Unified Gross Substitutes and Inverse Isotonicity for Equilibrium Problems*

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Abstract. We introduce a notion of substitutability for correspondences and establish a monotone comparative static result. More precisely, we introduce the notions of *unified gross substitutes* and *nonreversingness* and show that if $\mathtt{Q} : P \rightrightarrows Q$ is a supply correspondence defined on a set of prices P which is a sublattice of $\mathbb{R}^N$, and $\mathtt{Q}$ satisfies these two properties, then the set of equilibrium prices $\mathtt{Q}^{-1}(q)$ associated with a vector of quantities $q \in Q$ is a sublattice of P and is increasing (in the strong set order) in q. We establish connections between our notion of substitutes and existing notions, and examine applications such as the structure of inverse demand, profit maximization, the structure of competitive equilibria, matching games, hedonic pricing, and routing problems.

Keywords: Substitutes, inverse isotonicity, correspondence, M-function, monotone comparative statics, equilibrium flow.

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1 Introduction

This paper proposes the notion of *unified gross substitutes* for a correspondence $\mathtt{Q} : P \rightrightarrows Q$. For concreteness we often interpret $\mathtt{Q}$ as a supply correspondence mapping from a set of prices P to a set of quantities Q, though our analysis applies to correspondences in general. Our analysis encompasses the familiar case in which $\mathtt{Q}$ arises from the optimization problem of a single agent, but unified gross substitutes need not refer to a single agent's decision problem, and we are especially interested in its potential for the study of equilibrium problems.

For functions, the notion of unified gross substitutes is equivalent to the familiar notion of weak gross substitutes. For correspondences, the notion of unified gross substitutes implies (but is not equivalent to) Kelso and Crawford's [22] notion of gross substitutes for correspondences, which does not suffice for our central result.

Our focus is the inverse isotonicity of the correspondence $\mathtt{Q}$.[1] We show that if the correspondence $\mathtt{Q}$ satisfies unified gross substitutes as well as a mild condition called *nonreversingness*, then the set $\mathtt{Q}^{-1}(q)$ of parameters p associated with an element $q \in Q$ is a sublattice of P and is increasing (in the strong set order, a.k.a. Veinott's order, Veinott [36]) in q.

Section 2 presents the framework, the notion of unified gross substitutes, and the main results. Section 3 develops the following applications of these results.

First, Berry, Gandhi and Haile [3] identify conditions under which a *function* $\mathtt{q} : P \to Q$ has an inverse $\mathtt{q}^{-1}$ that is point-valued and inverse isotone (i.e., $\mathtt{q}(p) \leq \mathtt{q}(p')$ implies $p \leq p'$). Berry, Gandhi and Haile explain that their result is important for theoretical models that rely on the existence of an inverse demand function or the uniqueness of equilibrium prices, such as Cournot models of imperfect competition, and is also important for empirical work on the estimation or identification of demand and for the testing of revealed preference models.

There are important cases in which demand is specified by a correspondence rather than a function. For example, quantity-setting models of markets with in-

[1] See Ortega and Rheinboldt [26] for related work involving functions.

divisible goods inevitably give rise to demand correspondences, reflecting consumer indifference between indivisible consumption bundles. The literature typically takes it for granted that a point-valued inverse demand function exists and exhibits the desired monotonicity properties (e.g., Davis [7]). Section 3.1 shows that in this setting, unified gross substitutes implies not only that the demand correspondence $\mathtt{Q}$ is inverse isotone, but is point-valued. This gives us a foundation, in the form of the existence and isotonicity of inverse demand functions, for quantity-setting models with indivisibilities. More generally, this result applies to any model of firm competition with limited dependent variables.

Second, Section 3.2 shows that unified gross substitutes for the argmax correspondence of a maximization problem is equivalent to the submodularity of the maximand. This is a familiar result in the case of argmax *functions*. Auctions for multiple indivisible goods inevitably give rise to demand correspondences. Ausubel and Milgrom [2] show that submodularity of the bidders' maximand is a building block for showing that sincere bidding is an equilibrium in an ascending proxy auction. Our result provides alternative conditions for such submodularity.

Third, Section 3.3 supposes $\mathtt{Q}$ is the excess supply correspondence of an economy. If preferences and production sets are only weakly (rather than strictly) convex, than $\mathtt{Q}$ will not in general be a function.[2] Let $\tilde{q}$ be the aggregate endowment, and note that p is a competitive equilibrium price if $\mathtt{Q}(p) = -\tilde{q}$, which is to say that $p \in \mathtt{Q}^{-1}(-\tilde{q})$. We can then conclude that the set of equilibrium prices is a sublattice. Moreover, we can derive comparative static results either for the set of equilibrium prices, using the strong set order, or for the smallest and/or largest equilibrium price vector.[3] For example, an increase in the endowment of (say) good 1 increases (at least weakly) the equilibrium prices of all other goods. This generalizes results familiar for economies that generate (point-valued) excess supply functions, and illustrates how the correspondence $\mathtt{Q}$ can be used to generate comparative static results for equilibrium problems.

Fourth, Section 3.4 introduces a new equilibrium framework, referred to as

[2]Alternatively, the best response correspondence for an equilibrium problem with finite pure strategy sets, such as a market entry problem, is a correspondence.

[3]More precisely, the sublattice $\mathtt{Q}^{-1}(q)$ may not be complete, and so may have no smallest or largest element, but the infimum and supremeum of a sublattice $\mathtt{Q}^{-1}(q)$ of $\mathbb{R}^N$ is contained in the closure of $\mathtt{Q}^{-1}(q)$ (which is not the case in general), and so remain useful tools for characterizing the set.

the equilibrium flow problem, that contains a number of familiar settings as special cases and whose equilibrium correspondence satisfies unified gross substitutes. Section 3.5 shows that a one-to-one matching problem with imperfectly transferable utility is a special case of the equilibrium flow problem. This gives us versions of lattice and comparative static results, derived by Demange and Gale [9] for economies with a finite number of agents, that allow one to accommodate a continuum of each type of agent. The continuum of types in turn leads to a convenient model of competitive matching markets (cf. Nöldeke and Samuelson [28]).

Fifth, Section 3.6 shows that hedonic price problems (cf. Rosen [30] and Ekeland, Heckman and Nesheim [13]) are special cases of the equilibrium flow problem. This allows us to extend the basic results of Chiappori, McCann and Nesheim [6] beyond quasilinear utilities, addressing what Ekeland [12] identifies as the primary limitation of hedonic models and establishing an inverse isotonicity result for such models.

Section 3.7 notes that time-dependent routing problems are also a special case of the equilibrium flow problem, allowing us to provide foundations for this literature.

2 Theory

2.1 Framework

When u and v are two elements of a partially ordered set, we use $u < v$ to express that $u \leq v$ and $u \neq v$. Recall that a set P is a *sublattice of* $\mathbb{R}^N$ if for any pair of vectors $p, p' \in P$, the set P also contains their coordinate-wise maximum (denoted by $p \vee p'$) and their coordinate-wise minimum (denoted by $p \wedge p'$) (Topkis [32, p. 307]).

Let $P \subseteq \mathbb{R}^N$ and $Q \subseteq \mathbb{R}^N$, for some finite N, with generic elements $p \in P$ and $q \in Q$. Let $\mathtt{Q} : P \rightrightarrows Q$ be a correspondence. We maintain the following assumption throughout, without explicit mention:

Assumption 1. *P is a sublattice of $\mathbb{R}^N$.*

In one of our leading interpretations of the correspondence $\mathtt{Q}$, we view the dimensions of $\mathbb{R}^N$ as identifying goods and interpret $\mathtt{Q}$ as a supply correspondence. An element $p \in P = \mathbb{R}^N$ is then a price vector, with p_z denoting the price of good

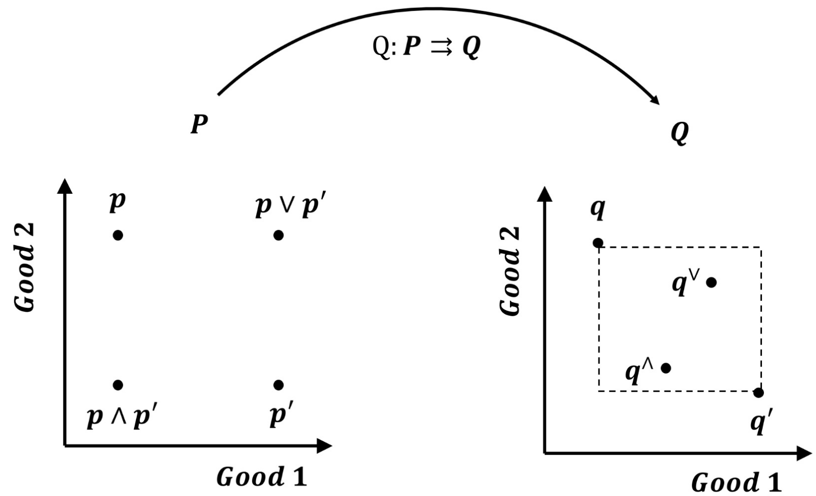

Figure 1: An illustration of the unified gross substitutes property with two goods. $\mathtt{Q}$ is a correspondence between the set of prices on the left and the set of quantities on the right with $q \in \mathtt{Q}(p)$ and $q' \in \mathtt{Q}(p')$. Definition 1 imposes that there exists a $q^{\vee} \in \mathtt{Q}(p \vee p')$, smaller than the top-right border of the rectangle drawn in the set of quantities, and a $q^{\wedge} \in \mathtt{Q}(p \wedge p')$ larger than the bottom-left border.

$z \in \{1, \ldots, N\}$. An element $q \in \mathtt{Q}(p)$ is an allocation, with q_z denoting the quantity of good z supplied at price vector p. No matter what the interpretation, we typically refer to elements of Q as quantities and elements of P as prices.

2.2 Unified Gross Substitutes

Our basic notion of substitutability for correspondences is:

Definition 1 (Unified Gross Substitutes). *The correspondence* $\mathtt{Q} : P \rightrightarrows Q$ *satisfies* unified gross substitutes *if, for every* $p \in P, p' \in P$, $q \in \mathtt{Q}(p)$ *and* $q' \in \mathtt{Q}(p')$, *there exist* $q^{\wedge} \in \mathtt{Q}(p \wedge p')$ *and* $q^{\vee} \in \mathtt{Q}(p \vee p')$ *such that*

$$p_z \leq p'_z \quad \Longrightarrow \quad q_z \leq q^{\wedge}_z \text{ and } q^{\vee}_z \leq q'_z \tag{1}$$

$$p'_z < p_z \quad \Longrightarrow \quad q'_z \leq q^{\wedge}_z \text{ and } q^{\vee}_z \leq q_z. \tag{2}$$

This definition is appropriate for our interpretation of $\mathtt{Q}$ as a supply correspondence, and would need to be adjusted in a straightforward way for applications to demand correspondences.[4]

Figure 1 gives an illustration of the unified gross substitutes property with two goods, for the case in which p and p' are not ordered. Intuitively, there must be at least one vector $q^{\vee} \in (p \vee p')$ such that, when moving from either price vector p (and quantity vector q) or price vector p' (and quantity vector q') to price vector $p \vee p'$, the quantity does not increase of the good whose price has *not* increased.

