon^{*,\zeta}$, whose unique zero must have degree one, to the original $\Upsilon^{\infty,\zeta}$, and show that the homotopy has no zero on the boundary of Θ^ζ. This establishes that $\Upsilon^{\infty,\zeta}$ has degree one. Lemma 4.1 then guarantees that $\Upsilon^{n,\zeta}$ also has degree one, and hence has a zero for n large enough. This establishes the existence of equilibria of $\Gamma^{n,\zeta}$. Step 5 concludes the proof by verifying that the constructed equilibria are close to the equilibrium of Γ^∞, and that the corresponding prices and allocations converge to the REE price and allocation.

For each $x = (x_{\mathrm{I}}, x_{\mathrm{II}}) \in [0,1]$, define $M^*(x)$ as follows: (0) $M^*(x) = 0$ if $\mu_0 P(x_{\mathrm{I}} \,|\, 0) + \mu_1 P(x_{\mathrm{II}} \,|\, 0) < \mu_0$; (1) $M^*(x) = 1$ if $\mu_0 P(x_{\mathrm{I}} \,|\, 1) + \mu_1 P(x_{\mathrm{II}} \,|\, 1) > \mu_0$; (2) otherwise, $M^*(x)$ is the unique ω such that $\mu_0 P(x_{\mathrm{I}} \,|\, \omega) + \mu_1 P(x_{\mathrm{II}} \,|\, \omega) = \mu_0$. Let $B = \sigma_{\mathrm{I}}^*([0,1])$ (or, equivalently, the range of σ^*_{II}). Let $\theta^*_i : B \to [0,1]$ be the inverse of σ^*_i, for $i = \mathrm{I}, \mathrm{II}$. Let $\theta^*_0 : B \to \Omega$ be the function such that $\theta^*_0(b)$ is the unique state ω such that $v(\omega, \theta^*_{\mathrm{I}}(b)) = b$.

To help navigate the arguments, we collected a list of the most important symbols used below in the following table.

Symbol	Meaning
α^k	Probability of trade at bid b^k
$B(\zeta)$	Bid grid
b^k	k-th bid in the grid, that is, $b^k = k\zeta$; also referred to as simply k
k_0	Bid in $B(\zeta)$ that is just below $v(0,0)$
k^*_0	Bid in $B(\zeta)$ that is just below $\phi^*(0)$
k_1	Bid in $B(\zeta)$ that is just below $v(1,1)$
k^*_1	Bid in $B(\zeta)$ that is just below $\phi^*(1)$
$O(\zeta^2)$	Big-O notation: bounded in absolute value by $K\zeta^2$, for some $K > 0$
$\bar{\pi}^{n\zeta,k}_i(\theta, x_i)$	Payoff difference from b^k and b^{k-1} at strategy induced by θ and signal x_i
σ^*	The symmetric profile that implements that REE ϕ^*.
$\underline{\theta}_i(k)$	Signal close to the signal x_i with $\sigma^*_i(x_i) = b^{k-1}$
$\overline{\theta}_i(k)$	Signal close to the signal x_i with $\sigma^*_i(x_i) = b^k$
$\theta_0(k)$	First coordinate of $\theta(k)$, that is, the state in the tuple $(\omega, x_{\mathrm{I}}, x_{\mathrm{II}})(k)$
Θ^ζ	Closed neighborhood of graph of σ^* with corresponding clearing state ω
t^k	Length of the interval of states with clearing bid b^k
$\bar{t}^k$	Average t^k under the linear approximations
$\hat{t}^k$	t^k in bid terms according to ϕ^*
$w^-(k)$	Average state where b^{k-1} clinches trade
$w^+(k)$	Average state where b^k clinches trade
$\Upsilon^{n,\zeta}(\theta)$	Payoff differences at strategy induced by θ and signal $\theta_i(k)$
$\Upsilon^{*,\zeta}(\theta)$	Difference between v at θ and certain values of b

Step 1. Fix now a grid size ζ. The set of admissible bids in the game $\Gamma^{n,\zeta}$ is $\{\, 0, \zeta, 2\zeta, \ldots, \,\}$. We use $b^k(\zeta)$ to denote $k\zeta$, and when the ζ we are using is unambiguous, we simply write b^k. Let b^{k_0} and b^{k_1} be the highest bids in $\Gamma^{n,\zeta}$ that are weakly below $v(0,0)$ and $v(1,1)$, respectively. Let $B(\zeta) = \{\, (k_0+1)\zeta, \ldots, (k_1+1)\zeta \,\}$. Let $b^{k^*_0}$ and $b^{k^*_1}$ be the highest bids in $B(\zeta)$ that are weakly below $\phi^*(0)$ and $\phi^*(1)$, respectively.

Fix $0 < \eta < \min\{\, b^{k_0+1} - v(0,0), b^{k_1+1} - v(1,1) \,\}$. For each $i = \mathrm{I}, \mathrm{II}$, let $\underline{\theta}_i(k)$ be the unique x_i for which:[7]

(1) $\sigma^*_i(x_i) = v(0,0)) + \eta\zeta$ if $k = k_0 + 1$

[7]Uniqueness follows from strict monotonicity of σ^*.

(2) $\sigma_i^*(x_i) = b^{k-1} + \zeta/8$ for $k_0 + 1 < k \leqslant k_1^*$

(3) $\sigma_i^*(x_i) = b^{k-1} - \zeta/12$ if $k_1^* + 1 \leqslant k \leqslant k_1 + 1$.

