Stochastic Impatience and the Separation of Time and Risk Preferences*

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Abstract

We study how the separation of time and risk preferences relates to a property called Stochastic Impatience. We show that, within a broad class of models, Stochastic Impatience holds if and only if risk aversion and the inverse elasticity of intertemporal substitution are sufficiently close. In the models of Epstein and Zin (1989) and Hansen and Sargent (1995), Stochastic Impatience is violated for *all* commonly used parameters. Our result also provides a simple, one-question test for the separation of time and risk preferences.

Key words: Stochastic Impatience, Epstein-Zin preferences, Separation of Time and Risk preferences, Risk Sensitive preferences, Non-Expected Utility

JEL: D81, D90, G11, E7.

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1 Introduction

In standard Expected Discounted Utility, the inverse of the elasticity of intertemporal substitution (EIS) coincides with the coefficient of relative risk aversion. However, an enormous literature in macroeconomics, finance, and behavioral economics has pointed to the need to separate them on empirical and conceptual grounds. Empirically, observations from lab experiments, longitudinal microdata, and the desire to fit macroeconomic and financial data require a higher coefficient of risk aversion than the inverse of EIS.[1] Conceptually, attitudes toward risk and intertemporal smoothing belong to different domains, and there is no compelling reason why they must be equal. These observations led to models that separate risk attitudes from EIS, such as the CRRA-CES version of Epstein and Zin (1989) and the Risk Sensitive preferences of Hansen and Sargent (1995). But does allowing for this separation have other implications? Can we develop simple, one-question tests of such separation?

This paper shows that a large enough difference between risk aversion and the inverse of EIS violates a behavioral postulate we call *Stochastic Impatience*. To illustrate, consider the choice between:

A. With equal probability, permanently increase consumption by either 20% starting today, or by 10% starting next year; and

B. With equal probability, permanently increase consumption by either 10% starting today, or by 20% starting next year.

Both options involve identical benefits, odds, and dates. However, option A pairs the highest increase (20%) with the earlier date, whereas B pairs it with the later date. If an individual prefers to associate the higher payment with the earlier date, she may choose option A.

One way to see this is by decomposing each alternative into two parts. Both A and B offer a basic lottery with an increase of 10% either today or next year. The difference between them is when the additional increase of 10% is made. In Option A, the prize is increased if the realization is today, whereas in Option B it is increased if the realization is next year. An individual who prefers to associate the higher payment with the sooner realization would prefer option A. This property is a

[1]For example, Barsky et al. (1997) study a cross-section of American households and find that risk aversion and EIS are uncorrelated. See Bansal and Yaron (2004); Hansen et al. (2007); Barro (2009); Andreoni and Sprenger (2012); Nakamura et al. (2017) and references therein.

version of impatience (preference for earlier payments) for risky environments, hence the name *Stochastic Impatience*. Indeed, it coincides with impatience in the standard model: when preferences are represented by the expectation of discounted utility, $\mathbb{E}\big[\sum_{t\in\mathbb{N}} D(t)u(x_t)\big]$, Stochastic Impatience holds if (and only if) D is decreasing. However, the two notions may differ when time and risk preferences are separated. Our main result shows that, under very general conditions, Stochastic Impatience is violated if risk aversion and the inverse of EIS are sufficiently different.

We first consider the widely used CRRA-CES version of Epstein and Zin (1989) (henceforth EZ). We show that Stochastic Impatience fails if the coefficient of risk aversion exceeds both the inverse of EIS and one; or if risk aversion is below the inverse of EIS and that is also below one. *All* applications of EZ we are aware of use parameters in this range—indeed, with typical parameters, individuals would prefer option B in the example above, violating Stochastic Impatience. For the Risk Sensitive preferences of Hansen and Sargent (1995) (henceforth HS), we show that Stochastic Impatience fails if the range of utilities of consumption is large enough. For example, with the parameters of Tallarini Jr (2000), option B is again chosen.

We then establish a more general result. Consider any preference relation over lotteries over consumption streams.[2] Assume that (*i*) without risk, preferences admit a Discounted Utility representation $\sum D(t)u(x_t)$ for a D decreasing over time and u increasing over consumption; and (*ii*) with risk, preferences satisfy the Expected Utility postulates. These assumptions hold for most models, including EZ and HS. In the space we consider, any preference relation that satisfies these two assumptions can be represented by the expectation of $\phi\big(\frac{\sum_t D(t)u(x_t)}{\sum_t D(t)}\big)$, where ϕ is some increasing function over discounted utils. Known as the Kihlstrom-Mirman (KM) representation, this model is similar to Expected Discounted Utility except for the additional curvature ϕ. Since ϕ affects risk aversion but not EIS, it captures the separation between the two. The KM representation thus gives a convenient way to discuss the gap between time and risk attitudes.[3]

Our general result shows that Stochastic Impatience imposes a bound on the

[2]Instead of considering dynamic preferences over temporal lotteries, as in EZ, it suffices to consider their static implications—how they evaluate lotteries over consumption streams at a given point in time and for a given date of resolution of uncertainty. This contains all relevant information for our analysis.

[3]The preferences implied in our setup by EZ and HS admit a KM representation with convenient functional forms (ϕ being CRRA or CARA, respectively). Of course, this does *not* mean that EZ and HS are special cases of KM—see below for a discussion.

curvature of ϕ, that is, on the separation of risk aversion and inverse EIS: ϕ cannot be either too convex or too concave.[4] We also provide our results in terms of a behavioral notion that captures the separation of risk and intertemporal preferences, which we call *Residual Risk Aversion*.

Our results have two implications. First, if Stochastic Impatience is taken as appealing, our results point to an issue with all common parametrizations of all leading approaches. In Section 5 we discuss possible ways to maintain Stochastic Impatience without sacrificing the fit of empirical data.

Second, independently of the normative appeal of Stochastic Impatience, our results provide a convenient, one-question empirical test. To verify that risk aversion is sufficiently different from the inverse of EIS, it is sufficient to document a violation of Stochastic Impatience. This is a more direct test than typical ones that involve estimating the two parameters separately using multiple questions and assuming specific functional forms.

We are not the first to point out implications of how the separation between time and risk attitudes is modeled. Epstein et al. (2014) argue that common parameterizations of EZ imply unrealistic preferences for early resolution of uncertainty. We show that they also violate Stochastic Impatience, a property distinct from preference over the timing of resolution of uncertainty. Bommier et al. (2017) show that many models that separate time and risk preferences, including common specifications of EZ, violate a monotonicity property. The latter is unrelated to Stochastic Impatience: for example, EZ with both risk aversion and the inverse of EIS less than 1 satisfies Stochastic Impatience but not Monotonicity; conversely, HS always satisfies Monotonicity but violates Stochastic Impatience when the utility range is large enough. Lastly, a companion paper, DeJarnette et al. (2020), studies theoretically and experimentally risk attitudes towards time lotteries, including their relationship with Stochastic Impatience.[5] That paper does not discuss the implications of this property on the separation of time and risk preferences, which is instead the main goal of the present paper.

[4] In Appendix A, we provide an extension to continuous time, showing that equivalent results hold. In Appendix B we extend to non-Expected Utility, where we show that Stochastic Impatience is violated with either First-Order Risk Aversion or Seeking.

[5] DeJarnette et al. (2020) show experimentally that the majority of individuals are not risk seeking over time lotteries, as implied by EDU with convex discounting (e.g., exponential or quasi-exponential discounting); but also establish that this is incompatible with Stochastic Impatience within a broad class of models. They further suggest generalizations to accommodate both.

2 Framework and Stochastic Impatience

We study preferences on lotteries over consumption streams. With per-period consumption in the interval $C \subseteq \mathbb{R}_+$, a consumption program $\mathbf{x} = (x_t)_{t\in\mathbb{N}}$ is a sequence in C corresponding to the consumption in each period.[6] Let $\mathcal{X} := C^{\mathbb{N}}$ be the set of all consumption programs and Δ be the set of all probability measures over $\mathcal{X}$ with finite support. Let $\succcurlyeq$ be a complete and transitive preference relation over Δ.

We consider the static space of lotteries over streams, and not the more complex space of temporal lotteries used in Kreps and Porteus (1978) or EZ, because this subdomain is much simpler yet sufficient for our purposes. Any model over temporal lotteries induces preferences over Δ and, crucially, these static preferences contain all the information on the separation of time and risk preferences relevant to our analysis.[7] Moreover, restricting attention to this subdomain allows us to derive results for a richer class of models, independently of how they are defined dynamically.

Abusing notation, let $\mathbf{x} \in \mathcal{X}$ denote both the consumption program and the lottery that gives this program with certainty. Let $\langle x, t; c, \delta\rangle$ denote the stream that gives x starting in period t and lasting for δ periods and c otherwise. For example, $\langle x, 3; c, 5\rangle$ is the program

$$(c, c, \underbrace{\overbrace{x}^{3}, \overbrace{x}^{4}, ..., \overbrace{x}^{7}}_{\delta=5}, c, c, ...).$$

We now introduce the central property of this paper:

Definition 1 (Stochastic Impatience)**.** The relation $\succcurlyeq$ satisfies Stochastic Impatience if for any $t, t' \in \mathbb{N}$ with $t < t'$, any $\delta \in \mathbb{N} \cup \{\infty\}$, and any $c, x, x' \in C$ with $x > x'$,

$$\frac{1}{2}\langle x, t; c, \delta\rangle + \frac{1}{2}\langle x', t'; c, \delta\rangle \succcurlyeq \frac{1}{2}\langle x, t'; c, \delta\rangle + \frac{1}{2}\langle x', t; c, \delta\rangle. \tag{1}$$

Stochastic Impatience states that the individual prefers the lottery that associates the higher prize with the earlier date. It can be seen as a stochastic counterpart of standard impatience—associating the higher payment with the sooner date. As mentioned in the introduction, Stochastic Impatience can be interpreted by decomposing

[6]We focus on discrete time and real-valued consumption for simplicity. Appendix A presents the extension to continuous time. We start from date one, but when required, we also consider an additional consumption at time zero that is not subject to uncertainty.

[7]Formally, lotteries over streams are embedded within temporal lotteries once we fix a start date and a time of resolution of uncertainty. For example, if we start from preferences over temporal lotteries, hold the time-zero consumption fixed, and assume that uncertainty is resolved between periods zero and one, we obtain preferences over Δ.

each lottery into two parts. Both options offer the same basic lottery that replaces c with x' for δ periods starting at either t or t', as well as an increment of $x - x'$. The difference is about when the increment is paid: on the left hand side of (1), it is paired with the earlier-dates realization $\{t, t+1, ...t+\delta-1\}$; on the right, with the later-dates realization $\{t', t'+1, ..., t'+\delta-1\}$. Insofar as the individual prefers to associate this increment with the sooner dates, the option on the left is preferred; this is what Stochastic Impatience prescribes. Note that x and x' can be above or below c, meaning that the changes $x - c$ and $x' - c$ can be positive or negative.[8]

Stochastic Impatience is related to multivariate risk aversion (Richard, 1975; Wakker et al., 2004) and, more generally, to supermodularity. What distinguishes it is the specification of one dimension as prize and the other as time. A version of Stochastic Impatience appears in DeJarnette et al. (2020).[9] Existing evidence appears to support Stochastic Impatience. Lanier et al. (2020) find evidence in favor of it in an experiment with assets that pay in different dates and states.

In Expected Discounted Utility, Stochastic Impatience is equivalent to impatience—a preference for earlier rewards:

Observation 1. Suppose $\succcurlyeq$ is represented by $\mathbb{E}\big[\sum_{t\in\mathbb{N}} D(t)u(x_t)\big]$ with u strictly increasing. Then, $\succcurlyeq$ satisfies Stochastic Impatience if and only if D is weakly decreasing.

We henceforth refer to Expected Discounted Utility (as in Observation 1) with any strictly decreasing discount function as EDU.

3 Stochastic Impatience in EZ and HS

3.1 Epstein-Zin preferences

We first consider the most widely used model that separates time and risk preferences: the CRRA-CES version of Epstein-Zin preferences (EZ). Letting $C = \mathbb{R}_{++}$ and fixing

[8]One may want to consider a weaker version of Stochastic Impatience where changes are permanent ($\delta = \infty$). All our results below remain true as stated with this weaker version. This is because the two versions of the axiom are equivalent in many special cases we consider (e.g., under the assumptions of Proposition 1 or 2). They are not equivalent under our Assumptions 1 and 2 only because of the discreteness of time. When we consider continuous time in Appendix A, we show that the two versions of the axioms are equivalent under these assumptions (Proposition 6).

[9]DeJarnette et al. (2020) consider a setup of prize-date pairs instead of streams (hence $\delta = 1$ and $c = 0$) and their version of Stochastic Impatience assumes only positive prizes ($x, x' > 0$).

(any) deterministic consumption $x_0 \in C$ for time 0, consider the following recursive representation:

$$V_t = \left\{ (1-\beta)\, x_t^{1-\frac{1}{\psi}} + \beta \left[\mathbb{E}_t \left(V_{t+1}^{1-\alpha} \right) \right]^{\frac{1-\frac{1}{\psi}}{1-\alpha}} \right\}^{\frac{1}{1-\frac{1}{\psi}}} \tag{2}$$

where $\alpha \in \mathbb{R}_+ \backslash \{1\}$ is the coefficient of relative risk aversion and $\psi \in \mathbb{R}_+ \backslash \{1\}$ is the EIS. When $\alpha = \frac{1}{\psi}$, the model is a special case of EDU.

The following result characterizes Stochastic Impatience in this model.

Proposition 1. *Suppose $\succcurlyeq$ admits the representation in (2). Stochastic Impatience holds if and only if either $1 > \alpha \geq \frac{1}{\psi}$ or $1 < \alpha \leq \frac{1}{\psi}$.*

Proposition 1 shows that Stochastic Impatience constrains how risk aversion and inverse EIS can differ. If $\alpha > 1$, as commonly assumed, Stochastic Impatience implies that inverse IES must be at least as high as risk aversion—the opposite of what is typically assumed. Under Stochastic Impatience, Risk aversion can be above the inverse EIS only when both are below 1. As discussed below, this implies that Stochastic Impatience is violated in all common parametrizations of the model.

For an intuition, consider again the choice between lotteries A and B given in the introduction:

A. With equal probability, permanently increase consumption by either 20% starting today, or by 10% starting next year; or

B. With equal probability, permanently increase consumption by either 10% starting today, or by 20% starting next year.

Stochastic Impatience prescribes that A should be preferred. Of the four possible outcomes, the best is 20% starting today, the worst is 10% next year, while the other two are intermediate. Option A involves the best and the worst outcomes; option B the two 'intermediate' ones. Thus, option A has more spread but also a higher expected discounted utility—since the higher discounting is applied to the smaller amount. When $\alpha = \frac{1}{\psi}$, i.e., under EDU, the agent cares only about the expected discounted utility, thus strictly prefers option A. But when risk aversion is increased fixing EIS, the individual dislikes spread in discounted utilities. When risk aversion is high enough, this second effect prevails, leading the individual to prefer option B and violate Stochastic Impatience. This shows how α cannot be much higher than $\frac{1}{\psi}$.

For an intuition of why α cannot be much lower than $\frac{1}{\psi}$, consider the following options, which now decrease rather than increase the baseline consumption:

C. With equal probability, permanently decrease consumption by either 20% starting today, or by 10% starting next year; or

D. With equal probability, permanently decrease consumption by 10% starting today, or by 20% starting next year.

Now Stochastic Impatience prescribes that D should be preferred. In this case, D has a higher expected discounted utility and a lower spread. If α is below $\frac{1}{\psi}$, the agent likes the spread, increasing the appeal of C. If it is sufficiently below $\frac{1}{\psi}$, this effect dominates and Stochastic Impatience is violated.

The relevance of this result should be understood in light of the parameters used in the wide literature that adopts EZ. *All* applications we are aware of assume $\alpha > \max\{\frac{1}{\psi}, 1\}$. Stochastic Impatience is therefore violated. Indeed, the possibility of incorporating risk aversion greater than the inverse of EIS is a primary reason for adopting this model, and relative risk aversion above one is also typically imposed to fit macroeconomics and finance data. For example, Bansal et al. (2016) estimate $\alpha = 9.67$ and $\psi = 2.18$ (see Example 1 below for other references). Another strand of the literature (typically not adopting EZ) has instead argued for $\psi < 1$, thus $\frac{1}{\psi} > 1$.[10] With this restriction, EZ necessarily violates Stochastic Impatience if $\alpha > \frac{1}{\psi}$.

Proposition 1 shows when there exist violations of Stochastic Impatience. We now provide an example of such a violation.

Example 1. Consider again lotteries A and B described above. Stochastic Impatience implies A preferred to B. However, B is preferred adopting the EZ model with the parameters of many known papers: Bansal and Yaron (2004) ($\alpha = 10$, $\beta = 0.998$, $\psi = 1.5$), Bansal et al. (2016) ($\alpha = 9.67$, $\beta = .999$, $\psi = 2.18$), Nakamura et al. (2017) ($\alpha = 9$, $\beta = 0.99$, $\psi = 1.5$), and Colacito et al. (2018) ($\alpha = 10$, $\beta = 0.97$, $\psi = 1.1$).[11]

[10]See Campbell (1999), Attanasio and Weber (2010), Campbell (2003) and, more recently, Gruber (2013); Ortu et al. (2013); Crump et al. (2015); Best et al. (2017).

