Weight-Ranked Divide-and-Conquer Contracts

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Abstract

This paper studies a large class of multi-agent contracting models with the property that agents' payoffs constitute a weighted potential game. Multiple equilibria arise due to agents' strategic interactions. I fully characterize a contracting scheme that is optimal for the principal for all equilibrium selection criteria that are more pessimistic than potential maximization. This scheme ranks agents in ascending order of their weights in the weighted potential game and then induces them to accept their offers in a dominance-solvable way, starting from the first agent. I apply the general results to networks, public goods/bads, and a class of binary-action applications.

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1 Introduction

Many contracting situations involve multiple agents, and in most of these situations, an agent's payoff depends on other agents' actions. For example, the value of joining a platform increases with the number of users; the return from an investment is affected by others' investment decisions; the incentive to work varies with co-workers' efforts. A natural question arises: How does the principal's optimal contracting scheme take into account these (potentially very complex) interactions among agents? Moreover, agents' strategic interactions often generate multiple equilibria. All the above examples may have (at least) a high- and a low-participation/investment/effort equilibrium. The principal's payoff typically differs across equilibria. In other words, one contracting scheme possibly maps to multiple payoff levels. This raises a more fundamental issue: How should we define the *optimality* of a contracting scheme when there are multiple equilibria? Ultimately, what contracts should the principal offer when there are multiple agents?

To deal with the fundamental issue, the conventional approach is to specify an equilibrium selection criterion and get rid of multiple equilibria. However, this approach is not fully satisfactory because it replaces the issue of multiple equilibria with the issue of multiple equilibrium selection criteria. Specifically, the optimal contracts for the best-case scenario are likely rejected by agents in less favorable scenarios. On the other hand, the optimal contracts for the worst-case scenario likely forgo huge profits in more favorable scenarios. What if the principal (or we as researchers) is uncertain about the equilibrium selection criterion? In this situation, can we still confidently recommend (or predict) what contracts the principal should (or would) offer?

This paper provides a partial positive answer to the above question for a large class of multi-agent contracting models with complete information. The timing is standard: The principal offers each agent a menu of publicly observable bilateral contracts in stage 1, and then each agent simultaneously chooses a contract (or rejects all contracts) in stage 2. Each agent has a general (i.e., possibly multi-dimensional) action set. Regarding their interactions, some agents' actions can be strategic complements while others can be strategic substitutes.

The principal has a general preference (say, can be self-interested or benevolent). The only key assumption is that agents' payoffs constitute a *weighted potential game*. Unlike the conventional approach, I do not pre-specify the equilibrium selection criterion but instead examine the principal's optimal contracting scheme under various equilibrium selection criteria. One criterion is said to be *more pessimistic* than the other if, in every subgame, the selected equilibrium gives the principal a weakly lower payoff.

The main result of this paper (Theorem 1) is that for all equilibrium selection criteria that are more pessimistic than *potential maximization* (a criterion originated from potential game theory), the principal optimally offers *weight-ranked divide-and-conquer* (w-DC) *contracts*. The w-DC contracts rank agents in increasing order of their *weights* in the weighted potential game and then offer each agent one contract asking him to take a specified action. The associated contract prices/subsidies are set in a way that the first agent has a weakly dominant strategy to accept his offer; given the first agent accepts, the second agent has an (iterated) weakly dominant strategy to accept as well, and so on. Thus, the w-DC contracts induce all agents to accept their offers as a dominance-solvable equilibrium in this particular order. Moreover, Proposition 2 shows that the w-DC contracts are possibly suboptimal for all equilibrium selection criteria that are *not more pessimistic* than potential maximization. Therefore, I have identified the complete set of equilibrium selection criteria for which the principal always offers the w-DC contracts.

Section 4 applies the general results to three special cases: networks, public goods/bads (hereafter goods for simplicity), and a class of binary-action games. The general undirected network I consider allows for multi-dimensional actions for agents and the coexistence of positive and negative links. In addition, agents can be heterogeneous, among others, in their (i) network positions, (ii) valuations of network benefits/costs, and (iii) importance to their linked agents regarding network benefits/costs. The w-DC contracts rank agents in increasing valuation-to-importance ratio. In contrast to the conventional wisdom in the economics of networks literature (see, e.g., the surveys by Bloch 2016 and Section 7 of Jackson et al. 2017) that the principal should offer more favorable contracts to agents with high centrality, the network structure plays no role in agents' ranking. I further derive a natural network formation process and show that under this process, agents with many (positive) links are those with high valuations and with either high or low importance.

In the public good application, agents can be heterogeneous, among others, in their (i) valuations of the public good and (ii) importance of their contributions to the public good. In contrast to the network application, the w-DC contracts rank agents in increasing valuation regardless of their importance. The reason for these opposing results is that the public good is non-excludable whereas the "network good" is excludable. The class of binary-action games I consider is applicable to most traditional applications such as network externalities, exclusive dealing, takeovers, and vertical contracting.

This paper primarily contributes to the literature on multi-agent contracting. Various contracting schemes are derived under different equilibrium selection criteria in the literature. For example, the seminal works of Segal (1999, 2003) study the same model but derive very different contracting schemes under the most optimistic and pessimistic criteria, respectively (Section 3.3 closely compares his latter work with mine). The primary contribution of this paper is to show that one contracting scheme—the w-DC contracts—is relatively robust because it is optimal for a large class of criteria. This result helps us make better predictions and policy advice on multi-agent contracting problems, especially when we as researchers do not know which equilibrium selection criterion prevails. Despite some model variations, specific divide-and-conquer (DC) contracts are derived (e.g., Segal 2003; Winter 2004; Bernstein and Winter 2012; Sakovics and Steiner 2012; Halac et al. 2020; Nora and Winter 2024) under (i) binary/one-dimensional actions for agents, (ii) strategic complementarities among all agents, and (iii) the most pessimistic criterion. This paper contributes by uncovering the generality and robustness of DC contracts by relaxing all three restrictions substantially and deriving the general form of DC contracts. As a further contribution, this paper is the first to derive the optimal ranking for DC contracts in a general framework and, thus, fully characterize the optimal contracting scheme: the w-DC contracts.

This paper also advances the analysis of multi-agent contracting problems. Although my primary focus is the w-DC contracts, the general framework and tools developed are applicable to all such problems. In particular, the novel interaction structure among agents—derived from two binary relations—is considerably more flexible than the conventional strategic complementarity/substitutability structure, enabling us to study a wider range of contracting environments. In addition, a methodological contribution of this paper is to fully exploit potential game theory in the study of multi-agent contracting. The concept of potential games was introduced by Rosenthal (1973) and formalized by Monderer and Shapley (1996). Potential maximization refines Nash equilibrium in (weighted) potential games.[1] Sections 2 and 3.1 will explain both concepts. As Section 4 reveals, agents' payoffs constitute a weighted potential game in many contracting models. By exploiting this meaningful property, one may be able to derive stronger results as this paper does.

Beyond the above contributions to multi-agent contracting, Section 4 contributes to the economics of networks and public goods. One contribution common to both strands of literature is to show the optimality and robustness of the corresponding w-DC contracts in each environment. As an independent contribution, to my knowledge this paper (Lemma 4) is the first to show that a general class of undirected networks with multi-dimensional actions are weighted potential games.[2] Overall, Section 4 demonstrates how seemingly contradictory findings across applications are reconciled with the unifying theories developed in this paper.

[1] Although inconsequential to my main result, this refinement is justified by many theoretical and experimental studies; see Chan (2021, Related Literature) for a summary of established justifications. In particular, it coincides with global-game selection in supermodular weighted potential games (Frankel et al. 2003) and risk dominance in two-agent two-action games.

[2] Bramoulle et al. (2014) show that a class of undirected networks with one-dimensional actions are exact potential games (i.e., weighted potential games with all weights equal to one) and then pioneeringly utilize potential game theory to analyze networks. Bourles et al. (2017) show that a class of directed networks with multi-dimensional actions are best-response potential games (Voorneveld 2000), which generalize weighted potential games. However, many important properties of weighted potential games, in particular the validity of potential maximization as a refinement of Nash equilibrium (which is proved in Appendix B for weighted potential games), do not carry over to best-response potential games.

2 Model

A principal ("she") contracts with $n \geq 1$ agents ("he"). Let $N \equiv \{1, \dots, n\}$ denote the set of agents. Let A_i denote agent i's set of actions with $o_i \in A_i$ denoting the outside option of rejecting the principal's offers. To avoid technical complications (mainly existence issues), assume each A_i is finite. Let $a \equiv (a_i)_i \in \prod_i A_i \equiv A$ denote agents' action profile, and $a_{-i} \in \prod_{j \neq i} A_j \equiv A_{-i}$ and $a_{-ij} \in \prod_{k \notin \{i,j\}} A_k \equiv A_{-ij}$ are defined in the usual way. The game has two stages. In stage 1, the principal sets a price function $p_i \in P_i \equiv \{p_i : A_i \rightarrow \mathbb{R} | p_i(o_i) = 0\}$ for each agent i. This is equivalent to offering each agent a menu of bilateral[3] contracts $(a_i, p_i(a_i))$ given that she can always prevent an agent from taking a certain action $a_i \in A_i \backslash \{o_i\}$ by charging an arbitrarily high price $p_i(a_i)$ for that action.[4] In stage 2, every agent observes the menu profile $p \in \prod_i P_i \equiv P$ and simultaneously chooses an action $a_i \in A_i$. Each agent i's payoff is linear in money $u_i(a) - p_i(a_i)$ where $u_i : A \rightarrow \mathbb{R}$ measures his gross utility. The principal's payoff is $U(a, \sum_i p_i(a_i))$ where $U : A \times \mathbb{R} \rightarrow \mathbb{R}$ is non-decreasing in her total revenue. The function U is sufficiently general to represent a self-interested (e.g., $U = \sum_i p_i(a_i)$) or benevolent (e.g., $U = \sum_i u_i(a)$) principal.