If two price vectors contain no common elements (as in Figure 1), then the implications of (1)–(2) are the same no matter which price vector is chosen to be p and which to be p'. When two price vectors exhibit common prices for some dimensions, unified gross substitutes gains strength from requiring (1)–(2) to hold for *both* of the ways that these vectors can be designated as p and p'. To illustrate, consider the price vectors $(4, 5, 6)$ and $(4, 6, 7)$. When we take $p = (4, 5, 6)$ and $p' = (4, 6, 7)$, then (1)–(2) hold trivially (the consequent of (1) can be satisfied by taking $q^{\wedge} = q$ and $q^{\vee} = q'$, and the antecedent of (2) is empty). When we take $p' = (4, 5, 6)$ and $p = (4, 6, 7)$, then (1)–(2) have substantive implications, most notably that as the prices of the other goods increase, the quantity supplied of good 1 does not increase.

Suppose condition (1) holds. Then by reversing the roles of the price vectors in (1), we obtain (2) (modulo the appearance of the weak inequality in only one antecedent). Then why bother to include condition (2)? For a fixed p and p', we require that both (1) and (2) hold for a *single* pair of quantity vectors $q^{\wedge} \in \mathtt{Q}(p \wedge p')$ and $q^{\vee} \in \mathtt{Q}(p \vee p')$. To illustrate, suppose $N = 2$ and we have the price vectors $(1, 2)$ and $(2, 1)$ as candidates for p and p'. The correspondence

$$\mathtt{Q}(1,2) = \{(3,9)\} \qquad \mathtt{Q}(2,2) = \{(8,8)\} \tag{3}$$

$$\mathtt{Q}(1,1) = \{(4,4)\} \qquad \mathtt{Q}(2,1) = \{(9,3)\} \tag{4}$$

satisfies unified gross substitutes. In contrast, consider the correspondence

$$\mathtt{Q}(1,2) = \{(3,9)\} \qquad \mathtt{Q}(2,2) = \{(7,10),(10,7)\} \tag{5}$$

$$\mathtt{Q}(1,1) = \{(4,4)\} \qquad \mathtt{Q}(2,1) = \{(9,3)\}. \tag{6}$$

[4]The counterpart of (1)–(2) would then be

$$\begin{aligned} p_z \le p'_z \quad &\Longrightarrow \quad q_z \ge q^{\wedge}_z \text{ and } q^{\vee}_z \ge q'_z \\ p'_z < p_z \quad &\Longrightarrow \quad q'_z \ge q^{\wedge}_z \text{ and } q^{\vee}_z \ge q_z. \end{aligned}$$

Let $(1,2) = p$ and $(2,1) = p'$. Then (1) is satisfied with $q^\vee = (7,10)$ and (2) is satisfied with $q^\vee = (10,7)$, but there is no single value $q^\vee$ satisfying (1) and (2), and so unified gross substitutes fails. Imposing (2) as well as (1) for a fixed pair of price vectors p and p' thus tightens our definition by comparing $q \in \mathtt{Q}(p)$ and $q' \in \mathtt{Q}(p')$ to *common* (and hence the "unified") values $q^\wedge$ and $q^\vee$.

One might be tempted to strengthen the definition of unified gross substitutes yet further by making the antecedents of (1)–(2) both weak inequalities. The resulting, stronger notion vitiates many of our equivalence results and often fails to hold. Galichon, Samuelson and Vernet [17, Appendix B.1] provide an example. Notice that, given the ability to interchange the prices p and p', it is irrelevant which antecedent carries the strict inequality.

2.3 Connections

We connect unified gross substitutes to two familiar properties and then provide a characterization.

First, recall that a function $\mathtt{q} : \mathbb{R}^N \to \mathbb{R}^N$ satisfies *weak gross substitutes* if $\mathtt{q}_i(p)$ is nonincreasing in p_j for $i \neq j$ (e.g., Mas-Colell, Whinston and Green [23, Definition 17.F.2, p. 611]).[5] When $\mathtt{Q}$ is a function, the notions of unified gross substitutes and weak gross substitutes coincide:

Lemma 1. *For functions, unified gross substitutes and weak gross substitutes are equivalent.*

Proof. We show that the function $\mathtt{q}$ satisfies weak gross substitutes if and only if the correspondence $\mathtt{Q}(p) = \{\mathtt{q}(p)\}$ satisfies unified gross substitutes. First let $\mathtt{q}$ satisfy weak gross substitutes, take $q = \mathtt{q}(p)$ and $q' = \mathtt{q}(p')$, and let $q^\wedge = \mathtt{q}(p \wedge p')$ and $q^\vee = \mathtt{q}(p \vee p')$. Then $p_z \leq p'_z \implies (p \wedge p')_z = p_z$, which combines with $p \wedge p' \leq p$ and weak gross substitutes to give $\mathtt{q}_z(p) \leq \mathtt{q}_z(p \wedge p')$ and hence the first part of (1). The other requirements in (1)–(2) are dealt with similarly, and hence $\mathtt{Q}$ satisfies unified gross substitutes. Conversely, let $\mathtt{Q}(p) = \{\mathtt{q}(p)\}$ satisfy unified gross substitutes, let

[5] Requiring $\mathtt{q}_i(p)$ to be strictly decreasing in p_j gives *strict* gross substitutes. Mas-Colell, Whinston and Green [23] provide the definition of weak gross substitutes for a demand function, namely that $\mathtt{q}_i(p)$ is nondecreasing in p_j.

$p_j \geq p'_j$ and $p_i = p'_i$ for all $i \neq j$. Then $p^\wedge = p'$ and $p^\vee = p$, so that applying (1) to some $i \neq j$ gives $\mathtt{q}_i(p) \leq \mathtt{q}_i(p')$, and hence $\mathtt{q}$ satisfies weak gross substitutes. ■

Second, translating Kelso and Crawford's [22, p. 1486] well-known definition of gross substitutes to our setting of supply correspondences, the correspondence $\mathtt{Q}$ has the *Kelso-Crawford gross substitutes* property if, given two price vectors p and p' with $p' \leq p$, (1) for any $q \in \mathtt{Q}(p)$ there exists $q' \in \mathtt{Q}(p')$ such that $p_z = p'_z \implies q'_z \geq q_z$ and (2) for any $q' \in \mathtt{Q}(p')$ there exists $q \in \mathtt{Q}(p)$ such that $p_z = p'_z \implies q'_z \geq q_z$.[6] The following is immediate from the definitions:

Lemma 2. *Unified gross substitutes implies the Kelso and Crawford gross substitutes property.*

Turning to the other direction, the correspondence given by (5)–(6) satisfies Kelso and Crawford's [22] notion of gross substitutes but does not satisfy unified gross substitutes.

We can provide an intuitive idea of how unified gross substitutes strengthens Kelso-Crawford's gross substitutes notion. Continuing with the correspondence given by (5)–(6), suppose we decompose the movement from price $(1, 2)$ to price $(2, 1)$ into an increase in the price of good 1 followed by a decrease in the price of good 2. Unified gross substitutes requires that there exists a corresponding sequence of quantities (e.g., from $(3, 9)$ to $(8, 8)$ to $(9, 3)$ in (3)–(4)) that first decreases the quantity of good 2 and then increases the quantity of good 1. In contrast, under the Kelso-Crawford gross substitutes as in example (5)–(6), constructing a similar sequence requires either responding to the increase in the price of good 1 by increasing the quantity of good 2 (the sequence $(3, 9)$ to $(7, 10)$ to $(9, 3)$) or responding to the decrease in the price of good 2 by decreasing the quantity of good 1 (the sequence $(3, 9)$ to $(10, 7)$ to $(9, 3)$). In effect, unified gross substitutes requires some consistency of the quantities verifying the substitutes conditions that Kelso-Crawford gross substitutes does not impose. Galichon, Samuelson and Vernet [17, Appendix B.2] show that Kelso and Crawford's notion does not suffice for our inverse isotonicity result (Theorem 1).

Galichon, Samuelson and Vernet [17, Appendix B.3] show that a correspondence $\mathtt{Q}$ satisfying uniform gross substitutes (and hence Kelso-Crawford gross

[6]Kelso and Crawford [22, p. 1486] impose only the first condition, which suffices for the price adjustment process they consider.

substitutes) need not admit a selection satisfying weak gross substitutes. Say that a finite sublattice $\tilde{P}$ is of *degree* n if n is the smallest number such that each element of $\tilde{P}$ has at most n immediate predecessors and at most n immediate successors. Let $\mathtt{Q}_{\tilde{P}}$ denote the restriction of $\mathtt{Q}$ to the sublattice $\tilde{P}$. Then a straightforward reformulation of the definition establishes that the Kelso-Crawford gross substitutes property is equivalent to the requirement that for any finite sublattice $\tilde{P}$ of degree one and any point in the graph of $\tilde{P}$, there exists a selection from $\mathtt{Q}_{\tilde{P}}$ containing the selected point and satisfying weak gross substitutes.[7] The following, whose conceptually straightforward but tedious proof is given in Galichon, Samuelson and Vernet [17, Appendix B.4], shows that unified gross substitutes has an analogous characterization in terms of finite sublattices of degree two rather than one:

Lemma 3. *The correspondence* $\mathtt{Q}$ *satisfies unified gross substitutes if and only if, for any finite sublattice* $\tilde{P}$ *of degree two and any two points drawn from the images of two nonordered elements of* $\tilde{P}$*, there exists a selection from* $\mathtt{Q}_{\tilde{P}}$ *containing those points and satisfying weak gross substitutes.*

We thus view unified gross substitutes as the natural strengthening of Kelso-Crawford gross substitutes.

Polterovich and Spivak [27, Definition 1, p. 118] propose a notion of gross substitutes for correspondences (see Howitt [21] for an intermediate notion), which stipulates that if the prices of some set of goods increase while others remain constant, it cannot be the case that every one of the quantities associated with the latter set strictly increases. Polterovich and Spivak [27, Lemma 1, p. 123] show that if the correspondence $\mathtt{Q}$ maps from the interior of $\mathbb{R}^N_+$ into $\mathbb{R}^N$, is convex valued and closed valued, and maps compact sets into nonempty bounded sets, then their gross substitutability condition implies (1)–(2). Galichon, Samuelson and Vernet [17, Appendix B.5] show that our requirement that P be a sublattice of $\mathbb{R}^N$ does not suffice for this result, and that in general neither notion implies the other.

2.4 Nonreversingness

Our second condition is a requirement that the correspondence $\mathtt{Q}$ cannot completely reverse the order of two points:

[7] The correspondence given by (5)–(6) shows that Kelso-Crawford's gross substitutes does not ensure sublattices of degree higher than one admit a selection satisfying weak gross substitutes.

Definition 2 (**Nonreversing correspondence).** *The correspondence* $\mathtt{Q} : P \rightrightarrows Q$ *is* nonreversing *if*

$$\begin{pmatrix} q \in \mathtt{Q}(p) \\ q' \in \mathtt{Q}(p') \\ q \leq q' \\ p \geq p' \end{pmatrix} \implies \begin{pmatrix} q \in \mathtt{Q}(p') \\ q' \in \mathtt{Q}(p) \end{pmatrix}. \tag{7}$$

Nonreversingness is implied by the (stronger but common) assumption that $\mathtt{Q}$ is increasing in the strong set order.

We view unified gross substitutes as the more substantive of the two conditions, with nonreversingness typically being innocuous. We will subsequently make use of the following two circumstances under which nonreversingness naturally holds.