Likewise, let $\overline{\theta}_i(k)$ be the unique x_i for which:

(4) $\sigma_i^*(x_i) = b^k + \zeta/12$ if $k_0 < k \leqslant k_0^* + 1$

(5) $\sigma_i^*(x_i) = b^k - \zeta/8$ if $k_0^* + 1 < k < k_1 + 1$

(6) $\sigma_i^*(x_i) = b^k - \eta\zeta$ if $k = k_1 + 1$.

Let Θ^ζ be the closure of the set of all functions $\theta : B(\zeta) \to \Omega \times [0,1] \times [0,1]$ such that for each k, writing $\theta(k)$ as short for $\theta(b^k)$, with $\theta(k) = (\theta_0(k), \theta_{-0}(k))$:

(i) $\theta_i(k) \in (\underline{\theta}_i(k), \overline{\theta}_i(k))$ for $i = \mathrm{I}, \mathrm{II}$ and

(ii) $\theta_0(k) = M^*(\theta_{-0}(k))$.

In words, the k-th coordinate of an element of Θ^ζ is a tuple $(\omega, x_{\mathrm{I}}, x_{\mathrm{II}})$ such that ω is the clearing state given $(x_{\mathrm{I}}, x_{\mathrm{II}})$ and $b = b^k$, and $(x_{\mathrm{I}}, x_{\mathrm{II}})$ lie within the signals just to the right of signals that would have bid b^{k-1} and just to left of signals that would have bid b^k, according to σ^*.[8] The particular amounts of $\zeta/8$, $\zeta/12$, etc, of "just to the right/left" are chosen so that a homotopy that we will define below avoids the boundary of Θ^ζ. By construction, Θ^ζ is a compact $2|B(\zeta)|$-dimensional (topological) manifold with boundary points consisting of θ where for some i, $\theta_i(k)$ is in $\{\, \underline{\theta}_i(k), \overline{\theta}_i(k) \,\}$.

Each function $\theta^\zeta \in \Theta^\zeta$ induces a strategy profile σ^ζ in the game $\Gamma^{n,\zeta}$ for all n, as follows. For $i = \mathrm{I}, \mathrm{II}$, $\sigma_i^\zeta(x_i) = b^k$, where k is such that $x_i \in [\theta_i(k), \theta_i(k+1))$ for $k_0 \leqslant k \leqslant k_1 + 1$, with the convention that $\theta_i(k_0) = 0$ and $\theta_i(k_1 + 2) = 1$. The strategy profile is clearly monotone and pure.

Step 2. We define a function $\bar{\pi}_i^{n,\zeta,k} : \Theta^\zeta \times X_i \to \mathbb{R}$ for each n (including $n = \infty$), ζ and $b^k \in B(\zeta)$ as follows. Fix $\theta \in \Theta^\zeta$ and let σ be the strategy profile induced by θ. For each i, $k_0 + 1 \leqslant k \leqslant k_1 + 1$, x_i, and $n \neq \infty$, define

$$\bar{\pi}_i^{n,\zeta,k}(\theta, x_i) = \frac{1}{\int_\Omega [\tau_i^n(b^k, \omega, \sigma) - \tau_i^n(b^{k-1}, \omega, \sigma)]\, p(\omega \mid x_i) d\omega} \left[\pi_i^n(b^k, \sigma; x_i) - \pi_i^n(b^{k-1}, \sigma; x_i)\right]$$

[8]With modified constructions for "boundary" k's, with $(x_{\mathrm{I}}, x_{\mathrm{II}})$ switching from "just to left (right)" to "just to the right (left)", as is clear from points (1), (3), and (4) above.

which is the expected payoff difference between bidding b^k and b^{k-1} for type x_i normalized by the difference in probabilities of clinching trade by switching from b^{k-1} to b^k. For $n = \infty$, we can use the same formula if with positive probability either b^{k-1} or b^k is a clearing price under σ. Otherwise, either b^k is below the clearing price at state $\omega = 0$ and we let this difference be $v(0, x_i) - b^k$; or b^{k-1} is above the clearing price at state $\omega = 1$ and we let this formula be $v(1, x_i) - b^{k-1}$.

The following lemma, whose proof can be found in the Appendix, gives a continuity property for the payoff differences $\bar{\pi}_i^{n,\zeta,k}(\theta, x_i)$.

Lemma 4.1. *For each k, $\bar{\pi}_i^{n,\zeta,k}(\theta, y_i)$ converges to $\bar{\pi}_i^{\infty,\zeta,k}(\theta, y_i)$ uniformly in $(\theta, y_i) \in \Theta^\zeta \times X_i$; and the same is true of its derivative w.r.t. y_i.*

Step 3. We now compute a good approximation for $\bar{\pi}_i^{\infty,\zeta,k}$ when either b^k or b^{k-1} is the clearing price for a non-null set of states. The error term in our approximation is denoted $O(\zeta^2)$, where $O(h)$ is bounded in absolute value by Kh for some $K > 0$ that is a function of the data of the model, i.e., the game Γ.

Let $t^k = \theta_0(k+1) - \theta_0(k)$, $t^{k-1} = \theta_0(k) - \theta_0(k-1)$ denote the lengths of the intervals of states where the market clearing bids are b^k and b^{k-1}, respectively.

Let α^k be equal to $\tau_i^\infty(b^k, \theta_0(k+1), \sigma)$ if $i = \mathrm{I}$ and to $1 - \tau_i^\infty(b^k, \theta_0(k+1), \sigma)$ if $i = \mathrm{II}$.

Let α^{k-1} be equal to $1 - \tau_i^\infty(b^{k-1}, \theta_0(k-1), \sigma)$ if $i = \mathrm{I}$ and to $\tau_i^\infty(b^{k-1}, \theta_0(k-1), \sigma)$ if $i = \mathrm{II}$.