The results in the next section hold for all agents' (gross) utilities $u \equiv (u_i)_i$ satisfying the following three assumptions.

Assumption 1 (weighted potential game) There exists a (weight) vector $w \equiv (w_i)_i \in \mathbb{R}^n_{++}$ and a (potential) function $\Phi : A \rightarrow \mathbb{R}$ such that for all $i \in N$,

$$u_i(a_i, a_{-i}) - u_i(a'_i, a_{-i}) = w_i[\Phi(a_i, a_{-i}) - \Phi(a'_i, a_{-i})] \quad \text{for all } a_i, a'_i \in A_i \text{ and } a_{-i} \in A_{-i}. \quad (1)$$

Assumption 1 (hereafter A1; similarly for A2 and A3) states that agents' utilities u constitute a *weighted potential game*. Verbally, there exists a real-valued function Φ defined

[3] As pointed out by the literature (e.g., Bernstein and Winter 2012; Halac et al. 2020), the principal can only rely on bilateral contracts in many real-world contracting situations. If the principal is allowed to offer multilateral contracts (i.e., contracts that can condition on others' actions), she can easily induce a unique equilibrium that fully extracts all agents' surplus in most such models.

[4] For example, if she offers only one contract to i asking him to take action $\hat{a}_i$ at price/subsidy $\hat{p}_i$, the corresponding price function is $p_i(o_i) = 0$, $p_i(\hat{a}_i) = \hat{p}_i$, and $p_i(a_i) \rightarrow \infty$ for all $a_i \notin \{o_i, \hat{a}_i\}$.

on the set of agents' action profiles such that the change in any agent's utility by unilaterally switching actions is proportional (with proportion w_i for i) to the corresponding change in Φ. Thus, all agents' strategic considerations, which concern only unilateral deviations, are summarized by Φ. Observe that (1) holds if and only if there exists a (pure externality) function $\xi_i : A_{-i} \to \mathbb{R}$ such that

$$u_i(a) = w_i\Phi(a) + \xi_i(a_{-i}) \quad \text{for all } a \in A.^5 \tag{2}$$

Many contracting models satisfy A1. For example, suppose $o_i = \mathbf{0} \in A_i \subseteq \mathbb{R}^n_+$ ($\mathbf{0}$ is a vector of n zeros; analogously for $\mathbf{1}$) and u_i takes the following form:

$$u_i(a) = c_i(a_i) + v_i \sum_j g_{ij}\theta_j a_{ij} a_{ji}, \tag{3}$$

where $g_{ij} = g_{ji} \in \{-1, 0, 1\}$ indicates i and j are negatively, not, or positively linked, respectively ($g_{ii} = 0$ by convention); $a_i \equiv (a_{ij})_j$ where a_{ij} is i's action on j; $c_i : A_i \to \mathbb{R}$ measures his stand-alone benefit/cost; $v_i \in \mathbb{R}_{++}$ measures his valuation of network benefits/costs; and $\theta_j \in \mathbb{R}_{++}$ measures the relative importance of j's actions to his linked agents (aka *neighbors*). Agents can differ in five dimensions in this general example: $(A_i, c_i, v_i, \theta_i)$ and how they are linked as described by $g \equiv (g_{ij})_{i,j}$. Thus, this example in turn covers a wide variety of contracting environments. In particular, if $A_i \subseteq \{a_i \in \mathbb{R}^n_+ | a_{ij} = a_{ii} \ \forall j\}$ (or equivalently, $c_i(a_i) \to -\infty$ if $a_{ij} \neq a_{ii}$ for some j) then it reduces to a network with one-dimensional actions. Section 4.1 analyzes (3) and (Lemma 4) shows that it satisfies A1. Sections 4.2 and 4.3 study two other examples that satisfy A1.

To state the other two assumptions on agents' utilities u, first we need to define two binary relations C and S between any two distinct agents' action sets A_j and A_i.

Definition 1 The expression $a_j C a_i$ ($a_j S a_i$) stands for

$$u_i(a_i, o_j, a_{-ij}) - u_i(o_i, o_j, a_{-ij}) \leq (\geq)\ u_i(a_i, a_j, a_{-ij}) - u_i(o_i, a_j, a_{-ij}) \quad \forall a_{-ij} \in A_{-ij}. \tag{4}$$

[5] The "if" part is trivial. For the "only if" part, the function $\xi_i(a_{-i}) \equiv u_i(a) - w_i\Phi(a)$ is well defined because, by (1), $u_i(a_i, a_{-i}) - w_i\Phi(a_i, a_{-i}) = u_i(a'_i, a_{-i}) - w_i\Phi(a'_i, a_{-i})$ for all $a_i, a'_i \in A_i$.

In words, a_jCa_i (a_jSa_i) means a_j always strategically complements (substitutes) a_i relative to the outside option. For the example (3), observe that a_jCa_i if $g_{ij} \neq -1$ and a_jSa_i if $g_{ij} \neq 1$. The second assumption (clearly satisfied by (3)) is stated as follows.

Assumption 2 (sign independence of others' actions) For each $a \in A$ and distinct $i, j \in N$, a_jCa_i or a_jSa_i.

To better understand A2, think of a scenario in which agents i and j consider between a particular action (say, a_i for i and a_j for j) and the outside option, and all other agents decide to take the outside option (i.e., $a_{-ij} = o_{-ij}$). In this scenario, a_j either strategically complements or substitutes a_i because $u_i(a_i, o_j, o_{-ij}) - u_i(o_i, o_j, o_{-ij})$ is either less or greater than $u_i(a_i, a_j, o_{-ij}) - u_i(o_i, a_j, o_{-ij})$. A2 implies if a_j complements (substitutes) a_i in this scenario then a_j complements (substitutes) a_i regardless of others' actions $a_{-ij} \in A_{-ij}$.

Observe from (1) and (4) that A1 implies C and S are symmetric, i.e., a_jCa_i (a_jSa_i) if and only if a_iCa_j (a_iSa_j). In other words, any two agents' actions either strategically complement or substitute each other relative to the outside option. This is also the reason for restricting (3) to undirected networks: A1 is violated if $g_{ij} \neq g_{ji}$ for some i, j. I write $a_j\bar{C}a_i$ if a_jCa_i and not a_jSa_i. Clearly, $\bar{C}$ is also symmetric. The last assumption is stated as follows.

Assumption 3 (weak transitivity for C) For each $a \in A$ and distinct $i_1, i_2, \ldots, i_m \in N$ ($m \leq n$), if $a_{i_1}\bar{C}a_{i_2}\bar{C}\cdots\bar{C}a_{i_m}$ then $a_{i_1}Ca_{i_m}$.

Observe that A3 is weaker than assuming C ($\bar{C}$) is transitive, which replaces all $\bar{C}$ (C) in A3 with C ($\bar{C}$). Analogously, (3) satisfies A3 if $g_{i_1i_m} \neq -1$ whenever $g_{i_1i_2} = g_{i_2i_3} = \cdots = g_{i_{m-1}i_m} = 1$. A2 and A3 are relatively weak. They are vacuous if there are only two agents. For more agents, they impose no restrictions on any two actions $a_i, a_i' \in A_i \backslash \{o_i\}$ from the same agent. In particular, they allow a_jCa_i for some a_i but a_jSa_i' for some other a_i'. Furthermore, even if A2 and A3 are strengthened to a_jCa_i (analogously for a_jSa_i) for all $i, j \in N$ and $a \in A$, this is still much weaker than the following strategic complementarity (analogously for substitutability) assumption as conventionally imposed by most of this literature.

Condition 1 (strategic complementarities) For all $i \in N$, $o_i = 0 \in A_i \subseteq \mathbb{R}_+$ and for all $a_i, a'_i \in A_i$ with $a_i > a'_i$, $u_i(a_i, a_{-i}) - u_i(a'_i, a_{-i})$ is non-decreasing in $a_{-i} \in A_{-i}$.

Observe that Condition 1 (hereafter C1) restricts agents' actions to be one-dimensional and imposes restrictions on every two pairs of actions (a_i, a'_i) and (a_j, a'_j) from any two agents. Unlike A2 and A3, C1 is far from vacuous when $n = 2$. Therefore, the extra flexibility of A2 and A3 enables us to study many more contracting environments such as the network application (3).[6]

3 Analysis

In this two-stage model, each menu profile $p \in P$ leads to a different subgame, and some subgames exhibit multiple equilibria. Therefore, depending on her beliefs about agents' behavior in stage 2, the principal may offer different menu profiles in stage 1. For example, if she believes agents always coordinate on the best (worst) equilibrium for her whenever multiple equilibria are present, she will charge them a lot (less). Section 3.1 (Proposition 1) first derives a menu profile (10) that is optimal under an equilibrium selection criterion called *potential maximization*. This result is the stepping stone to prove the main result (Theorem 1) in Section 3.2, which states that (10) is actually optimal for a large class of beliefs held by the principal.

[6] In a binary-action setting, Bernstein and Winter (2012, Section III.D) also allow some actions to be strategic substitutes but require C to be transitive. This strengthening of A3 is quite restrictive when applied to (3). Consider a three-agent example where 1 and 2 are negatively linked and 3 is not linked. It (does not) violates (weak) transitivity for C because both 1 and 2's actions weakly complement 3's. Overall, my interaction structure generalizes (beyond binary actions) and relaxes theirs.

3.1 Potential Maximization

It is known that for a weighted potential game, the maximizer of the potential function (which exists because A is finite) is generically unique and is a Nash equilibrium.[7] This equilibrium refinement is called potential maximization. It is also known to coincide with *risk dominance* (Harsanyi and Selten 1988) in 2×2 games (i.e., $n = |A_1| = |A_2| = 2$). As the first step of the analysis, I show that if agents' utilities u constitute a weighted potential game, any subgame with an arbitrary menu profile $p \in P$ offered by the principal is also a weighted potential game. Thus, we can (and this subsection will) select the potential maximizer in every subgame.[8] All omitted proofs are in Appendix A.