[11]Even with $\psi < 1$, B is preferred to A if α is high enough. With $\alpha = 10$ and $\beta = 0.998$, B is preferred if $\psi > 0.2576$. With lower risk aversion, violations of Stochastic Impatience require higher prizes. For example, with the parameters of Nakamura et al. (2013) ($\alpha = 6.4$, $\beta = 0.967$, $\psi = 2$), a violation is observed with low prize of 20% and high prize of 30% of per-period consumption. With the parameters of Barro (2009) ($\alpha = 4$, $\beta = 0.948$, and $\psi = 2$), with low prize of 35% and high prize of 40%. With even less risk aversion, closer to one, violations require higher and higher prizes.

Finally, in EZ α and ψ also determine the preference over the timing of resolution of uncertainty: The individual prefers early (late) resolution of uncertainty whenever α is higher (smaller) than $\frac{1}{\psi}$ (Epstein and Zin, 1989). By imposing a bound on α given ψ, Stochastic Impatience limits the strength of the preference for early resolution of uncertainty, even though these are conceptually independent notions. This links our results to Epstein et al. (2014), who argue that the parameters used in much of the literature generate an implausibly high preference for early resolution of uncertainty. Here we show that these same parameters imply a violation of Stochastic Impatience.

3.2 Risk Sensitive preferences

In the same setting of the previous section, consider the Risk Sensitive preferences of Hansen and Sargent (HS), which admit the recursive representation:

$$V_t = u(x_t) - \beta \cdot \frac{1}{k} \cdot \log\left(\mathbb{E}_t\left[e^{-kV_{t+1}}\right]\right), \tag{3}$$

where $k > 0$ increases risk aversion relative to standard expected utility.

Proposition 2. *Suppose $\succcurlyeq$ admits the representation in (3). Stochastic Impatience holds if and only if* $\sup\{u(x)\}_{x\in C} - \inf\{u(x)\}_{x\in C} \leq -\frac{\log(\beta)}{k\beta}$.

Under HS, Stochastic Impatience is violated if the utility range of prizes is large enough. This is necessarily the case if u is unbounded above or below on an unbounded domain (such as with a CARA utility and an unbounded consumption space). Otherwise, Stochastic Impatience requires both the utility range ($\sup\{u(x)\} - \inf\{u(x)\}$) and risk sensitivity k to be small enough.

We now illustrate examples of violations of Stochastic Impatience using an influential parameterization:

Example 2. Tallarini Jr (2000) uses HS preferences with $C = \mathbb{R}_{++}$, $u(x) = \log(x)$, and $k = (1-\beta)(\xi-1)$. Since u is unbounded, Stochastic Impatience fails. Tallarini Jr (2000) shows that the model can match key moments in asset pricing for some $(\xi, \beta) \in [46, 180] \times [.991, .999]$. Consider again options A and B used in Example 1. With any of the parameters above, option B is preferred, violating Stochastic Impatience.

4 General results

We now generalize the previous results beyond EZ and HS. We first present a convenient functional form to analyze the separation of time and risk preferences in a general setup and show the bounds imposed by Stochastic Impatience. Next, we introduce a behavioral counterpart of this separation, which we call Residual Risk Aversion, and show the bounds imposed by Stochastic Impatience in this context.

4.1 The KM model

For simplicity, in the remainder we assume that the interval of per-period consumption is compact: $C = [\underline{x}, \bar{x}]$.[12] We focus on preferences that satisfy the following two assumptions.

Assumption 1 (Discounted Utility without risk)**.** *There exist a strictly increasing and continuous function $u : C \to \mathbb{R}_+$ and a strictly decreasing function $D : \mathbb{N} \to [0,1]$ with $\sum_{t\in\mathbb{N}} D(t) < +\infty$ such that for all $\mathbf{x}, \mathbf{y} \in \mathcal{X}$*

$$\mathbf{x} \succcurlyeq \mathbf{y} \quad \Leftrightarrow \quad \sum_{t\in\mathbb{N}} D(t)u(x_t) \geq \sum_{t\in\mathbb{N}} D(t)u(y_t).$$

Assumption 2 (Expected Utility)**.** *The following hold:*

(i) For all $p, q, r \in \Delta$ and $\lambda \in (0,1)$, $p \succcurlyeq q \Leftrightarrow \lambda p + (1-\lambda)r \succcurlyeq \lambda q + (1-\lambda)r$;

(ii) For all $p, q, r \in \Delta$ with $p \succ q \succ r$, there exist $\lambda, \gamma \in (0,1)$ such that $\lambda p + (1-\lambda)r \succ q \succ \gamma p + (1-\gamma)r$.

Assumption 1 posits that in the absence of risk, preferences can be modeled using Discounted Utility with a generic discount function D. This allows for many types of discounting (e.g., exponential and quasi-hyperbolic). Assumption 2 posits the postulates of Expected Utility, satisfied by most models in the literature.

Assumptions 1 and 2 yield the following representation:

Observation 2. The relation $\succcurlyeq$ satisfies Assumptions 1 and 2 if and only if there exists a strictly increasing and continuous $u : C \to \mathbb{R}$, a strictly decreasing $D : \mathbb{N} \to [0,1]$ with $\sum_{t\in\mathbb{N}} D(t) < +\infty$, and a strictly increasing $\phi : u(C) \to \mathbb{R}$ such that $\succcurlyeq$ is represented by

$$V(p) = \mathbb{E}_p\Big[\phi\Big(\frac{\sum_{t\in\mathbb{N}} D(t)u(x_t)}{\sum_{t\in\mathbb{N}} D(t)}\Big)\Big]. \tag{4}$$

[12]This assumption is for convenience only. All our results hold if C is unbounded, as long as the sums of discounted utilities in Assumption 1 are well-defined.

Conditional on u and D, ϕ is unique up to a positive affine transformation.[13]

This representation is known as the Kihlstrom-Mirman (KM) representation, as it can be seen as an application to time of the multi-attribute function of Kihlstrom and Mirman (1974).[14] Fixing D, the curvature of u captures EIS; risk aversion is captured by $\phi \circ u$. Thus, ϕ is the additional curvature that separates between risk aversion and EIS.

Observation 2 highlights that the KM model is characterized by Assumptions 1 and 2 and thus provides a convenient functional form to study the static implications of commonly used models, including EZ and HS—as their static implications satisfy both assumptions. Importantly, the function ϕ contains all the information on the separation of time and risk preferences. Note that EDU corresponds to the case in which ϕ is an affine function. Note also that whenever preferences admit a recursive formulation of the form

$$V_t = \phi\left((1-\beta)u(c) + \beta\phi^{-1}\left(\mathbb{E}[V_{t+1}]\right)\right),$$

then the corresponding preferences over Δ admit a KM representation with the same ϕ, that is, $\mathbb{E}\left[\phi\left((1-\beta)\sum \beta^t u(x_t)\right)\right]$. This shows how one can map many recursive representations into a KM one on Δ and use our results.

As we show next, EZ and HS correspond to CRRA and CARA functions ϕ.[15]

[13] Assumption 1 guarantees a Discounted Utility representation without risk; Assumption 2 guarantees an Expected Utility representation with a given Bernoulli utility v over consumption programs. Since v and the Discounted Utility representation must be ordinally equivalent, there exists a strictly increasing function ϕ that makes them equal.

[14] See, for example, Epstein and Zin (1989). This functional form is derived, in a different setup, in DeJarnette et al. (2020). A similar functional form is used in Edmans and Gabaix (2011); Garrett and Pavan (2011); Abdellaoui et al. (2017); Andersen et al. (2017); Apesteguia et al. (2019).

[15] We should stress that this does *not* imply that the KM model includes EZ and HS as special cases. Rather, it implies that the *static implications* of EZ and HS on Δ admit a KM representation. If applied dynamically on the space of temporal lotteries with different current-period consumption, the models are not nested (e.g., EZ and HS are dynamically consistent, while KM is not if applied dynamically without modification). These differences are inconsequential in our setup, and the observation highlights that one can view EZ or HS as being composed of a *collection* of KM representations, where ϕ varies with the timing of resolution of uncertainty and with current consumption in order to allow for recursivity and dynamic consistency. As mentioned above, we confine our attention to KM because of its simplicity and because the space of lotteries over consumption streams at a given date of resolution of uncertainty contains all relevant information for our analysis, namely tying Stochastic Impatience to parametric restrictions.

Example 3 (EZ with CRRA-CES)**.** Fix a consumption at time zero and consider a preference relation $\succcurlyeq$ over Δ generated by (2). As we show in the Online Appendix, $\succcurlyeq$ admits a KM representation with $u(x) = \frac{x^{1-\frac{1}{\psi}}}{1-\frac{1}{\psi}}$, $D(t) = \beta^{t-1}$, and

$$\phi(z) = \begin{cases} z^{\frac{1-\alpha}{1-\frac{1}{\psi}}} & \text{if } \alpha < 1 < \psi \\ -(-z)^{\frac{1-\alpha}{1-\frac{1}{\psi}}} & \text{if } \alpha > 1 > \psi \\ -z^{\frac{1-\alpha}{1-\frac{1}{\psi}}} & \text{if } 1 < \alpha,\ 1 < \psi \\ (-z)^{\frac{1-\alpha}{1-\frac{1}{\psi}}} & \text{if } \alpha < 1,\ \psi < 1 \end{cases}. \tag{5}$$

Example 4 (HS)**.** Fix a consumption at time zero and consider a preference relation $\succcurlyeq$ over Δ generated by (3). As we show in the Online Appendix, $\succcurlyeq$ admits a KM representation with $D(t) = \beta^{t-1}$ and $\phi(z) = -\exp\left(-\frac{kz}{1-\beta}\right)$.

4.2 General Implications of Stochastic Impatience

We now present our first general result. Recall that $\underline{x}$ and $\bar{x}$ are the lower and upper bounds of the consumption space C. As usual, we say that ϕ is (strictly) more concave/convex than g if $\phi = f \circ g$ for some (strictly) concave/convex f. Lastly, let $\bar{\phi}$ and $\underline{\phi}$ be the functions given by $\bar{\phi}(v) := -\log(u(\overline{x}) - v)$ and $\underline{\phi}(v) := \log(v - u(\underline{x}))$, respectively.

Proposition 3. *Let $\succcurlyeq$ be a preference relation that satisfies Assumptions 1 and 2 and let (ϕ, u, D) be a KM representation. The following are true:*

(i) The relation $\succcurlyeq$ satisfies Stochastic Impatience if the function ϕ is more convex than $\underline{\phi}$ and more concave than $\bar{\phi}$.

(ii) There exist $v_1, v_2, v_3, v_4 \in u(C)$ such that preferences violate Stochastic Impatience if either

(a) ϕ is strictly more concave than $\underline{\phi}$ on $[v_1, v_2]$, or

(b) ϕ is strictly more convex than $\bar{\phi}$ on $[v_3, v_4]$.

Proposition 3 shows that Stochastic Impatience restricts the curvature of ϕ, which determines how risk aversion and EIS can differ. The intuition generalizes the one from EZ. In KM, individuals' tastes over spreads in discounted utilities depend on the curvature of ϕ. When ϕ is concave, they dislike this spread. If it is sufficiently concave, they violate Stochastic Impatience with positive changes ($x, x' > c$). When ϕ is convex, they like this spread. If it is sufficiently convex, they violate Stochastic Impatience with negative changes ($x, x' < c$).

Note that parts (i) and (ii) of the proposition together do not provide an if and only if statement: (ii) shows that Stochastic impatience fails if the curvature of ϕ is too high in specific ranges, while the negation of (i) suggests it should be enough for it to hold in (the neighborhood of) a single point. This is due to the discreteness of time intervals. In Appendix A, we consider a continuous time version of KM and show that, in that case, the sufficient conditions in part (i) are also necessary for Stochastic Impatience. That is, in continuous time, Stochastic Impatience holds if and only if ϕ is more convex than $\underline{\phi}$ and more concave than $\bar{\phi}$.

We conclude this discussion by connecting Proposition 3 with Propositions 1 and 2. In particular, we show that the sufficient conditions from Proposition 3 are also necessary in the case of EZ but not in the case of HS:

Example 5 (EZ)**.** Consider the KM representation of EZ in Example 3. Using Equation (5), the coefficient of absolute risk aversion of ϕ is $-\frac{\phi''(v)}{\phi'(v)} = \frac{1}{v} \cdot \frac{\alpha - \frac{1}{\psi}}{1 - \frac{1}{\psi}}$. Comparing with the coefficients of $\underline{\phi}$ and $\bar{\phi}$, it follows that ϕ is more convex than $\underline{\phi}$ and more concave than $\bar{\phi}$ if either $1 > \alpha \geq \frac{1}{\psi}$ or $1 < \alpha \leq \frac{1}{\psi}$ (see Online Appendix for detailed calculations). Therefore, the sufficient conditions from Proposition 3 coincide with the necessary and sufficient conditions found in Proposition 1.

Example 6 (HS)**.** Consider the KM representation of HS in Example 4. The coefficient of absolute risk aversion of $\phi(z) = -\exp\left(-\frac{kz}{1-\beta}\right)$ is $-\frac{\phi''(z)}{\phi'(z)} = \frac{k}{1-\beta} > 0$. Since ϕ is concave, it is always more concave than the convex function $\bar{\phi}$. Comparing with the coefficient of absolute risk aversion of $\underline{\phi}$, we find that ϕ is more convex than $\underline{\phi}$ if and only if

$$\sup\{u(x)\}_{x\in C} - \inf\{u(x)\}_{x\in C} \leq \frac{1-\beta}{k}.$$

This (sufficient) condition from Proposition 3 (part i) is stronger than the necessary and sufficient condition from Proposition 2.

4.3 Residual Risk Aversion

We now introduce a behavioral notion that captures the additional risk aversion relative to EIS.

Definition 2. The relation $\succcurlyeq$ displays Residual Risk Aversion if for any $a, b, c, d, x \in C$ such that

$$(a, d, x, x, \dots) \sim (b, b, x, x, \dots) \text{ and } (d, a, x, x, \dots) \sim (c, c, x, x, \dots)$$

we have

$$\frac{1}{2}(b, b, \dots) + \frac{1}{2}(c, c, \dots) \succcurlyeq \frac{1}{2}(a, a, \dots) + \frac{1}{2}(d, d, \dots).$$

Similarly, $\succcurlyeq$ displays Residual Risk Seeking/Neutrality if the above is instead $\preccurlyeq/\sim$.

Residual Risk Aversion is an adaptation to our framework of Traeger (2014)'s notion of Intertemporal Risk Aversion.[16] Suppose $\succcurlyeq$ satisfies Assumption 1 with u and D. If $(a, d, x, x, \dots) \sim (b, b, x, x, \dots)$ and $(d, a, x, x, \dots) \sim (c, c, x, x, \dots)$,

$$D(1)u(a)+D(2)u(d) = [D(1)+D(2)]u(b) \text{ and } D(1)u(d)+D(2)u(a) = [D(1)+D(2)]u(c).$$

Thus, $u(a) + u(d) = u(b) + u(c)$. Moreover, either $a > b > c > d$ or $a < b < c < d$. Consider a lottery that returns with equal chances the constant streams a or d; and a lottery that returns with equal chances the constant streams b or c. If all risk aversion is included in the curvature of u, these two lotteries must be indifferent, since $u(a) + u(d) = u(b) + u(c)$. With additional risk aversion, because $a > b > c > d$ or $a < b < c < d$, the individual should prefer the lottery between b and c, in which the utility spread is smaller.

We first link Residual Risk Aversion to the curvature of ϕ in the KM representation:

Proposition 4. *Suppose $\succcurlyeq$ admits a KM representation (ϕ, D, u). Then, $\succcurlyeq$ displays Residual Risk Aversion/Seeking/Neutrality if and only if ϕ is concave/convex/affine.*

It follows that Residual Risk Neutrality characterizes EDU given Assumptions 1 and 2, and that EZ allows for Residual Risk Aversion/Seeking:

[16] Similar to Proposition 4 and Observation 5 below, Traeger also gives a functional characterization of attitudes towards intertemporal risk aversion in his framework. Relative Risk Aversion is related to *correlation aversion* (Richard 1975; Epstein and Tanny 1980; see also Stanca 2023), which postulates

$$\frac{1}{2}(a, d, x, x, \dots) + \frac{1}{2}(d, a, x, x, \dots) \succcurlyeq \frac{1}{2}(a, a, x, x, \dots) + \frac{1}{2}(d, d, x, x, \dots)$$

for all a, d, x. While the two notions are in general distinct (Traeger, 2014), it can be shown they are equivalent within the KM model. Therefore, all results below could be equivalently stated using correlation aversion. On the relation between correlation aversion and the separation of time and risk preferences, see also Bommier 2007.

Observation 3 (EDU is characterized by Residual Risk Neutrality)**.** Suppose $\succsim$ satisfies Assumptions 1 and 2. Then, it admits an EDU representation if and only if it displays Residual Risk Neutrality.

Observation 4 (EZ preferences)**.** Suppose $\succsim$ admits a representation as in (2). Then $\succsim$ displays Residual Risk Aversion/Neutrality/Seeking if and only if $\alpha \geq / = / \leq \frac{1}{\psi}$.[17]

Finally, we can introduce a comparative notion.

Definition 3. Let $\succsim_1$ and $\succsim_2$ be preference relations over Δ. We say that $\succsim_1$ has more Residual Risk Aversion than $\succsim_2$ if they coincide on degenerate lotteries and if, for all $a > b > c > d$,

$$\frac{1}{2}(b, b, \dots) + \frac{1}{2}(c, c, \dots) \succsim_2 \frac{1}{2}(a, a, \dots) + \frac{1}{2}(d, d, \dots)$$

implies

$$\frac{1}{2}(b, b, \dots) + \frac{1}{2}(c, c, \dots) \succsim_1 \frac{1}{2}(a, a, \dots) + \frac{1}{2}(d, d, \dots).$$

This comparative notion parallels standard ones for risk and ambiguity (Epstein, 1999; Ghirardato and Marinacci, 2002). It has an immediate counterpart in KM representations.