Either t^k or t^{k-1} will be positive by assumption (and then in $O(\zeta)$). Also α^k is zero if $\theta_0(k+1) < 1$ and α^{k-1} is zero if $\theta_0(k-1) > 0$, so that one of these two variables will be zero for small ζ. We can approximate $\tau_{\mathrm{I}}^\infty(b^k, \omega, \sigma)$ (resp. $1 - \tau_{\mathrm{II}}^\infty(b^k, \omega, \sigma)$) on $[\theta_0(k), \theta_0(k+1)]$ by a linear function that is 1 at $\theta_0(k)$ and α^k at $\theta_0(k+1)$, and $1 - \tau_{\mathrm{I}}^\infty(b^{k-1}, \omega, \sigma)$ (resp. $\tau_{\mathrm{II}}^\infty(b^{k-1}, \omega, \sigma)$) on $[\theta_0(k-1), \theta_0(k)]$ by a linear function that is $1 - \alpha^{k-1}$ at $\theta_0(k-1)$ and 0 at $\theta_0(k)$. Also, using

$$v(\theta_0(k) + s, x_i) = v(\theta_0(k), x_i) + \frac{\partial v(\theta_0(k), x_i)}{\partial \omega} s + O(s^2)$$

and

$$p(\theta_0(k)+s\,|\,x_i) = p(\theta_0(k)\,|\,x_i) + \frac{\partial p(\theta_0(k)\,|\,x_i)}{\partial \omega} s + O(s^2),$$

we can write

$$\bar{\pi}_i^{\infty,\zeta,k}(\theta,x_i) = [v(w^-(k),x_i) - b^{k-1}]\bar{t}^{k-1} + [v(w^+(k),x_i) - b^k]\bar{t}^k + O(\zeta^2)$$

where

$$w^-(k) = \theta_0(k) - \frac{t^{k-1}(1+2\alpha^{k-1})}{3+3\alpha^{k-1}}$$

if $\frac{\partial v(\theta_0(k),x_i)}{\partial \omega} > 0$, and $w^-(k) = \theta_0(k)$ otherwise;

$$\bar{t}^{k-1} = \frac{t^{k-1}(1+\alpha^{k-1})}{t^k(1+\alpha^k)+t^{k-1}(1+\alpha^{k-1})};$$

$$w^+(k) = \theta_0(k) + \frac{t^k(1+2\alpha^k)}{3+3\alpha^k}$$

if $\frac{\partial v(\theta_0(k),x_i)}{\partial \omega} > 0$, and $w^+(k) = \theta_0(k)$ otherwise; and

$$\bar{t}^k = \frac{t^k(1+\alpha^k)}{t^k(1+\alpha^k)+t^{k-1}(1+\alpha^{k-1})}.$$

Since the integrals involved in the computations above using ω can also be performed by a change of variable using b, we have:

$$\bar{t}^{k-1} = \frac{\hat{t}^{k-1}(1+\alpha^{k-1})}{\hat{t}^k(1+\alpha^k)+\hat{t}^{k-1}(1+\alpha^{k-1})} + O(\zeta),$$

where $\hat{t}^{k-1} = \phi^*(\theta_0(k)) - \phi^*(\theta_0(k-1))$ and $\hat{t}^k = \phi^*(\theta_0(k+1)) - \phi^*(\theta_0(k))$, and similarly for $\bar{t}^k$. For the same reason,

$$v(w^+(k),x_i) < v(\theta_0(k),x_i) + \frac{\hat{t}^k(1+2\alpha^{k-1})}{3+3\alpha^{k-1}}$$

and

$$v(w^-(k),x_i) > v(\theta_0(k),x_i) - \frac{\hat{t}^{k-1}(1+2\alpha^{k-1})}{3+3\alpha^{k-1}}.$$

Finally, given θ, if either b^k or b^{k-1} is a clearing price, then for x with $x_{\mathrm{I}} = x_{\mathrm{II}}$, since $\frac{\partial v(\theta_0(k),x_i)}{\partial \omega}$ is independent of i, we have that $\bar{\pi}_i^{\infty,\zeta,k}(\theta,x_i) - \bar{\pi}_j^{\infty,\zeta,k}(\theta,x_j) \in O(\zeta^2)$

so if $(\theta_i^*)^{-1}(\theta_i(k)) - (\theta_j^*)^{-1}(\theta_j(k))$ is negative but not in $O(\zeta^2)$, then $\bar{\pi}_i^{\infty,\zeta,k}(\theta, \theta_i(k)) - \bar{\pi}_j^{\infty,\zeta,k}(\theta, \theta_j(k)) < 0$.

Lemma 4.2. *For all sufficiently small ζ, there exists $N(\zeta)$ with the following property. For $n \geqslant N(\zeta)$ if there is θ in Θ^ζ such that $\bar{\pi}_i^{n,\zeta,k}(\theta, \theta_i(b^k)) = 0$ for each k and i, then the corresponding strategy profile $\sigma^\zeta = (\sigma_I^\zeta, \sigma_{II}^\zeta)$ is an equilibrium of $\Gamma^{n,\zeta}$.*

Proof. The result follows if we establish a single-crossing property for the payoffs (c.f. Athey [1]) i.e. if we show the existence of $\delta > 0$ such that $\frac{\partial \bar{\pi}_i^{n,\zeta,k}(\theta, x_i)}{\partial x_i} > \delta$ for all $\theta \in \Theta^\zeta$, $i = \mathrm{I}, \mathrm{II}$, $x_i \in X_i$ and k, if n is large. In light of Lemma 4.1, it is sufficient to get this bound when $n = \infty$. If neither b^{k-1} or b^k is a clearing price, then the derivative of $\bar{\pi}_i^{\infty,\zeta,k}$ is the derivative of the value, which is strictly positive by assumption. Otherwise, using the approximation of $\bar{\pi}_i^{\infty,\zeta,k}$ above, the derivative of $\bar{\pi}_i^{\infty,\zeta,k}$ is strictly positive with a lower bound that is independent of θ (and indeed also of ζ when it is sufficiently small) and the conclusion follows. □

Step 4. The previous lemmas combine to give the following penultimate step in the proof of Theorem 3.3.