First, the correspondence $\mathtt{Q}$ satisfies *constant aggregate output* if there exists $k \in \mathbb{R}^N_{++}$ such that $\sum_{z=1}^N k_z q_z = 0$ holds for all $p \in P$ and $q \in \mathtt{Q}(p)$. One obvious circumstance in which aggregate output is constant is that in which the bundle of goods is augmented by an "outside good" whose quantity is the negative of the sum of the quantities of the original set of goods, as in Berry, Gandhi and Haile [3]. Alternatively, the correspondence may be describing market shares in a model of competition, probabilities in a prediction problem, or budget shares in a model of consumption. Constant aggregate output ensures that the antecedent of (7) can hold only if $q = q'$, rendering the consequent immediate, giving:

Lemma 4. *Constant aggregate output implies nonreversingness.*

Second, the correspondence $\mathtt{Q}$ satisfies *constant-value output* if for all p, and all $q \in \mathtt{Q}(p)$, we have $p^\top q = 0$. If $-\mathtt{Q}$ is the excess demand function of an economy (allowing us to maintain our interpretation of $\mathtt{Q}$ as a supply correspondence), then Walras' law ensures that $\mathtt{Q}$ will satisfy constant-value output.

Given a correspondence $\mathtt{Q}$, consider the associated correspondence measured in monetary terms,

$$\mathtt{Q}^{\$}(p) = \left\{ (p_z q_z)_{z \in \{1,\ldots,N\}} \text{ with } q \in \mathtt{Q}(p) \right\}.$$

Then $\mathtt{Q}$ satisfies constant-value output if and only if for all p and all $\tilde{q} \in \mathtt{Q}^{\$}(p)$, we have $1^\top \tilde{q} = 0$. This gives the first statement of the following; the second is straightforward.

Lemma 5. *If* $\mathtt{Q}$ *satisfies constant-value output, then* $\mathtt{Q}^{\$}$ *has constant aggregate output, and is therefore nonreversing. When the domain* P *is contained in* $\mathbb{R}^N_+$*, so that prices are positive, then* $\mathtt{Q}$ *satisfies unified gross substitutes if and only if* $\mathtt{Q}^{\$}$ *satisfies unified gross substitutes.*

2.5 Inverse Isotonicity

We are interested in conditions under which $\mathtt{Q}$ has an isotone (i.e., weakly increasing) inverse.[8]

Definition 3 (**Totally Isotone Inverse**). *A correspondence* $\mathtt{Q} : P \rightrightarrows Q$ *has* totally isotone inverse *if, whenever* $q \in \mathtt{Q}(p)$ *and* $q' \in \mathtt{Q}(p')$ *are such that there exists* $B \subseteq \{1, \ldots, N\}$ *with* $p_z \leq p'_z$ *for all* $z \in B$ *and* $q_z \leq q'_z$ *for all* $z \notin B$*, we have*

$$q \in \mathtt{Q}(p \wedge p') \text{ and } q' \in \mathtt{Q}(p \vee p'). \tag{8}$$

If we require (8) *to hold only for the case* $B = \emptyset$*, then we say simply that* $\mathtt{Q}$ *is* inverse isotone*.*

The weaker property of inverse isotonicity, implied by total inverse isotonicity, is the condition that the inverse correspondence $\mathtt{Q}^{-1}$ is isotone in the strong set order, i.e, whenever $p \in \mathtt{Q}^{-1}(q)$ and $p' \in \mathtt{Q}^{-1}(q')$ where $q \leq q'$, it follows that

$$p \wedge p' \in Q^{-1}(q) \text{ and } p \vee p' \in Q^{-1}(q').$$

The following inverse isotonicity result is the building block for subsequent applications.

Theorem 1. *Let* $\mathtt{Q} : P \rightrightarrows Q$ *satisfy unified gross substitutes. Then the following conditions are equivalent:*

(i) $\mathtt{Q}$ *is nonreversing, and*

(ii) $\mathtt{Q}$ *has totally isotone inverse.*

[8] A linear function $p = Mq$ satisfies our condition of totally isotone inverse if and only if the inverse matrix M^{-1} is totally positive (i.e., every minor is positive). The analogue between a nonempty B and a nontrivial minor of the matrix M motivates our "totally" modifier in the following definition.

Proof Assume (i) and consider $q \in \mathtt{Q}(p)$, $q' \in \mathtt{Q}(p')$, and $B \subseteq \{1, ..., N\}$ such that $p_z \leq p'_z$ for all $z \in B$ and $q_z \leq q'_z$ for all $z \notin B$. By unified gross substitutes, one has the existence of $q^\wedge \in \mathtt{Q}(p \wedge p')$ and $q^\vee \in \mathtt{Q}(p \vee p')$ such that

$$\begin{aligned} p_z \leq p'_z &\implies q_z \leq q^\wedge_z \\ p_z > p'_z &\implies q'_z \leq q^\wedge_z. \end{aligned}$$

We have $p_z > p'_z \implies z \notin B$, which implies $q_z \leq q'_z$. Combining with the inequalities above, we see that $q_z \leq q^\wedge_z$ holds for any $1 \leq z \leq N$. But then $q \leq q^\wedge$ and $p \geq p \wedge p'$, so by nonreversingness it follows that $q \in \mathtt{Q}(p \wedge p')$. A similar reasoning shows that $q' \in \mathtt{Q}(p \vee p')$. We have therefore shown statement (ii).

Conversely, we assume statement (ii) holds and show $\mathtt{Q}$ is nonreversing. Take $p \geq p'$ and $q \leq q'$ such that $q \in \mathtt{Q}(p)$ and $q' \in \mathtt{Q}(p')$. Letting $B = \emptyset$, because $\mathtt{Q}$ has totally isotone inverse, we have that $q \in \mathtt{Q}(p \wedge p')$, which is equivalent to $q \in \mathtt{Q}(p')$, and $q' \in \mathtt{Q}(p \vee p')$, which is equivalent to $q' \in \mathtt{Q}(p)$, which shows (i). ■

It is an immediate implication that:

Corollary 1. *Let $\mathtt{Q}$ satisfy unified gross substitutes and be nonreversing. Then the set of prices $\mathtt{Q}^{-1}(q)$ associated with an allocation q is a sublattice of P.*

Proof Take $p \in \mathtt{Q}^{-1}(q)$ and $p' \in \mathtt{Q}^{-1}(q)$. Then $q \leq q$ yields $q \in \mathtt{Q}(p \wedge p')$ and $q' \in \mathtt{Q}(p \vee p')$. ■

3 Applications

3.1 Inverse Demand

Berry, Gandhi and Haile [3], abbreviated as BGH, examine functions $\mathtt{q} : P \to Q$. Given such a function, they introduce an additional good 0, defined by letting $\mathtt{q}_0(p) = -\sum_{z=1}^{N} \mathtt{q}_z(p)$, and direct attention to the function $\tilde{\mathtt{q}} : \{1\} \times P \to \mathbb{R} \times Q$, written $\tilde{q} = \tilde{\mathtt{q}}(\tilde{p}) = (q_0(p), q(p))$, with $\tilde{p} = (1, p)$ and $\tilde{q} = (q_0, q)$. They make the following assumptions, which allow them to show that an inverse demand function is isotone and point-valued:

BGH, Assumption 1: $\mathtt{q}$ is defined on a Cartesian product of sets.

BGH, Assumption 2.a: q has the gross substitutes property.

BGH, Assumption 2.b: The function q_0 is weakly decreasing in each p_z for $z \in \{1, ..., N\}$.

BGH, Assumption 3: For all $B \subseteq \{1, \ldots, N\}$ and all p and p' such that $p_z = p'_z$ for all $z \in B$ and $p_z > p'_z$ for all $z \notin B$, there exists $\tilde{z} \in B \cup \{0\}$ such that $\mathsf{q}_{\tilde{z}}(p) < \mathsf{q}_{\tilde{z}}(p')$.

Berry, Gandhi and Haile's Assumption 1, that q is defined on a Cartesian product of Euclidean spaces [3, p. 2094], implies our Assumption 1 that P is a sublattice of $\mathbb{R}^N$. Their second assumption [3, p. 2094] (in our notation, and translating from their framing in terms of demand functions to our framing in terms of supply functions) is split here into Assumptions by 2.a and 2.b. Lastly, their Assumption 3 [3, p. 2095] is a connected strict substitutes assumption which expresses that one cannot partition the set of goods into two set of products such that no good in the first set is a substitute for some good in the second set. We have stated this assumption in the equivalent form established in their Lemma 1.

Applying our Theorem 1 gives a variant of Berry, Gandhi and Haile's [3] Theorem 1 which operates under weaker assumptions (more precisely, without their Assumption 3) but delivers a weaker conclusion:

Corollary 2. *Under Berry, Gandhi and Haile's Assumptions 1, 2.a, and 2.b, the inverse* q^{-1} *of a function* q *is isotone in the strong set order, that is* $\mathsf{q}(p) \leq \mathsf{q}(p')$ *implies* $\mathsf{q}(p) = \mathsf{q}(p \wedge p')$ *and* $\mathsf{q}(p') = \mathsf{q}(p \vee p')$*.*

Proof Recall that Lemma 1 establishes that if the function q satisfies weak gross substitutes, which it does under Assumption 2.a, then it satisfies unified gross substitutes. Using Assumption 2.b and Lemma 4 (taking k to be the unit vector), we get that q is also nonreversing. Theorem 1 then gives the result. ∎

To generalize Berry, Gandhi and Haile's [3] Theorem 1 and Corollary 1 to correspondences, we introduce the idea of strong nonreversingness.[9]

Definition 4 (Strongly nonreversing correspondence). *The correspondence* $\mathsf{Q} : P \rightrightarrows Q$ *is* strongly nonreversing *if*

[9] Strong nonreversingness implies Sabarwal's [31] *never decreasing* property, which he uses to establish comparative static results.

$$\begin{pmatrix} q \in \mathtt{Q}(p) \\ q' \in \mathtt{Q}(p') \\ q \leq q' \\ p \geq p' \end{pmatrix} \implies p = p'.$$

Strong nonreversingness allows us to strengthen Theorem 1. adding the conclusion that $\mathtt{Q}^{-1}$ is point-valued as well as isotone. Appendix A.1 proves:

Theorem 2. *Let $\mathtt{Q} : P \rightrightarrows Q$ satisfy unified gross substitutes. Then the following conditions are equivalent:*

(i) $\mathtt{Q}$ is strongly nonreversing

(ii) $\mathtt{Q}^{-1}$ is point-valued and isotone where not empty, i.e. $q \in \mathtt{Q}(p)$, $q' \in \mathtt{Q}(p')$ and $q \leq q'$ imply $p \leq p'$.

We note in passing that for functions, this gives us an alternative proof of Berry, Gandhi and Haile's central result:

Corollary 3. *Under Berry, Gandhi and Haile's Assumptions 1, 2.a, 2.b and 3, the function $\mathtt{q}$ has a point-valued inverse and is inverse isotone.*

Proof We show that under the additional Assumption 3, the function $\mathtt{q}$ is in addition strongly nonreversing. Indeed, assume $\mathtt{q}(p) \leq \mathtt{q}(p')$ and $p \geq p'$. By nonreversingness, one has $\mathtt{q}(p) = \mathtt{q}(p')$. Assume $p > p'$. Let $B = \{z : p_z = p'\} \neq \{1, \ldots, N\}$. By BGH Assumption 3, there exists $z \in B \cup \{0\}$ such that $\mathtt{q}_z(p) < \mathtt{q}_z(p')$, a contradiction. Hence $p = p'$, giving strong nonreversingness. Theorem 2 then gives the result. ■

Turning now to a correspondence $\mathtt{Q}$, let $\tilde{\mathtt{Q}} : \{1\} \times P \rightrightarrows \mathbb{R} \times Q$ be the correspondence, written $\tilde{q} \in \tilde{\mathtt{Q}}(\tilde{p})$, constructed from $\mathtt{Q}$ by letting $\tilde{p} = (1, p)$ and $\tilde{q} = (-\sum_{z=1}^{N} q_z, q)$. By construction, the correspondence $\tilde{\mathtt{Q}}$ satisfies constant aggregate output and hence is nonreversing (Lemma 4). An argument analogous to the proof of Corollary 3 establishes that $\mathtt{Q}$ is strongly nonreversing. It is straightforward (Galichon, Samuelson and Vernet [17, Appendix B.6]) that if $\tilde{\mathtt{Q}}$ satisfies unified gross substitutes, then so does $\mathtt{Q}$. Theorem 2 then immediately gives the following extension of Berry, Gandhi and Haile's [3] inverse isotonicity result to correspondences:

Corollary 4. *Let $\tilde{\mathtt{Q}}$ satisfy unified gross substitutes. Then $\mathtt{Q}$ is inverse isotone ($\mathtt{q}(p) \le \mathtt{q}(p')$ implies $p \le p'$) and $\mathtt{Q}^{-1}$ is point-valued.*

Section 1 explained how Corollary 4 can be applied to quantity-setting models of markets with indivisibilities, or to other models of competition with limited dependent variables.