Observation 5. Let $\succsim_1$ and $\succsim_2$ be two preferences with KM representations (ϕ_1, u, D) and (ϕ_2, u, D). Then, $\succsim_1$ has more Residual Risk Aversion than $\succsim_2$ if and only if there exists a strictly increasing and concave function $f : \mathbb{R} \to \mathbb{R}$ such that $\phi_1 = f \circ \phi_2$.

We are now ready to discuss the implications of Stochastic Impatience on Residual Risk attitude.

Proposition 5. *Let $\succsim$, $\succsim_1$, and $\succsim_2$ be preference relations over Δ that satisfy Assumptions 1 and 2 and coincide on degenerate lotteries. Suppose $\succsim_1$ displays Residual Risk Aversion and $\succsim_2$ displays Residual Risk Seeking. The following are true:*

(i) Suppose both $\succsim_1$ and $\succsim_2$ satisfy Stochastic Impatience. If $\succsim$ has less Residual Risk Aversion than $\succsim_1$ and more Residual Risk Aversion than $\succsim_2$, then it also satisfies Stochastic Impatience.

[17] Recall that the CRRA-CES version of EZ displays a preference for early (late/neutrality towards) resolution of uncertainty if $\alpha > (< / =) \frac{1}{\psi}$. Therefore, in this model, $\succsim$ displays Residual Risk Aversion (Seeking/Neutrality) if and only if, in the space of temporal lotteries, there is a preference for early (a preference for late/neutrality towards) resolution of uncertainty.

(ii) Suppose both $\succcurlyeq_1$ *and* $\succcurlyeq_2$ *violate Stochastic Impatience. If* $\succcurlyeq$ *has more Residual Risk Aversion than* $\succcurlyeq_1$ *or has less Residual Risk Aversion than* $\succcurlyeq_2$*, then it also violates Stochastic Impatience.*

(iii) There exist $\succcurlyeq_3$ *and* $\succcurlyeq_4$ *that satisfy Assumptions 1 and 2 and coincide with* $\succcurlyeq$ *on degenerate lotteries such that:*

(a) $\succcurlyeq_3$ *has more Residual Risk Aversion than* $\succcurlyeq$ *and violates Stochastic Impatience, and*

(b) $\succcurlyeq_4$ *has less Residual Risk Aversion than* $\succcurlyeq$ *and violates Stochastic Impatience.*

Proposition 5 is the behavioral counterpart of Proposition 3, showing how Stochastic Impatience restricts Residual Risk Aversion/Seeking. If $\succcurlyeq$ has Residual Risk Aversion/Seeking in between that of two preferences that satisfy Stochastic Impatience, it will also satisfy it (part i). If it is more extreme than preferences that violate it, it will also violate it (part ii). Finally, part (iii) shows that large enough changes in Residual Risk Aversion alone generate violations of Stochastic Impatience.

5 Discussion

We have shown that within a broad class of models, Stochastic Impatience restricts risk aversion relative to EIS, ruling out the parameters used in virtually all applications of EZ and HS. This has two implications. First, independently of the appeal of Stochastic Impatience, it provides a simple way to test common parametrizations of existing models: they imply that Stochastic Impatience must be violated. Second, it implies that if we want to preserve Stochastic Impatience, we need to either consider a class of models where our results do not hold or find different ways to match empirical patterns while keeping risk aversion close enough to the inverse EIS. We conclude with a discussion of these two possibilities.

Beyond our assumptions. Our results assume Expected Utility under risk and Discounted Utility without risk. Do they hold more broadly?

There are two natural ways to extend beyond Expected Utility. First, adopting models of non-Expected Utility developed in the atemporal environment. In Appendix B, we consider a very broad class that includes probability weighting and

Disappointment Aversion.[18] We show that, in a continuous time setting, these models violate Stochastic Impatience whenever they exhibit (even a local version of) First-Order Risk Attitudes (Segal and Spivak, 1990)—as is the case in most specifications that use them.

Alternatively, one can consider models that maintain Expected Utility within each period but violate it across periods. While this avenue has received little attention in the literature on non-Expected Utility, there are models that fit into this category. One example is the Dynamic Ordinal Certainty Equivalent model (Selden, 1978; Selden and Stux, 1978), where the individual first calculates the per-period certainty equivalents using one utility function, and then calculates their discounted value using a different utility. This model satisfies Stochastic Impatience while allowing for a separation of time and risk preferences (DeJarnette et al., 2020; Selden and Wei, 2019). A discussion of the appeal of this model is beyond our scope, although some papers pointed to concerns with dynamic consistency and notions of monotonicity (Epstein and Zin, 1989; Chew and Epstein, 1990; Bommier et al., 2017).

Our other assumption is Discounted Utility without risk. Going beyond it requires dropping additive separability, as in models of habit formation or memorable consumption. Our results qualitatively extend, in the sense that, for any such model, Stochastic Impatience imposes a bound on risk aversion for a fixed EIS.[19] However, this bound depends on the specifics of the preferences considered.

Beyond high risk aversion. Models in finance and macroeconomics often require high risk aversion to capture the unwillingness to take financial risks. Other features have the same effect, like ambiguity aversion/robustness, incorrect beliefs about stock returns, rational inattention, or inertia. If these aspects are relevant but omitted from the model, risk aversion may be overestimated.

For example, if some equity premium is due to ambiguity aversion, incorporating it may allow for much lower coefficients of risk aversion (Barillas et al., 2009). This would reduce the preference for early resolution of uncertainty to more realistic levels (Epstein et al., 2014) and allow for Stochastic Impatience: since the latter is based on objective lotteries, it is unaffected by ambiguity aversion. In general, any feature

[18] Applications of these models have been suggested, starting in the original paper of Epstein and Zin (1989). See Backus et al. (2004) and references therein.

[19] Even weakening Assumption 1 while maintaining Assumption 2, a concave enough ϕ makes the value of any lottery arbitrarily close to the value of its worst outcome, thus generating a violation of Stochastic Impatience.

that reduces the individual's willingness to undertake financial risk without modifying her attitude towards objective lotteries, as discussed in the surveys of Backus et al. (2004), Epstein and Schneider (2010), and Hansen and Sargent (2014), could provide a way to reconcile the empirical fit of the model with more moderate levels of risk aversion and—as we show in this paper—also with Stochastic Impatience.

Appendices

A Continuous time and additional results

We now consider a continuous time formulation of the model from Section 4. A consumption program is now a function: $\mathbf{x} : \mathbb{R}_+ \to [\underline{x}, \overline{x}]$. For each $\delta > 0$, let $\langle x, t; c, \delta \rangle$ denote the consumption function that gives the constant consumption c except for dates in $[t, t+\delta)$, where it gives x.

We consider preferences $\succcurlyeq$ over lotteries over streams that can be represented by

$$V(p) = \mathbb{E}_p\Big[\phi\Big(\frac{\int_0^{+\infty} D(t)u(x(t))dt}{\int_0^{+\infty} D(t)dt}\Big)\Big],$$

where $u : [\underline{x}, \overline{x}] \to \mathbb{R}$ is continuous and strictly increasing, $D : \mathbb{R}_+ \to [0,1]$ is continuous and strictly decreasing, $\int_0^{+\infty} D(t)dt < +\infty$, and $\phi : [u(\underline{x}), u(\overline{x})]$ is strictly increasing. Stochastic Impatience then becomes:

Definition 4 (Stochastic Impatience'). The relation $\succcurlyeq$ satisfies Stochastic Impatience' if for any $t, t' \in \mathbb{R}_+$ with $t < t'$, $\delta \in \mathbb{R}_+ \cup \{\infty\}$ and any $c, x, x' \in C$ with $x > x'$,

$$\frac{1}{2}\langle x, t; c, \delta\rangle + \frac{1}{2}\langle x', t'; c, \delta\rangle \succcurlyeq \frac{1}{2}\langle x, t'; c, \delta\rangle + \frac{1}{2}\langle x', t; c, \delta\rangle. \tag{6}$$

In this context, we can also define a weaker version of stochastic impatience in which we restrict δ to be ∞.

Definition 5 (Stochastic Impatience"). The relation $\succcurlyeq$ satisfies Stochastic Impatience" if for any $t, t' \in \mathbb{R}_+$ with $t < t'$, and any $c, x, x' \in C$ with $x > x'$,

$$\frac{1}{2}\langle x, t; c, \infty\rangle + \frac{1}{2}\langle x', t'; c, \infty\rangle \succcurlyeq \frac{1}{2}\langle x, t'; c, \infty\rangle + \frac{1}{2}\langle x', t; c, \infty\rangle. \tag{7}$$

Recall that $\bar{\phi}(v) := -\log(u(\overline{x}) - v)$ and $\underline{\phi}(v) := \log(v - u(\underline{x}))$. We now show that with continuous time, the bounds on the curvature of ϕ are both necessary and sufficient for Stochastic Impatience, and that the two versions of the axiom are equivalent to each other.

Proposition 6. *Suppose time is continuous and let (ϕ, u, D) be a KM representation of $\succcurlyeq$. The following statements are equivalent:*

(i) The relation $\succcurlyeq$ satisfies Stochastic Impatience';

(ii) The relation $\succcurlyeq$ satisfies Stochastic Impatience";

(iii) The function ϕ is weakly more convex than $\underline{\phi}$ and weakly more concave than $\bar{\phi}$.

B Beyond Expected Utility

In this section, we show that the tension between Stochastic Impatience and the separation of time and risk preferences goes beyond Expected Utility.

We extend beyond Expected Utility by assuming that preferences are at least *locally bilinear* at $\frac{1}{2}$. This generalization includes as special cases popular models such as those of probability weighting (Rank-Dependent Utility, Quiggin 1982, and Cumulative Prospect Theory, Tversky and Kahneman 1992) and Disappointment Aversion (Gul, 1991).[20] In general, bilinearity holds if there is an increasing onto function $\pi : [0,1] \to [0,1]$, and a function f that evaluates (arbitrary) prizes, such that, for any x, y such that $f(x) > f(y)$, the prospect that yields x with probability λ and y otherwise is evaluated by $\pi(\lambda) f(x) + [1 - \pi(\lambda)] f(y)$. Since our goal is to be as general as possible, we only require preferences to be bilinear for equally likely binary lotteries ($\lambda = \frac{1}{2}$)—the *local* bilinear model (Dean and Ortoleva, 2017).[21] Applying it to our setting, we obtain the following generalization of the KM model using the continuous time setup of Appendix A.

Definition 6. We say that $\succcurlyeq$ admits a *local bilinear KM* representation if there exist strictly increasing and continuous $u : C \to \mathbb{R}_+$, a strictly decreasing $D : \mathbb{R}_+ \to \mathbb{R}_+$ with $\int_0^{+\infty} D(t)dt < +\infty$, a strictly increasing and differentiable $\phi : u(C) \to \mathbb{R}$, and $\pi(\frac{1}{2}) \in (0,1)$, such that for all $\mathbf{x}, \mathbf{y} \in \mathcal{X}$, $p = \frac{1}{2}\mathbf{x} + \frac{1}{2}\mathbf{y}$ with $\int_0^{+\infty} D(t)u(x(t))dt \geq \int_0^{+\infty} D(t)u(y(t))dt$ is evaluated according to:

$$V(p) = \pi\Big(\frac{1}{2}\Big)\phi\Big(\frac{\int_0^{+\infty} D(t)u(x(t))dt}{\int_0^{+\infty} D(t)dt}\Big) + \Big[1 - \pi\Big(\frac{1}{2}\Big)\Big]\phi\Big(\frac{\int_0^{+\infty} D(t)u(y(t))dt}{\int_0^{+\infty} D(t)dt}\Big).$$

In a local bilinear KM representation, Residual Risk Aversion/Seeking can be achieved either by adding curvature to ϕ, as in the KM representation, or by adding local First-Order Risk Attitudes ($\pi(\frac{1}{2}) \neq \frac{1}{2}$; see Segal and Spivak 1990).[22] Preferences display local First-Order Risk Aversion if $\pi(\frac{1}{2}) < \frac{1}{2}$ (the best outcome is under-

[20] It also allows for generalizations of Rank-Dependent Expected Utility, e.g., the minimum from a set of probability distortions (Dean and Ortoleva, 2017). On the other hand, it does not encompass all models of risk preferences (e.g., it does not encompass Cautious Expected Utility, Cerreia-Vioglio et al. 2015).

[21] This is a local specification of the bilinear (or biseparable) model of Ghirardato and Marinacci (2001) for objective risk. Here, preferences are not restricted to be bilinear in general, but only that there is some bilinear representation for 50/50 lotteries.

[22] Global First-Order Risk Attitude is implied whenever $\pi(\gamma) \neq \gamma$ for all $\gamma \in (0,1)$.

weighted) and local First-Order Risk Seeking if $\pi(\frac{1}{2}) > \frac{1}{2}$ (the best outcome is over-weighted).

Proposition 7. *Suppose time is continuous and let (π, ϕ, u, D) be a local bilinear KM representation of $\succcurlyeq$. If $\pi(\frac{1}{2}) \neq \frac{1}{2}$, then $\succcurlyeq$ violates Stochastic Impatience'.*

The result above shows that, in continuous time, in the broad class of local bilinear models, displaying local First-Order Risk Attitudes *always* leads to violations of Stochastic Impatience, independently of the shape of ϕ. Intuitively, local First-Order Risk Aversion ($\pi(\frac{1}{2}) < \frac{1}{2}$) implies extreme amounts of risk aversion in a neighborhood around certainty and, as seen before, Stochastic Impatience fails if risk aversion is high enough. Similarly, local First-Order Risk Seeking ($\pi(\frac{1}{2}) > \frac{1}{2}$) implies extreme amounts of risk seeking in a neighborhood around certainty, which also leads to violations of Stochastic Impatience.

C Proofs

C.1 Proof of Observation 1

To simplify notation, let $d_\delta(\tilde{t}) \equiv \sum_{t=\tilde{t}}^{\tilde{t}+\delta-1} D(t)$ and $\bar{d} \equiv \sum_{t=1}^{\infty} D(t)$. Take $x_1 > x_2$ and $t_1 < t_2$. The utility of lottery $\frac{1}{2}\langle c, x_1, t_1, \delta\rangle + \frac{1}{2}\langle c, x_2, t_2, \delta\rangle$ is

$$\frac{u(x_1)\cdot d_\delta(t_1) + u(c)\cdot\left[\bar{d} - d_\delta(t_1)\right]}{2} + \frac{u(x_2)\cdot d_\delta(t_2) + u(c)\cdot\left[\bar{d} - d_\delta(t_2)\right]}{2},$$

and the utility of lottery $\frac{1}{2}\langle c, x_1, t_2, \delta\rangle + \frac{1}{2}\langle c, x_2, t_1, \delta\rangle$ equals

$$\frac{u(x_2)\cdot d_\delta(t_1) + u(c)\cdot\left[\bar{d} - d_\delta(t_1)\right]}{2} + \frac{u(x_1)\cdot d_\delta(t_2) + u(c)\cdot\left[\bar{d} - d_\delta(t_2)\right]}{2}.$$

Stochastic Impatience holds if the first expression is weakly greater than the second one. Algebraic manipulations show that this is true if and only if:

$$[u(x_1) - u(x_2)]\cdot[d_\delta(t_1) - d_\delta(t_2)] \geq 0,$$

which holds if and only if $d_\delta(t_1) \geq d_\delta(t_2)$ (since u is strictly increasing). Using the definition of d_δ, we find that Stochastic Impatience holds if and only if $\sum_{t=t_1}^{t_1+\delta-1} D(t) \geq \sum_{t=t_2}^{t_2+\delta-1} D(t)$ for all $t_1 < t_2$ and all δ, which is true if and only if $D(t)$ is weakly decreasing. ■

C.2 Proof of Proposition 1

To simplify notation, let $\rho \equiv \frac{1}{\psi}$ denote the inverse of EIS. It is without loss of generality to consider lotteries in which the earliest prize is paid in period $t = 1$ (making the earliest date $t > 1$ only adds to each prospect a discounted sum of some deterministic stream, that will be canceled out in all comparisons). Let c_0 denote an arbitrary but fixed consumption in period 0.