Lemma 1 *Suppose A1 holds. Every subgame is a weighted potential game with the same weight vector w given in A1 and the potential function*

$$\Phi_p(a) = \Phi(a) - \sum_i \frac{p_i(a_i)}{w_i}. \tag{5}$$

Under potential maximization, the principal's problem can be formulated as the following two-step optimization problem. In step 1, given a target action profile $\hat{a} \in A$, she chooses the optimal menu profile $p^* \in P$ such that $\hat{a}$ is the potential maximizer in the subgame, i.e.,

$$\max_{p \in P} U(\hat{a}, \sum_i p_i(\hat{a}_i)) \quad \text{s.t.} \quad \Phi_p(\hat{a}) \geq \Phi_p(a) \quad \text{for all } a \in A. \tag{6}$$

In step 2, she chooses the optimal action profile $a^* \in A$, i.e.,

$$\max_{a \in A} U(a, \sum_i p_i^*(a_i)). \tag{7}$$

[7] See Appendix B for generic uniqueness of the potential maximizer. It is a Nash equilibrium: If someone deviates from the potential maximizer, the potential will decrease, and by (1), the deviator will have a lower payoff. For more interpretations of weighted potential games, see Chan (2021, Section 2).

[8] Note however that some non-generic subgames admit multiple potential maximizers (just as risk dominance works only generically). I resolve this technical issue by allowing the principal to select among potential maximizers; see footnote 12 for the formal treatment.

The main analysis is on the step-1 problem (6). Observe that given a fixed $\hat{a}$, the principal's objective is to maximize her total revenue. Also, the constraints can be simplified with (5). Thus, (6) is simplified to

$$\max_{p \in P} \sum_i p_i(\hat{a}_i) \quad \text{s.t.} \quad \sum_i \frac{p_i(\hat{a}_i) - p_i(a_i)}{w_i} \leq \Phi(\hat{a}) - \Phi(a) \quad \text{for all } a \in A. \tag{8}$$

Now observe that charging arbitrarily high prices $p_i(a_i)$ (equivalent to not offering the contract $(a_i, p_i(a_i))$ to agent i) for all $a_i \notin \{o_i, \hat{a}_i\}$ of every $i \in N$ relaxes all constraints involving $a_i \notin \{o_i, \hat{a}_i\}$ and has no impact on her total revenue. Therefore, we have the following lemma (p. 17 explains why this result fails in general).

Lemma 2 *Suppose A1 holds. Under potential maximization, the principal can restrict herself to offering (at most) one contract to each agent without loss of optimality.*

Under this restriction, agent i only chooses between the contract offer $(\hat{a}_i, p_i(\hat{a}_i))$ and the outside option $(o_i, p_i(o_i) = 0)$ (if $\hat{a}_i = o_i$ then the principal is not offering him any contract). Denote $\hat{N} \equiv \{i \in N | \hat{a}_i \neq o_i\}$ as the set of agents whom she wants to contract with, $\hat{p}_i \equiv p_i(\hat{a}_i)$ as the contract price, and $\hat{p} \equiv (\hat{p}_i)_{i \in \hat{N}}$ as the price vector. The step-1 problem (8) is further simplified as follows.

Lemma 3 *Suppose A1 holds. For any target action profile $\hat{a} \in A$, the principal's optimal contracts under potential maximization solve the following linear program:*

$$\max_{\hat{p} \in \mathbb{R}^{|\hat{N}|}} \sum_{i \in \hat{N}} \hat{p}_i \quad s.t. \quad \sum_{i \in \hat{N}: a_i = o_i} \frac{\hat{p}_i}{w_i} \leq \Phi(\hat{a}) - \Phi(a) \quad \text{for all } a \in \prod_i \{o_i, \hat{a}_i\}. \tag{9}$$

The proof of the following proposition shows that, together with A2 and A3, the set of feasible solutions of (9) is equivalent to a submodular polyhedron. Therefore, the optimal solution can be obtained by Lovasz's (1983) result.

Proposition 1 *Suppose A1–A3 hold and $w_1 \leq \cdots \leq w_n$. For any target action profile $\hat{a} \in A$, the principal's optimal contracts under potential maximization are*

$$\hat{p}_i^* = u_i(\hat{a}_1, \ldots, \hat{a}_{i-1}, \hat{a}_i, \hat{b}_{i+1}, \ldots, \hat{b}_n) - u_i(\hat{a}_1, \ldots, \hat{a}_{i-1}, o_i, \hat{b}_{i+1}, \ldots, \hat{b}_n) \quad \text{for all } i \in N, \tag{10}$$

where $\hat{b}_j = o_j$ *if* $\hat{a}_j C \hat{a}_i$ *and* $\hat{b}_j = \hat{a}_j$ *otherwise.*[9] *If* $w_1 < \cdots < w_n$*, the above contracts are the unique optimal solution to (9).*

For submodular polyhedra, Lovasz proves that a feasible solution generated by the *greedy algorithm* is optimal. I now illustrate this algorithm with $\hat{N} = N$ for convenience. Observe from (9) that agents with higher weights are less sensitive to price changes. Therefore, first choose $\hat{p}_n$ as large as possible. There is an upper bound given by the inequality with $a = (\hat{a}_1, \ldots, \hat{a}_{n-1}, o_n)$, i.e., $\hat{p}_n / w_n \leq \Phi(\hat{a}) - \Phi(\hat{a}_1, \ldots, \hat{a}_{n-1}, o_n)$, so set $\hat{p}_n$ at this bound. Observe that by (1) $\hat{p}_n = \hat{p}^*_n$ in (10), implying n is indifferent between accepting and rejecting his offer given all others accept theirs. Next choose $\hat{p}_{n-1}$ as large as possible. There are now two upper bounds given by the inequalities with $a = (\hat{a}_1, \ldots, \hat{a}_{n-2}, o_{n-1}, \hat{a}_n)$ and $a = (\hat{a}_1, \ldots, \hat{a}_{n-2}, o_{n-1}, o_n)$, so set $\hat{p}_{n-1}$ at the minimum of these two. Observe that with some simplifications $\hat{p}_{n-1} = \hat{p}^*_{n-1}$, implying $n-1$ has a weakly dominant strategy to accept given agents 1 to $n-2$ accept. The remaining prices $\hat{p}_{n-2}, \ldots, \hat{p}_1$ are chosen analogously. Note however that without A2 and A3, possibly $\hat{p}_i \neq \hat{p}^*_i$ for some $i \leq n-2$ (p. 18 provides further details).[10] Nevertheless, A2 and A3 guarantee $\hat{p}_i = \hat{p}^*_i$ for all i, implying every agent has a weakly dominant strategy to accept given all his preceding agents accept. In other words, the contracts in (10) implement $\hat{a}$ as a dominance-solvable equilibrium in a particular order, and I call them *weight-ranked divide-and-conquer* (w-DC) *contracts.* If $\hat{a}_j C \hat{a}_i$ for all $i, j \in N$, the w-DC contracts reduce to

$$\hat{p}^*_i = u_i(\hat{a}_1, \ldots, \hat{a}_i, o_{i+1}, \ldots, o_n) - u_i(\hat{a}_1, \ldots, \hat{a}_{i-1}, o_i, \ldots, o_n) \quad \text{for all } i \in N. \tag{11}$$

I call the above *weight-ranked simple divide-and-conquer* (w-SDC) *contracts.*[11]

The proof of Proposition 1 shows that under A1–A3, agents can be partitioned into several groups where their target actions $\hat{a}_i$ are strategic complements within a group and

[9] For notational convenience, the optimal contracts also include agents with $\hat{a}_i = o_i$. For these agents, we have $\hat{p}^*_i = 0$, which coincides with the requirement that $p_i(o_i) = 0$.

[10] A2 and A3 are not needed to show $\hat{p}_{n-1} = \hat{p}^*_{n-1}$ because when the algorithm chooses $\hat{p}_{n-1}$, it is as if there are only two agents $n-1$ and n. Recall from p. 8 that A2 and A3 are vacuous in this case.

[11] Recall that most of the literature imposes C1, which implies all target actions strategically complement each other. Therefore, their derived DC contracts take (variants of) this simplified form.

substitutes across groups. The idea behind the rest of the proof (written slightly differently for brevity) is that the greedy algorithm can then be optimally applied to each group, and the collection of these solutions yields the w-DC contracts.

We now proceed to the principal's step-2 problem (7). By implementing $\hat{a}$ with the w-DC contracts, she receives a payoff of

$$\begin{aligned} & U(\hat{a}, \sum_i [u_i(\hat{a}_1, \ldots, \hat{a}_{i-1}, \hat{a}_i, \hat{b}_{i+1}, \ldots, \hat{b}_n) - u_i(\hat{a}_1, \ldots, \hat{a}_{i-1}, o_i, \hat{b}_{i+1}, \ldots, \hat{b}_n)]) \\ = \; & U(\hat{a}, \sum_i w_i [\Phi(\hat{a}_1, \ldots, \hat{a}_{i-1}, \hat{a}_i, \hat{b}_{i+1}, \ldots, \hat{b}_n) - \Phi(\hat{a}_1, \ldots, \hat{a}_{i-1}, o_i, \hat{b}_{i+1}, \ldots, \hat{b}_n)]). \quad \text{(by (1))} \end{aligned}$$

She then chooses the optimal action profile $a^* \in A$ to maximize the above. If $\hat{a}_j C \hat{a}_i$ for all $i, j \in N$ and all agents' weights are equal to w, the above is simplified to $U(\hat{a}, w[\Phi(\hat{a}) - \Phi(o)])$. Hence, we have the following corollary.

Corollary 1 *If $a_j C a_i$ and $w_i = w$ for all $i, j \in N$ and $a \in A$, the principal's optimal action profile is*

$$a^* \in \arg\max_{a \in A} U(a, w[\Phi(a) - \Phi(o)]).$$

When all agents have the same weight, w can be normalized to 1 by rescaling the potential function Φ to $w\Phi$; after normalization it is called an *exact potential game*.