3.2 Profit Maximization

Suppose a competitive multiproduct firm faces output price vector $p \in \mathbb{R}^N$ and convex (and hence continuous) cost function $c : \mathbb{R}^N \to \mathbb{R}$. Given a price p, the set of optimal production vectors $\mathtt{Q}(p) \subseteq \mathbb{R}^N$ is given by

$$\mathtt{Q}(p) = \arg\max_{q \in \mathbb{R}^N} \left\{ p^\top q - c\left(q\right) \right\}. \tag{9}$$

The profit function c^* is the conjugate of the cost function c, given by

$$c^*(p) = \max_{q \in \mathbb{R}^N_+} \left\{ p^\top q - c\left(q\right) \right\}.$$

We note that $\partial c^*(p)$ is the subdifferential of c^* at p,[10] and the inverse correspondence $\mathtt{Q}^{-1}$ is given by

$$\mathtt{Q}^{-1}(q) = \arg\max_{p \in \mathbb{R}^N_+} \left\{ p^\top q - c^*\left(p\right) \right\} \;=\; \partial c(q),$$

where $\partial c(q)$ is the subdifferential of c at p.

The inverse correspondence $\mathtt{Q}^{-1}$ is isotone in the strong set order if $\mathtt{Q}$ is nonreversing and exhibits unified gross substitutes. Appendix A.2 shows that $\mathtt{Q}$ is nonreversing and proves the following result. The underlying principle is that convex functions are submodular if and only if their subdifferentials satisfy unified gross substitutes.

Theorem 3. *The following conditions are equivalent:*
(i) the profit function c^* is submodular, and
(ii) the supply correspondence $\mathtt{Q}(p) = \partial c^*\left(p\right)$ satisfies unified gross substitutes.

[10] Given the convexity of the cost function c, the general form of Shephard's lemma (Rockafellar [34, Theorem 23.5, p. 218]) gives $\mathtt{Q}(p) = \partial c^*(p)$, where $\partial c^*(p)$ is the subdifferential of the profit function.

Invoking Theorem 1, we thus have that the supply correspondence $\mathtt{Q}$ is inverse isotone if and only if the profit function is submodular.[11] Galichon, Samuelson and Vernet [17, Appendix B.7] extend this result to an imperfectly competitive market.

Ausubel and Milgrom [2, Theorem 10] argue that a demand correspondence satisfies a weak gross substitutes property (namely that $\mathtt{q}_i(p)$ is nonincreasing in p_j for $i \neq j$ and for those p for which $\mathtt{q}(p)$ is single-valued) if and only if the associated indirect utility function is submodular, and use this connection to show that sincere bidding strategies are an equilibrium of an ascending proxy auction. Theorem 3 provides alternative conditions for such submodularity.

3.3 Competitive Equilibria

If $\mathtt{Q}$ is the excess supply correspondence (equivalently, $-\mathtt{Q}$ is the excess demand correspondence) of an economy, then $\mathtt{Q}$ satisfies constant-value output. Since constant-value output implies nonreversingness (Lemma 5), we can conclude that if the excess supply correspondence $\mathtt{Q}$ satisfies unified gross substitutes, then the set of equilibrium prices is a sublattice.[12] As we noted in Section 2.5, this lattice structure allows one to pursue comparative static analyses of the extreme equilibria or (using the strong set order) the set of equilibria. For example, if the endowment of good 1 increases, then the (smallest, or largest, or set of, in the strong set order) relative equilibrium price of each other good increases.

It is a familiar result that if each agent's (point-valued) demand function in an exchange economy satisfies strict gross substitutes, then that economy has a unique equilibrium (e.g., Arrow and Hahn [1, Chapter 9]), with corresponding comparative statics (cf. Nachbar [25, Section 3.3]). We thus have the counterpart of this result to cases in which individual demands are not point-valued, such as when preferences are

[11]Alternatively, given a submodular profit function c^*, we can note that the function $p^\top q - c^*(p)$ is then supermodular in p and exhibits increasing differences, and so Theorem 6.1 of Topkis [32, p. 317] (see also Topkis [33] and Milgrom and Shannon [24]) ensures that $\mathtt{Q}^{-1}$ is increasing in the strong set order. The key to the applicability of Topkis's result in this setting is that the correspondence $\mathtt{Q}^{-1}$ is itself the solution to a maximization problem whose maximand exhibits the needed properties. We can thus view Theorem 1 as extending familiar monotone comparative statics results beyond the case in which $\mathtt{Q}^{-1}$ is the solution to a maximization problem.

[12]Polterovich and Spivak [27, Corollary 1, p. 125] similarly show that the set of equilibrium prices in an exchange economy satisfying their gross substitutes condition (cf. Section 2.3) is a lattice. Gul and Stacchetti [19, Corollary 1, p. 105] have a similar result for economies with indivisible goods.

weakly but not strictly convex.

3.4 An Equilibrium Flow Problem

We introduce a general structure that naturally leads to a correspondence satisfying unified gross substitutes. We then illustrate three applications in the following three subsections.

Consider a network $(\mathcal{Z}, \mathcal{A})$ where $\mathcal{Z}$ is a finite set of nodes and $\mathcal{A} \subseteq \mathcal{Z} \times \mathcal{Z}$ is the set of directed arcs. If $xy \in \mathcal{A}$, we say that xy is the arc with starting node x and end node y. We assume there is no arc in $\mathcal{A}$ whose starting and ending nodes coincide.

Let $p \in \mathbb{R}^{\mathcal{Z}}$ be a price vector, where we interpret p_z as the price at node z. To have a concrete description, though we do not require this interpretation, one may consider a trader who is able to purchase one unit of a commodity at node x, ship it along arc xy toward node y, and resell it at node y. Given the resale price at node y, there is a certain threshold value of the price at node x such that the trader is indifferent between engaging in the trade or not. This value is an increasing and continuous (but not necessarily linear) function of the price at node y, and can be expressed as $G_{xy}(p_y)$, where for each arc $xy \in \mathcal{A}$, the *connection function* $G_{xy} : \mathbb{R} \to \mathbb{R}$ is continuous and increasing.

Let $q \in \mathbb{R}^{\mathcal{Z}}$ attach a net flow to each node $z \in \mathcal{Z}$. If $q_z > 0$, then the net quantity $|q_z|$ must flow into node z, while $q_z < 0$ indicates that the net quantity $|q_z|$ must flow away from node z. We let $\mu \in \mathbb{R}_+^{\mathcal{A}}$ be the vector of *internal flows* along arcs, so that μ_{xy} is the flow through arc xy.

The triple $(q, \mu, p) \in \mathbb{R}^{\mathcal{Z}} \times \mathbb{R}_+^{\mathcal{A}} \times \mathbb{R}^{\mathcal{Z}}$ is an *equilibrium flow outcome* if it satisfies three conditions. The first is the feasibility condition that, for any node $z \in \mathcal{Z}_0$, the total internal flow that arrives at z minus the total internal flow that leaves z equals the net flow at z, that is

$$\sum_{x:xz \in \mathcal{A}} \mu_{xz} - \sum_{y:zy \in \mathcal{A}} \mu_{zy} = q_z. \tag{10}$$

The second condition is that there there is no positive rent on any arc, that is:

$$p_x \geq G_{xy}(p_y) \quad \forall xy \in \mathcal{A}. \tag{11}$$

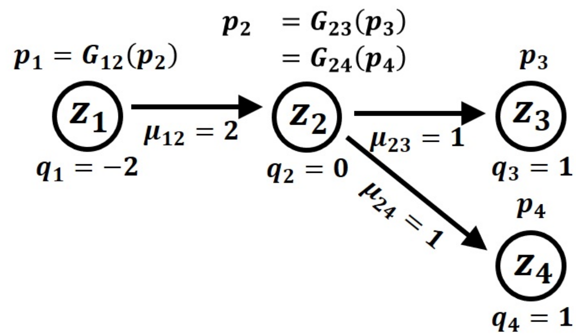

Figure 2: A simple example of an equilibrium flow outcome. The three equilibrium conditions are satisfied as (i) the flow balance condition is satisfied at each node of the network (ii) there is no positive rent on the network (iii) when there is a trade on an arc (μ larger than zero on this arc), rent on this arc is nonnegative.

Our third condition is that arcs with negative rents carry no flow, or

$$\mu_{xy} > 0 \implies p_x \leq G_{xy}(p_y). \tag{12}$$

Figure 2 presents a simple example of an equilibrium flow outcome. Notice that if (q, μ, p) is a equilibrium flow outcome, then so is $(\lambda q, \lambda \mu, p)$ for any nonnegative scalar λ. Hence, there will either be no equilibrium flow outcome (if there is no p satisfying (11)) or there will be multiple equilibrium flows outcomes.

Given an equilibrium flow problem, let the *equilibrium flow correspondence* associate with a price p the quantities q that appear as part of an equilibrium flow given p:

Definition 5 (Equilibrium flow correspondence)**.** *The equilibrium flow correspondence is the correspondence* $\mathtt{Q} : \mathbb{R}^{\mathcal{Z}} \rightrightarrows \mathbb{R}^{\mathcal{Z}}$ *defined by the fact that for* $p \in \mathbb{R}^{\mathcal{Z}}$, $\mathtt{Q}(p)$ *is the set of* $q \in \mathbb{R}^{\mathcal{Z}}$ *such that there is a flow* μ *such that* (q, μ, p) *is an equilibrium flow outcome.*

Note that $\mathtt{Q}$ may be empty valued. Appendix A.3 proves:

Theorem 4. *The equilibrium flow correspondence* $\mathtt{Q} : \mathbb{R}^{\mathcal{Z}} \rightrightarrows \mathbb{R}^{\mathcal{Z}}$ *satisfies unified gross substitutes.*

It is immediate that the correspondence $\mathtt{Q}$ is nonreversing, since $q \in \mathtt{Q}(p)$ implies $\sum_{z \in \mathcal{Z}} q_z = 0$. Hence it follows from Theorem 4, the inverse isotonicity Theorem 1, and Corollary 1 that:

Corollary 5. *The equilibrium flow correspondence* $\mathtt{Q}(p)$ *has totally isotone inverse and the set of equilibrium prices* $\mathtt{Q}^{-1}$ *is a sublattice of* $\mathbb{R}^{\mathcal{Z}}$*.*

Galichon, Samuelson and Vernet [16] identify conditions under which an equilibrium flow outcome exists, and hence $\mathtt{Q}(p)$ is nonempty, invoking a generalization of Hall's [20] conditions and an analogue of Rochet's [29] cyclical monotonicity condition.