As we show in the online appendix, using the EZ recursion, we can write the value of the lottery

$$\frac{1}{2}(\overbrace{c_0}^{0}, \underbrace{\overbrace{x}^{1}, \overbrace{x}^{2}, ..., \overbrace{x}^{\delta}}_{\delta}, \overbrace{c}^{\delta+1}, c, ...) + \frac{1}{2}(\overbrace{c_0}^{0}, \overbrace{c}^{1}, ..., \overbrace{c}^{\tau-1}, \underbrace{\overbrace{y}^{\tau}, y, ..., \overbrace{y}^{\tau+\delta-1}}_{\delta}, c, ...)$$

as

$$V_0 = \left\{ (1-\beta)c_0^{1-\rho} + \beta \left[\frac{\begin{array}{c} \{(1-\beta^\delta)x^{1-\rho} + \beta^\delta c^{1-\rho}\}^{\frac{1-\alpha}{1-\rho}} \\ + \left\{c^{1-\rho} + \beta^{\tau-1}(1-\beta^\delta)\left(y^{1-\rho} - c^{1-\rho}\right)\right\}^{\frac{1-\alpha}{1-\rho}} \end{array}}{2} \right]^{\frac{1-\rho}{1-\alpha}} \right\}^{\frac{1}{1-\rho}}.$$

Using Definition 1, Stochastic Impatience holds if and only if

$$\left\{ (1-\beta)c_0^{1-\rho} + \beta \left[\frac{\begin{array}{c} \{(1-\beta^\delta)x^{1-\rho} + \beta^\delta c^{1-\rho}\}^{\frac{1-\alpha}{1-\rho}} \\ + \left\{c^{1-\rho} + \beta^{\tau-1}(1-\beta^\delta)\left(y^{1-\rho} - c^{1-\rho}\right)\right\}^{\frac{1-\alpha}{1-\rho}} \end{array}}{2} \right]^{\frac{1-\rho}{1-\alpha}} \right\}^{\frac{1}{1-\rho}}$$

$$\geq$$

$$\left\{ (1-\beta)c_0^{1-\rho} + \beta \left[\frac{\begin{array}{c} \{(1-\beta^\delta)y^{1-\rho} + \beta^t c^{1-\rho}\}^{\frac{1-\alpha}{1-\rho}} \\ + \left\{c^{1-\rho} + \beta^{\tau-1}(1-\beta^\delta)\left(x^{1-\rho} - c^{1-\rho}\right)\right\}^{\frac{1-\alpha}{1-\rho}} \end{array}}{2} \right]^{\frac{1-\rho}{1-\alpha}} \right\}^{\frac{1}{1-\rho}}$$

for all $\tau, \delta \in \mathbb{N}$ and for all $x, y, c \in \mathbb{R}_+$ with $x > y$.

In the online appendix, we use the previous inequality to obtain the following result:

Lemma 1. *Suppose $\succcurlyeq$ admits the representation in (2). Stochastic Impatience holds if and only if*

$$\left\{(1-\beta^{\delta})z^{1-\rho}+\beta^{\delta}\right\}^{\frac{\rho-\alpha}{1-\rho}} \geq \beta^{\tau}\left\{1+\beta^{\tau}(1-\beta^{\delta})\left(z^{1-\rho}-1\right)\right\}^{\frac{\rho-\alpha}{1-\rho}}$$

for all $\delta,\tau \in \mathbb{N}$ and all $z \in \mathbb{R}_{+}$.

Straightforward calculations show that the condition in Lemma 1 hold if and only if either $1 > \alpha \geq \rho$ or $1 < \alpha \leq \rho$ (the online appendix includes a detailed calculation).

C.3 Proof of Proposition 2

As in the proof of Proposition 1, it is without loss of generality to consider lotteries in which the earliest price is paid at $t = 1$. Let c_0 denote an arbitrary but fixed consumption in period 0.

Let $p_{x,x';c,\delta,t'}$ denote the lottery that pays either the prize x starting in period 1 or x' starting in period t', where each prize lasts for δ periods. That is:

$$p_{x,x';c,\delta,t'} \equiv \frac{1}{2}(\overbrace{c_0}^{0},\underbrace{\overbrace{x}^{1},\overbrace{x}^{2},...,\overbrace{x}^{\delta}}_{\delta},c,...)+\frac{1}{2}(\overbrace{c_0}^{0},\overbrace{c}^{1},...,\overbrace{c}^{t'-1},\underbrace{\overbrace{x'}^{t'},x',...,\overbrace{x'}^{t'+\delta-1}}_{\delta},c,...).$$

Stochastic Impatience holds if and only if $p_{x,x';c,t,t'} \succsim p_{x',x;c,t,t'}$ for all $\delta \in \mathbb{N}$, all $t' \in \{2,3,...\}$ and all $x,x',c \in C$ with $x > x'$.

By the additive separability of current-period consumption in the HS representation and the fact that all lotteries considered here have the same period-0 consumption, we can omit the the period-0 utility $u(c_0)$ from the expressions, focusing only in the continuation utility starting in period 1:

$$-\frac{\beta}{k}\log \mathbb{E}_0\left[e^{-kV_1}\right].$$

Moreover, because $\frac{\beta}{k} > 0$ and the logarithm is an increasing function, it suffices to evaluate the following expression for each of the lotteries:

$$\mathbb{E}_0\left[-e^{-kV_1}\right].$$

For the lottery $p_{x,x';c,\delta,t'}$, we have:

$$\mathbb{E}_0\left[-e^{-kV_1}\right] = -\frac{1}{2}\left\{\begin{array}{c}\exp\left(-k\cdot\left[\sum_{\tilde{t}=1}^{\delta}\beta^{\tilde{t}}u\left(x\right)+\sum_{\tilde{t}=\delta+1}^{\infty}\beta^{\tilde{t}}u\left(c\right)\right]\right)\\ +\exp\left(-k\cdot\left(\sum_{\tilde{t}\notin\{t',...,t'+\delta-1\}}\beta^{\tilde{t}}u\left(c\right)+\sum_{\tilde{t}\in\{t',...,t'+\delta-1\}}\beta^{\tilde{t}}u\left(x'\right)\right)\right)\end{array}\right\},$$

which, with some algebraic manipulations, can be rewritten as

$$-\frac{\exp\left\{ku\left(c\right)\frac{\beta}{1-\beta}\right\}}{2}\left\{\begin{array}{c}\exp\left\{-k\cdot\frac{\beta}{1-\beta}\cdot\left[u\left(x\right)-u\left(c\right)\right]\cdot\left(1-\beta^{\delta}\right)\right\}\\+\exp\left\{-k\cdot\frac{\beta}{1-\beta}\cdot\left[u\left(x'\right)-u\left(c\right)\right]\cdot\left(\beta^{\tau}-\beta^{\tau+\delta}\right)\right\}\end{array}\right\},$$

where $\tau\equiv t'-1\in\mathbb{N}$.

Performing the same calculations for lottery $p_{x',x;c,\delta,t'}$, we obtain:

$$\mathbb{E}_0\left[-e^{-k\tilde{V}_1}\right]=-\frac{\exp\left\{ku\left(c\right)\frac{\beta}{1-\beta}\right\}}{2}\left\{\begin{array}{c}\exp\left\{-k\cdot\frac{\beta}{1-\beta}\cdot\left[u\left(x'\right)-u\left(c\right)\right]\cdot\left(1-\beta^{\delta}\right)\right\}\\+\exp\left\{-k\cdot\frac{\beta}{1-\beta}\cdot\left[u\left(x\right)-u\left(c\right)\right]\cdot\left(\beta^{\tau}-\beta^{\tau+\delta}\right)\right\}\end{array}\right\}.$$

Dividing both expressions by $-\frac{\exp\left\{ku(c)\frac{\beta}{1-\beta}\right\}}{2}<0$, we find that $p_{x,x';c,\delta,t'}\precsim p_{x',x;c,\delta,t'}$ if and only if

$$\exp\left\{-k\cdot\frac{\beta}{1-\beta}\cdot\left[u\left(x\right)-u\left(c\right)\right]\cdot\left(1-\beta^{\delta}\right)\right\}+\exp\left\{-k\cdot\frac{\beta}{1-\beta}\cdot\left[u\left(x'\right)-u\left(c\right)\right]\cdot\left(\beta^{\tau}-\beta^{\tau+\delta}\right)\right\}$$

$$\leq\exp\left\{-k\cdot\frac{\beta}{1-\beta}\cdot\left[u\left(x'\right)-u\left(c\right)\right]\cdot\left(1-\beta^{\delta}\right)\right\}+\exp\left\{-k\cdot\frac{\beta}{1-\beta}\cdot\left[u\left(x\right)-u\left(c\right)\right]\cdot\left(\beta^{\tau}-\beta^{\tau+\delta}\right)\right\}.$$

Rearranging this expression gives

$$\exp\left\{-k\cdot\frac{\beta}{1-\beta}\cdot\left[u\left(x\right)-u\left(c\right)\right]\cdot\left(1-\beta^{\delta}\right)\right\}-\exp\left\{-k\cdot\frac{\beta}{1-\beta}\cdot\left[u\left(x\right)-u\left(c\right)\right]\cdot\left(\beta^{\tau}-\beta^{\tau+\delta}\right)\right\}$$

$$\leq\exp\left\{-k\cdot\frac{\beta}{1-\beta}\cdot\left[u\left(x'\right)-u\left(c\right)\right]\cdot\left(1-\beta^{\delta}\right)\right\}-\exp\left\{-k\cdot\frac{\beta}{1-\beta}\cdot\left[u\left(x'\right)-u\left(c\right)\right]\cdot\left(\beta^{\tau}-\beta^{\tau+\delta}\right)\right\},$$

which holds for all $x>x'$ if and only if

$$\frac{\partial}{\partial x}\left[\begin{array}{c}\exp\left\{-k\cdot\frac{\beta}{1-\beta}\cdot\left[u\left(x\right)-u\left(c\right)\right]\cdot\left(1-\beta^{\delta}\right)\right\}\\-\exp\left\{-k\cdot\frac{\beta}{1-\beta}\cdot\left[u\left(x\right)-u\left(c\right)\right]\cdot\left(\beta^{\tau}-\beta^{\tau+\delta}\right)\right\}\end{array}\right]\leq 0$$

for all x. Evaluating the derivative and rearranging, we find that Stochastic Impatience holds if and only if for all $c,x\in C$ and all $\tau,\delta\in\mathbb{N}$,

$$-\log\beta\geq k\cdot\frac{\beta}{1-\beta}\cdot\left[u\left(x\right)-u\left(c\right)\right]\cdot\frac{\left(1-\beta^{\delta}\right)\left(1-\beta^{\tau}\right)}{\tau}.$$

Since $\sup_{\delta\in\mathbb{N}}\left(1-\beta^{\delta}\right)=\lim_{\delta\nearrow+\infty}1-\beta^{\delta}=1$, Stochastic Impatience holds if and only if for all c, x, and τ,

$$-\log\beta\geq k\cdot\frac{\beta}{1-\beta}\cdot\left[u\left(x\right)-u\left(c\right)\right]\cdot\frac{1-\beta^{\tau}}{\tau}.$$

Note also that $\sup_{\tau\in\mathbb{N}} \frac{1-\beta^\tau}{\tau} = 1-\beta$ (which is obtained by taking $\tau = 1$). Thus, the condition becomes

$$-\frac{\log\beta}{k\cdot\beta} \geq u(x) - u(c)$$

for all c, x, and τ, which holds if and only if

$$-\frac{\log\beta}{k\cdot\beta} \geq \sup\{u(x)\}_{x\in C} - \inf\{u(x)\}_{x\in C}.$$

■

C.4 Proof of Observation 2

For necessity, note that, when restricted to degenerate streams, the representation is a monotone transformation of $\sum_{t=1}^{\infty} D(t)u(x_t)$, so preferences must satisfy Assumption 1. Moreover, since risky lotteries are evaluated by taking expectations, preferences satisfy Assumption 2 as in Expected Utility Theory.

For sufficiency, by Assumption 1, there exist a strictly increasing and continuous $u : [\underline{x}, \overline{x}] \to \mathbb{R}$ and a strictly decreasing $D : \mathbb{N} \to [0,1]$ with $\sum_{t=1}^{\infty} D(t) < +\infty$ such that $\succcurlyeq$ restricted to $\mathcal{X}$ is represented by

$$F^*(\mathbf{x}) := \sum_{t=1}^{\infty} D(t)u(x_t).$$

Applying a positive transformation, the same preference is also represented by

$$F(\mathbf{x}) := \frac{\sum_{t=1}^{\infty} D(t)u(x(t))}{\sum_{t=1}^{\infty} D(t)},$$

Note that $F(\mathcal{X}) = u(C)$. By Assumption 2, there exists $U : \mathcal{X} \to \mathbb{R}$ such that $\succcurlyeq$ is represented by

$$V(p) := \mathbb{E}_p\big[U\big].$$

It follows that U and F represent the same preferences over $\mathcal{X}$, i.e., for all $\mathbf{x}, \mathbf{y} \in \mathcal{X}$,

$$U(\mathbf{x}) \geq U(\mathbf{y}) \Leftrightarrow \mathbf{x} \succcurlyeq \mathbf{y} \Leftrightarrow F(\mathbf{x}) \geq F(\mathbf{y}). \tag{8}$$

Therefore, there must exist an increasing $\phi : u(C) \to \mathbb{R}$ such that $U = \phi \circ F$.

We claim that ϕ must be strictly increasing. Suppose not. Then, there are $a, b \in u(C)$ with $a > b$ and $\phi(a) = \phi(b)$. Consider the streams $\mathbf{x}$ and $\mathbf{y}$ that return $u^{-1}(a)$ and $u^{-1}(b)$ each period, respectively. Since $a > b$ we must have $F(\mathbf{x}) = a > b = F(\mathbf{y})$.

At the same time, since $\phi(a) = \phi(b)$, we have $U(\mathbf{x}) = \phi(F(\mathbf{x})) = \phi(a) = \phi(b) = \phi(F(\mathbf{y})) = U(\mathbf{y})$, violating (8).

The uniqueness claims follow from the same arguments as in the Expected Utility Theorem. ■

C.5 Proof of Proposition 3

To simplify notation, let $\underline{u} := u(\underline{x})$ and $\bar{u} := u(\bar{x})$, so the space of per-period utility is $u(C) = [\underline{u}, \bar{u}]$. Let $\mathcal{D} \equiv \left\{ \frac{\sum_{\tilde{t}=t}^{t+\tau} D(\tilde{t})}{\sum_{\tilde{t}=1}^{\infty} D(\tilde{t})} : \ t, \tau \in \mathbb{N} \right\}$ denote the space of discount factors. By definition, Stochastic Impatience holds if

$$\begin{gathered}\phi\left(d_1 H + (1-d_1)\,\bar{C}\right) + \phi\left(d_2 L + (1-d_2)\,\bar{C}\right) \geq \\ \phi\left(d_1 L + (1-d_1)\,\bar{C}\right) + \phi\left(d_2 H + (1-d_2)\,\bar{C}\right)\end{gathered} \tag{9}$$

for all $H, L, \bar{C} \in [\underline{u}, \bar{u}]$ with $H > L$, and all $d_1, d_2 \in \mathcal{D}$ with $d_1 > d_2$.

Proof of part (i).

Since $\mathcal{D} \subset [0,1]$, it suffices to show that (9) holds for all $d_1, d_2 \in (0,1]$ with $d_1 > d_2$ and all $H, L, \bar{C} \in [\underline{u}, \bar{u}]$ with $H > L$. Since ϕ is less concave than $\ln(v - \underline{u})$ and more concave than $-\ln(\bar{u} - v)$, there exist increasing functions f and g, with f convex and g concave, such that

$$\phi(v) = f(\log(v - \underline{u})) = g(-\log(\bar{u} - v)).$$

There are three cases depending on the value of $\bar{C}$.

Case 1: $H > L \geq \bar{C}$**.**

Substitute $\phi(v) = f(\log(v - \underline{u}))$ in inequality (9) to obtain:

$$\begin{gathered}f\left[\log\left(d_1 H + (1-d_1)\,\bar{C} - \underline{u}\right)\right] + f\left[\log\left(d_2 L + (1-d_2)\,\bar{C} - \underline{u}\right)\right] \geq \\ f\left[\log\left(d_1 L + (1-d_1)\,\bar{C} - \underline{u}\right)\right] + f\left[\log\left(d_2 H + (1-d_2)\,\bar{C} - \underline{u}\right)\right]\end{gathered}.$$

Since the sum of logarithms is equal to the logarithm of the product, algebraic manipulations show that sum of terms being evaluated on the LHS is greater than those on the RHS:

$$\log\left(d_1 H + (1-d_1)\,\bar{C} - \underline{u}\right) + \log\left(d_2 L + (1-d_2)\,\bar{C} - \underline{u}\right)$$

$$\geq \log\left(d_1 L + (1-d_1)\,\bar{C} - \underline{u}\right) + \log\left(d_2 H + (1-d_2)\,\bar{C} - \underline{u}\right)$$

$$\iff (d_1 - d_2)\,(H - L)\,\bar{C} \geq 0.$$

Moreover, the terms on the LHS also have a higher spread than the ones on the RHS:

$$d_1 H + (1-d_1)\,\bar{C} - \underline{u} > d_1 L + (1-d_1)\,\bar{C} - \underline{u} \iff H > L$$

$$d_1 H + (1-d_1)\,\bar{C} - \underline{u} > d_2 H + (1-d_2)\,\bar{C} - \underline{u} \iff H > \bar{C}$$

$$d_2 H + (1-d_2)\,\bar{C} - \underline{u} > d_2 L + (1-d_2)\,\bar{C} - \underline{u} \iff H > L$$

$$d_1 L + (1-d_1)\,\bar{C} - \underline{u} \geq d_2 L + (1-d_2)\,\bar{C} - \underline{u} \iff L \geq \bar{C}$$

Since $\phi(v) = f(\log(v))$ where f is convex, it follows from Jensen's inequality that (9) holds.