3.2 General Beliefs

The previous analysis restricts attention to potential maximization because we need Proposition 1 to prove the main result. This subsection considers the general case where the principal holds a *belief* $\beta : P \to \Delta(A)$ that assigns a probability distribution over agents' action profiles for each menu profile p. This formulation is more flexible and thus often more plausible than imposing an *equilibrium selection criterion* (e.g., potential maximization), in which $\beta(p)$ must assign probability 1 to a Nash equilibrium of that subgame. In particular, a belief allows the principal to consider the possibility of miscoordination among agents, which is reasonable especially when multiple equilibria exist. A weak and natural restriction maintained throughout is that beliefs are *rationalizable*, i.e., the support of $\beta(p)$ contains only rationalizable action profiles of that subgame.

Intuitively, the principal is pessimistic (optimistic) if she believes action profiles that are bad (good) for her are likely to arise. I now define a preorder on her beliefs; this relation will help us state the main result concisely.

Definition 2 One belief β is *more pessimistic* than the other β' if the principal receives a weakly lower expected payoff at $\beta(p)$ than at $\beta'(p)$ for each $p \in P$.

Before proceeding, I first deal with a technical issue that frequently appears in contracting problems. This "open set" issue can be easily seen in a one-agent example where $A_1 = \{o_1 = 0, 1\}$, $u_1(0) = 0$, $u_1(1) = 1$, and $U(a_1, p_1(a_1)) = p_1(a_1)$. Every belief puts probability 1 on $a_1 = 1$ ($a_1 = 0$) for all $p_1(1) < 1$ ($p_1(1) > 1$). However, if a belief does not put probability 1 on $a_1 = 1$ when $p_1(1) = 1$ then an optimal contracting scheme does not exist. The solution is standard: To guarantee existence, I allow the principal to implement $a_1 = 1$ also with $p_1(1) = 1$. This footnote provides the formal treatment.[12]

We are ready to derive the main result. Recall from p. 12 that the w-DC contracts implement the target action profile $\hat{a}$ as a dominance-solvable equilibrium. This is a *property* of the w-DC contracts, i.e., unrelated to the principal's belief. In other words, regardless of her belief, she always has the option to offer the w-DC contracts and implement $\hat{a}$ in the same dominance-solvable way.[13] In addition, for any belief that is more pessimistic than potential maximization, by definition the principal cannot do better than she does under

[12]Let $B(\sigma) \equiv \{p \in P | \beta(p) = \sigma\}$ denote the set of menu profiles implementing $\sigma \in \Delta(A)$ given the belief is β. Thus, any belief can be expressed as an implementation requirement, which is fully characterized by $B \equiv (B(\sigma))_{\sigma \in \Delta(A)}$. An optimal contracting scheme may not exist because $B(\sigma)$ is not always closed. To guarantee existence, I slightly relax each requirement B by enlarging $B(\sigma)$ to its closure for every σ. Note that after applying this enlargement to potential maximization, the principal can select among potential maximizers in a non-generic subgame.

[13]Notice that $\hat{a}$ becomes the unique rationalizable action profile if she charges each agent a price slightly lower than that of the w-DC contracts; any belief must then assign probability 1 to $\hat{a}$. Enlarging $B(\hat{a})$ to its closure in footnote 12 guarantees she can implement $\hat{a}$ with the w-DC contracts for any belief.

potential maximization. Given the w-DC contracts are optimal under potential maximization (Proposition 1), they remain optimal for all these beliefs.

Theorem 1 *Suppose A1–A3 hold. The w-DC contracts are optimal for the principal for all beliefs that are more pessimistic than potential maximization.*

In the presence of multiple equilibria, most of this literature (including all those cited in this paper) specifies the principal's belief and then searches for the corresponding optimal contracting scheme like what Section 3.1 does. However, in practice and applications we as researchers often have no access to her actual belief. To overcome this challenge, which is also the conceptual novelty of this paper, Theorem 1 does not pre-specify her belief but instead derives a contracting scheme that is optimal for a large class of her beliefs. With this main result, we can confidently predict/advise that the principal would/should offer the w-DC contracts as long as she is relatively pessimistic. The literature (see p. 4) derives specific divide-and-conquer contracts under the requirement of *unique implementation*, i.e., the principal can only choose from contracting schemes p that induce a unique Nash equilibrium in the stage-2 subgame. This requirement is generally more demanding than (but often equivalent to) the most pessimistic criterion. Theorem 1 reveals that, for a large class of contracting models, the use of the w-DC contracts is actually robust to various beliefs.[14] In other words, these contracts need not be justified by extreme pessimism or very demanding implementation requirements.

Theorem 1 does not exclude the possibility that the w-DC contracts remain optimal even if the principal is more optimistic (or not more pessimistic) than potential maximization. A natural question is then whether the main result can be further generalized to include more beliefs. It turns out this is difficult without making additional assumptions. In particular, the following proposition shows that this is infeasible if we restrict attention to beliefs being (pure- or mixed-strategy) equilibrium selection criteria.

[14] Given the w-DC contracts implement $\hat{a}$ as a dominance-solvable equilibrium, they remain feasible and thus optimal under unique implementation (after enlarging to its closure as in footnote 12).

Proposition 2 *There exists a game satisfying A1–A3 in which the w-DC contracts are suboptimal for the principal for all equilibrium selection criteria that are not more pessimistic than potential maximization.*

The example is a symmetric 2×2 coordination game with two pure equilibria (namely, acceptance and rejection equilibria) and one mixed as long as neither action is dominant. The key is to construct the principal's payoff U in a way that both the acceptance and mixed equilibria give her the same expected payoff. Thus, for any criterion not more pessimistic than potential maximization, there must exist a price vector p for which the potential maximizer is the rejection equilibrium but this criterion selects either the acceptance or mixed equilibrium; either gives her a payoff higher than what the w-DC contracts can achieve.

Theorem 1 and Proposition 2 together imply we have identified the entire set of equilibrium selection criteria for which the principal always offers the w-DC contracts. The watershed—potential maximization—is typically not too optimistic or pessimistic because, like risk dominance, this criterion is dictated by agents' payoffs u but not the principal's payoff U, whereas "more pessimistic" is defined according to U but not u. This also implies how large is the set of beliefs more pessimistic than potential maximization depends on the relationship between u and U. This set is small (precisely, a singleton) in the practically rare case where $w_i = 1$ for all i and $U = -\Phi(a) + \sum_i p_i(a_i) = -\Phi_p(a)$ by (5), i.e., the principal and agents (proxied by Φ_p) have exactly opposite preferences. In this case, $\arg\max_{a \in A} \Phi_p(a)$ always corresponds to the worst action profile for the principal, i.e., potential maximization coincides with the most pessimistic belief. When applied to this case, Theorem 1 is not more robust than the unique implementation approach in the literature. The set of beliefs more pessimistic than potential maximization becomes larger when departing from the above case; Appendix C illustrates this with the (generalized) example in Proposition 2.

3.3 Discussion

Comparison to Segal's (2003) Lemma 4 The closest work is Segal (2003), who analyzes the same model under C1 (he does not impose A1 except for Lemma 4) and unique implementation. His Lemma 3 first shows that under unique implementation, any feasible contracting scheme p must implement the target action profile $\hat{a}$ via a divide-and-conquer path, i.e., to induce $\hat{a}$ as a dominance-solvable equilibrium in stage 2. There are many choices of DC paths, and his Lemma 4 shows that all paths (e.g., the w-DC contracts) are equally optimal if agents' utilities u constitute an exact potential game (i.e., A1 with $w_i = 1$ for all i). However, his results are silent on the set of feasible contracting schemes if the principal holds other beliefs. In particular, observe from (9) that the feasible set is much larger under potential maximization, i.e., she need not restrict attention to DC contracting schemes. By solving a linear program (no similar problem arises in Segal's analysis), Proposition 1 finds that a DC contracting scheme—the w-DC contracts—remains optimal in this larger feasible set.

Single Contract Lemma 2 and Theorem 1 together imply the principal can, without loss of optimality, offer one contract to each agent for a large class of beliefs. This is not a trivial result: Segal's principal is generally better off offering multiple contracts to each agent because these contracts enable her to induce complex DC paths where some agents "move" more than once (see Appendix D for an example).[15] It turns out imposing A1–A3 rules out this possibility.[16]

[15] Solving for the optimal DC contracting scheme is therefore intractable, motivating Bernstein and Winter (2012) to solve for that in Segal's special case where actions are binary $A_i = \{0, 1\}$ and externalities are linear, i.e., $u_i(1, a_{-i}) - u_i(0, a_{-i}) = f_{ii} + \sum_{j \neq i} f_{ij} a_j$ for some f_{ij}. Note that externalities are linear in the network application (3) but not necessarily in the other two applications.

[16] Intuition can be gained by building on Segal's Lemma 4: All DC paths are equally optimal in exact potential games. When instead all agents have different weights, the total revenue is maximized with the DC path that has agents with lower weights (who are more sensitive to price changes) moving before those with higher weights. This further implies each agent moves only once (i.e., receives a single contract): All his moves occur after those of agents with lower weights and before those of agents with higher weights, and thus they can be combined into a single move.

Violation of A2 or A3 For Proposition 1, A2 and A3 guarantee the greedy algorithm is optimal and generates DC contracts. Without these assumptions, Appendix E demonstrates with examples that the algorithm need not be optimal. Moreover, even when it is optimal, the generated contracts need not take a DC form. Nevertheless, there are also cases where the algorithm remains optimal and generates DC contracts, and therefore Theorem 1 remains true in these cases. Hence, a future research direction is to identify other meaningful assumptions that ensure the validity of Theorem 1.

4 Applications

This section applies the previous results (in particular, Theorem 1) to (i) networks, (ii) public goods, and (iii) a class of binary-action applications. The issue of equilibrium multiplicity naturally arises in each application. Hence, one contribution shared by all applications is to partially resolve this issue by showing the optimality and robustness of the corresponding w-DC contracts in each setting. Additional application-specific contributions are discussed in each subsection. These applications together demonstrate how the general results in the previous section can unify apparently conflicting results across applications.