Lemma 4.3. *For each sufficiently small $\zeta > 0$, there exists $N(\zeta)$ such that for each $n \geqslant N(\zeta)$, there exists $\theta^{n,\zeta} \in \Theta^\zeta$ such that the induced strategy profile $\sigma^{n,\zeta}$ is an equilibrium of the game $\Gamma^{n,\zeta}$.*

Proof. Fix ζ. For each n, define a map $\Upsilon^{n,\zeta} : \Theta^\zeta \to \mathbb{R}^{\{\mathrm{I},\mathrm{II}\} \times B(\zeta)}$ by:

$$\Upsilon_{i,k}^{n,\zeta}(\theta) = \bar{\pi}_i^{n,\zeta,k}(\theta, \theta_i(k)).$$

for each $i = \mathrm{I}, \mathrm{II}$ and $b^k \in B(\zeta)$. We will now show that for all small ζ, $\Upsilon^{\infty,\zeta}$ has no zeros on the boundary of Θ^ζ and that the degree of zero over Θ^ζ is one. The result then follows. Indeed, by Lemma 4.1, $\Upsilon^{n,\zeta}$ has a zero $\theta^{n,\zeta} \in \Theta^\zeta$ for large n; and Lemma 4.2 shows that $\theta^{n,\zeta}$ induces an equilibrium of $\Gamma^{n,\zeta}$.

To prove that the degree of zero over Θ^{ζ} under the map $\Upsilon^{\infty,\zeta}$ is one, we proceed as follows. Define $\Upsilon^{*,\zeta} : \Theta^{\zeta} \to \mathbb{R}^{\{\mathrm{I},\mathrm{II}\}\times B(\zeta)}$ by:

$$\Upsilon^{*,\zeta}_{i,k}(\theta) = \begin{cases} v(\theta_0(k), \theta_i(k)) - b^k & \text{if } k \leqslant k_0^* + 1 \\ v(\theta_0(k), \theta_i(k)) - .5b^{k-1} - .5b^k & \text{if } k_0 + 1 < k < k_1^* + 1 \\ v(\theta_0(k), \theta_i(k)) - b^{k-1} & \text{o.w.} \end{cases}$$

Obviously $\Upsilon^{*,\zeta}$ has a unique zero. Moreover, by Assumption 3.2, this map is a homeomorphism onto its image and hence has degree one. To obtain our result, we show that for each $\lambda \in [0, 1)$, $\lambda\Upsilon^{*,\zeta} + (1-\lambda)\Upsilon^{\infty,\zeta}$ has no zero on the boundary of Θ^{ζ}.[9] Take θ in the boundary of Θ^{ζ}. Let $\vartheta^{*,\zeta}$ and $\vartheta^{\infty,\zeta}$ be its image under $\Upsilon^{*,\zeta}$ and $\Upsilon^{\infty,\zeta}$ respectively; we will show that $\lambda\vartheta^{\infty,\zeta} + (1-\lambda)\vartheta^{*,\zeta} \neq 0$ for all $\lambda \in [0, 1)$.

Since $\theta \in \partial\Theta^{\zeta}$, there exists some k such that $\theta_i(k) \in \{\, \underline{\theta}_i(k), \overline{\theta}_i(k) \,\}$ for some $i = \mathrm{I}, \mathrm{II}$. Suppose $\theta_i(k) = \underline{\theta}_i(k)$. (The case where $\theta_i(k) = \overline{\theta}_i(k)$ is similar and therefore omitted.)[10] $\vartheta^{*,\zeta}_{i,k}$ is negative and, therefore, it suffices to show that $\vartheta^{\infty,\zeta}_{i,k}$ is negative as well. Suppose that neither b^{k-1} nor b^k is a clearing price. Then, $\theta_0(k)$ is either 0 or 1; in both cases, $\vartheta^{\infty,\zeta}_{i,k} = \vartheta^{*,\zeta}_{i,k} < 0$. If either b^{k-1} or b^k is a clearing price then $|(\theta_i^*)^{-1}(\theta_i(k)) - (\theta_j^*)^{-1}(\theta_j(k))|$ is in $O(\zeta^2)$, as otherwise both $\Upsilon^{\infty,\zeta}_{i,k}(\theta) - \Upsilon^{\infty,\zeta}_{j,k}(\theta)$ and $\Upsilon^{*,\zeta}_{i,k}(\theta) - \Upsilon^{*,\zeta}_{j,k}(\theta)$ are strictly negative. Therefore,

$$v(\theta_0(k), \theta_i(k)) = b^{k-1} + \frac{\zeta}{8} + O(\zeta^2)$$

if $k < k_1^* + 1$ and

$$v(\theta_0(k), \theta_i(k)) < b^{k-1}$$

otherwise. It follows that, if $k = k_1^* + 1, k_1^* + 2$, then $\theta_i(k)$ gets a negative payoff from both ties at b^{k-1} and b^k, and thus $\vartheta^{\infty,\zeta}_{i,k}$ is negative. If $k = k_0^*$ and b^k is a clearing price, then

$$v(w^+(k), \theta_i(k)) < v(\theta_0(k), \theta_i(k)) + \frac{\hat{t}^k}{3},$$

[9] For $\lambda = 1$, the homotopy is equal to $\Upsilon^{*,\zeta}$, which by construction is not equal to zero on the boundary of Θ^{ζ}.