Case 2: $\bar{C} \geq H > L$**.**

Using $\phi(v) = g(-\log(\bar{u} - v))$, inequality (9) becomes:

$$g\left[-\log\left(\bar{u} - \left(d_1 H + (1-d_1)\,\bar{C}\right)\right)\right] + g\left[-\log\left(\bar{u} - \left(d_2 L + (1-d_2)\,\bar{C}\right)\right)\right]$$

$$\geq g\left[-\log\left(\bar{u} - \left(d_1 L + (1-d_1)\,\bar{C}\right)\right)\right] + g\left[-\log\left(\bar{u} - \left(d_2 H + (1-d_2)\,\bar{C}\right)\right)\right].$$

As in case 1, using the fact that the sum of logarithms equals the logarithm of the product and algebraic manipulations, we find that the sum of terms on the LHS is higher than the sum of terms on the RHS:

$$\left[-\log\left(\bar{u} - \left(d_1 H + (1-d_1)\,\bar{C}\right)\right)\right] + \left[-\log\left(\bar{u} - \left(d_2 L + (1-d_2)\,\bar{C}\right)\right)\right]$$

$$\geq \left[-\log\left(\bar{u} - \left(d_1 L + (1-d_1)\,\bar{C}\right)\right)\right] + \left[-\log\left(\bar{u} - \left(d_2 H + (1-d_2)\,\bar{C}\right)\right)\right].$$

Moreover, the terms on the RHS have a higher spread since:

$$-\log\left(\bar{u} - \left(d_2 H + (1-d_2)\,\bar{C}\right)\right) > -\log\left(\bar{u} - \left(d_1 H + (1-d_1)\,\bar{C}\right)\right) \iff H < \bar{C}$$

$$-\log\left(\bar{u} - \left(d_2 H + (1-d_2)\,\bar{C}\right)\right) > -\log\left(\bar{u} - \left(d_2 L + (1-d_1)\,\bar{C}\right)\right) \iff L < H$$

$$-\log\left(\bar{u} - \left(d_1 L + (1-d_1)\,\bar{C}\right)\right) < -\log\left(\bar{u} - \left(d_2 L + (1-d_2)\,\bar{C}\right)\right) \iff L < \bar{C}$$

$$-\log\left(\bar{u} - \left(d_1 L + (1-d_1)\,\bar{C}\right)\right) < -\log\left(\bar{u} - \left(d_1 H + (1-d_1)\,\bar{C}\right)\right) \iff L < H$$

Since $\phi(v) = g(-\log(\bar{u} - v))$, where g is concave, Jensen's inequality implies that (9) holds.

Case 3: $H \geq \bar{C} \geq L$**.**

Note that $H \geq \bar{C}$ implies

$$d_1 H + (1 - d_1)\,\bar{C} \geq d_2 H + (1 - d_2)\,\bar{C},$$

whereas $\bar{C} \geq L$ implies

$$d_2 L + (1 - d_2)\,\bar{C} \geq d_1 L + (1 - d_1)\,\bar{C}.$$

Since ϕ is increasing, it follows that (9) holds.

Proof of part (ii).

(a). Suppose there exist $H, L \in [\underline{u}, \bar{u}]$ and $d_1, d_2 \in \mathcal{D}$ with $H > L$ and $d_1 > d_2$, such that $\phi(z)$ is strictly more concave than $\log(z - \underline{u})$ on the interval $[v_1, v_2] \equiv [d_2 L + (1 - d_2)\underline{u},\ d_1 H + (1 - d_1)\underline{u}]$. Then, there exists a strictly increasing and strictly concave function $f : [v_1, v_2] \to \mathbb{R}$ such that

$$\phi(z) = f(\log(z - \underline{u}))$$

for all $z \in [v_1, v_2]$. By the properties of the log,

$$\begin{aligned}
\log\left(d_1 H + (1 - d_1)\underline{u} - \underline{u}\right) + \log\left(d_2 L + (1 - d_2)\underline{u} - \underline{u}\right) &= \\
\log\left(d_1 (H - \underline{u})\right) + \log\left(d_2 (L - \underline{u})\right) &= \\
\log\left(d_2 (H - \underline{u})\right) + \log\left(d_1 (L - \underline{u})\right) &= \\
\log\left(d_2 H + (1 - d_2)\underline{u} - \underline{u}\right) + \log\left(d_1 L + (1 - d_1)\underline{u} - \underline{u}\right).
\end{aligned}$$

By the strict concavity of f and the fact that $d_1 > d_2$ and $H > L$, it follows that:

$$f\left(\log(d_1 (H - \underline{u})\right) + f\left(\log\left(d_2 (L - \underline{u}\right)\right) < f\left(\log\left(d_2 (H - \underline{u})\right) + f\left(\log\left(d_1 (L - \underline{u}\right)\right),$$

establishing that Stochastic Impatience fails.

(b). Suppose there exist $H, L \in [\underline{u}, \bar{u}]$ and $d_1, d_2 \in \mathcal{D}$ with $H > L$ and $d_1 > d_2$, such that $\phi(z)$ is strictly more convex that $-\log(\bar{u} - z)$ on the interval $[v_3, v_4] \equiv [d_1 L + (1 - d_1)\bar{u},\ d_2 H + (1 - d_2)\bar{u}]$. Then, there exists a strictly increasing and strictly convex function $f : [v_3, v_4] \to \mathbb{R}$ such that

$$\phi(z) = g(-\log(\bar{u} - z))$$

for all $z \in [v_3, v_4]$. Again, by the properties of the log,

$$- \log \left[\bar{u} - \left(d_1 H + (1 - d_1) \, \bar{u} \right) \right] - \log \left[\bar{u} - \left(d_2 L + (1 - d_2) \, \bar{u} \right) \right]$$

$$= - \log \left[\bar{u} - \left(d_1 L + (1 - d_1) \, \bar{u} \right) \right] - \log \left[\bar{u} - \left(d_2 H + (1 - d_2) \, \bar{u} \right) \right].$$

Note that, for any $A \in [\underline{u}, \bar{u}]$, we have

$$- \log \left[\bar{u} - \left(d_2 A + (1 - d_2) \, \bar{u} \right) \right] > - \log \left[\bar{u} - \left(d_1 A + (1 - d_1) \, \bar{u} \right) \right]$$

$$\iff - \log \left(\frac{d_2 (\bar{u} - A)}{d_1 (\bar{u} - A)} \right) > 0 \iff d_2 < d_1,$$

which is true by assumption. It follows that, for $d \in \{d_1, d_2\}$ and $A \in \{H, L\}$, the expression

$$- \log \left[u(\bar{x}) - \left(dA + (1 - d) \, u(\bar{x}) \right) \right]$$

takes its highest value when $d = d_2$ and $A = H$ and its lowest value when $d = d_1$ and $A = L$. Then, by the strict convexity of g, we have

$$g \left(- \log \left[\bar{u} - \left(d_1 H + (1 - d_1) \, \bar{u} \right) \right] \right) + g \left(- \log \left[\bar{u} - \left(d_2 L + (1 - d_2) \, \bar{u} \right) \right] \right)$$

$$< g \left(- \log \left[\bar{u} - \left(d_1 L + (1 - d_1) \, \bar{u} \right) \right] \right) + g \left(- \log \left[\bar{u} - \left(d_2 H + (1 - d_2) \, \bar{u} \right) \right] \right),$$

showing that Stochastic Impatience fails. ■

C.6 Proof of Proposition 4

The proof will be presented in three lemmas.

Lemma 2. *Preferences are Residual Risk Averse (Seeking) if for all* $v_1, v_2 \in [u(\underline{x}), u(\overline{x})]$,

$$\phi \left(\gamma v_1 + (1 - \gamma) \, v_2 \right) + \phi \left(\gamma v_2 + (1 - \gamma) \, v_1 \right) \geq (\leq) \phi \left(v_1 \right) + \phi \left(v_2 \right) \tag{10}$$

where $\gamma \equiv \frac{D(1)}{D(1)+D(2)} \in (\frac{1}{2}, 1)$.

Proof. Note that by Definition 2 and the KM representation, preferences display Residual Risk Aversion whenever:

$$\phi \left(\frac{D(1) u(a) + D(2) u(d) + \sum_{t \notin \{1,2\}} D(t) u(x)}{\sum_{t=1}^{\infty} D(t)} \right)$$
$$= \phi \left(\frac{\left[D(1) + D(2) \right] u(b) + \sum_{t \notin \{1,2\}} D(t) u(x)}{\sum_{t=1}^{\infty} D(t)} \right)$$

and

$$\phi\left(\frac{D(1)u(d)+D(2)u(a)+\sum_{t\notin\{1,2\}}D(t)u(x)}{\sum_{t=1}^{\infty}D(t)}\right)$$
$$=\phi\left(\frac{[D(1)+D(2)]\,u(c)+\sum_{t\notin\{1,2\}}D(t)u(x)}{\sum_{t=1}^{\infty}D(t)}\right)$$

imply

$$\frac{\phi\left(u(b)\right)+\phi\left(u(c)\right)}{2}\geq\frac{\phi\left(u(a)\right)+\phi\left(u(d)\right)}{2}.$$

Since ϕ is strictly increasing, the first two equations can be simplified as:

$$u(b)=\frac{D(1)u(a)+D(2)u(d)}{D(1)+D(2)}\text{ and }u(c)=\frac{D(1)u(d)+D(2)u(a)}{D(1)+D(2)}.$$

Therefore, Residual Risk Aversion holds if and only if, for all a and all d,

$$\phi\left(\frac{D(1)u(a)+D(2)u(d)}{D(1)+D(2)}\right)+\phi\left(\frac{D(1)u(d)+D(2)u(a)}{D(1)+D(2)}\right)\geq\phi\left(u(a)\right)+\phi\left(u(d)\right). \quad (11)$$

Letting $\gamma\equiv\frac{D(1)}{D(1)+D(2)}$, $v_1\equiv u(a)$, and $v_2\equiv u(d)$ concludes the proof. □

Lemma 3. *Let (ϕ,u,D) be a KM representation of $\succcurlyeq$.*

- *If ϕ is discontinuous at any point $v\neq u\left(\underline{x}\right)$, then $\succcurlyeq$ is not Residual Risk Averse.*
- *If ϕ is discontinuous at any point $v\neq u\left(\overline{x}\right)$, then $\succcurlyeq$ is not Residual Risk Seeking.*

Proof. Suppose ϕ is discontinuous at $v>u\left(\underline{x}\right)$. Let $\{h_n\}\searrow v$ be a non-increasing sequence that converges to v and $\{l_n\}\nearrow v$ be a non-decreasing sequence that converges to v. Let

$$\phi_+:=\lim_{n\to\infty}\phi(h_n)>\lim_{n\to\infty}\phi(l_n)=\phi_-.$$

For each n, let $u_{a_n}:=h_n$ and $u_{d_n}:=\frac{l_n-\gamma h_n}{1-\gamma}$. Note that

$$u_{d_n}<\gamma u_{d_n}+(1-\gamma)\,u_{a_n}<\gamma u_{a_n}+(1-\gamma)\,u_{d_n}=l_n<v<h_n=u_{a_n}.$$

Since ϕ is bounded (by $\phi(u(\underline{x})$ and $\phi(u(\overline{x}))$, we can assume that the sequences $\{\phi\left(\gamma u_{a_n}+(1-\gamma)\,u_{d_n}\right)\}$, $\{\phi\left(\gamma u_{d_n}+(1-\gamma)\,u_{a_n}\right)\}$, $\{\phi\left(u_{a_n}\right)\}$, and $\{\phi\left(u_{d_n}\right)\}$ are convergent (taking a subsequence if necessary). Therefore,

$$\lim_{n\to\infty}\phi(u_{d_n})=\lim_{n\to\infty}\phi\left(\gamma u_{d_n}+(1-\gamma)\,u_{a_n}\right)=\lim_{n\to\infty}\phi\left(\gamma u_{a_n}+(1-\gamma)\,u_{d_n}\right)=\phi_-,$$

and

$$\lim_{n\to\infty} \phi(u_{a_n}) = \phi_+ > \phi_-.$$

Therefore, there exists $\bar{n}$ such that for all $n > \bar{n}$,

$$\phi\left(\gamma u_{a_n} + (1-\gamma)\, u_{d_n}\right) + \phi\left(\gamma u_{d_n} + (1-\gamma)\, u_{a_n}\right) < \phi\left(u_{a_n}\right) + \phi\left(u_{d_n}\right),$$

which, by (10), shows that preferences are not Residual Risk Averse.

Next, suppose ϕ is discontinuous at $v < u\left(\overline{x}\right)$. Let $\{h_m\} \searrow v$ be a non-increasing sequence that converges to v, let $\{l_m\} \nearrow v$ be an increasing sequence that converges to v. As before, let

$$\phi_+ := \lim_{m\to\infty} \phi(h_m) > \lim_{m\to\infty} \phi(l_m) = \phi_-,$$

where the limits exist by the Monotone Convergence Theorem.

For each m, take $u_{d_m} := l_m$ and take $u_{a_m} = \frac{h_m - \gamma l_m}{1-\gamma}$. Note that

$$u_{a_m} > \gamma u_{a_m} + (1-\gamma)\, u_{d_m} > \gamma u_{d_m} + (1-\gamma)\, u_{a_m} = h_m > x > l_m = u_{d_m}.$$

As before (taking a subsequence if necessary), we have

$$\lim_{m\to\infty} \phi\left(u_{a_m}\right) = \lim_{m\to\infty} \phi\left(\gamma u_{a_m} + (1-\gamma)\, u_{d_m}\right) = \lim_{m\to\infty} \phi\left(\gamma u_{d_m} + (1-\gamma)\, u_{a_m}\right) = \phi_+,$$

and

$$\lim_{m\to\infty} \phi\left(u_{d_m}\right) = \phi_- < \phi_+.$$

Thus, there exists $\bar{m}$ such that for all $m > \bar{m}$,

$$\phi\left(\gamma u_{a_m} + (1-\gamma)\, u_{d_m}\right) + \phi\left(\gamma u_{d_m} + (1-\gamma)\, u_{a_m}\right) > \phi\left(u_{a_m}\right) + \phi\left(u_{d_m}\right),$$

showing that preferences are not Residual Risk Seeking. □

Lemma 4. *Let (ϕ, u, D) be a KM representation of $\succcurlyeq$.*

- *$\succcurlyeq$ is Residual Risk Averse if and only if ϕ is concave.*
- *$\succcurlyeq$ is Residual Risk Seeking if and only if ϕ is convex.*

Proof. To establish sufficiency, suppose, without loss of generality, that $v > w$, so that

$$v > \gamma v + (1-\gamma) w > \gamma w + (1-\gamma) v > w.$$

It follows from the definition of concavity (convexity) and inequality (10) that preferences are Residual Risk Averse (Seeking) if ϕ is concave (convex). We now establish necessity. Suppose preferences are Residual Risk Averse. By the Lemma 3, ϕ must be continuous at any point $v > u(\underline{x})$. We need to show that ϕ is concave. Suppose not. Then, there exist $v, w \in [u(\underline{x}), u(\overline{x})]$ with $v > w$ and $\lambda \in (0,1)$ such that

$$\lambda\phi(v) + (1-\lambda)\,\phi(w) > \phi\left(\lambda v + (1-\lambda)\,w\right). \tag{12}$$

Let $F : [0,1] \to \mathbb{R}$ given by

$$F(\tilde{\lambda}) \equiv \phi\left(\tilde{\lambda}v + \left(1-\tilde{\lambda}\right)w\right) - \left[\tilde{\lambda}\phi(v) + \left(1-\tilde{\lambda}\right)\phi(w)\right],$$

and note that $F(\lambda) < 0$, while $F(1) = F(0) = 0$. Since ϕ can only be discontinuous at $u(\underline{x})$, $F(\tilde{\lambda})$ is continuous at all $\tilde{\lambda} > 0$. It is continuous at $\tilde{\lambda} = 0$ if either $w > u(\underline{x})$ or if ϕ is continuous at $u(\underline{x})$.

Let

$$L \equiv \left\{\tilde{\lambda} \in [0,\lambda] :\; F(\tilde{\lambda}) \leq 0\right\} \text{ and } H \equiv \left\{\tilde{\lambda} \in [\lambda,1] :\; F(\tilde{\lambda}) \geq 0\right\}.$$

Let

$$l \equiv \sup L \text{ and } h \equiv \inf H.$$

Because $F(\tilde{\lambda})$ is continuous at all $\tilde{\lambda} > 0$ and $F(\lambda) < 0$, it follows that $l < \lambda < h$. Moreover, it follows from the definitions of the supremum and infimum that

$$F\left(\tilde{\lambda}\right) < 0 \quad \forall \tilde{\lambda} \in (l,h).$$

We claim that $F(l) = 0$. There are two cases to consider. If F is continuous at 0, then L is a compact and non-empty set ($0 \in L$), which implies that $F(l) = 0$. Suppose, instead, that F is discontinuous at 0, which can only happen if $w = u(\underline{x})$ and ϕ is discontinuous at $u(\underline{x})$. Because ϕ is increasing, the discontinuity must correspond to an upwards jump: $\phi(u(\underline{x})) < \lim_{z \searrow u(\underline{x})} \phi(z) =: \phi(u(\underline{x})_+)$. Since

$$\lim_{\tilde{\lambda}\searrow 0} F(\tilde{\lambda}) = \lim_{\tilde{\lambda}\searrow 0} \left\{\phi\left(\tilde{\lambda}v + \left(1-\tilde{\lambda}\right)w\right) - \left[\tilde{\lambda}\phi(v) + \left(1-\tilde{\lambda}\right)\phi(w)\right]\right\}$$
$$= \phi\left(u(\underline{x})_+\right) - \phi(u(\underline{x})) > 0,$$

and F is continuous at any $\tilde{\lambda} > 0$, there exists $\bar{\lambda} > 0$ such that $F(\tilde{\lambda}) > 0$ for all $\tilde{\lambda} \in (0, \bar{\lambda})$. Hence,again by continuity of F for $\tilde{\lambda} > 0$, $l \geq \bar{\lambda}$ by the definition of supremum. Therefore,

$$l \equiv \sup L = \sup\left\{\tilde{\lambda} \in [\bar{\lambda}, \lambda] :\; F(\tilde{\lambda}) \leq 0\right\}.$$

Because $\left\{\tilde{\lambda} \in [\bar{\lambda}, \lambda] : \ F(\tilde{\lambda}) \le 0\right\}$ is compact ($F(\tilde{\lambda})$ is continuous for all $\tilde{\lambda} > 0$) and non-empty (λ belongs to it), it again follows that $F(l) = 0$.