3.5 Matching with (Im)perfectly Transferable Utility

To put our equilibrium flow formulation to work, consider the following one-to-one matching market. Let $\mathcal{X}$ be a finite set of types of workers and $\mathcal{Y}$ a finite set of types of firms. There are n_x workers of each type $x \in \mathcal{X}$, and m_y firms of each type $y \in \mathcal{Y}$. A match between worker type x and firm type y is characterized by a wage w_{xy}, in which case it gives rise to the utilities $\mathcal{U}_{xy}(w_{xy})$ for the worker and $\mathcal{V}_{xy}(w_{xy})$ for the firm. An unmatched worker receives utility $\mathcal{U}_{x0}$ and an unmatched firm receives utility $\mathcal{V}_{0y}$. A matching is a pair (μ, w), where w is a vector specifying the wage w_{xy} attached to each pair $xy \in \mathcal{X} \times \mathcal{Y}$ and μ is a vector identifying the mass μ_{xy} of matches between workers of type x and firms of type y, for each $xy \in \mathcal{X} \times \mathcal{Y}$.

We reformulate this matching problem as an equilibrium flow problem. Let

$$\begin{aligned}
\mathcal{Z} &= \mathcal{X} \cup \mathcal{Y} \cup \{0\} \\
\mathcal{A} &= ((\mathcal{X} \cup \{0\}) \times (\mathcal{Y} \cup \{0\})) \setminus \{(0,0)\} \\
q_z &= -n_z, \qquad z \in \mathcal{X} \\
q_z &= m_z, \qquad z \in \mathcal{Y} \\
q_0 &= \sum_{x \in \mathcal{X}} n_x - \sum_{y \in \mathcal{Y}} m_y.
\end{aligned}$$

The nodes thus include the sets of types of workers and firms as well as an additional node 0 designed to allow agents to be unmatched. Each node corresponding to a type of worker carries a negative net flow equal to the mass of such types, while each node corresponding to a type of firm carries a positive net flow corresponding to the mass of such types. There is an arc from every worker-type node to every firm-type node, as well as arcs from workers to the "unmatched node" and from the unmatched node to firms. Figure 3 illustrates a simple case. A flow along an arc signifies the mass of matches between the types residing at each node connected by the arc. The price attached to a node identifies the utility of the agent at that node (if a worker) or the negative of the utility of the agent at that node (if a firm). Given a price vector p, we define the connection function

$$\begin{aligned} G_{xy}(p_y) &= \mathcal{U}_{xy} \circ \mathcal{V}_{xy}^{-1}(-p_y) \text{ for } x \in \mathcal{X}, y \in \mathcal{Y} \\ G_{x0}(p_0) &= p_0 + \mathcal{U}_{x0} \text{ for } x \in \mathcal{X} \\ G_{0y}(p_y) &= p_y + \mathcal{V}_{0y} \text{ for } y \in \mathcal{Y}. \end{aligned}$$

As required, $G_{xy}(p_y)$ is increasing in p_y. We adopt the normalization $p_0 = 0$.[13]

The matching (μ, w) is stable if

$$\sum_{y \in \mathcal{Y}} \mu_{xy} + \mu_{x0} = n_x, \qquad \sum_{x \in \mathcal{X}} \mu_{xy} = \mu_{0y} m_y,$$

as well as

$$\mu_{xy} > 0 \implies \left\{ \begin{array}{c} \mathcal{U}_{xy} = \max\left\{\max_{\tilde{y} \in \mathcal{Y}} \mathcal{U}_{x\tilde{y}}(w_{x\tilde{y}}), \mathcal{U}_{x0}\right\} \\ \mathcal{V}_{xy} = \max\left\{\max_{\tilde{x} \in \mathcal{X}} \mathcal{V}_{\tilde{x}y}(w_{\tilde{x}y}), \mathcal{V}_{0y}\right\} \end{array} \right\},$$

and

$$\begin{aligned} \mu_{x0} > 0 &\implies \mathcal{U}_{x0} = \max\left\{\max_{\tilde{y} \in \mathcal{Y}} \mathcal{U}_{x\tilde{y}}(w_{x\tilde{y}}), \mathcal{U}_{x0}\right\} \\ \mu_{0y} > 0 &\implies \mathcal{V}_{0y} = \max\left\{\max_{\tilde{x} \in \mathcal{X}} \mathcal{V}_{\tilde{x}y}(w_{\tilde{x}y}), \mathcal{V}_{0y}\right\}. \end{aligned}$$

The first condition provides the feasibility condition that the number of each type of worker and firm that is matched is no more than the number present in the market (excess agents are unmatched), while the second provides the stability condition that no worker and firm can improve their utilities by matching with one another at an appropriate wage or remaining unmatched.

Appendix A.4 proves the following:

Lemma 6. *The matching (μ, w) is stable if and only the associated outcome (q, μ, p) is an equilibrium flow.*

[13]The quantity vector q identifies the negative of the quantity of each type of workers and the quantity of each type of firm, allowing us to represent a match between a work and a firm as a flow along the arc connecting the node containing that type of worker to the node representing the firm. Similarly, the payoff vector p identifies the payoffs of workers and the negative of the payoffs of firms, so that an increase in the payoff of a firm at node y corresponds to a smaller utility requirement from the firm at that node, making it more attractive for workers to traverse the arc terminating at the node, which corresponds to the increasingness of the connection function $G_{xy}(p_y)$.

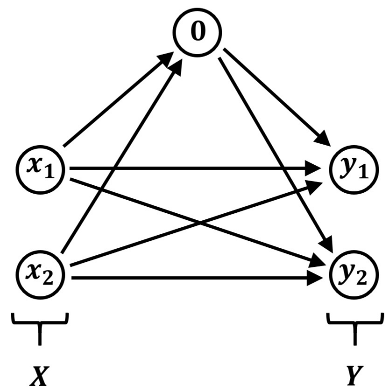

Figure 3: Network associated with a bipartite matching problem.

The wages in the stable match are related to the prices the equilibrium flow problem via $\mathcal{V}_{xy}^{-1}(-p_y) \leq w_{xy} \leq \mathcal{U}_{xy}^{-1}(p_{xy})$, with equality on links exhibiting positive flows.

The equilibrium flow correspondence $\mathtt{Q}$ then associates with each utility vector p the set of of vectors q, i.e., the specifications of the sets of firms and workers, for which there exists a stable outcome yielding the utilities specified by p. Theorem 4 immediately gives:

Theorem 5. *The correspondence that maps the vector of payoffs p to vectors q of populations is nonreversing and satisfies unified gross substitutes.*

It follows from Corollary 5 that, given a specification q of workers and firms, the set of equilibrium utilities consistent with a stable match constitutes a lattice. The inverse isotonicity of the equilibrium correspondence then gives comparative static results. For example, as the number of firms increases, the set of equilibrium payoffs of firms decreases (in the strong set order) while the set of equilibrium payoffs of workers increases. An increase in the number of workers has the reverse effects. Galichon, Samuelson and Vernet [17, Appendix B.8] provide a similar result for the case of matching without transfers, pioneered by Gale and Shapley [14].

Demange and Gale [9, Lemma 2 and Property 2] (see also Demange, Gale and Sotomayor [10] and Decker, Lieb, McCann and Stephens [8] for a quasilinear models) establish similar results for a model in which the quantities attached to the various nodes must take integer values, which we do not require. Dropping this requirement is useful in allowing one to accommodate a continuum of each type of agent, as is common in models of competitive matching (cf. Nöldeke and Samuelson [28]).

3.6 Hedonic Pricing

The simplest models of a competitive economy assume that each of a finite number of goods is perfectly divisible and perfectly homogeneous. The hedonic pricing model, introduced by Rosen [30] and developed by Ekeland, Heckman and Nesheim [13], examines the opposite extreme, an economy filled with indivisible, idiosyncratic goods. To reduce the dimensionality of the prices in the latter case, one typically assumes the goods can be described by the extent to which they exhibit certain characteristics, with prices determined by these characteristics, thus giving rise to the term hedonic pricing.

Chiappori, McCann and Nesheim [6] show that the hedonic pricing problem with quasilinear utilities is equivalent to a matching problem with quasilinear utility, which is in turn equivalent to an optimal transport problem[14]. Ekeland [12, p. 295] comments that the primary limitation of existing hedonic pricing models is the restriction to quasilinear utilities. Carlier and Ekeland [5] extend the hedonic pricing analysis by relaxing quasilinearity on one side of the market. In markets providing the original impetus for hedonic pricing, such as residential housing, one would expect income effects to arise (and hence quasilinearity to fail) on both sides.

We extend the analysis of hedonic pricing beyond the quasilinear case. Chiappori, McCann and Nesheim [6] invoke a twist condition to establish the uniqueness for equilibrium for the quasilinear case with a continuum of characteristics. Our contribution to this edifice is to establish that for any equilibrium allocation, the set of utilities and prices supporting this equilibrium allocation is a lattice. We avoid a host of technical difficulties, addressed by Chiappori, McCann and Nesheim, by working with finite sets of types of agents and goods, though again allowing a continuum of each type.

The basic elements of the model are a finite set $\mathcal{X}$ of types of producers and a finite set $\mathcal{Y}$ of types of consumers. There are n_x producers of each type $x \in \mathcal{X}$ and m_y consumers of each type $y \in \mathcal{Y}$. There is a finite set $\mathcal{W}$ of qualities, also sometimes referred to as contracts or characteristics. Each producer must choose to produce one of the qualities in $\mathcal{W}$, or to remain inactive. Each consumer must choose to consume one quality in $\mathcal{W}$ or remain inactive.

[14]In the same vein, Queyranne showed that the problem is equivalent to a min cost flow problem (see `https://www.mit.edu/~dimitrib/PTseng/optsem/optsemWI06_4_rtr.MPG`).

Let $p \in \mathbb{R}^{\mathcal{W}}$ be a price vector assigning prices to qualities, with p_w denoting the price of quality w. A producer of type x who produces a quality w that bears price p_w earns the profit $\pi_{xw}(p_w)$. A consumer of type y who consumes a quality w bearing price p_w reaps surplus $s_{yw}(p_w)$. Inactive producers and consumers receive payoff 0.

Let μ_{xw} be the quantity of producers of type x producing quality w, and μ_{wy} be the number of consumers of type y consuming quality w. Similarly, μ_{x0} and μ_{0y} are the quantity of producers of type x and consumers of type y opting out.

A *hedonic pricing equilibrium* is a price vector p_w and a specification of the production flows μ_{xw} and consumption flows μ_{wy} such that the quantities supplied and demanded are feasible,

$$\sum_{w \in \mathcal{W}} \mu_{xw} + \mu_{x0} = n_x \text{ and } \sum_{w \in \mathcal{W}} \mu_{wy} + \mu_{0y} = m_y, \tag{13}$$

markets balance,

$$\sum_{x \in \mathcal{X}} \mu_{xw} - \sum_{y \in \mathcal{Y}} \mu_{wy} = 0, \tag{14}$$

and agents maximize,

$$\begin{array}{rcc} \mu_{xw} > 0 & \Longrightarrow & \pi_{xw}(p_w) = \max_{\tilde{w} \in W} \pi_{x\tilde{w}}(p_{\tilde{w}}) \geq 0 \\ \mu_{wy} > 0 & \Longrightarrow & s_{yw}(p_w) = \max_{\tilde{w} \in W} s_{y\tilde{w}}(p_{\tilde{w}}) \geq 0 \\ \mu_{x0} > 0 & \Longrightarrow & 0 \geq \max_{\tilde{w} \in W} \pi_{x\tilde{w}}(p_{\tilde{w}}) \\ \mu_{0y} > 0 & \Longrightarrow & 0 \geq \max_{\tilde{w} \in W} s_{y\tilde{w}}(p_{\tilde{w}}). \end{array} \tag{15}$$

The hedonic pricing problem can be reformulated as an equilibrium flow problem. Let the set of nodes be $\mathcal{Z} = \mathcal{X} \cup \mathcal{Y} \cup \mathcal{W}_0$, where $\mathcal{W}_0 = \mathcal{W} \cup \{0\}$. The set of arcs $\mathcal{A}$ is given by

$$\mathcal{A} = (\mathcal{X} \times \mathcal{W}_0) \cup (\mathcal{W}_0 \times \mathcal{Y}).$$

Figure 4 illustrates.