[10] The arguments for $k_1^* + 1$ are similar to those below for k_0^*, and the arguments for k_0^* and $k_0^* - 1$ are similar to those below for $k_1^* + 1$ and $k_2^* + 2$.

where $\hat{t}^k = \phi^*(\theta_0(k+1)) - \phi^*(\theta_0(k))$. Since $\theta_0(k) = 0$, we have

$$v(\theta_0(k), \theta_i(k)) = b^{k-1} + \frac{\zeta}{8}.$$

As $\phi^*(\theta_0(k)) = \phi^*(0) \geqslant b^k$ and $\phi^*(\theta_0(k+1)) \leqslant b^{k+1} + \zeta/12$, $\hat{t}^k$ is no more than $13\zeta/12$. Therefore,

$$v(w^+(k), \theta_i(k)) \leqslant b^{k-1} + \frac{35\zeta}{72}$$

and thus

$$\vartheta_{i,k}^{\infty,\zeta} \leqslant -\frac{37\zeta}{72} + O(\zeta^2).$$

Finally, for all other k's, similar computations give us

$$v(w^+(k), \theta_i(k)) - b^k < -\frac{7\zeta}{24} + O(\zeta^2)$$

and

$$v(w^-(k), \theta_i(k)) - b^{k-1} \leqslant \frac{\zeta}{8} + O(\zeta^2).$$

Also

$$\frac{\bar{t}^k}{\bar{t}^{k-1}} = \frac{\hat{t}^k}{\hat{t}^{k-1}} + O(\zeta^2) \geqslant 1 + O(\zeta^2),$$

Therefore,

$$\vartheta_{i,k}^{\infty,\zeta} \leqslant -\frac{\zeta}{12} + O(\zeta^2),$$

which completes the proof. □

Step 5. We are now ready to wrap up the proof of Theorem 3.3. Fix $\varepsilon > 0$. Choose $\zeta_0 < \varepsilon/2$ such that $7\overline{p}\zeta_0\|\theta^*\|_1 < \varepsilon$, where $\overline{p} = \max_{x_t,\omega} p(x_t \,|\, \omega)$ and $\|\cdot\|_1$ is the C^1-norm restricted to the interval $(\phi^*(0), \phi^*(1))$, and that is sufficiently small in the sense of both Lemma 4.2 and Lemma 4.3. Fix $0 < \zeta \leqslant \zeta_0$. Choose n_0 such that for each $n \geqslant n_0$, Lemmas 4.2 and 4.3 apply, and $\max\{\frac{\|\sigma^*\|_1^2}{4n_0 m\zeta^2\underline{p}^2}, \frac{\|\theta^*\|_1^{-2}}{4n_0 m\zeta^2\overline{p}^2}\} \leqslant \frac{\varepsilon}{3}$, where $\underline{p} = \min_{x_t,\omega} p(x_t \,|\, \omega) > 0$. Now fix $n \geqslant n_0$. By Lemmas 4.2 and 4.3, there exists $\theta^{n,\zeta} \in \Theta^\zeta$ that induces a strategy profile $\sigma^{n,\zeta}$ that is an equilibrium of $\Gamma^{n,\zeta}$. For $i = \mathrm{I}, \mathrm{II}$ and $x_i \in [0,1]$, $\sigma_i^{n,\zeta}(x_i) \in [b^{k-1}, b^k) \pm \zeta$ where k is the unique integer such that $x_i \in (\theta_i^*(b^{k-1}), \theta_i^*(b^k)]$. Therefore, Property (1) of Theorem 3.3 is satisfied.

As for Property (2), fix $\omega \in \Omega$. Let A^n_ε be the set of $x^n \in X^n$ such that:

(a) $\frac{1}{mn}\sum_{t\in T^n} \mathbb{1}(\sigma^*(x^n_t) - \phi^*(\omega) \geqslant \zeta) \geqslant \mu_1$; or
(b) $\frac{1}{mn}\sum_{t\in T^n} \mathbb{1}(\phi^*(\omega) - \sigma^*(x^n_t) \geqslant \zeta) \geqslant \mu_0$; or
(c) $\frac{1}{mn}\sum_{t\in T^n} \mathbb{1}(|\phi^*(\omega) - \sigma^*(x^n_t)| \leqslant 3\zeta) \geqslant 7\overline{p}\zeta\|\theta^*\|_1$.

Chebyshev's inequality gives a bound of the probability of each of the three events to be less than $\varepsilon/3$ by our choice of n_0 and ζ. Indeed, for (a) consider mn trials of a binomial where the probability of success is $1 - P(\theta^*_i(\zeta + \phi^*(\omega)) \,|\, \omega)$, and hence bounded from above by $\mu_1 - \underline{p}\zeta\|\sigma^*\|_1^{-1}$. By Chebyshev's inequality, the probability of the event (a) is bounded by $\frac{\|\sigma^*\|_1^2}{4nm\zeta^2\underline{p}^2}$. The other two cases are similar. For x^n not in A^n_ε: the clearing price is one of the prices in the grid that are in the interval $\phi^*(\omega) \pm \zeta$, which proves 2(a); the misallocation under $\sigma^{n,\zeta}$ is only for signals x_i whose bid under σ^* is within ζ of $\phi^*(\omega)$ and the fraction of such signals is at most 2ζ, which proves 2(b).