Next, we show that $F(h) = 0$. Because $F(\lambda) < 0$ and $F(0) = 0$, (12) implies that $\lambda > 0$. Therefore, $F(\tilde{\lambda})$ is continuous at $[\alpha, 1]$, implying that H is a compact set. Because it is also non-empty ($1 \in H$), we must have $F(h) = 0$.

Substituting the definition of F, we have shown:

$$l\phi(v) + (1-l)\,\phi(w) = \phi\left(lv + (1-l)\,w\right), \tag{13}$$

$$h\phi(v) + (1-h)\,\phi(w) = \phi\left(hv + (1-h)\,w\right), \tag{14}$$

and

$$\tilde{\lambda}\phi(v) + \left(1-\tilde{\lambda}\right)\phi(w) > \phi\left(\tilde{\lambda}v + \left(1-\tilde{\lambda}\right)w\right) \tag{15}$$

for all $\tilde{\lambda} \in (l, h)$.

Let $w' \equiv lv + (1-l)w$ and $v' \equiv hv + (1-h)w$, so that $w < w' < v' < v$. Note that, for all $\lambda \in (0,1)$, we have

$$\begin{aligned} \lambda w' + (1-\lambda)\,v' \ &= \lambda\left[lv + (1-l)w\right] + (1-\lambda)\left[hv + (1-h)w\right] \\ &= \left[\lambda l + (1-\lambda)\,h\right]v + \left\{1 - \left[\lambda l + (1-\lambda)\,h\right]\right\}w \end{aligned}. \tag{16}$$

Since $\lambda l + (1-\lambda)\,h \in (l, h)$, we have

$$\begin{aligned} \phi\left(\lambda w' + (1-\lambda)\,v'\right) \ &= \phi\left(\left[\lambda l + (1-\lambda)\,h\right]v + \left\{1 - \left[\lambda l + (1-\lambda)\,h\right]\right\}w\right) \\ &< \left[\lambda l + (1-\lambda)\,h\right]\phi(v) + \left\{1 - \left[\lambda l + (1-\lambda)\,h\right]\right\}\phi\left(w\right) \\ &= \lambda\left[l\phi(v) + (1-l)\phi(w)\right] + (1-\lambda)\left[h\phi(v) + (1-h)\phi(w)\right] \\ &= \lambda\phi\left(lv + (1-l)\,w\right) + (1-\lambda)\,\phi\left(hv + (1-h)\,w\right) \\ &= \lambda\phi\left(w'\right) + (1-\lambda)\,\phi\left(v'\right) \end{aligned}$$

for all $\lambda \in (0,1)$, where the first line uses (16), the second line uses equation (15), the third line follows from algebraic manipulations, the fourth line uses (13) and (14), and the last line substitutes the definitions of v' and w'. Since this inequality holds for all $\lambda \in (0,1)$, in particular, it must hold for γ and $1-\gamma$:

$$\phi\left(\gamma w' + (1-\gamma)\,v'\right) < \gamma\phi\left(w'\right) + (1-\gamma)\,\phi\left(v'\right)$$

and

$$\phi\left(\gamma v' + (1-\gamma)\,w'\right) < \gamma\phi\left(v'\right) + (1-\gamma)\,\phi\left(w'\right).$$

Combining these two inequalities, gives

$$\phi\left(\gamma w' + (1-\gamma)\,v'\right) + \phi\left(\gamma v' + (1-\gamma)\,w'\right) < \phi\left(w'\right) + \phi\left(v'\right),$$

showing that Residual Risk Aversion fails. The proof for Residual Risk Seeking is analogous. □

This concludes the proof of Proposition 4. ■

C.7 Proof of Proposition 5

Recall from Proposition 3 that Stochastic impatience holds if

$$\begin{aligned}\phi\left(d_1 H+(1-d_1)\bar{C}\right)+\phi\left(d_2 L+(1-d_2)\bar{C}\right)\geq\\ \phi\left(d_1 L+(1-d_1)\bar{C}\right)+\phi\left(d_2 H+(1-d_2)\bar{C}\right)\end{aligned}\tag{17}$$

for all $H, L, \bar{C} \in \mathcal{U} \equiv u(C)$ with $H > L$, and all $d_1, d_2 \in \mathcal{D}$ with $d_1 > d_2$.

Proof of part (i). Let (ϕ_i, u_i, D_i) and (ϕ, u, D) be the KM representations of $\succcurlyeq_i$, for $i = 1, 2$, and $\succcurlyeq$. Since all relations agree on the ranking of degenerate lotteries, we have that $u_i = u$ and $D_i = D$. We need to show that (17) holds. There are three cases: (1) $H > L > \bar{C}$, (2) $\bar{C} > H > L$, and (3) $H > \bar{C} > L$.

Suppose first that $H > L > \bar{C}$. Since $\succcurlyeq$ has less residual risk aversion than $\succcurlyeq_1$, it must be the case that $\phi = g \circ \phi_1$ for an increasing and convex function g. Since $\succcurlyeq_1$ satisfies Stochastic Impatience, we have

$$\begin{aligned}\phi_1\left(d_1 H+(1-d_1)\bar{C}\right)+\phi_1\left(d_2 L+(1-d_2)\bar{C}\right)\geq\\ \phi_1\left(d_1 L+(1-d_1)\bar{C}\right)+\phi_1\left(d_2 H+(1-d_2)\bar{C}\right)\end{aligned}\tag{18}$$

for all $d_1, d_2 \in \mathcal{D}$ with $d_1 > d_2$.

We need to show that

$$\begin{aligned}g(\phi_1\left(d_1 H+(1-d_1)\bar{C}\right))+g(\phi_1\left(d_2 L+(1-d_2)\bar{C}\right))\geq\\ g(\phi_1\left(d_1 L+(1-d_1)\bar{C}\right))+g(\phi_1\left(d_2 H+(1-d_2)\bar{C}\right)).\end{aligned}$$

As in the proof of Proposition 3, it can be checked that the terms on the LHS of the expression have a higher mean (by equation (18)) and a higher spread than the terms on the RHS (since $H > L > \bar{C}$ and $d_1 > d_2$). Then, the result follows by Jensen's inequality (since g is convex).

Next, suppose that $\bar{C} > H > L$. Since $\succcurlyeq$ has more residual risk aversion than $\succcurlyeq_2$, it must be the case that $\phi = g \circ \phi_2$ for an increasing and concave function g. Since $\succcurlyeq_2$ satisfies Stochastic Impatience, we have

$$\begin{aligned}\phi_2\left(d_1 H+(1-d_1)\bar{C}\right)+\phi_2\left(d_2 L+(1-d_2)\bar{C}\right)\geq\\ \phi_2\left(d_1 L+(1-d_1)\bar{C}\right)+\phi_2\left(d_2 H+(1-d_2)\bar{C}\right)\end{aligned}\tag{19}$$

for all $d_1, d_2 \in \mathcal{D}$ with $d_1 > d_2$.

We need to show that

$$g(\phi_2\left(d_1 H + (1-d_1)\,\bar{C}\right)) + g(\phi_2\left(d_2 L + (1-d_2)\,\bar{C}\right)) \geq \\ g(\phi_2\left(d_1 L + (1-d_1)\,\bar{C}\right)) + g(\phi_2\left(d_2 H + (1-d_2)\,\bar{C}\right)).$$

As in the proof of Proposition 3, the terms on the LHS of the expression have a higher mean (by equation (19)) and a lower spread than the terms on the RHS (since $\bar{C} > H > L$ and $d_1 > d_2$). The result then follow by Jensen's inequality (since g is concave).

Finally, note that when $H > \bar{C} > L$, we have

$$d_1 H + (1-d_1)\,\bar{C} \geq d_2 H + (1-d_2)\,\bar{C} \iff H \geq \bar{C}$$

and

$$d_2 L + (1-d_2)\,\bar{C} \geq d_1 L + (1-d_1)\,\bar{C} \iff \bar{C} \geq L.$$

Therefore, for any increasing ϕ, equation (17) holds.

Proof of part (ii). Let (ϕ_1, u_1, D_1) and (ϕ, u, D) be the KM representations of $\succcurlyeq_1$ and $\succcurlyeq$. Since both relations agree on degenerate lotteries, we have that $u_1 = u$ and $D_1 = D$. We need to show that (17) fails.

Suppose first that $\succcurlyeq$ has more residual risk aversion than $\succcurlyeq_1$, so that $\phi = g \circ \phi_1$ for an increasing and concave function g. Since $\succcurlyeq_1$ does not satisfy Stochastic Impatience, we have

$$\phi_1\left(d_1 H + (1-d_1)\,\bar{C}\right) + \phi_1\left(d_2 L + (1-d_2)\,\bar{C}\right) < \\ \phi_1\left(d_1 L + (1-d_1)\,\bar{C}\right) + \phi_1\left(d_2 H + (1-d_2)\,\bar{C}\right) \tag{20}$$

for some $H > L$ and some $d_1, d_2 \in \mathcal{D}$ with $d_1 > d_2$. Take $\bar{C} < L$. By Jensen's inequality, it follows that

$$g(\phi_1\left(d_1 H + (1-d_1)\,\bar{C}\right)) + g(\phi_1\left(d_2 L + (1-d_2)\,\bar{C}\right)) < \\ g(\phi_1\left(d_1 L + (1-d_1)\,\bar{C}\right)) + g(\phi_1\left(d_2 H + (1-d_2)\,\bar{C}\right)),$$

where we used the fact that the terms on the LHS of the expression have a lower mean (by equation (20)), a higher spread than the terms on the RHS (since $H > L > \bar{C}$ and $d_1 > d_2$), and g is concave. The proof of (ϕ_2, u_1, D_1) is analogous.

Proof of part (iii). For (iii.a), let $\phi_3 = g \cdot \phi$. If $g(v)$ is more concave than $\phi^{-1}(\log(v - u((\underline{x})))$ on $\mathcal{U}$, then by Proposition 3 (part ii) it violates Stochastic Impatience. The result for (iii.b) is analogous.

This concludes the proof of Proposition 5. ■

C.8 Proof of Observations 3, 4, and 5

Observation 3 is due to Proposition 4 and the fact that KM coincides with EDU if and only if ϕ is affine. Observation 4 follows from Proposition 4 and the KM representation of EZ given in Example 3. Observation 5 follows directly from Proposition 4. ■

C.9 Proof of Proposition 6

Let $\mathcal{D}' \equiv \left\{ \frac{\int_{\tilde{t}=t}^{t+\tau} D(\tilde{t})d\tilde{t}}{\int_{\tilde{t}=0}^{\infty} D(\tilde{t})d\tilde{t}} \,:\, t,\tau \in \mathbb{R}_+ \right\}$ and note that, by definition, Stochastic impatience' holds if

$$\begin{aligned} \phi\left(d_1 H + (1-d_1)\bar{C}\right) + \phi\left(d_2 L + (1-d_2)\bar{C}\right) \geq \\ \phi\left(d_1 L + (1-d_1)\bar{C}\right) + \phi\left(d_2 H + (1-d_2)\bar{C}\right) \end{aligned} \tag{21}$$

for all $H, L, \bar{C} \in [0,1]$ with $H > L$, and all $d_1, d_2 \in \mathcal{D}'$ with $d_1 > d_2$. Similarly, let $\mathcal{D}'' \equiv \left\{ \frac{\int_{\tilde{t}=t}^{\infty} D(\tilde{t})d\tilde{t}}{\int_{\tilde{t}=0}^{\infty} D(\tilde{t})d\tilde{t}} \,:\, t \in \mathbb{R}_+ \right\}$ and note that Stochastic impatience" holds if

$$\begin{aligned} \phi\left(d_1 H + (1-d_1)\bar{C}\right) + \phi\left(d_2 L + (1-d_2)\bar{C}\right) \geq \\ \phi\left(d_1 L + (1-d_1)\bar{C}\right) + \phi\left(d_2 H + (1-d_2)\bar{C}\right) \end{aligned} \tag{22}$$

for all $H, L, \bar{C} \in [0,1]$ with $H > L$, and all $d_1, d_2 \in \mathcal{D}''$ with $d_1 > d_2$.

Observe that $\mathcal{D}' = (0,1) = \mathcal{D}''$. This implies that Stochastic Impatience' and Stochastic Impatience" are equivalent, proving (i) $\Leftrightarrow$ (ii).

Following the exact same steps as in the proof of part (i) of Proposition 3 but replacing $\mathcal{D}$ with $\mathcal{D}'$, we find that Stochastic Impatience' holds if ϕ is more convex than $\underline{\phi}$ and more concave than $\bar{\phi}$.

To establish that the converse is also true, note that, unlike in Proposition 3, $\mathcal{D}'$ is now an open interval. Therefore, whenever there is a point v in which ϕ is either strictly more concave than $\underline{\phi}$ or strictly more convex than $\bar{\phi}$, we can find $d_1, d_2 \in (0,1)$ and $H, L \in \mathcal{U} \equiv u(C)$ such that ϕ is either strictly more concave than $\underline{\phi}$ on the interval $[d_2 L + (1-d_2)u(\underline{x}),\ d_1 H + (1-\ d_1)u(\underline{x})]$ or strictly more convex than $\bar{\phi}$ on the interval $[d_1 L + (1-d_1)\bar{u},\ d_2 H + (1-d_2)\bar{u}]$. The result then follows from part (ii) of the proof of Proposition 3. ■

C.10 Proof of Proposition 7

Let $x \in (\underline{x}, \bar{x})$ be such that $\phi'(u(x)) > 0$ (which exists because ϕ is differentiable and strictly increasing). Without loss of generality, let $u(\underline{x}) = 0$. Pick an arbitrary $\delta > 0$

and an arbitrary $t' > 0$ and let x' be the value such that

$$u(x') = u(x) \cdot \frac{\int_{t'}^{t'+\delta} D(s)ds}{\int_0^{\delta} D(s)ds}. \tag{23}$$

Note that $x' \in (\underline{x}, x)$ exists by the continuity of $u(\cdot)$ and the fact that $D(\cdot)$ is decreasing.

Lemma 5. *Stochastic Impatience' fails if* $\pi(\frac{1}{2}) < \frac{1}{2}$.

Proof. We claim that if t' is close enough to zero, then

$$\frac{1}{2}(x, 0; \underline{x}, \delta) + \frac{1}{2}(x', t'; \underline{x}, \delta) \prec \frac{1}{2}(x, t'; \underline{x}, \delta) + \frac{1}{2}(x', 0; \underline{x}, \delta),$$

violating Stochastic Impatience' (with $c = \underline{x}$ and $t = 0$). Writing in terms of the representation, this means that for t' close enough to 0,

$$\pi\left(\frac{1}{2}\right)\phi\left(u(x)\frac{\int_0^{\delta} D(s)ds}{\int_0^{+\infty} D(s)ds}\right) + \left[1 - \pi\left(\frac{1}{2}\right)\right]\phi\left(u(x')\frac{\int_{t'}^{t'+\delta} D(s)ds}{\int_0^{+\infty} D(s)ds}\right) < \phi\left(u(x) \cdot \frac{\int_{t'}^{t'+\delta} D(s)ds}{\int_0^{+\infty} D(s)ds}\right),$$

where we use (23) for the RHS (note that with the normalization of $u(\underline{x})$, we have that $(x, t'; \underline{x}, \delta) \sim (x', 0; \underline{x}, \delta)$). Substitute (23) on the LHS of this inequality to rewrite it as:

$$\pi\left(\frac{1}{2}\right)\phi\left(u(x)\frac{\int_0^{\delta} D(s)ds}{\int_0^{+\infty} D(s)ds}\right) + \left[1 - \pi\left(\frac{1}{2}\right)\right]\phi\left(u(x) \cdot \frac{\left[\int_{t'}^{t'+\delta} D(s)ds\right]^2}{\int_0^{\delta} D(s)ds \cdot \int_0^{+\infty} D(s)ds}\right) < \phi\left(u(x) \cdot \frac{\int_{t'}^{t'+\delta} D(s)ds}{\int_0^{+\infty} D(s)ds}\right). \tag{24}$$

First, note that both sides of (24) equal $\phi\left(u(x)\frac{\int_0^{\delta} D(s)ds}{\int_0^{+\infty} D(s)ds}\right)$ when $t' = 0$. We now show that the LHS falls faster than RHS when we increase t' slightly, so that (24) holds for $t' \approx 0$.