Let u_x be a vector of producer utilities and u_y a vector of consumer utilities. Normalize the price $p_0 = 0$ and let the prices of the nodes in X be given by $p_x = u_x$ for all $x \in \mathcal{X}$, and the prices of the nodes in Y be given by $p_y = -v_y$ for all $y \in \mathcal{Y}$. Then the connection functions are given by $G_{xw}\left(p_w\right) = \pi_{xw}\left(p_w\right)$, $G_{x0}\left(p_0\right) = p_0$, $G_{wy}\left(p_y\right) = s_{yw}^{-1}\left(-p_y\right)$ and $G_{0y}\left(p_y\right) = p_y$. The net flows are given by $q_w = \sum_{x \in \mathcal{X}} \mu_{xx} - \sum_{y \in \mathcal{Y}} \mu_{xy}$ for all $w \in \mathcal{W}$, $q_x = -n_x$ for all $x \in \mathcal{X}$, and $q_y = m_y$ for all $y \in \mathcal{Y}$.

It is a straightforward inspection of the definitions to confirm that hedonic pricing equilibria correspond to equilibria of the equilibrium flow problem. We then

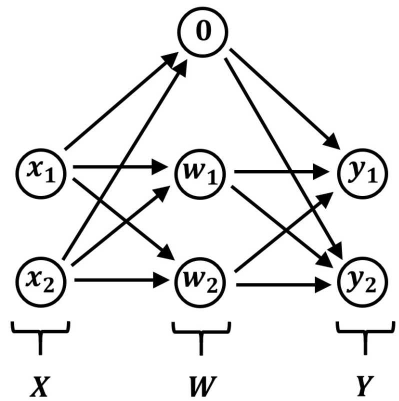

Figure 4: Network associated with a hedonic model.

examine a correspondence Q that takes in a price vector p specifying the utilities of the various buyers, the prices of the various qualities and (the negative of) the utilities of the sellers, and identifies the set of net flows for which there is an internal flow for which the hedonic pricing equilibrium conditions (13), (14) and (15) are satisfied. It follows from Theorem 4 that:

Theorem 6. *The correspondence* Q*, associating equilibrium net flows with vectors* $(u, p, -v)$ *specifying utilities and prices, is nonreversing and satisfies unified gross substitutes.*

This equilibrium therefore exhibits the lattice structure established in Corollary 1. We again immediately have some comparative static results. An increase in the quantity of any one of the types of customers increases (in the strong set order) the set of equilibrium prices and profits of all of the firms, while an increase in the quantity of any type of firm decreases prices and increases the utilities of all of the customers.[15]

3.7 Time-Dependent Routing

Time dependent routing problems, a variant of the classic vehicle routing problem (Toth and Vigo [35]) in which completion times vary over the course of the planning horizon, have recently attracted great attention (see Gendreau, Ghiani and Guerriero [18] for a survey). The time-dependent routing literature has focused on methods for

[15] Edlefsen [11] refers to a nonlinear constrain as a hedonic price function and examines utility maximization problems subject to exogenously given hedonic price functions.

computing solutions. We show that time-dependent routing problems are special cases of the equilibrium flow problems, adding existence and comparative static results to the literature.

There is a finite set of nodes $\mathcal{Z}$ and arcs $\mathcal{A}$. We interpret a node with a net flow $q_z < 0$ as a source firm with quantity $|q_z|$ of a commodity to be supplied to the market. A node with a net flow $q_z > 0$ is interpreted as an end firm with a demand q_z to be filled. A node with $q_z = 0$ is a processing firm, that can receive a good, perform an operation on the good, and pass the processed good on to the market. We introduce a distinguished node 0, with $\mathcal{A}$ containing an arc $z0$ for every node with $q_z < 0$ and an arc $0z$ for every node with $q_z > 0$. The interpretation is that a shipment to node 0 represents an unsold good, while a shipment from node 0 represents an unfilled demand. It is then innocuous to assume that the demand equal the supply in the economy: $\sum_z q_z = 0$. In alternative interpretations the flows along the arcs might be interpreted as traveling vehicles or messages.

We interpret prices as times, with p_z being the time at which the commodity reaches node z. The differences of prices $p_y - p_x$ is then interpreted as the duration that it takes node x to process the good and deliver it to node y. We define $G_{xy}(p_y)$ as the latest time at which one can to leave node x and arrive at node y at time p_y. In the simplest formulation, the travel duration through arc xy would be constant across time and denoted by c_{xy}, and therefore $G_{xy}(p_y) = p_y - c_{xy}$. In the time-dependent formulation, G_{xy} is a general function of p_y, such as $p_y - c_{xy}(p_y)$ for some $c_{xy}(p_y)$ which varies with p_y, to capture travel duration that may vary with time. We let G_{z0} be the identify function.

We assume that there is free disposal of time, but no time travel, so that if $\mu_{xy} > 0$, then $p_x \leq G_{xy}(p_y)$ holds, meaning that the commodity cannot leave an arc before being available there. In our previous terms, condition (12) must hold, and hence arcs with negative rent will carry no flow. We assume that the nodes prefer to sell as late as possible and buy as early as possible, captured in the form of the no-positive-rent condition (11). For example, if $p_x < G_{xy}(p_y)$ for an arc $xy \in \mathcal{A}$, then it is possible for the commodity to arrive at node x at time p_x and then arrive at node y at time $G_{xy}^{-1}(p_x) = p'_y < p(y)$, and hence the prices p_x and p_y are inconsistent with an equilibrium. We normalize $p_0 = 0$.

Given the demands and supplied specified by q, an equilibrium flow is a triple (q, μ, p) with

$$\begin{cases} \forall xy,\, p_x \geq G_{xy}\left(p_y\right), \text{ with equality if } \mu_{xy} > 0 \\ \forall z \in \mathcal{Z},\ \sum_x \mu_{xz} - \sum_y \mu_{zy} = q_z \end{cases}. \tag{16}$$

We can again formulate an equilibrium flow correspondence $\mathtt{Q}$ that associates with any collection of prices (or times, in this instance) p the set of demands and supplies q for which there exists an equilibrium consistent with these arrival times. The correspondence $\mathtt{Q}$ satisfies unified gross substitutes and so is inverse isotone. As before, Theorem 1 and Corollary 1 tell us that for a given market configuration, the set of equilibrium times p will constitute a lattice, and allows us to derive comparative statics. For example, suppose we increase the quantity demanded of some end firm. Then the arrival times of all end firms are delayed (more precisely, the set of vectors of equilibrium arrival times for end firms shifts upward in the strong set order). Increasing the supply of some source firm has the opposite effect. Notice that these effects arise out of competition rather than congestion. In this simple formulation, the processing times of the various nodes can depend on time in complicated ways, but do not depend the volume of flow through that node. Instead, increasing the demand of a end firm causes that seller to bid flows away from other end firms, forcing all to settle for later arrivals.

4 Conclusion

The concept of weak gross substitutes plays a prominent role in economic theory. We view the concept of unified gross substitutes as the natural generalization of weak gross substitutes to correspondences. It connects to the literature in multiple points, unifying some results and generalizing others.

The concept of unified gross substitutes allows one to derive the inverse isotonicity and lattice-valued-inverse properties of general correspondences. This provides a tool that should be useful in a number directions. In some settings it allows one to establish the existence of an inverse demand function or its functional equivalent, while in other settings it allows empirical or theoretical research to proceed without the uniqueness of inverse demand by focusing on the largest or smallest elements of the inverse. It provides a route to comparative statics when the traditional methods of monotone comparative statics do not apply, presumably because one is dealing with an equilibrium rather than optimization problem. It should be useful in extending familiar results developed under the assumption of quasilinearity to more general settings. Finally, we believe there is great potential for formulating a variety of problems as special cases of the equilibrium flow problem, allowing immediate application of the implications of unified gross substitutes.

Unified gross substitutes is a sufficient condition for our central results. A weaker condition would suffice for our inverse isotonicity result, just as a weakening of Kelso-Crawford gross substitutes would typically suffice in applications where it is used. We view the sufficiency of unified gross substitutes as analogous to the sufficiency of single crossing in incentive problems, in the sense that a necessary and sufficient condition is some version of "single crossing where it counts," while the standard sufficient condition is simply single crossing. Section 2.3 explained why we think of unified gross substitutes as the natural strengthening of Kelso-Crawford substitutes. In addition, we are induced to regard unified gross substitutes as the condition of interest by the observation that it often appears as in implication, being implied by the submodularity of the competitive profit function, equivalent to a continuous version of Gul and Stacchetti's [19] no complementarities (cf. Galichon et al. [15]), and implied by the equilibrium flow correspondence.

A Appendix

A.1 Proof of Theorem 2

Assume (ii) holds, hence $\mathtt{Q}$ is inverse isotone, and assume for $q \in \mathtt{Q}(p)$, $q' \in \mathtt{Q}(p')$, $q \leq q'$ and $p \geq p'$. By inverse isotonicity one has $p \leq p'$, and thus $p = p'$, as needed.

Assume (i) holds and so $\mathtt{Q}$ is strongly nonreversing and hence nonreversing. Assume $q \in \mathtt{Q}(p)$ and $q \in \mathtt{Q}(p')$. By unified gross substitutes, we have $q^{\vee} \in \mathtt{Q}(p \vee p')$ with $q^{\vee} \leq q$. Because $p \vee p' \geq p, p'$, strong nonreversingness gives $p = p \vee p' = p'$. Hence, $p = p'$ and thus $\mathtt{Q}$ is point valued. To show it is isotone, assume $q \in \mathtt{Q}(p)$, $q' \in \mathtt{Q}(p')$ and $q \leq q'$. By Theorem 1, we have $q \in \mathtt{Q}(p \wedge p')$ and $q' \in \mathtt{Q}(p \vee p')$. Because $\mathtt{Q}$ is injective, it follows that $p = p \wedge p'$ and thus $p \leq p'$.

A.2 Proof of Theorem 3

We first show that as soon as $\mathtt{Q}$ is defined as (9), $\mathtt{Q}$ is nonreversing. Let $p \geq p'$, $q \leq q'$, $q \in \mathtt{Q}(p)$ and $q' \in \mathtt{Q}(p')$. Because $\mathtt{Q}$ is the argmax correspondence for the objective $p^\top q - c(q)$, we have $p^\top q - c(q) \geq p^\top q' - c(q')$ and $p'^\top q' - c(q') \geq p'^\top q - c(q)$. Rearranging gives

$$p'^\top (q' - q) \geq c(q') - c(q) \geq p(q' - q).$$

Using $p \geq p'$ and $q \leq q'$, this can only hold if the two weak inequalities are in fact equalities, which suffices for $q \in \mathtt{Q}(p')$ and $q' \in \mathtt{Q}(p)$, as needed for nonreversingness.