Appendix

This appendix first lays out a few key computations concerning asymptotic probabilities, both in the case where the Central Limit Theorem applies as well as in the case of large deviations. The asymptotics in the text derive from getting n draws from a (2-fold) sum of trinomial or binomial variables. These results are then used to prove Lemma 4.1. See Gray [5] for related material.

Trinomial Probabilities. For each $i = \mathrm{I}, \mathrm{II}$, a pair of signals $0 \leqslant x^1_i < x^2_i \leqslant 1$ with $x^2_i - x^1_i < 1$ generates a trinomial distribution with outcomes 0, 1, and 2, with their respective probabilities being $P(x^1_i \,|\, \omega)$, $P([x^1_i, x^2_i] \,|\, \omega)$, and $1 - P(x^2_i \,|\, \omega)$. For each n, and a triple of integers $k_i = (k_{i,0}, k_{i,1}, k_{i,2})$ that sum to n, let $P(\omega, x^1, x^2, k_i, n)$ be the probability that in n trials, exactly $k_{i,0}$ draws are below x^1_i, and exactly $k_{i,2}$ draws are above x^2_i. Letting $\kappa_i = n^{-1}k_i$, we have:

$$P(\omega, x^1_i, x^2_i, k_i, n) = \binom{n!}{k_{i,0}!k_{i,1}!k_{i,2}!} \exp(-nD(\omega, x^1_i, x^2_i, \kappa_i)) \exp(-nH(\kappa_i)),$$

where $H(\cdot)$ is the entropy of the trinomial distribution $\kappa_i = (\kappa_{i,0}, \kappa_{i,1}, \kappa_{i,2})$ and $D(\cdot)$ is its relative entropy w.r.t. to the given trinomial.[11] When $x_i^1 = 0$ or $x_i^2 = 1$, we have a binomial distribution which we view as a special case of the trinomial just defined, with the convention that $0! = 1$ and $0 \ln 0 = 0$.

Relative-Entropy Minimization. Let K be the set of $\kappa \in [0,1]^6$ such that: for each i, $\sum_{l=0}^{2} \kappa_{i,l} = 1$, with $\kappa_{i,l} = 0$ if l is not in the support of the trinomial given by (x_i^1, x_i^2); $\sum_i \kappa_{i,0} < \mu_0$; and $\sum_i \kappa_{i,2} < \mu_1$. Let $\bar{K}$ be its closure. For each κ, let $D(\omega, x^1, x^2, \kappa) = \sum_i D(\omega, x_i^1, x_i^2, \kappa_i)$ and consider the minimization problem:

$$\min_{\kappa \in \bar{K}} D(\omega, x^1, x^2, \kappa).$$

Let $D^*(\omega, x^1, x^2)$ be the optimal value and $\kappa^*(\omega, x^1, x^2)$ the minimizer. The norm of the gradient $d^*(\omega, x^1, x^2)$ at the optimal solution is non-negative and finite. Denote by $\Omega(x^1, x^2)$ the set of ω such that $P(x \,|\, \omega) = \mu_0$ for some $x \in [x^1, x^2]$. $\Omega(x^1, x^2)$ is then an interval. If $\Omega(x^1, x^2)$ is a singleton, then it is either the state $\omega = 0$ or $\omega = 1$. If $\Omega(x^1, x^2)$ is an interval, then for each $\omega \in \Omega(x^1, x^2)$, in the optimal solution $\kappa^*(\omega, x^1, x^2)$, the i-th coordinates, for each i, are the trinomial probabilities of the distribution derived from (x_i^1, x_i^2) and the optimal relative entropy is zero. Outside this interval of states, the relative entropy $D^*(\cdot, x^1, x^2)$ is strictly increasing in the distance from $\Omega(x^1, x^2)$ and convex. If $\Omega(x^1, x^2)$ is empty then $D^*(\cdot, x^1, x^2)$ is minimized at either $\omega = 0$ or $\omega = 1$ depending on whether $P(x_i^1 \,|\, 1) < \mu_0$ or $(1 - P(x^2 \,|\, 0)) > \mu_1$ and then $D^*(\cdot, x^1, x^2)$ is C^2 and accordingly either strictly increasing or decreasing.

Expected Probability of Winning in a Tie at a Bid. Take pairs $0 \leqslant x_i^1 \leqslant x_i^2 \leqslant 1$ for each i as in the previous subsection. Suppose types in $[x_i^1, x_i^2]$ bid b, types above (resp. below) x_i^2 (resp. x_i^1) bid above b (resp. below b) in the game Γ^n. A tie at bid b in state ω occurs if the total number of traders with signals above x^2 is strictly less than $m_{\mathrm{II}} n$ and those with signals less than x^1 is strictly less than $m_{\mathrm{I}} n$. Thus, it occurs when the

[11]The entropy of a discrete random variable with probability mass $p(x)$ is $-\sum_{x \in x} p(x) \ln(p(x))$; relative to another random variable with probability mass $q(x)$, the entropy is $-\sum_{x \in X} p(x) \ln(\frac{p(x)}{q(x)}$.