Use Liebniz's rule to write the derivative of the expression on the LHS of (24) with respect to t' evaluated at 0 equals

$$-\left[1 - \pi\left(\frac{1}{2}\right)\right]\phi'\left(u(x) \cdot \frac{\int_0^{\delta} D(s)ds}{\int_0^{+\infty} D(s)ds}\right)\frac{2u(x)}{\int_0^{+\infty} D(s)ds}\left[D(0) - D(\delta)\right],$$

and the derivative of the expression on the RHS of (24) with respect to t' evaluated at 0 equals

$$-\phi'\left(u(x)\cdot\frac{\int_0^{\delta}D(s)ds}{\int_0^{+\infty}D(s)ds}\right)\frac{u(x)}{\int_0^{+\infty}D(s)ds}\left[D(0)-D(\delta)\right].$$

Thus, (24) holds for $t'>0$ small enough if

$$-\left[1-\pi\left(\frac{1}{2}\right)\right]\phi'\left(u(x)\cdot\frac{\int_0^{\delta}D(s)ds}{\int_0^{+\infty}D(s)ds}\right)\frac{2u(x)}{\int_0^{+\infty}D(s)ds}\left[D(0)-D(\delta)\right]$$
$$<-\phi'\left(u(x)\cdot\frac{\int_0^{\delta}D(s)ds}{\int_0^{+\infty}D(s)ds}\right)\frac{u(x)}{\int_0^{+\infty}D(s)ds}\left[D(0)-D(\delta)\right],$$

which can be simplified to $\pi\left(\frac{1}{2}\right)<\frac{1}{2}$. Therefore, Stochastic Impatience' fails whenever $\pi\left(\frac{1}{2}\right)<\frac{1}{2}$. □

Lemma 6. *Stochastic Impatience' fails if* $\pi(\frac{1}{2})>\frac{1}{2}$.

Proof. The value of the lottery $\frac{1}{2}(x,0;\bar{x},\delta)+\frac{1}{2}(x',t';\bar{x},\delta)$ is

$$\pi\left(\frac{1}{2}\right)\phi\left(u(x)\frac{\int_0^{\delta}D(s)ds}{\int_0^{+\infty}D(s)ds}+u(\bar{x})\frac{\int_{\delta}^{+\infty}D(s)ds}{\int_0^{+\infty}D(s)ds}\right)$$
$$+\left[1-\pi\left(\frac{1}{2}\right)\right]\phi\left(u(x')\frac{\int_{t'}^{t'+\delta}D(s)ds}{\int_0^{+\infty}D(s)ds}+u(\bar{x})\frac{\int_{s\notin[t',t'+\delta]}D(s)ds}{\int_0^{+\infty}D(s)ds}\right),$$

which, using (23), can be written as:

$$\pi\left(\frac{1}{2}\right)\phi\left(u(x)\frac{\int_0^{\delta}D(s)ds}{\int_0^{+\infty}D(s)ds}+u(\bar{x})\frac{\int_{\delta}^{+\infty}D(s)ds}{\int_0^{+\infty}D(s)ds}\right)+$$
$$\left[1-\pi\left(\frac{1}{2}\right)\right]\phi\left(u(x)\cdot\frac{\left[\int_{t'}^{t'+\delta}D(s)ds\right]^2}{\int_0^{\delta}D(s)ds\cdot\int_0^{+\infty}D(s)ds}+u(\bar{x})\frac{\int_{s\notin[t',t'+\delta]}D(s)ds}{\int_0^{+\infty}D(s)ds}\right).\quad(25)$$

The value of the lottery $\frac{1}{2}(x,t';\bar{x},\delta)+\frac{1}{2}(x',0;\bar{x},\delta)$ is

$$\pi\left(\frac{1}{2}\right)\phi\left(u(x')\frac{\int_0^{\delta}D(s)ds}{\int_0^{+\infty}D(s)ds}+u(\bar{x})\frac{\int_{\delta}^{+\infty}D(s)ds}{\int_0^{+\infty}D(s)ds}\right)+$$
$$\left[1-\pi\left(\frac{1}{2}\right)\right]\phi\left(u(x)\frac{\int_{t'}^{t'+\delta}D(s)ds}{\int_0^{+\infty}D(s)ds}+u(\bar{x})\frac{\int_{s\notin[t',t'+\delta]}D(s)ds}{\int_0^{+\infty}D(s)ds}\right),$$

which, again using (23), becomes

$$\pi\left(\frac{1}{2}\right)\phi\left(u(x)\frac{\int_{t'}^{t'+\delta}D(s)ds}{\int_0^{+\infty}D(s)ds}+u(\bar{x})\frac{\int_{\delta}^{+\infty}D(s)ds}{\int_0^{+\infty}D(s)ds}\right)+$$
$$\left[1-\pi\left(\frac{1}{2}\right)\right]\phi\left(u(x)\frac{\int_{t'}^{t'+\delta}D(s)ds}{\int_0^{+\infty}D(s)ds}+u(\bar{x})\frac{\int_{s\notin[t',t'+\delta]}D(s)ds}{\int_0^{+\infty}D(s)ds}\right). \quad (26)$$

At $t'=0$, both expressions equal $\phi\left(u(x)\frac{\int_0^{\delta}D(s)ds}{\int_0^{+\infty}D(s)ds}+u(\bar{x})\frac{\int_{\delta}^{\infty}D(s)ds}{\int_0^{+\infty}D(s)ds}\right)$. As in the proof of Lemma 5, it can be verified that if $\pi\left(\frac{1}{2}\right)>\frac{1}{2}$ then (25) increases slower than (26) when we increase t' slightly. Therefore, for sufficiently small $t'>0$,

$$\frac{1}{2}(x,0;\bar{x},\delta)+\frac{1}{2}(x',t';\bar{x},\delta)\prec\frac{1}{2}(x,t';\bar{x},\delta)+\frac{1}{2}(x',0;\bar{x},\delta),$$

violating Stochastic Impatience' (with $c=\bar{x}$ and $t=0$). □

This concludes the proof of Proposition 7. ■

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Online Appendix

Detailed Calculations in Examples 3 and 5

As in the proof of Proposition 1, let $\rho \equiv \frac{1}{\psi}$. Recall that, in EZ, lotteries are evaluated according to the recursion

$$V_t = \left\{ (1-\beta) C_t^{1-\rho} + \beta \left[E_t \left(V_{t+1}^{1-\alpha} \right) \right]^{\frac{1-\rho}{1-\alpha}} \right\}^{\frac{1}{1-\rho}}.$$

It can be verified that the EZ value of the stream $(x_2, x_3, x_4, ...)$ is:

$$V_2 = \left\{ (1-\beta) \cdot \left[x_2^{1-\rho} + \sum_{t=3}^{\infty} \beta^{t-2} x_t^{1-\rho} \right] \right\}^{\frac{1}{1-\rho}}.$$

In our domain, all uncertainty is resolved in period 1 and all streams have the same period-1 consumption ($x_1 = c$). So, the EZ utility of the lottery with random payments $\{\tilde{x}_t\}$ in periods $t \geq 2$ is:

$$V_1 = \left\{ (1-\beta) c^{1-\rho} + \beta \left[E_1 \left(\left\{ (1-\beta) \cdot \sum_{t=2}^{\infty} \beta^{t-2} x_t^{1-\rho} \right\}^{\frac{1-\alpha}{1-\rho}} \right) \right]^{\frac{1-\rho}{1-\alpha}} \right\}^{\frac{1}{1-\rho}}. \quad (27)$$

It is convenient to split in two cases depending on whether $\rho < 1$ or $\rho > 1$.

Case 1: $\rho < 1$

Since $x^{\frac{1}{1-\rho}}$ is a strictly increasing function of x for $\rho < 1$, it follows that preferences can be represented by

$$\tilde{V}_1 = (1-\beta) c^{1-\rho} + \beta \left[E_1 \left(\left\{ (1-\beta) \cdot \sum_{t=2}^{\infty} \beta^{t-2} x_t^{1-\rho} \right\}^{\frac{1-\alpha}{1-\rho}} \right) \right]^{\frac{1-\rho}{1-\alpha}}.$$

Since all lotteries have the same period-1 consumption c and $\beta > 0$, they are also represented by

$$\tilde{\tilde{V}}_1 = \left[E_1 \left(\left\{ (1-\beta) \cdot \sum_{t=2}^{\infty} \beta^{t-2} x_t^{1-\rho} \right\}^{\frac{1-\alpha}{1-\rho}} \right) \right]^{\frac{1-\rho}{1-\alpha}}. \quad (28)$$

There are two subcases.

Case 1a: $\alpha, \rho < 1$**.**

If $\alpha < 1$, so that $\frac{1-\rho}{1-\alpha} > 0$, we can raise the expression above by $\frac{1-\alpha}{1-\rho} > 0$ (which is a monotone transformation) to obtain the following equivalent representation for EZ:

$$\hat{V}_1 = E_1 \left\{ \left[(1-\beta) \cdot \sum_{t=2}^{\infty} \beta^{t-2} x_t^{1-\rho} \right]^{\frac{1-\alpha}{1-\rho}} \right\}.$$

Dividing this expression by $(1-\rho)^{\frac{1-\alpha}{1-\rho}} > 0$, we obtain

$$\hat{\hat{V}}_1 = E_1 \left\{ \left[(1-\beta) \cdot \sum_{t=2}^{\infty} \beta^{t-2} \cdot \frac{x_t^{1-\rho}}{1-\rho} \right]^{\frac{1-\alpha}{1-\rho}} \right\},$$

which is a KM representation with $\phi(z) = z^{\frac{1-\alpha}{1-\rho}}$ and $u(x) = \frac{x^{1-\rho}}{1-\rho}$. Note that ϕ is indeed increasing and its coefficient of absolute risk aversion of ϕ is $-\frac{\phi''(z)}{\phi'(z)} = \frac{1}{z} \cdot \frac{\alpha-\rho}{1-\rho}$.

Case 1b: $\rho < 1 < \alpha$**.**

Next, suppose $\alpha > 1 > \rho$. Applying the increasing transformation $g(z) \equiv -\frac{\left(z^{\frac{1-\alpha}{1-\rho}}\right)}{(1-\rho)^{\frac{1-\alpha}{1-\rho}}}$ to (28), we find that preferences can be represented by:

$$\hat{V} = E_1 \left\{ - \left[(1-\beta) \cdot \sum_{t=2}^{\infty} \beta^{t-2} \frac{x_t^{1-\rho}}{1-\rho} \right]^{\frac{1-\alpha}{1-\rho}} \right\},$$

giving a KM representation with $\phi(z) = -\left(z^{\frac{1-\alpha}{1-\rho}}\right)$ and $u(x) = \frac{x^{1-\rho}}{1-\rho}$. Note that ϕ is increasing (since $\frac{1-\alpha}{1-\rho} < 0$) and its coefficient of absolute risk aversion is $-\frac{\phi''(z)}{\phi'(z)} = \frac{1}{z} \cdot \frac{\alpha-\rho}{1-\rho}$.

Case 2: $\rho > 1$

We now consider the case of $\rho > 1$. Since $f(x) = x^{\frac{1}{1-\rho}}$ is a decreasing function when $\rho > 1$, it follows from (27) that preferences can be represented by

$$\tilde{V}_1 = -(1-\beta)\, c^{1-\rho} - \beta \left[E_1 \left(\left\{ (1-\beta) \cdot \sum_{t=2}^{\infty} \beta^{t-2} x_t^{1-\rho} \right\}^{\frac{1-\alpha}{1-\rho}} \right) \right]^{\frac{1-\rho}{1-\alpha}}.$$

As before, since the first term, $-(1-\beta)\,c^{1-\rho}$, is the same in all lotteries in our domain (the first-period consumption c is constant) and since $\beta > 0$ is a constant, preferences in this case can be represented by

$$-\left[E_1\left(\left\{(1-\beta)\cdot\sum_{t=2}^{\infty}\beta^{t-2}x_t^{1-\rho}\right\}^{\frac{1-\alpha}{1-\rho}}\right)\right]^{\frac{1-\rho}{1-\alpha}}. \tag{29}$$

There are two subcases: $\alpha, \rho > 1$ and $\rho > 1 > \alpha$.

Case 2a: $\alpha, \rho > 1$

Suppose first $\alpha, \rho > 1$, so that $\frac{1-\rho}{1-\alpha} > 0$. Applying the increasing transformation $f(x) = x^{\frac{1-\alpha}{1-\rho}}$, we find that preferences can also be represented by

$$-\left[E_1\left(\left\{(1-\beta)\cdot\sum_{t=2}^{\infty}\beta^{t-2}x_t^{1-\rho}\right\}^{\frac{1-\alpha}{1-\rho}}\right)\right].$$

Dividing by the constant $(\rho-1)^{\frac{1-\alpha}{1-\rho}} > 0$, establishes that preferences can be represented by

$$E_1\left\{-\left[-(1-\beta)\cdot\sum_{t=2}^{\infty}\beta^{t-2}\frac{x_t^{1-\rho}}{1-\rho}\right]^{\frac{1-\alpha}{1-\rho}}\right\},$$

which is a KM representation with $\phi(z) = -(-z)^{\frac{1-\alpha}{1-\rho}}$ and $u(x) = \frac{x^{1-\rho}}{1-\rho}$. Note that ϕ is increasing (since $\frac{1-\rho}{1-\alpha} > 0$) and the coefficient of absolute risk aversion is $-\frac{\phi''(z)}{\phi'(z)} = \frac{1}{z}\cdot\frac{\alpha-\rho}{1-\rho}$.

Case 2b: $\rho > 1 > \alpha$

Since $\frac{1-\rho}{1-\alpha} < 0$, it follows from (29) that preferences can be represented by

$$E_1\left(\left\{(1-\beta)\cdot\sum_{t=2}^{\infty}\beta^{t-2}x_t^{1-\rho}\right\}^{\frac{1-\alpha}{1-\rho}}\right).$$

Dividing this expression by $(\rho-1)^{\frac{1-\alpha}{1-\rho}} > 0$, we obtain

$$E_1\left\{\left[-(1-\beta)\cdot\sum_{t=2}^{\infty}\beta^{t-2}\frac{x_t^{1-\rho}}{1-\rho}\right]^{\frac{1-\alpha}{1-\rho}}\right\},$$

which is a KM representation with $\phi(z) = (-z)^{\frac{1-\alpha}{1-\rho}}$ and $u(x) = \frac{x^{1-\rho}}{1-\rho}$. Again, the coefficient of absolute risk aversion is $-\frac{\phi''(z)}{\phi'(z)} = \frac{1}{z}\cdot\frac{\alpha-\rho}{1-\rho}$.

SI conditions for EZ using the KM representation

From Proposition 3, a sufficient condition for SI is that ϕ is more convex than $\underline{\phi}(z) = \log(z - \underline{u})$ and more concave than $\bar{\phi}(z) \equiv -\log(\bar{u} - z)$. As calculated previously, the coefficients of relative risk aversion of ϕ equals $-\frac{\phi''(z)}{\phi'(z)} = \frac{1}{z} \cdot \frac{\alpha-\rho}{1-\rho}$, whereas:

$$-\frac{\bar{\phi}''(z)}{\bar{\phi}'(z)} = -\frac{1}{\bar{u} - z} \quad \text{and} \quad -\frac{\underline{\phi}''(z)}{\underline{\phi}'(z)} = \frac{1}{z - \underline{u}}.$$

Therefore, the sufficient condition for SI from from Proposition 3 is

$$-\frac{1}{\bar{u} - z} \leq \frac{1}{z} \cdot \frac{\alpha - \rho}{1 - \rho} \leq \frac{1}{z - \underline{u}} \tag{30}$$

for all $z \in u(\mathbb{R}_+)$, where $\bar{u} \equiv \sup\{u(c) : \; c \in \mathbb{R}_+\}$ and $\underline{u} \equiv \inf\{u(c) : \; c \in \mathbb{R}_+\}$.

Note that when $\rho < 1$, we have $u(\mathbb{R}_+) = [0, +\infty)$, so that $\bar{u} = +\infty$ and $\underline{u} = 0$. Then, condition (30) becomes

$$0 \leq \frac{\alpha - \rho}{1 - \rho} \leq 1 \iff \rho \leq \alpha \leq 1.$$

When, instead, $\rho > 1$, we have $u(\mathbb{R}_+) = (-\infty, 0]$, so that $\bar{u} = 0$ and $\underline{u} = -\infty$. Then, condition (30) becomes

$$0 \leq \frac{\alpha - \rho}{1 - \rho} \leq 1 \iff \rho \geq \alpha \geq 1.$$

Noting that since in EZ $\alpha \neq 1$, these are the same as the necessary and sufficient conditions from Proposition 1.

Detailed Calculations in Examples 4 and 6

Recall that the Risk Sensitive preferences of Hansen and Sargent (HS) admit the following recursive representation:

$$V_t = u(x_t) - \frac{\beta}{k} \log \left[E_t \left(e^{-kV_{t+1}} \right) \right].$$

In our setting, all lotteries have the same consumption in period 0 and all uncertainty is resolved in period 1. Since consumption is deterministic after the realization of uncertainty at the start of period 1, we have:

$$V_t = u(x_t) + \beta V_{t+1}$$

for all $t \geq 1$. It can be verified that the following expression solves this equation:

$$V_1 = \sum_{t=1}^{\infty} \beta^{t-1} u(x_t).$$

Taking expectations in period 0 (before uncertainty is resolved), we obtain the following expression:

$$V_0 = u(x_0) - \frac{\beta}{k} \log \left[E_0 \left(e^{-k \sum_{t=1}^{\infty} \beta^{t-1} u(x_t)} \right) \right].$$

Since all lotteries have the same consumption in period 0 in the domain we consider, we can omit the period-0 consumption. Moreover, since $\frac{\beta}{\kappa} > 0$ is a constant and the logarithm function is strictly increasing, HS preferences over lotteries in our domain can be also represented by:

$$\tilde{V}_0 = E_0 \left(-e^{-k \sum_{t=1}^{\infty} \beta^{t-1} u(x_t)} \right) = E_0 \left(-e^{-\kappa(1-\beta) \sum_{t=1}^{\infty} \beta^{t-1} u(x_t)} \right),$$

where $\kappa \equiv \frac{k}{1-\beta}$. This coincides with the KM representation for $\phi(z) \equiv -\exp\left(-\frac{kz}{1-\beta}\right)$.