4.1 Networks

We now revisit the network game (3). For tractability, the economics of networks literature typically assumes all agents' actions are both one-dimensional and either strategic complements (i.e., C1) or strategic substitutes.[17] By contrast, (3) can accommodate multidimensional actions and a mix of complements and substitutes. A classic application that

[17]Regarding one-dimensional actions, the survey by Bramoulle and Kranton (2016, p. 109) writes: "In some contexts, players' actions are naturally multidimensional. [...] multidimensional strategies emerge when players can play different actions with different neighbors. Little research has been conducted to date on such games." Regarding strategic complementarities (or substitutabilities) among all agents, the survey by Jackson and Zenou (2015, p. 97) writes: "Without focusing in on specific structures in terms of the games, it is hard to draw any conclusions. The literature has primarily taken three approaches to this challenge, [...] One involves looking at games of strategic complements and strategic substitutes..."

can exploit the full generality of (3) is R&D networks (e.g., Goyal and Moraga-Gonzalez 2001; Goyal et al. 2008; Konig et al. 2019): Each firm i may exert different efforts a_{ij} on different joint projects with his positively linked firms j; these efforts intensify global competition and thus reduce the profits of distant firms (who are therefore negatively linked to i).[18] Recall from p. 8 that agents' utilities u satisfy A2; they also satisfy A3 if positive links are weakly transitive. For example, A3 holds if the links among R&D firms are non-negative within a city and non-positive across cities. The following lemma shows that u also satisfy A1.

Lemma 4 *Agents' utilities constitute a weighted potential game with $w = (v_i/\theta_i)_i$ and*

$$\Phi(a) = \sum_i \frac{\theta_i c_i(a_i)}{v_i} + \frac{1}{2}\sum_{i,j} g_{ij}\theta_i\theta_j a_{ij}a_{ji}. \tag{12}$$

Therefore, as long as A3 holds then all previous results apply to this network game. Theorem 1 implies the following.

Corollary 2 *Suppose A3 holds. A principal holding beliefs more pessimistic than potential maximization offers the w-DC contracts where $w = (v_i/\theta_i)_i$.*

The w-DC contracts rank agents in increasing order of valuation-to-importance ratio v_i/θ_i regardless of A_i, c_i, or g. Perhaps surprisingly, the entire network structure g of how agents are positively/negatively linked plays no role in the ranking. This is in stark contrast to the conventional wisdom that the principal should prioritize agents with important network positions (e.g., the center agent in a star network), especially in the absence of negative links. Intuitively, although a central agent delivers network benefits (or costs) to many neighbors, at the same time he receives network benefits (or costs) from all neighbors; these two effects offset each other perfectly in the ranking decision.

[18] There are functional forms differing from (3) that also involve multi-dimensional actions in networks; see, e.g., Chen et al. (2018) and Demange (2024). The general results remain applicable as long as A1–A3 hold.

When all agents have the same weight, they can still differ in all other three dimensions (A_i, c_i, g). This literature often assumes for simplicity that $v_i = \theta_i = 1$ for all i. This assumption implies $w_i = 1$ for all i, and therefore the principal's optimal action profile a^* is characterized by Corollary 1 if all links are non-negative. Moreover, observe from Proposition 1 that the principal can also rank agents in an arbitrary order. This echoes the previous finding: She has no strict incentive to prioritize and offer more favorable contracts to agents with high centrality.[19]

With respect to the literature, Corollary 2 multi-dimensionally generalizes and thus unifies Proposition 1 of Nora and Winter (2024) and Proposition 2 of Sakovics and Steiner (2012). The former study a special case of (3) with no negative links, binary actions $A_i = \{\mathbf{0}, \mathbf{1}\}$, $c_i = 0$, $v_i = f(\sum_j g_{ij})$ with non-increasing $f > 0$, and $\theta_i = 1$. They show that under unique implementation (which is equivalent to the most pessimistic equilibrium selection criterion as stated on p. 15), the principal optimally ranks agents in decreasing degree centrality $\sum_j g_{ij}$ (which corresponds to increasing v_i) and then offers SDC contracts (see also footnote 11). The latter study a variant[20] of (3) with a complete positive network (i.e., $g_{ij} = 1$ for all $j \neq i$) and binary actions. They show that under global-game selection (which is equivalent to potential maximization as stated in footnote 1), the principal optimally ranks agents in increasing v_i/θ_i and then offers SDC contracts.

I now discuss some implications on network formation. For expositional convenience, assume no negative links, $A_i = \{\mathbf{0}, \mathbf{1}\}$, $v_1/\theta_1 < \cdots < v_n/\theta_n$ (implying $w_1 < \cdots < w_n$), and

[19] My network irrelevance result is fundamentally different from those of Candogan et al. (2012, Corollary 1) and Bloch and Querou (2013, Proposition 3.2). In their base models, a monopoly charges symmetric consumers (i.e., they differ only in network positions) the same price regardless of the network structure. By contrast, the w-DC contracts charge symmetric consumers different prices.

[20] In their main text, each agent's payoff is binary (their common project either succeeds with a high payoff or fails with a low payoff). But in the online appendix, they prove their results with more general payoff functions. To apply global-game selection, they consider the limiting case where every agent observes the state of the world almost perfectly, i.e., their agents play a nearly complete information subgame in stage 2.

$\hat{a} = a^* = (\mathbf{1}, \ldots, \mathbf{1})$. The w-SDC contracts (11) are given by

$$p_i^*(\mathbf{1}) = c_i(\mathbf{1}) - c_i(\mathbf{0}) + v_i \sum_{j: w_j < w_i} g_{ij}\theta_j \quad \text{for all } i \in N.$$

Hence, agent i's equilibrium payoff is

$$u_i(\mathbf{1}, \ldots, \mathbf{1}) - p_i^*(\mathbf{1}) = c_i(\mathbf{0}) + v_i \sum_{j: w_j > w_i} g_{ij}\theta_j.$$

Now consider a scenario in which i is linked to an additional agent j (i.e., switching from $g_{ij} = 0$ to $g_{ij} = 1$). Agent i is strictly better off if $w_j > w_i$: He pays the same price but receives additional network benefits. Conditional on $w_j > w_i$, he most prefers the additional agent with the highest importance θ_j. By contrast, i is just as well off if $w_j < w_i$: The principal raises his price by an amount equal to his additional network benefits. In either case, i does not mind having more neighbors. Therefore, if the network is endogenously formed in stage 0, a natural formation process is that each agent unilaterally forms a few links. Under this process, agents with high weights v_i/θ_i and/or importance θ_i end up having many neighbors in equilibrium. In other words, popular agents are those who value the network a lot and with either high or low importance. If all agents have the same valuation and can only form one link, an assortative line network is formed in which i chooses $i+1$ (agent n is indifferent between choosing any agent). To my knowledge, these findings are novel to the literature on network formation.

4.2 Public Goods

Consider a public good/bad (hereafter good for simplicity) application in which each agent i's utility takes the following form:

$$u_i(a) = c_i(a_i) + v_i h(a), \tag{13}$$

where $c_i : A_i \to \mathbb{R}$ measures his stand-alone benefit/cost; $v_i \in \mathbb{R}_{++}$ measures his valuation of the public good; and $h : A \to \mathbb{R}$ measures the size of the public good. Agents can differ in four dimensions: (A_i, c_i, v_i) and how each agent's actions A_i affect h. For example, if

$o_i = 0 \in A_i \subseteq \mathbb{R}_+$ and $h(a) = (\sum_j \theta_j a_j)^2$ then $\theta_j \in \mathbb{R}_{++}$ measures the relative importance of j's actions as in the network application (3).

For tractability, most of the literature on public goods (see, e.g., the book by Batina and Ihori 2005) assumes actions are one-dimensional $A_i \subseteq \mathbb{R}_+$ and the function h takes certain aggregate forms (e.g., summation $h(\sum_i a_i)$ in the canonical model of Bergstrom et al. 1986; weakest-link $h(\min_i\{a_i\})$ and best-shot $h(\max_i\{a_i\})$ pioneered by Hirshleifer 1983). I make neither assumption but make other assumptions on h instead, as stated shortly. Thus, my formulation enables the study of public goods problems (e.g., air pollution) involving multi-dimensional actions (e.g., emissions of different pollutants such as particulate matter, CO, O_3, NO_2, and SO_2) in practice. The following lemma shows that agents' utilities u satisfy A1.

Lemma 5 *Agents' utilities constitute a weighted potential game with $w = (v_i)_i$ and*

$$\Phi(a) = \sum_i \frac{c_i(a_i)}{v_i} + h(a).$$

To state the condition for u to satisfy A2 and A3, I first define the modified binary relations C_h and S_h as follows.

Definition 3 The expression $a_j C_h a_i$ ($a_j S_h a_i$) stands for

$$h(a_i, o_j, a_{-ij}) - h(o_i, o_j, a_{-ij}) \leq (\geq)\ h(a_i, a_j, a_{-ij}) - h(o_i, a_j, a_{-ij}) \quad \forall a_{-ij} \in A_{-ij}.$$

We can easily verify that $a_j C a_i$ ($a_j S a_i$) if and only if $a_j C_h a_i$ ($a_j S_h a_i$). Therefore, A2 and A3 hold if and only if they remain true when C and S are replaced by C_h and S_h, respectively. For one-dimensional actions, they hold if (but *not* only if) h is supermodular (i.e., C1) or submodular or (recall from p. 8 that) there are only two agents. Theorem 1 implies the following.

Corollary 3 *Suppose A2 and A3 hold. A principal holding beliefs more pessimistic than potential maximization offers the w-DC contracts where $w = (v_i)_i$.*

The w-DC contracts rank agents in increasing order of valuation v_i regardless of A_i, c_i, or h. In contrast to the network application where the optimal ranking depends crucially on agents' importance θ_i (Corollary 2), the ranking is now independent of their importance to the public good (as captured by h). The reason for these opposing results is that the public good is *non-excludable* whereas the "network good" is *excludable*, in that an agent receives zero network benefit/cost whenever he rejects the offer.