empirical frequency of the trinomial draws falls in K. The probability $G^n(\omega, x^1, x^2)$ of a tie (at bid b) in state ω is computed as:

$$G^n(\omega, x^1, x^2) \equiv \sum_{\kappa \in K^n} \prod_i P(\omega, x^1, x^2, \kappa_i)$$

where K^n is the subset of K consisting of κ such that $n\kappa$ is a vector of integers. As we only have to move at most n^{-1} in each coordinate in K to be in K^n, and K is convex, the usual inequalities from the method of types employed in proving Sanov's Theorem give the following bounds:

$$\frac{1}{(n+1)^6} \exp(-\sqrt{6} d^*(\omega, x^1, x^2) + O(n^{-2})) \leqslant \frac{G^n(\omega, x^1, x^2)}{\exp(-nD^*(\omega, x^1, x^2))} \leqslant 1,$$

where $d^*(\omega, x^1, x^2)$ is the norm of the gradient of $D(\omega, x^1, x^2)$ at the entropy minimizer. Moreover, suppose $(x^{1,n}, x^{2,n}) \to (x^1, x^2)$. If ω is in the interior of $\Omega(x^1, x^2)$, then by the Uniform Law of Large Numbers, $\lim_n G^n(\omega, x^{1,n} x^{2,n}) = 1$. If ω falls outside this interval, then we get $\lim_n n^{-1} \ln(G^n(\omega, x^{1,n}, x^{2,n})) = -D^*(\omega, x^1, x^2)$ by Sanov's Theorem. For a boundary point ω of the set $\Omega(x^1, x^2)$, the limit probability falls somewhere in $[0, 1]$.

When there is a tie, the probability of winning is determined by the number of agents involved in the tie and is thus a random variable defined as follows. Let $\bar{\tau}^\infty : \bar{K} \to [0, 1]$ be given by:

$$\bar{\tau}^\infty(\kappa) = \frac{m_1 - \bar{\kappa}_2}{\bar{\kappa}_1}$$

where $\bar{\kappa}_l = \sum_i \kappa_{i,l}$ for $l = 1, 2$. Then, the expected probability of winning a tie in state ω is:

$$\bar{\tau}^n(\omega, x^1, x^2) = \sum_{\kappa \in K^n} \prod_i P(\omega, x^1, x^2, \kappa_i) \bar{\tau}^\infty(\kappa).$$

Also, let

$$\hat{\tau}^n(\omega, x^1, x^2) = \sum_{\kappa \in K^n} \prod_i P(\omega, x^1, x^2, \kappa_i)(1 - \bar{\tau}^\infty(\kappa)).$$

As $\bar{\tau}^\infty(\kappa)$ is at least $\frac{1}{2n}$ and, of course, no more than $\frac{1}{2}$, we have the following bounds for this probability:

$$\frac{1}{2n}G^n(\omega, x^1, x^2) \leqslant \bar{\tau}^n(\omega, x^1, x^2) \leqslant \frac{1}{2}G^n(\omega, x^1, x^2),$$

and similarly,

$$\frac{1}{2}G^n(\omega, x^1, x^2) \leqslant \hat{\tau}^n(\omega, x^1, x^2) \leqslant \frac{2n-1}{2n}G^n(\omega, x^1, x^2),$$

Suppose $(x^{1,n}, x^{2,n}) \to (x^1, x^2)$. If ω is in the interior of $\Omega(x^1, x^2)$, then by the Uniform Law of Large Numbers we have $\lim_n \bar{\tau}^n(\omega, x^{1,n}, x^{2,n}) = \bar{\tau}^\infty(\kappa^*(\omega, x^1, x^2))$.

Finally, let $\tilde{K}^n$ be the subset of $\bar{K}$ consisting of κ such that $n\kappa$ is a vector of integers, $\sum_i \kappa_{i,0} = m_0$, and $\sum_i \kappa_{i,2} = m_1$. Using $\tilde{K}^n$ in the place of K^n we compute $\tilde{G}^n(\omega, x^1, x^2)$, the probability of the event that the number of players with signals less than x^1 is equal to $m_0 n$ and that the number of players with signals above x^2 is equal to $m_1 n$. As above, $\tilde{G}^n$ is driven by the minimum relative entropy $\tilde{D}^*(\omega, x^1, x^2)$. Observe that $\tilde{D}^*(\omega, x^1, x^2) > D^*(\omega, x^1, x^2)$, and hence that $\frac{\tilde{G}^n(\omega,x^1,x^2)}{G^n(\omega,x^1,x^2)} \to 0$ exponentially in n and uniformly in (ω, x^1, x^2).

For the first-order difference equations for a player type i, we need to compute the probability $G^n_i(\omega, x^1, x^2)$ of a tie at a bid b if one trader t of type i were to submit it, as well as the corresponding expected probabilities $\bar{\tau}^n_i(\omega, x^1, x^2)$ and $\hat{\tau}^n_i(\omega, x^1, x^2)$. The probabilities are obtainable by a small modification of the above computations. Fix i. We get n trinomial trials for $j \neq i$ and $n-1$ for i. Let K^n_i be the set of $\kappa \in K$ such that: (a) for $j \neq i$, $n\kappa_j$ is a vector of integers; (b) $(n-1)\kappa_i$ is a vector of integers; $\sum_{j\neq i} n\kappa_{j,0} + (n-1)\kappa_{i,0} \leqslant m_{\mathrm{I}}n - 1$, $\sum_{j\neq i} n\kappa_{j,2} + (n-1)\kappa_{i,2} \leqslant m_{\mathrm{II}}n - 1$, $\sum_{j\neq i} n\kappa_{j,1} + (n-1)\kappa_{i,1} \geqslant 1$. Replace K^n with K^n_i to get the probability $G^n_i(\omega, x^1, x^2)$ of a tie involving i at bid b and also $\bar{\tau}^\infty_i$, with the denominator being $\bar{\kappa}_1 + n^{-1}$ (to include i). One way to leverage the previous computations is to take $n-1$ trials for all j (including i) and then have an extra trial for players $(j, 1)$, $j \neq i$. Thus, we get the bound below for $G^n_i(\omega, x^1, x^2)$:

$$\frac{1}{n^6}\exp(-\sqrt{6}d^*(\omega, x^1, x^2) + O(n^{-2})) \leqslant \frac{G^n_i(\omega, x^1, x^2)}{\exp(-(n-1)D^*(\omega, x^1, x^2))} \leqslant 1,$$

and $\bar{\tau}_i^n(\omega, x^2, x^1)$ is derived as before, using now $G_i^n(\omega, x^1, x^2)$ instead of $G^n(\omega, x^1, x^2)$. We can similarly compute the probability $\tilde{G}_i^n(\omega, x^1, x^2)$ that i is the only player with a signal in $[x^1, x^2]$.