We next present a proof of Theorem 3 that establishes the equivalence between unified gross substitutes of ∂c^* and the submodularity of c. In order to do this, we prove a series of lemmas regarding convex functions and how to characterize their submodularity. Let f be a convex function. Our interpretation will be that f is a profit function associated with a cost function $c(q)$, but we will not use that interpretation in the lemmas.

The first result is a well-known result in convex analysis (Theorem 23.4 in Rockafellar [34]), which essentially asserts that the support function of the subdifferential of a convex function coincides with the directional derivatives.

Lemma 7. *Let $f : \mathbb{R}^N \to \mathbb{R}$ be a convex function. We have*

$$\partial f\left(p\right) = \left\{ q \in \mathbb{R}^N : q^\top b \leq \frac{d}{dt} f\left(p + tb\right)|_{0^+}, \forall b \in \mathbb{R}^N \right\}.$$

Next, for $X \subseteq \mathbb{R}^N$, define $\tilde{X}$ as the set of vectors of $\mathbb{R}^N$ that are dominated by some vector in X, or more formally:

$$\tilde{X} = \left\{ \tilde{q} \in \mathbb{R}^N : \exists q \in X \ s.t.\ \tilde{q} \leq q \right\}, \tag{17}$$

and define the *support function* of X as

$$h_X\left(b\right) = \sup_{q \in X} \left\{ q^\top b \right\}. \tag{18}$$

Let $cch\left(X\right)$ be the convex closure of X, which is the closure of the convex hull of X. It is well-known (Rockafellar [34, Theorem 13.1]) that $cch\left(X\right)$ is the set of elements $x \in \mathbb{R}^N$ such that

$$x^\top b \leq h_X(b) \quad \forall b \in \mathbb{R}^N. \tag{19}$$

The next result states that the support function of $\tilde{C}$ is the support function of C whose domain has been restricted to nonnegative coordinates.

Lemma 8. *If C is a closed convex set of $\mathbb{R}^N$, then $\tilde{C}$ as defined in expression* (17) *can be expressed as*

$$\tilde{C} = \left\{ q : q^\top b \leq h_C(b), \forall b \in \mathbb{R}^N_+ \right\}$$

where h_C has been defined in (18).

Proof of Lemma 8. By the supporting hyperplane theorem, for any convex set C and any boundary point $q_0 = \arg\max_{q \in C} q^\top b$ of the boundary of C, there exists a supporting hyperplane for C at q_0. Therefore one has

$$C = \left\{ q : q^\top b \leq h_C(b), \forall b \in \mathbb{R}^N \right\}.$$

Further, note that $\hat{C} = \left\{ \hat{q} : \hat{q}^\top b \leq \hat{h}_C(b), \forall b \in \mathbb{R}^N \right\}$, where $\hat{h}_C(b) = h(b)$ if $b \in \mathbb{R}^N_+$ and $\hat{h}_C(b) = +\infty$ otherwise, and thus

$$\hat{h}_C(b) = \max_{\hat{q} \in \hat{C}} \hat{q}^\top b.$$

Now compute $\tilde{h}(b) = \max_{\tilde{q} \in \tilde{C}} \tilde{q}^\top b$. One has $\tilde{C} = \left\{ q - \delta : q \in C, \delta \in \mathbb{R}^N_+ \right\}$, and so

$$\tilde{h}(b) = \max_{q \in C} \max_{\delta \geq 0} (q - \delta)^\top b.$$

Thus, if $b \in \mathbb{R}^N_+$, one has $\tilde{h}(b) = \max_{q \in C} q^\top b = h(b)$. Now if $b_z < 0$ for some z, one has clearly $\tilde{h}(b) = +\infty$. Hence

$$\tilde{h}(b) = \hat{h}(b) = h(b) + \iota_{\mathbb{R}^N_+}(b),$$

where $\iota_K(b) = 0$ if $b \in K$ and $\iota_K(b) = +\infty$ otherwise. This implies that $\hat{C}$ and $\tilde{C}$ have the same support function, and thus coincide. □

From Lemma 8, it follows that:

Lemma 9. *The inequality $b^\top x \leq h_X(b)$ holds for all $b \in \mathbb{R}^N_+$ if and only if there is $\tilde{x} \in cch(X)$ with $x \leq \tilde{x}$.*

Proof of Lemma 9. First, we assume $b^\top x \leq h_X(b)$ for all $b \geq 0$, and show that $x \in Y$ where $Y = \{x' : \exists \tilde{x} \in cch(X) : x' \leq \tilde{x}\}$. One has $X \subseteq Y$. Consider $h_Y(b)$ for $b \in R^Z$. First, if $b_z < 0$ for some z, then $h_Y(b) = +\infty$. Next, if $b \geq 0$, then $h_Y(b) = h_X(b)$. Indeed, one has $h_X(b) \leq \iota^*_Y(b)$, but taking $y \in Y$ such that $b^\top y$ attains $h_Y(b)$, we have $\iota^*_Y(b) = b^\top y$ and by definition of $y \in Y$, there is $\tilde{x} \in cch(X)$ such that $y \leq \tilde{x}$. Hence as $b \geq 0$, $b^\top y \leq b^\top \tilde{x}$, and as $\tilde{x} \in cch(X)$, we get $b^\top \tilde{x} \leq h_X(b)$, thus $h_Y(b) \leq h_X(b)$. As a result $h_Y(b) = h_X(b)$ as soon as $b \geq 0$, and we have that

$$b^\top x \leq h_Y(b) \text{ for all } b \in R^Z$$

and therefore, given that Y is a closed convex set, this implies that $x \in Y$.

Conversely, assume there is $\tilde{x} \in cch(X)$ with $x \leq \tilde{x}$. Then $\tilde{x} = \int_0^1 x_t d\mu(t)$ where μ is a probability measure on $[0,1]$ and $x_t \in X$. Then we have $x_t^\top b \leq h_X(b)$ and thus $x^\top b \leq \tilde{x}^\top b = \int_0^1 x_t^\top b d\mu(t) \leq \int_0^1 h_X(b)\, d\mu(t) = h_X(b)$. □

By combining Lemma 7 and Lemma 8, we get:

Lemma 10. *For a convex function $f : \mathbb{R}^N \to \mathbb{R}$, one has*

$$\left\{\tilde{q} \in \mathbb{R}^N : \exists q \in \partial f(p) \text{ s.t. } \tilde{q} \leq q\right\} = \left\{q \in \mathbb{R}^N : q^\top b \leq \frac{d}{dt} f(p + tb)\,|_{0^+}, \forall b \in \mathbb{R}_+^N\right\}, \tag{20}$$

and both these values coincide with $\widetilde{\partial f(p)}$.

Proof of Lemma 10. Let the function $\hat{h}$ in the specification of $\hat{C}$ be given by $h(b) = \frac{d}{dt} f(p + tb)\,|_{0^+}$. From Lemma 7 and (19), we then have $h(b) = \max_{q \in \partial f(p)} q^\top b$ and hence can take $C = \partial f(p)$ in the specification of $\tilde{C}$. The equality of $\hat{C}$ and $\tilde{C}$ established in Lemma 8 then gives the required identity (20). □

In the sequel, we shall consider a pair of prices p and p' in $\mathbb{R}^N$, and for a vector $b \in \mathbb{R}^N$, we define two vectors $b^>$ and $b^\leq$ in $\mathbb{R}^N$ such that

$$b_z^\leq = b_z \mathbf{1}_{\{p_z \leq p'_z\}} \text{ and } b_z^> = b_z \mathbf{1}_{\{p_z > p'_z\}}. \tag{21}$$

Lemma 11. *A function $f : \mathbb{R}^N \to \mathbb{R}$ is submodular if and only if for any p, p' in $\mathbb{R}^N$, $b \in R_+^N$ such that $b^> \leq (p - p')^+$, one has*

$$f\left(p + b^\leq\right) + f\left(p' + b^>\right) + f\left(p \wedge p'\right) \leq f(p) + f(p') + f\left(p \wedge p' + b\right).$$

Proof of Lemma 11. Suppose f is submodular. Then we have, by the submodularity of f,

$$f(p) + f(p \wedge p' + b) \geq f(p \wedge (p \wedge p' + b)) + f(p \vee (p \wedge p' + b)).$$

However, $p \wedge (p \wedge p' + b) = p \wedge p' + b^>$ and $p \vee (p \wedge p' + b) = p + b^\leq$, and so

$$f(p) + f(p \wedge p' + b) \geq f(p \wedge p' + b^>) + f\left(p + b^\leq\right).$$

This implies

$$f(p) + f(p') + f(p \wedge p' + b) \geq f(p \wedge p' + b^{>}) + f(p') + f\left(p + b^{\leq}\right).$$

Again by the submodularity of f, we have

$$f(p \wedge p' + b^{>}) + f(p') \geq f((p \wedge p' + b^{>}) \wedge p') + f((p \wedge p' + b^{>}) \vee p'),$$

and hence

$$\begin{aligned} & f(p) + f(p') + f(p \wedge p' + b) \\ \geq \; & f((p \wedge p' + b^{>}) \wedge p') + f((p \wedge p' + b^{>}) \vee p') + f\left(p + b^{\leq}\right). \end{aligned} \tag{22}$$

But $(p \wedge p' + b^{>}) \wedge p' = p \wedge p'$ and $(p \wedge p' + b^{>}) \vee p' = p' + b^{>}$, and therefore (22) becomes

$$f(p) + f(p') + f(p \wedge p' + b) \geq f(p \wedge p') + f(p' + b^{>}) + f\left(p + b^{\leq}\right),$$

giving the required result.

Conversely, assume we have for all p, p' and $b \geq 0$ that

$$f(p) + f(p') + f(p \wedge p' + b) \geq f\left(p + b^{\leq}\right) + f(p' + b^{>}) + f(p \wedge p').$$

Choose b to be specified by $b^{\leq} = (p' - p)^{+}$ and $b^{>} = 0$. We have

$$\begin{aligned} p \wedge p' + b &= p' \\ p + b^{\leq} &= p \vee p' \end{aligned}$$

and thus

$$f(p) + f(p') \geq f(p \vee p') + f(p \wedge p'),$$

giving the submodularity of f. □

Lemma 12. *A convex function $f : \mathbb{R}^N \to \mathbb{R}$ is submodular if and only if for any $b \in \mathbb{R}^N_+$*

$$\frac{d}{dt} f\left(p + tb^{\leq}\right)|_{0^+} + \frac{d}{dt} f(p' + tb^{>})|_{0^+} \leq \frac{d}{dt} f(p \wedge p' + tb)|_{0^+}, \tag{23}$$

where $b^{\leq}$ and $b^{>}$ are defined in equation (21).

Proof of Lemma 12. Applying Lemma 11, if f is submodular it follows that for $t \geq 0$, we have

$$f\left(p + tb^{\leq}\right) - f(p) + f\left(p' + tb^{>}\right) - f(p') \leq f\left(p \wedge p' + tb\right) - f\left(p \wedge p'\right),$$

and thus, using the convexity of f, it follows that

$$\frac{d}{dt} f\left(p + tb^{\leq}\right)|_{0^+} + \frac{d}{dt} f\left(p' + tb^{>}\right)|_{0^+} \leq \frac{d}{dt} f\left(p \wedge p' + tb\right)|_{0^+}.$$

The converse holds by integration over $t \in [0, 1]$. □

Proof of Theorem 3, direct implication. We show that if f is submodular then $\partial f(p)$ exhibits unified gross substitutes.