If agents have different marginal utilities of money $\mu_i \in \mathbb{R}_{++}$ (the higher the poorer), after normalization (dividing u_i by μ_i) the w-DC contracts rank agents in increasing v_i/μ_i. This result unifies existing as well as offers new insights into various applications as demonstrated below.

Vote Buying (Dal Bo 2007; Dekel et al. 2008) An interest group bribes n voters to vote against their preferred party in a two-party election. Let $A_i = \{o_i = 0, 1, \ldots, \bar{a}_i\}$ where $\bar{a}_i \in \mathbb{Z}_{++}$ is the total number of votes i holds, and a_i is the number of bribed votes from i. Voter i's utility is given by (13), where $h(\sum_i a_i) \in [0, 1]$ is decreasing and is the winning probability of voters' preferred party; v_i measures his preference over the voting outcome; and c_i measures his expressive preference (i.e., caring about how he votes independent of the outcome). Corollary 3, adjusted accordingly, implies the group prioritizes voters who care little about the outcome (as in Lindbeck and Weibull 1987) and/or are poor (as in Dixit and Londregan 1996). Interestingly, the number of votes $\bar{a}_i$ and expressive preference c_i play no role in their ranking.[21]

[21] In fact, agents' action sets A_i play no role in the optimal ranking in the general model. In the context of capital raising, this implies the optimal ranking to offer DC contracts is independent of agents' capital endowments; this is opposite to the main finding of Halac et al. (2020). This is due to a few modeling differences. In particular, their principal has a budget constraint, and therefore their agents receive payments only if the principal's project succeeds (the success probability depends on all agents' investment decisions). By contrast, my principal has no budget constraint, and therefore her payments to agents do not depend on other agents' actions, i.e., payments are deterministic rather than stochastic.

Yellow Dog Contracts (Neeman 1999; Posner et al. 2010) An employer rewards n workers for not joining a union. Let $A_i = \{o_i = 0, 1\}$ where "0" ("1") represents (not) joining. Worker i's utility is again given by (13), where $h(a)$ is decreasing and measures the union's bargaining power; and v_i and c_i are interpreted analogously. Recall that h can capture the differences in workers' importance to the union (say, highly visible workers are more important). Corollary 3 implies the employer prioritizes those who benefit little from the union (say, highly skilled workers) and/or earn low wages but not necessarily those who are important to the union.

4.3 A Class of Binary-Action Applications

Consider the class of binary-action games $A_i = \{o_i = 0, 1\}$ where agents' utilities satisfy

$$u_i(1, a_{-i}) - u_i(0, a_{-i}) = c_i + v_i h(\sum_{j \neq i} a_j), \tag{14}$$

for some $c_i \in \mathbb{R}$, $v_i \in \mathbb{R}_{++}$, and $h : \mathbb{Z}_+ \to \mathbb{R}$. This class contains (but not limited to) all symmetric binary-action games, i.e., games satisfying Condition S in Segal (1999, 2003), and therefore can be applied to his 10 applications (e.g., network externalities, exclusive dealing, takeovers, vertical contracting) when (14) holds. The following lemma shows that agents' utilities u satisfy A1.

Lemma 6 *Agents' utilities constitute a weighted potential game with $w = (v_i)_i$ and*

$$\Phi(a) = \sum_i \frac{c_i a_i}{v_i} + \sum_{m=0}^{\sum_i a_i - 1} h(m).$$

We can easily verify that A2 and A3 (C1) hold if and only if h is monotone (increasing), which is the case in most of Segal's applications. Hence, for each application, the following corollary characterizes the respective contracting scheme that is optimal for a large class of beliefs.[22]

[22] Recall from p. 16 and footnote 15 that Segal solves for the optimal DC contracting scheme only in exact potential games, whereas Bernstein–Winter consider only linear externalities. Moreover, both consider only unique implementation.

Corollary 4 *Suppose h is monotone. A principal holding beliefs more pessimistic than potential maximization offers the w-DC contracts where $w = (v_i)_i$.*

One might wonder if $h(\sum_{j \neq i} a_j)$ could be generalized to $h(\sum_{j \neq i} \theta_j a_j)$ where θ_j measures j's importance as before. It turns out A1 would generally be violated with exceptions such as (i) h is linear or (ii) $n = 2$. The former essentially reduces to a special case of the network game (3), and the latter is analyzed in this footnote.[23] Both demonstrate that, unlike the public good game (13), the principal also prioritizes agents with high importance θ_i.

[23] Consider the more interesting case where h is (strictly) increasing. It is easy to verify that this 2×2 game is a weighted potential game with $w_i = v_i/[h(\theta_i) - h(0)]$ and $\Phi(0,0) = 0$, $\Phi(1,0) = [c_1 + v_1 h(0)]/w_1$, $\Phi(0,1) = [c_2 + v_2 h(0)]/w_2$, and $\Phi(1,1) = [c_1 + v_1 h(\theta_2)]/w_1 + [c_2 + v_2 h(0)]/w_2$.

Appendix

A Proofs

Proof of Lemma 1 For all $p \in P$, $i \in N$, and $a \in A$,

$$\begin{aligned} u_i(a) - p_i(a_i) &= w_i\Phi(a) + \xi_i(a_{-i}) - p_i(a_i) \quad \text{(by A1 and (2))} \\ &= w_i\Phi_p(a) + w_i\sum_{j\neq i}\frac{p_j(a_j)}{w_j} + \xi_i(a_{-i}) \quad \text{(by (5))} \\ &= w_i\Phi_p(a) + \xi_i'(a_{-i}). \quad \left(\xi_i'(a_{-i}) = w_i\sum_{j\neq i}\frac{p_j(a_j)}{w_j} + \xi_i(a_{-i})\right) \end{aligned}$$

Proof of Proposition 1 The target action profile $\hat{a}$ is fixed throughout the proof. For notational convenience, the optimal contracts (10) include all agents, but only those with $\hat{a}_i \neq o_i$ (i.e., belonging to $\hat{N}$) matter in the linear program (9); see also footnote 9. Wlog assume $\hat{N} = \{1, \ldots, |\hat{N}|\}$. I first re-express (9) in a well-known form. Define $\widetilde{p}_i \equiv \hat{p}_i/w_i$ and $\widetilde{\Phi} : 2^{\hat{N}} \to \mathbb{R}$ where $\widetilde{\Phi}(X) = \Phi(a)$ with $a_i = \hat{a}_i$ if $i \in X$ and $a_i = o_i$ otherwise. Further define $\Psi(X) \equiv \widetilde{\Phi}(\hat{N}) - \widetilde{\Phi}(\hat{N}\backslash X)$. Thus, (9) is re-expressed as

$$\max_{\widetilde{p}\in\mathbb{R}^{|\hat{N}|}} \sum_{i\in\hat{N}} w_i\widetilde{p}_i \quad \text{s.t.} \quad \sum_{i\in X}\widetilde{p}_i \leq \Psi(X) \quad \text{for all } X \subseteq \hat{N}. \tag{15}$$

The set of feasible solutions is now a polyhedron associated with the set function Ψ. Lovasz (1983) derives the the optimal solution $\widetilde{p}^*$ for submodular Ψ. Observe that Ψ is submodular if $\hat{a}_j C \hat{a}_i$ for all $i, j \in \hat{N}$ because, by (1), $\widetilde{\Phi}$ is supermodular in this case. However, Ψ is not submodular if $\hat{a}_j S \hat{a}_i$ (and not $\hat{a}_j C \hat{a}_i$) for some $i, j \in \hat{N}$. The idea of solving for $\widetilde{p}^*$ in the general case is to show that, under A1–A3, Ψ in (15) can be replaced by another function $\underline{\Psi}$, which is submodular, without altering the feasible set. Lovasz's result then applies.

First, I derive some properties of Ψ implied by A1–A3. Recall from p. 8 that A1 implies $\bar{C}$ is symmetric. Therefore, agents can be partitioned into several groups so that for any two group members i and j, there exist mutual group members $k_1, \ldots, k_m \in \hat{N}$ ($m \geq 0$) such that $\hat{a}_j \bar{C} \hat{a}_{k_1} \bar{C} \cdots \bar{C} \hat{a}_{k_m} \bar{C} \hat{a}_i$.[24] Let $L \geq 1$ be the number of groups and $\hat{N}_l$ denote the set of

[24] In graph theory terms, each vertex represents an agent, and agents i and j are linked iff $\hat{a}_j \bar{C} \hat{a}_i$. Every

group-l agents ($l = 1, \ldots, L$). Clearly, $\bigcup_l \hat{N}_l = \hat{N}$ and $\hat{N}_l \cap \hat{N}_{l'} = \emptyset$ for all $l \neq l'$. Analogously, we can express each $X = \bigcup_l X_l$ where $X_l \subseteq \hat{N}_l$. For each group l, A3 implies $\hat{a}_j C \hat{a}_i$ for all $i, j \in \hat{N}_l$. This in turn implies Ψ is submodular on the restricted domain $2^{\hat{N}_l}$, i.e.,

$$\Psi(X_l) + \Psi(X_l') \geq \Psi(X_l \cup X_l') + \Psi(X_l \cap X_l') \quad \text{for all } X_l, X_l' \subseteq \hat{N}_l. \tag{16}$$

For each $i \in \hat{N}_l$, A2 implies $\hat{a}_j S \hat{a}_i$ for all $j \in \hat{N} \backslash \hat{N}_l \equiv \hat{N}_{-l}$. This in turn implies for all $X_l \subseteq \hat{N}_l \backslash \{i\}$, $\Psi(X_{-l} \cup X_l \cup \{i\}) - \Psi(X_{-l} \cup X_l)$ is non-decreasing in $X_{-l} \subseteq \hat{N}_{-l}$, further implying for all $X_l \subseteq \hat{N}_l$,

$$\Psi(X_{-l} \cup X_l) - \Psi(X_{-l}) \quad \text{is non-decreasing in } X_{-l} \subseteq \hat{N}_{-l}. \tag{17}$$