Proof of Lemma 4.1. As $\Theta^\zeta \times X_i$ is compact and $\bar{\pi}_{i,k}^{n,\zeta}$ is continuous in (θ, y_i) for all n (including $n = \infty$), the result is proved if we show that for a sequence $(\theta^n, y_i^n) \to (\theta, y_i)$, we have $\bar{\pi}_i^{n,\zeta,k}(\theta^n, y_i^n) \to \bar{\pi}_i^{\infty,\zeta}(\theta, y_i, k)$ and similarly for the derivatives.

Let $(x^{0,n}, x^{1,n}, x^{2,n}) = (\theta_{-0}^n(k-1), \theta_{-0}^n(k), \theta_{-0}^n(k+1))$ for each n and let (x^0, x^1, x^2) be its limit. Let $y_i^n \to y_i$.

For each n, and for the case of a buyer, we decompose $\bar{\pi}_i^{n,\zeta,k}(\theta^n, y_i^n)$ as

$$\lambda^{0,n} \int_\Omega (v_i(\omega, y_i^n) - b^{k-1}) q^{0,n}(\omega) d\omega + \lambda^{1,n} \int_\Omega (v_i(\omega, y_i^n) - b^k) q^{1,n}(\omega) d\omega - (1 - \lambda^{0,n} - \lambda^{1,n}) \alpha\zeta$$

(for a seller we have $-(1-\alpha)\zeta$ in the place of $\alpha\zeta$),

$$q^{0,n}(\omega) = \frac{\hat{\tau}_i^n(\omega, x^{0,n}, x^{1,n}) p(\omega \mid y_i^n)}{\int_\Omega \hat{\tau}_i^n(\omega', x^{0,n}, x^{1,n}) p(\omega' \mid y_i^n) d\omega'};$$

$q^{1,n}$ is defined similarly using $(x^{1,n}, x^{2,n})$ and also replacing $\hat{\tau}_i^n$ with $\bar{\tau}_i^n$; and

$$\lambda^{\ell,n} = \frac{\int_\Omega q^{\ell,n}(\omega) d\omega}{\int_\Omega [q^{0,n}(\omega) + q^{1,n}(\omega) + \tilde{G}_i^n(\omega, x^{0,n}, x^{1,n})] d\omega}, \ \ell = 0, 1.$$

Suppose $\Omega(x^1, x^2)$ has a nonempty interior. Then, $q^{1,n}$ converges pointwise to the density q^1 given by: $q^1(\omega) = \bar{\tau}^\infty(\kappa^*(\omega, x^1, x^2)) p(\omega \mid y_i)$ if ω belongs to the interior of $\Omega(x^1, x^2)$ and is zero if it does not belong to $\Omega(x^1, x^2)$. Hence, the expectation under $q^{1,n}$ corresponds to the payoff from a tie at bid b^k for types in $[x^1, x^2]$. If $\Omega(x^1, x^2)$ has an empty interior, then $D^*(\omega, x^1, x^2)$ is the lowest at either $\omega = 0$ or $\omega = 1$. Assume the former. Then, for each $\omega < \omega'$,

$$\lim_{n\to\infty} \frac{q^{1,n}(\omega')}{q^{1,n}(\omega)} = \lim_{n\to\infty} \exp[-n(D^*(\omega', x^{1,n}, x^{2,n}) - D^*(\omega, x^{1,n}, x^{2,n}))] = 0.$$

and we have that the limit of the probability measures $Q^{1,n}$ is point mass at $\omega = 0$. Hence the expectation converges to $v_i(0, y_i) - b$, which is what we impute under $\bar{\pi}_i^{\infty,\zeta}$. A similar computation holds for $q^{0,n}$. To finish the proof, we need to get the convergences of $\lambda^{\ell,n}$.

Observe first that because $\tilde{G}_i^n(\omega, x^0, x^1)$ is dominated by $G_i^n(\omega, x^0, x^1)$, $\lambda^{0,n} + \lambda^{1,n} \to 1$. If both $\Omega(x^0, x^1)$ and $\Omega(x^1, x^2)$ have nonempty interiors, then the limit of $\lambda^{0,n}$ exists and is in $(0, 1)$ as $q^{0,n}$ and $q^{1,n}$ converge pointwise; if $\Omega(x^0, x^1)$ has an empty interior but $D^*(\omega, x^0, x^1)$ is lowest at $\omega = 0$, then

$$\lim_n \lambda^{0,n} = \lim_n \exp(-n(D^*(0, x^{0,n}, x^{1,n}) - D^*(0, x^{1,n}, x^{2,n})) = 0.$$

All other cases are handled similarly and we get the appropriate convergence.

The logic for the functions involving the derivatives is similar and, therefore, omitted.

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Department of Economics, University of Rochester, NY 14627, USA.
Email address: `paulo.barelli@rochester.edu`

Department of Economics, University of Rochester, NY 14627, USA.
Email address: `s.govindan@rochester.edu`