Assume f is submodular. Take $q \in \partial f(p)$ and $q' \in \partial f(p')$. We want to show that there exists $q^{\wedge} \in \partial f(p \wedge p')$ such that

$$\begin{aligned} p_z \leq p'_z \quad &\Longrightarrow \quad q_z \leq q^{\wedge}_z \\ p_z > p'_z \quad &\Longrightarrow \quad q'_z \leq q^{\wedge}_z. \end{aligned}$$

To show this, we need to show that $q\mathbf{1}_{\mathcal{Z}^{\leq}} + q'\mathbf{1}_{\mathcal{Z}^{>}} \leq q^{\wedge}$, where we have defined $\mathcal{Z}^{\leq} = \{z \in \mathcal{Z} : p_z \leq p'_z\}$ and $\mathcal{Z}^{>} = \{z \in \mathcal{Z} : p_z > p'_z\}$. Equivalently, we need to show that $q\mathbf{1}_{\mathcal{Z}^{\leq}} + q'\mathbf{1}_{\mathcal{Z}^{>}} \in \widetilde{\partial f}(p \wedge p')$, where the tilde notation $\widetilde{\partial f}$ was introduced in (17). By Lemma 10, it suffices to show that

$$\forall b \in \mathbb{R}^N_+ : \left(q\mathbf{1}_{\mathcal{Z}^{\leq}} + q'\mathbf{1}_{\mathcal{Z}^{>}}\right)^{\top} b \leq \frac{d}{dt} f\left(p \wedge p' + tb\right)|_{0^+}.$$

In order to do this, take $b \in \mathbb{R}^N_+$ and express that $q \in \partial f(p)$ and $q' \in \partial f(p')$ by writing

$$q^{\top} b^{\leq} \leq \frac{d}{dt} f\left(p + tb^{\leq}\right)|_{0^+} \text{ and } q'^{\top} b^{>} \leq \frac{d}{dt} f\left(p' + tb^{>}\right)|_{0^+},$$

and then note that, by summation of these two inequalities we get

$$\left(q\mathbf{1}_{\mathcal{Z}^{\leq}} + q'\mathbf{1}_{\mathcal{Z}^{>}}\right)^{\top} b \leq \frac{d}{dt} f\left(p + tb^{\leq}\right)|_{0^+} + \frac{d}{dt} f\left(p' + tb^{>}\right)|_{0^+},$$

and so by Lemma 12, we have

$$\forall b \in \mathbb{R}^N_+ : \left(q\mathbf{1}_{\mathcal{Z}^{\leq}} + q'\mathbf{1}_{\mathcal{Z}^{>}}\right)^{\top} b \leq \frac{d}{dt} f\left(p \wedge p' + tb\right)|_{0^+},$$

and hence (again, by Lemma 10) $q\mathbf{1}_{\mathcal{Z}^{\leq}} + q'\mathbf{1}_{\mathcal{Z}^{>}} \in \widetilde{\partial f}(p \wedge p')$ as required.

We set $\hat{f} = f(-p)$, and we introduce $\hat{p} = p'$ and $\hat{p}' = p$. We note that $\hat{f}$ is submodular, and we apply the previous claim to $\hat{p}$ and $\hat{p'}$ to get the existence of $q^{\vee} \in \partial f(p \vee p')$ such that

$$\begin{aligned} p_z \leq p'_z \quad &\Longrightarrow \quad q_z \geq q^{\vee}_z \\ p_z > p'_z \quad &\Longrightarrow \quad q'_z \geq q^{\vee}_z. \end{aligned}$$

Proof of Theorem 3, backward implication We show that if $\partial f\,(p)$ satisfies unified gross substitutes, then $f\,(p)$ is submodular.

Assume unified gross substitutes holds. Then for $q \in \partial f\,(p)$ and $q' \in \partial f\,(p')$, there exists $q^{\wedge} \in \partial f\,(p \wedge p')$ such that

$$q \mathbf{1}_{\mathcal{Z}^{\leq}} + q' \mathbf{1}_{\mathcal{Z}^{>}} \leq q^{\wedge}.$$

Hence $q \mathbf{1}_{\mathcal{Z}^{\leq}} + q' \mathbf{1}_{\mathcal{Z}^{>}} \in \widetilde{\partial f}\,(p \wedge p')$, and therefore (using Lemma 8), for all $q \in \partial f\,(p)$ and $q' \in \partial f\,(p')$

$$\forall b \in \mathbb{R}^{\mathcal{Z}}_{+} : q^{\top} b^{\leq} + q'^{\top} b^{<} \leq \frac{d}{dt} f\,(p \wedge p' + tb)\,|_{0^{+}},$$

and thus, by maximizing over $q \in \partial f\,(p)$ and $q' \in \partial f\,(p')$, we get

$$\frac{d}{dt} f\left(p + tb^{\leq}\right)|_{0^{+}} + \frac{d}{dt} f\,(p' + tb^{>})\,|_{0^{+}} \leq \frac{d}{dt} c^{*}\,(p \wedge p' + tb)\,|_{0^{+}}$$

for all $b \geq 0$, hence (by Lemma 12) f is submodular.

A.3 Proof of Theorem 4

Let $q \in \mathtt{Q}\,(p)$ and $q' \in \mathtt{Q}\,(p')$. Rewriting (1)–(2), it suffices for unified gross substitutes to show that there exists $q^{\vee} \in \mathtt{Q}\,(p \vee p')$ and $q^{\wedge} \in \mathtt{Q}\,(p \wedge p')$ such that for all $z \in \mathcal{Z}$,

$$\begin{aligned} \mathbf{1}_{\left\{z \in \mathcal{Z}^{\leq}\right\}} q_z + \mathbf{1}_{\{z \in \mathcal{Z}^{>}\}} q'_z \quad &\leq \quad q^{\wedge}_z \\ \mathbf{1}_{\left\{z \in \mathcal{Z}^{\leq}\right\}} q'_z + \mathbf{1}_{\{z \in \mathcal{Z}^{>}\}} q_z \quad &\geq \quad q^{\vee}_z, \end{aligned}$$

where we recall that $\mathcal{Z}^{\leq} = \{z \in \mathcal{Z} : p_z \leq p'_z\}$ and $\mathcal{Z}^{>} = \{z \in \mathcal{Z} : p_z > p'_z\}$.

First, note that $\mu_{xz} > 0$ and $p_z \leq p'_z$ implies $p'_x \geq G_{xz}(p'_z) \geq G_{xz}(p_z) = p_x$, which implies

$$\mu_{xz} \mathbf{1}_{\left\{z \in \mathcal{Z}^{\leq}\right\}} \leq \mu_{xz} \mathbf{1}_{\left\{x \in \mathcal{Z}^{\leq}\right\}}. \tag{24}$$

Similarly, $\mu'_{xz} > 0$ and $p_z > p'_z$ implies $p_x \geq G_{xz}(p_z) > G_{xz}(p'_z) = p'_x$, thus

$$\mu'_{xz}\mathbf{1}_{\{z\in\mathcal{Z}^>\}} \leq \mu'_{xz}\mathbf{1}_{\{x\in\mathcal{Z}^>\}}. \tag{25}$$

Given these results, set:

$$\begin{aligned} \mu^{\wedge}_{xz} &= \mathbf{1}_{\{x\in\mathcal{Z}^{\leq}\}}\mu_{xz} + \mathbf{1}_{\{x\in\mathcal{Z}^>\}}\mu'_{xz} \\ q^{\wedge}_z &= \sum_x \mu^{\wedge}_{xz} - \sum_y \mu^{\wedge}_{zy}. \end{aligned}$$

We have $\mu^{\wedge}_{xy} > 0$ implies $p_x = G_{xy}(p_y)$, and, using (24) and (25) for the inequality, we obtain

$$\begin{aligned} q^{\wedge}_z &= \sum_x (\mathbf{1}_{\{x\in\mathcal{Z}^{\leq}\}}\mu_{xz} + \mathbf{1}_{\{x\in\mathcal{Z}^>\}}\mu'_{xz}) - \sum_y (\mathbf{1}_{\{z\in\mathcal{Z}^{\leq}\}}\mu_{zy} + \mathbf{1}_{\{x\in\mathcal{Z}^>\}}\mu'_{zy}) \\ &\geq \sum_x (\mathbf{1}_{\{z\in\mathcal{Z}^{\leq}\}}\mu_{xz} + \mathbf{1}_{\{z\in\mathcal{Z}^>\}}\mu'_{xz}) - \sum_y (\mathbf{1}_{\{z\in\mathcal{Z}^{\leq}\}}\mu_{zy} + \mathbf{1}_{\{z\in\mathcal{Z}^>\}}\mu'_{zy}) \\ &= \mathbf{1}_{\{z\in\mathcal{Z}^{\leq}\}}q_z + \mathbf{1}_{\{z\in\mathcal{Z}^>\}}q'_z, \end{aligned}$$

giving the required result for $q^{\wedge}$. A similar argument shows that $\mathbf{1}_{\{z\in\mathcal{Z}^{\leq}\}}q'_z + \mathbf{1}_{\{z\in\mathcal{Z}^>\}}q'_z \geq q^{\vee}_z$.

A.4 Proof of Lemma 6

First, let (q, μ, p) be an equilibrium of the equilibrium flow problem. To obtain a stable matching (μ, w), choose w_{xy} so that $\mathcal{V}^{-1}_{xy}(-p_y) \leq w_{xy} \leq \mathcal{U}^{-1}_{xy}(p_x)$. Because we have $p_x \geq G_{xy}(p_y)$ in equilibrium, this is possible. Then for worker x we have $p_x = u_x \leq \mathcal{U}_{xy}(w_{xy})$ and

$$\begin{aligned} \mu_{xy} > 0 &\implies p_x = G_{xy}(p_y) \\ &\implies w_{xy} = \mathcal{U}^{-1}_{xy}(p_x) \\ &\implies y \in \arg\max_{y\in\mathcal{Y}} \mathcal{U}_{xy}(w_{xy}) \\ \mu_{x0} > 0 &\implies p_x = G_{x0}(p_0) = \mathcal{U}_{x0} \\ \mu_{0y} > 0 &\implies p_0 = G_{0y}(p_y) \\ &\implies -p_y = \mathcal{V}_{0y}. \end{aligned}$$

The argument for firms is similar, giving a stable matching.

Conversely, suppose we have a stable matching (μ, w). Define $q_z = -n_x$ for $z \in \mathcal{X}$ and $q_z = m_y$ for $z \in \mathcal{Y}$. We identify prices p such that (q, μ, p) is an equilibrium of the equilibrium flow problem. Define the indirect utilities

$$\begin{aligned} u_x &= \max_{x \in \mathcal{X}} \left\{ \mathcal{U}_{xy}\left(w_{xy}\right), \mathcal{U}_{x0} \right\} \\ v_y &= \max_{y \in \mathcal{Y}} \left\{ \mathcal{V}_{xy}\left(w_{xy}\right), \mathcal{V}_{0y} \right\}. \end{aligned}$$

Then define p by $p_x = u_x$ if $x \in \mathcal{X}$, $p_y = -v_y$ if $y \in \mathcal{Y}$ and $p_0 = 0$, and define $q_z = -n_x$ for $z \in \mathcal{X}$ and $q_z = m_y$ for $z \in \mathcal{Y}$. Define the connection functions G_{xy} as above. If $\mu_{xy} = 0$, then it follows from the stability condition for a stable matching that $p_x = G_{xy}(p_y)$. For other pairs (x, y), the stability condition implies that there is no wage w_{xy} at which x and y can match and obtain utilities in excess of u_x and v_y, which is equivalent to the statement that $p_x \geq G_{xy}(p_y)$. We thus have an equilibrium flow.

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