Next, I define $\underline{\Psi}(X) \equiv \sum_l \Psi(X_l)$ and show it is submodular. Notice that $\Psi(X_l) = \Psi(X \cap \hat{N}_l) \equiv \Psi_l(X)$. Given the sum of submodular functions is submodular, it suffices to show each Ψ_l is submodular: For all $X, X' \subseteq \hat{N}$,

$$\begin{aligned} & \Psi_l(X) + \Psi_l(X') - \Psi_l(X \cup X') - \Psi_l(X \cap X') \\ = \ & \Psi(X_l) + \Psi(X_l') - \Psi(X_l \cup X_l') - \Psi(X_l \cap X_l') \geq 0. \quad \text{(by (16))} \end{aligned}$$

Then, I show that the feasible set of (15), denoted by $F(\Psi)$, is the same as that with Ψ replaced by $\underline{\Psi}$, denoted by $F(\underline{\Psi})$. I first show $F(\underline{\Psi}) \subseteq F(\Psi)$ by showing $\underline{\Psi} \leq \Psi$. For each $X \subseteq \hat{N}$, (17) implies

$$\Psi(X_l) = \Psi(X_l) - \Psi(\emptyset) \leq \Psi(X_1 \cup \cdots \cup X_l) - \Psi(X_1 \cup \cdots \cup X_{l-1}) \quad \text{for } l = 1, \ldots, L.$$

Summing the above L inequalities yields $\underline{\Psi}(X) = \sum_l \Psi(X_l) \leq \Psi(X)$. I next show $F(\Psi) \subseteq F(\underline{\Psi})$. If $\widetilde{p} \notin F(\underline{\Psi})$, there exists $X \subseteq \hat{N}$ in which

$$\sum_l \sum_{i \in X_l} \widetilde{p}_i = \sum_{i \in X} \widetilde{p}_i > \underline{\Psi}(X) = \sum_l \Psi(X_l).$$

Therefore, $\sum_{i \in X_l} \widetilde{p}_i > \Psi(X_l)$ for some l, implying $\widetilde{p} \notin F(\Psi)$.

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undirected graph can be decomposed into several connected components, which are the groups I have described.

Finally, I apply Lovasz's (1983, Section 3) result to (15) with Ψ replaced by $\underline{\Psi}$. Wlog assume $w_1 \leq \cdots \leq w_{|\hat{N}|}$. The optimal solution $\widetilde{p}^*$ is given by his Equation (6):

$$\widetilde{p}_i^* = \underline{\Psi}(\{i, \ldots, |\hat{N}|\}) - \underline{\Psi}(\{i+1, \ldots, |\hat{N}|\}) \quad \text{for all } i \in \hat{N}.$$

It remains to simplify the above to see that $\widetilde{p}_i^* = \hat{p}_i^*/w_i$ where $\hat{p}_i^*$ is given by (10). Let $l(i)$ denote the group agent i belongs to. The above becomes

$$\begin{aligned}
\widetilde{p}_i^* &= \sum_l \Psi(\{i, \ldots, |\hat{N}|\} \cap \hat{N}_l) - \sum_l \Psi(\{i+1, \ldots, |\hat{N}|\} \cap \hat{N}_l) \\
&= \Psi(\{i, \ldots, |\hat{N}|\} \cap \hat{N}_{l(i)}) - \Psi(\{i+1, \ldots, |\hat{N}|\} \cap \hat{N}_{l(i)}) \\
&= \widetilde{\Phi}(\hat{N} \backslash (\{i+1, \ldots, |\hat{N}|\} \cap \hat{N}_{l(i)})) - \widetilde{\Phi}(\hat{N} \backslash (\{i, \ldots, |\hat{N}|\} \cap \hat{N}_{l(i)})) \\
&= \widetilde{\Phi}(\{1, \ldots, i\} \cup \hat{N}_{-l(i)}) - \widetilde{\Phi}(\{1, \ldots, i-1\} \cup \hat{N}_{-l(i)}).
\end{aligned}$$

Hence, by (1), it is easy to see that $\widetilde{p}_i^* = \hat{p}_i^*/w_i$.

Last, I show that $\widetilde{p}^*$ is the unique optimal solution to (15) (and therefore $\hat{p}^*$ is the unique optimal solution to (9)) if $w_1 < \cdots < w_{|\hat{N}|}$. The necessary and sufficient condition provided by Mangasarian (1979, Theorem 1) is that $\widetilde{p}^*$ remains optimal for all linear programs obtained from (15) by an arbitrary but sufficiently small perturbation of the vector $(w_i)_{i \in \hat{N}}$. This condition is satisfied because any sufficiently small perturbation does not alter the ranking of w_i and, therefore, $\widetilde{p}^*$ remains optimal.

Proof of Proposition 2 Consider a symmetric two-agent game with $A_i = \{o_i = 0, 1\}$, $u_i(a) = 1$ if $a = (1,1)$ and $u_i(a) = 0$ otherwise, and $U(a, p_1(a_1) + p_2(a_2)) = V(a) + p_1(a_1) + p_2(a_2)$ where $V(0,0) = V(1,1) = 0$ and $V(1,0) = V(0,1) = 1$. Recall from p. 8 that A2 and A3 are vacuous for $n = 2$; A1 also holds with $w_i = 1$ and $\Phi = u_i$. For notational convenience, denote $p_i \equiv p_i(1)$.

There are exactly three types of price vectors (p_1, p_2) leading to multiple equilibria in stage 2: (i) $p_i \leq 0$ and $p_j = 1$, (ii) $p_i \geq 1$ and $p_j = 0$, and (iii) $(p_1, p_2) \in [0,1] \times [0,1]$. For the first type, there is a continuum of mixed equilibria in which $a_i = 1$ with probability 1 and $a_j = 1$ with any probability. Similarly, for the second type, there is a continuum of mixed equilibria in which $a_i = 0$ with probability 1 and $a_j = 1$ with any probability. For the third type, there are three equilibria: $(0, 0)$, $(1, 1)$, and the mixed one in which $a_i = 1$ with probability p_j.

For the first (second) type, all equilibria have the same potential of $-p_i$ (0). Recall from footnote 8 that the principal can select among potential maximizers under potential maximization. Therefore, she can always select the best equilibrium for her for both types. For the third type, we can easily show that (i) her expected payoffs in those three equilibria are 0, $p_1 + p_2$, and $p_1 + p_2$, respectively, and (ii) the potential maximizer is $(0, 0)$ if $p_1 + p_2 \geq 1$ and $(1, 1)$ if $p_1 + p_2 \leq 1$. Observe that potential maximization selects the best equilibrium for her if and only if $p_1 + p_2 \leq 1$.

If an equilibrium selection criterion is not more pessimistic than potential maximization, there exists a non-empty subset of the third type of price vectors $\{(p_1, p_2) \in (0, 1] \times (0, 1] | p_1 + p_2 > 1\}$ in which either $(1, 1)$ or the mixed equilibrium is selected; both yield the same payoff of $p_1 + p_2 > 1$. If instead she offers the w-DC contracts, the optimal action profiles are $(1, 0)$, $(0, 1)$, and $(1, 1)$ by Corollary 1; in either case, her payoff is only 1.

Proof of Lemma 4 First, note that

$$
\begin{aligned}
& \sum_{j,k} g_{jk}\theta_j\theta_k a_{jk}a_{kj} = \sum_k g_{ik}\theta_i\theta_k a_{ik}a_{ki} + \sum_{j\neq i,k} g_{jk}\theta_j\theta_k a_{jk}a_{kj} \\
= \; & \sum_k g_{ik}\theta_i\theta_k a_{ik}a_{ki} + \sum_{j\neq i} g_{ji}\theta_j\theta_i a_{ji}a_{ij} + \sum_{j\neq i,k\neq i} g_{jk}\theta_j\theta_k a_{jk}a_{kj} \\
= \; & \sum_k g_{ik}\theta_i\theta_k a_{ik}a_{ki} + \sum_j g_{ij}\theta_i\theta_j a_{ij}a_{ji} + \sum_{j\neq i,k\neq i} g_{jk}\theta_j\theta_k a_{jk}a_{kj} \quad (g_{ji} = g_{ij} \text{ and } g_{ii} = 0) \\
= \; & 2\sum_j g_{ij}\theta_i\theta_j a_{ij}a_{ji} + \sum_{j\neq i,k\neq i} g_{jk}\theta_j\theta_k a_{jk}a_{kj}.
\end{aligned}
$$

For all $i \in N$ and $a \in A$,

$$w_i\Phi(a) = \frac{v_i}{\theta_i}(\sum_j \frac{\theta_j c_j(a_j)}{v_j} + \frac{1}{2}\sum_{j,k} g_{jk}\theta_j\theta_k a_{jk}a_{kj}) \quad \text{(by (12))}$$
$$= c_i(a_i) + \frac{v_i}{\theta_i}\sum_{j\neq i}\frac{\theta_j c_j(a_j)}{v_j} + v_i\sum_j g_{ij}\theta_j a_{ij}a_{ji} + \frac{v_i}{2\theta_i}\sum_{j\neq i,k\neq i} g_{jk}\theta_j\theta_k a_{jk}a_{kj}$$
$$= u_i(a) - \xi_i(a_{-i}). \quad \text{(by (3) and } \xi_i(a_{-i}) = -\frac{v_i}{\theta_i}\sum_{j\neq i}\frac{\theta_j c_j(a_j)}{v_j} - \frac{v_i}{2\theta_i}\sum_{j\neq i,k\neq i} g_{jk}\theta_j\theta_k a_{jk}a_{kj})$$

Proof of Lemma 5 For all $i \in N$ and $a \in A$,

$$w_i\Phi(a) = v_i(\sum_j \frac{c_j(a_j)}{v_j} + h(a)) = c_i(a_i) + v_i\sum_{j\neq i}\frac{c_j(a_j)}{v_j} + v_i h(a)$$
$$= u_i(a) - \xi_i(a_{-i}). \quad \text{(by (13) and } \xi_i(a_{-i}) = -v_i\sum_{j\neq i}\frac{c_j(a_j)}{v_j})$$

Proof of Lemma 6 For all $i \in N$ and $a_{-i} \in A_{-i}$,

$$w_i[\Phi(1,a_{-i}) - \Phi(0,a_{-i})] = v_i[\frac{c_i}{v_i} + h(1+\sum_{j\neq i}a_j - 1)] = c_i + v_i h(\sum_{j\neq i}a_j) = u_i(1,a_{-i}) - u_i(0,a_{-i}).$$

B Generic Uniqueness of the Weighted Potential Maximizer

This appendix proves that the potential maximizer of a weighted potential game is generically unique. Suppose a game $\Gamma \equiv \langle N, A, u\rangle$ is a weighted potential game. Given a potential function Φ (together with a weight vector $w \in \mathbb{R}^n_{++}$) of Γ, the maximizer of Φ is clearly generically unique. However, it is unclear whether another potential function Φ' (together with another weight vector $w' \in \mathbb{R}^n_{++}$) of Γ has the same maximizer(s). Therefore, the exact statement to prove is the following. To my knowledge, this paper is the first to give a direct proof of this statement.

Lemma 0 *The potential maximizer(s) of a weighted potential game is independent of the choice of the potential function.*

Proof. Suppose (w, Φ) and (w', Φ') are two choices of "weight-potential" pairs. By the definition of weighted potential games (A1), for all $i \in N$, $a_i, a_i' \in A_i$, and $a_{-i} \in A_{-i}$,

$$u_i(a_i, a_{-i}) - u_i(a_i', a_{-i}) = w_i[\Phi(a_i, a_{-i}) - \Phi(a_i', a_{-i})] = w_i'[\Phi'(a_i, a_{-i}) - \Phi'(a_i', a_{-i})]. \quad (18)$$

Denote $\widetilde{w}_i \equiv w_i'/w_i$. Clearly, $\widetilde{w}_i > 0$ for all i. Wlog assume $\widetilde{w}_1 \leq \cdots \leq \widetilde{w}_n$. It remains to show $\bar{a} \in \arg\max_{a \in A} \Phi'(a)$ implies $\bar{a} \in \arg\max_{a \in A} \Phi(a)$. To see this, for each $a \in A$,

$$\begin{aligned}
& \Phi(\bar{a}) - \Phi(a) \\
= & \sum_{i=1}^{n} [\Phi(\bar{a}_1, \dots, \bar{a}_i, a_{i+1}, \dots, a_n) - \Phi(\bar{a}_1, \dots, \bar{a}_{i-1}, a_i, \dots, a_n)] \\
= & \sum_{i=1}^{n} \widetilde{w}_i [\Phi'(\bar{a}_1, \dots, \bar{a}_i, a_{i+1}, \dots, a_n) - \Phi'(\bar{a}_1, \dots, \bar{a}_{i-1}, a_i, \dots, a_n)] \quad \text{(by (18))} \\
\geq & \sum_{i=1}^{n-1} \widetilde{w}_i [\Phi'(\bar{a}_1, \dots, \bar{a}_i, a_{i+1}, \dots, a_n) - \Phi'(\bar{a}_1, \dots, \bar{a}_{i-1}, a_i, \dots, a_n)] \quad (\bar{a} \in \arg\max_{a \in A} \Phi'(a)) \\
+ & \widetilde{w}_{n-1} [\Phi'(\bar{a}) - \Phi'(\bar{a}_1, \dots, \bar{a}_{n-1}, a_n)] \\
= & \sum_{i=1}^{n-2} \widetilde{w}_i [\Phi'(\bar{a}_1, \dots, \bar{a}_i, a_{i+1}, \dots, a_n) - \Phi'(\bar{a}_1, \dots, \bar{a}_{i-1}, a_i, \dots, a_n)] \\
+ & \widetilde{w}_{n-1} [\Phi'(\bar{a}) - \Phi'(\bar{a}_1, \dots, \bar{a}_{n-2}, a_{n-1}, a_n)] \\
\geq & \sum_{i=1}^{n-2} \widetilde{w}_i [\Phi'(\bar{a}_1, \dots, \bar{a}_i, a_{i+1}, \dots, a_n) - \Phi'(\bar{a}_1, \dots, \bar{a}_{i-1}, a_i, \dots, a_n)] \quad (\bar{a} \in \arg\max_{a \in A} \Phi'(a)) \\
+ & \widetilde{w}_{n-2} [\Phi'(\bar{a}) - \Phi'(\bar{a}_1, \dots, \bar{a}_{n-2}, a_{n-1}, a_n)] \\
\geq & \cdots \geq \widetilde{w}_1 [\Phi'(\bar{a}) - \Phi'(a)] \geq 0. \quad (\bar{a} \in \arg\max_{a \in A} \Phi'(a)) \quad \blacksquare
\end{aligned}$$

C Beliefs More Pessimistic than Potential Maximization: An Example

This appendix illustrates with an example how the set of beliefs more pessimistic than potential maximization varies with the principal's payoff function. Revisit the example on p. 28 and allow for a general V (normalize $V(0,0) = 0$). Restrict attention to $(p_1, p_2) \in [0,1] \times [0,1]$ where all four action profiles are rationalizable. Recall that the potential maximizer is $(0,0)$ if $p_1 + p_2 \geq 1$ and $(1,1)$ if $p_1 + p_2 \leq 1$. It is easy to verify that the potential maximizer is always the worst action profile for the principal if $V = -\Phi$. As $V(1,0)$ or $V(0,1)$

decreases, the set of beliefs more pessimistic than potential maximization becomes larger because those assigning positive probabilities to non-equilibria become more pessimistic. The set also becomes larger as $V(1,1)$ departs from $-\Phi(1,1) = -1$. In particular, if $V(1,1) \geq \max\{0, V(1,0), V(0,1)\}$ then the potential maximizer is the best for the principal for all $p_1+p_2 \leq 1$. Hence, any belief yielding non-positive expected payoffs for all $p_1+p_2 \geq 1$ is more pessimistic than potential maximization. Similarly, if $\max\{V(1,1), V(1,0)-1, V(0,1)-1\} \leq -2$ then the potential maximizer is the best for all $p_1 + p_2 \geq 1$. Hence, any belief yielding a payoff lower than $V(1,1) + p_1 + p_2$ for each $p_1 + p_2 \leq 1$ is more pessimistic than potential maximization.

D Suboptimality of Single Contract: An Example

This appendix illustrates with an example how offering one contract to each agent might be suboptimal. Consider a 2×3 example where the two agents' payoffs are given as follows:

	2	1	0
1	$1 - p_1(1), 2 - p_2(2)$	$1 - p_1(1), -p_2(1)$	$-p_1(1), 0$
0	$0, -p_2(2)$	$0, -p_2(1)$	$0, 0$

It is easy to verify that this example satisfies C1 but not A1. Suppose the principal wants to implement $\hat{a} = (1, 2)$ as a dominance-solvable (or the unique Nash) equilibrium. If she restricts herself to offering one contract to each agent (i.e., setting $p_2(1) \to \infty$), she optimally sets $p_1(1) = 0$ (first to make $a_1 = 1$ dominant) and $p_2(2) = 2$ and obtains a revenue of 2. The corresponding DC path is $(0, 0) \to (1, 0) \to (1, 2)$. However, she can obtain a revenue of 3 by setting $p_2(1) = 0$ (first to make $a_2 = 0$ dominated by $a_2 = 1$), $p_1(1) = 1$ (next to make $a_1 = 1$ iteratively dominant), and $p_2(2) = 2$. The corresponding DC path is $(0, 0) \to (0, 1) \to (1, 1) \to (1, 2)$ where agent 2 "moves" twice.

E Violation of A2 or A3: Examples

This appendix illustrates with examples how the main result of this paper might or might not break down if A2 or A3 is violated. Suppose $\hat{N} = \{1, 2, 3\}$ and $u_i = w_i\Phi$ for all i where $\widetilde{\Phi}$ (defined on p. 26) is given by $\widetilde{\Phi}(X) = 0$ if $|X| \leq 1$, $\widetilde{\Phi}(\{1, 2\}) = -1$, and $\widetilde{\Phi}(X) = 1$ otherwise. Hence, $\Psi(X) = 1$ if $|X| \geq 2$, $\Psi(\{3\}) = 2$, and $\Psi(X) = 0$ otherwise. Observe that $\hat{a}_1\bar{C}\hat{a}_3\bar{C}\hat{a}_2$ but $\hat{a}_1 S \hat{a}_2$ (and not $\hat{a}_1 C \hat{a}_2$), and therefore only A3 fails. Applying the greedy algorithm to (15) need not generate DC contracts: If $w_1 > w_2 > w_3$ then it (optimally) generates $\widetilde{p} = (0, 0, 1)$, but none has a dominant strategy to accept his offer. Furthermore, the algorithm need not be optimal: If $w_1 < w_2 < w_3$ then it generates $\widetilde{p} = (-1, -1, 2)$ (which are the w-DC contracts (10) because $\widetilde{p}_i = \hat{p}_i^*/w_i$), but the unique optimal solution to (15) is $\widetilde{p} = (0, 0, 1)$ if $w_1 + w_2 > w_3$.

Now make one modification to $\widetilde{\Phi}$: $\widetilde{\Phi}(\{1, 2\}) = 1$ (i.e., $\Psi(\{3\}) = 0$). Observe that A2 fails for every pair of target actions, and therefore A3 holds vacuously. It is easy to verify that the unique optimal solution to (15) is $\widetilde{p} = (0, 0, 0)$ for all w, and every agent has a weakly dominant strategy to accept. Hence, the greedy algorithm is trivially optimal and generates DC contracts.

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