SI conditions for HS using the KM representation

The coefficient of absolute risk aversion of ϕ equals:

$$-\frac{\phi''(z)}{\phi'(z)} = \frac{k}{1-\beta}.$$

Since $-\frac{\phi''(z)}{\phi'(z)} > 0$, the sufficient conditions from Proposition 3 hold if and only if ϕ is less concave than $\underline{\phi}$. Recall that the coefficient of absolute risk aversion of $\underline{\phi}$ equals:

$$-\frac{\underline{\phi}''(x)}{\underline{\phi}'(x)} = \frac{1}{x - \underline{u}}.$$

Therefore, the sufficient conditions from Proposition 3 hold if and only if:

$$\frac{k}{1-\beta} \leq \frac{1}{x - \underline{u}} \quad \forall x \in u(X) \iff \bar{u} - \underline{u} \leq \frac{1-\beta}{k}, \tag{31}$$

where $\bar{u} \equiv \sup\{u(x)\}_{x \in C}$ and $\underline{u} \equiv \inf\{u(x)\}_{x \in C}$.

Contrast (31) with the necessary and sufficient condition from Proposition 2:

$$\bar{u} - \underline{u} \leq -\frac{\log(\beta)}{\beta} \frac{1}{k}. \tag{32}$$

We claim that the sufficient condition from Proposition 3 is strictly weaker than the necessary and sufficient condition from Proposition 2, so there exist preferences that satisfy SI but do not satisfy the sufficient condition from Proposition 3. To establish this, we need to show that the bound in (32) is higher than the bound in (31):

$$-\frac{\log(\beta)}{\beta}\frac{1}{k} > \frac{1-\beta}{k} \iff \beta^2 - \beta - \log(\beta) > 0.$$

We claim that this inequality holds for all $\beta \in [0, 1)$. To see this first note that at $\beta = 1$, the LHS equals 0 so both bounds coincide. Moreover the derivative is negative for all $\beta \in [0, 1)$:

$$2\beta - 1 - \frac{1}{\beta} < 0 \iff \beta^2 - \frac{\beta}{2} - \frac{1}{2} < 0,$$

which is true since the expression on the LHS is an upward facing parabola with roots $-\frac{1}{2}$ and $+1$.

Proof of Proposition 1 (detailed calculations)

Recall that with EZ, lotteries are evaluated according to

$$V_t = \{(1-\beta)x_t^{1-\rho} + \beta[\mathbb{E}_t(V_{t+1}^{1-\alpha})]^{\frac{1-\rho}{1-\alpha}}\}^{\frac{1}{1-\rho}}. \tag{33}$$

Substitution verifies that the value of a constant stream that pays c is c:

$$V_0 = \{(1-\beta)c^{1-\rho} + \beta c^{1-\rho}\}^{\frac{1}{1-\rho}} = c.$$

Next, consider a stream that pays $(\underbrace{\overbrace{x}^{1}, x, ..., \overbrace{x}^{t}}_{t}, \overbrace{c}^{t+1}, c, c, ...)$. By the previous expression, the continuation value at $t+1$ is c. Using the expression in (33), we obtain:

$$V_t = \{(1-\beta)x^{1-\rho} + \beta c^{1-\rho}\}^{\frac{1}{1-\rho}}.$$

Substituting this expression for V_{t-1}, gives:

$$V_{t-1} = \{(1-\beta)x^{1-\rho} + \beta V_t^{1-\rho}\}^{\frac{1}{1-\rho}} = \{(1-\beta)(1+\beta)x^{1-\rho} + \beta^2 c^{1-\rho}\}^{\frac{1}{1-\rho}}.$$

Substituting recursively s times, gives the following expression:

$$V_{t-s} = \{(1-\beta)x^{1-\rho}(1+\beta+\beta^2+...+\beta^s) + \beta^{s+1}c^{1-\rho}\}^{\frac{1}{1-\rho}}.$$

In particular, taking $s = t-1$, gives value of the stream:

$$V_1 = \{(1-\beta)x^{1-\rho}(1+\beta+\beta^2+...+\beta^{t-1})+\beta^t c^{1-\rho}\}^{\frac{1}{1-\rho}} = \{(1-\beta^t)x^{1-\rho}+\beta^t c^{1-\rho}\}^{\frac{1}{1-\rho}}. \quad (34)$$

Next, consider the stream $(\overbrace{c}^{1}, ..., \overbrace{c}^{\tau-1}, \underbrace{\overbrace{x}^{\tau}, x, ..., \overbrace{x}^{\tau+t-1}}_{t}, c, c, ...)$. Note that the stream starting at τ is the same as the one evaluated in the previous parargaph. Therefore, by the previous calculations, we have

$$V_\tau = \{(1-\beta^t)x^{1-\rho} + \beta^t c^{1-\rho}\}^{\frac{1}{1-\rho}}.$$

Using the expression in (33), we obtain the value in period $\tau - 1$:

$$V_{\tau-1} = [(1-\beta)c^{1-\rho} + \beta V_\tau^{1-\rho}]^{\frac{1}{1-\rho}} = [c^{1-\rho} + \beta(1-\beta^t)(x^{1-\rho} - c^{1-\rho})]^{\frac{1}{1-\rho}}.$$

Substituting recursively s times, gives

$$V_{\tau-s} = \{c^{1-\rho} + \beta^s(1-\beta^t)(x^{1-\rho} - c^{1-\rho})\}^{\frac{1}{1-\rho}}.$$

Taking $s = \tau - 1$ gives

$$V_1 = \{c^{1-\rho} + \beta^{\tau-1}(1-\beta^t)(x^{1-\rho} - c^{1-\rho})\}^{\frac{1}{1-\rho}}. \quad (35)$$

Let c_0 be an arbitrary but fixed consumption in period 0. We are interested in the lottery that pays either

$$(\overbrace{c_0}^{0}, \underbrace{\overbrace{x}^{1}, x, ..., \overbrace{x}^{t}}_{t}, \overbrace{c}^{t+1}, c, c, ...)$$

or

$$(\overbrace{c_0}^{0}, \overbrace{c}^{1}, ..., \overbrace{c}^{\tau-1}, \underbrace{\overbrace{y}^{\tau}, y, ..., \overbrace{y}^{\tau+t-1}}_{t}, c, c, ...)$$

with 50-50 chance each. From the recursion in (33), the value of this lottery is:

$$V_0 = \left\{(1-\beta)c_0^{1-\rho} + \beta[\mathbb{E}_0(V_1^{1-\alpha})]^{\frac{1-\rho}{1-\alpha}}\right\}^{\frac{1}{1-\rho}}.$$

Using expressions in (34) and (35), we obtain

$$\mathbb{E}_0(V_1^{1-\alpha}) = \frac{\{(1-\beta^t)x^{1-\rho} + \beta^t c^{1-\rho}\}^{\frac{1-\alpha}{1-\rho}} + \{c^{1-\rho} + \beta^{\tau-1}(1-\beta^t)\,(x^{1-\rho} - c^{1-\rho})\}^{\frac{1-\alpha}{1-\rho}}}{2}.$$

Substituting in the expression for V_0, gives

$$V_0 = \left\{ (1-\beta)c_0^{1-\rho} + \beta \left[\frac{\begin{array}{l} \{(1-\beta^t)x^{1-\rho} + \beta^t c^{1-\rho}\}^{\frac{1-\alpha}{1-\rho}} \\ + \{c^{1-\rho} + \beta^{\tau-1}(1-\beta^t)\left(y^{1-\rho} - c^{1-\rho}\right)\}^{\frac{1-\alpha}{1-\rho}} \end{array}}{2} \right]^{\frac{1-\rho}{1-\alpha}} \right\}^{\frac{1}{1-\rho}}.$$

Using this formula, we can write the condition for Stochastic Impatience in EZ as:

$$\left\{ (1-\beta)c_0^{1-\rho} + \beta \left[\frac{\begin{array}{l} \{(1-\beta^t)x^{1-\rho} + \beta^t c^{1-\rho}\}^{\frac{1-\alpha}{1-\rho}} \\ + \{c^{1-\rho} + \beta^{\tau-1}(1-\beta^t)\left(y^{1-\rho} - c^{1-\rho}\right)\}^{\frac{1-\alpha}{1-\rho}} \end{array}}{2} \right]^{\frac{1-\rho}{1-\alpha}} \right\}^{\frac{1}{1-\rho}}$$

$$\geq$$

$$\left\{ (1-\beta)c_0^{1-\rho} + \beta \left[\frac{\begin{array}{l} \{(1-\beta^t)y^{1-\rho} + \beta^t c^{1-\rho}\}^{\frac{1-\alpha}{1-\rho}} \\ + \{c^{1-\rho} + \beta^{\tau-1}(1-\beta^t)\left(x^{1-\rho} - c^{1-\rho}\right)\}^{\frac{1-\alpha}{1-\rho}} \end{array}}{2} \right]^{\frac{1-\rho}{1-\alpha}} \right\}^{\frac{1}{1-\rho}}$$

for all $t \in \mathbb{N}$ all $\tau \in \{2, 3, ...\}$ and all $x, y, c \in \mathbb{R}_+$ with $x > y$. Letting $\tilde{\tau} \equiv \tau - 1$, we can rewrite this condition as:

$$\left\{ (1-\beta)c_0^{1-\rho} + \beta \left[\frac{\begin{array}{l} \{(1-\beta^t)x^{1-\rho} + \beta^t c^{1-\rho}\}^{\frac{1-\alpha}{1-\rho}} \\ + \left\{c^{1-\rho} + \beta^{\tilde{\tau}}(1-\beta^t)\left(y^{1-\rho} - c^{1-\rho}\right)\right\}^{\frac{1-\alpha}{1-\rho}} \end{array}}{2} \right]^{\frac{1-\rho}{1-\alpha}} \right\}^{\frac{1}{1-\rho}}$$

$$\geq$$

$$\left\{ (1-\beta)c_0^{1-\rho} + \beta \left[\frac{\begin{array}{l} \{(1-\beta^t)y^{1-\rho} + \beta^t c^{1-\rho}\}^{\frac{1-\alpha}{1-\rho}} \\ + \left\{c^{1-\rho} + \beta^{\tilde{\tau}}(1-\beta^t)\left(x^{1-\rho} - c^{1-\rho}\right)\right\}^{\frac{1-\alpha}{1-\rho}} \end{array}}{2} \right]^{\frac{1-\rho}{1-\alpha}} \right\}^{\frac{1}{1-\rho}}$$

for all $t, \tilde{\tau} \in \mathbb{N}$ and all $x, y, c \in \mathbb{R}_+$ with $x > y$.

First, suppose $\rho < 1$. The condition becomes

$$\left[\left\{(1-\beta^t)x^{1-\rho} + \beta^t c^{1-\rho}\right\}^{\frac{1-\alpha}{1-\rho}} + \left\{c^{1-\rho} + \beta^{\tilde{\tau}}(1-\beta^t)\left(y^{1-\rho} - c^{1-\rho}\right)\right\}^{\frac{1-\alpha}{1-\rho}}\right]^{\frac{1-\rho}{1-\alpha}}$$

$$\geq$$

$$\left[\left\{(1-\beta^t)y^{1-\rho} + \beta^t c^{1-\rho}\right\}^{\frac{1-\alpha}{1-\rho}} + \left\{c^{1-\rho} + \beta^{\tilde{\tau}}(1-\beta^t)\left(x^{1-\rho} - c^{1-\rho}\right)\right\}^{\frac{1-\alpha}{1-\rho}}\right]^{\frac{1-\rho}{1-\alpha}}.$$

Next, suppose $\rho > 1$. The condition becomes

$$\left[\left\{(1-\beta^t)x^{1-\rho} + \beta^t c^{1-\rho}\right\}^{\frac{1-\alpha}{1-\rho}} + \left\{c^{1-\rho} + \beta^{\tilde{\tau}}(1-\beta^t)\left(y^{1-\rho} - c^{1-\rho}\right)\right\}^{\frac{1-\alpha}{1-\rho}}\right]^{\frac{1-\rho}{1-\alpha}}$$

$$\leq$$

$$\left[\left\{(1-\beta^t)y^{1-\rho} + \beta^t c^{1-\rho}\right\}^{\frac{1-\alpha}{1-\rho}} + \left\{c^{1-\rho} + \beta^{\tilde{\tau}}(1-\beta^t)\left(x^{1-\rho} - c^{1-\rho}\right)\right\}^{\frac{1-\alpha}{1-\rho}}\right]^{\frac{1-\rho}{1-\alpha}}.$$

Note that c_0 does not enter this expressions, so the period-0 consumption does not affect the conditions for Stochastic Impatience.

It is straightforward to see that (by homotheticity) we can take $c = 1$ without loss of generality (express $x \equiv \lambda_x c$ and $y \equiv \lambda_y c$ for $\lambda_x, \lambda_y \in (0, +\infty)$, then note that $c^{1-\rho}$ cancels out in all expressions). So the conditions become

$$\left[\left\{(1-\beta^t)x^{1-\rho} + \beta^t\right\}^{\frac{1-\alpha}{1-\rho}} + \left\{1 + \beta^{\tilde{\tau}}(1-\beta^t)\left(y^{1-\rho} - 1\right)\right\}^{\frac{1-\alpha}{1-\rho}}\right]^{\frac{1-\rho}{1-\alpha}}$$

$$\geq$$

$$\left[\left\{(1-\beta^t)y^{1-\rho} + \beta^t\right\}^{\frac{1-\alpha}{1-\rho}} + \left\{1 + \beta^{\tilde{\tau}}(1-\beta^t)\left(x^{1-\rho} - 1\right)\right\}^{\frac{1-\alpha}{1-\rho}}\right]^{\frac{1-\rho}{1-\alpha}}$$

if $\rho < 1$, and

$$\left[\left\{(1-\beta^t)x^{1-\rho} + \beta^t\right\}^{\frac{1-\alpha}{1-\rho}} + \left\{1 + \beta^{\tilde{\tau}}(1-\beta^t)\left(y^{1-\rho} - 1\right)\right\}^{\frac{1-\alpha}{1-\rho}}\right]^{\frac{1-\rho}{1-\alpha}}$$

$$\leq$$

$$\left[\left\{(1-\beta^t)y^{1-\rho} + \beta^t\right\}^{\frac{1-\alpha}{1-\rho}} + \left\{1 + \beta^{\tilde{\tau}}(1-\beta^t)\left(x^{1-\rho} - 1\right)\right\}^{\frac{1-\alpha}{1-\rho}}\right]^{\frac{1-\rho}{1-\alpha}}$$

if $\rho > 1$.

There are 4 cases.

Case 1: $\alpha, \rho < 1$.

Here, the condition becomes

$$\left\{(1-\beta^t)x^{1-\rho}+\beta^t\right\}^{\frac{1-\alpha}{1-\rho}} - \left\{1+\beta^{\tilde{\tau}}(1-\beta^t)\left(x^{1-\rho}-1\right)\right\}^{\frac{1-\alpha}{1-\rho}}$$

$$\geq$$

$$\left\{(1-\beta^t)y^{1-\rho}+\beta^t\right\}^{\frac{1-\alpha}{1-\rho}} - \left\{1+\beta^{\tilde{\tau}}(1-\beta^t)\left(y^{1-\rho}-1\right)\right\}^{\frac{1-\alpha}{1-\rho}}$$

for all $x > y$ and all $t, \tilde{\tau}$. This holds iff

$$\frac{d}{dz}\left\{\left\{(1-\beta^t)z^{1-\rho}+\beta^t\right\}^{\frac{1-\alpha}{1-\rho}} - \left\{1+\beta^{\tilde{\tau}}(1-\beta^t)\left(z^{1-\rho}-1\right)\right\}^{\frac{1-\alpha}{1-\rho}}\right\} \geq 0$$

for all $z \in \mathbb{R}_+$.

Case 2: $\alpha, \rho > 1$.

$$\left\{(1-\beta^t)x^{1-\rho}+\beta^t\right\}^{\frac{1-\alpha}{1-\rho}} - \left\{1+\beta^{\tilde{\tau}}(1-\beta^t)\left(x^{1-\rho}-1\right)\right\}^{\frac{1-\alpha}{1-\rho}}$$

$$\leq$$

$$\left\{(1-\beta^t)y^{1-\rho}+\beta^t\right\}^{\frac{1-\alpha}{1-\rho}} - \left\{1+\beta^{\tilde{\tau}}(1-\beta^t)\left(y^{1-\rho}-1\right)\right\}^{\frac{1-\alpha}{1-\rho}}$$

for all $x > y$ and all $t, \tilde{\tau}$. This holds iff

$$\frac{d}{dz}\left\{\left\{(1-\beta^t)x^{1-\rho}+\beta^t\right\}^{\frac{1-\alpha}{1-\rho}} - \left\{1+\beta^{\tilde{\tau}}(1-\beta^t)\left(x^{1-\rho}-1\right)\right\}^{\frac{1-\alpha}{1-\rho}}\right\} \leq 0$$

for all $z \in \mathbb{R}_+$.

Case 3: $\alpha > 1 > \rho$.

$$\left\{(1-\beta^t)x^{1-\rho}+\beta^t\right\}^{\frac{1-\alpha}{1-\rho}} - \left\{1+\beta^{\tilde{\tau}}(1-\beta^t)\left(x^{1-\rho}-1\right)\right\}^{\frac{1-\alpha}{1-\rho}}$$

$$\leq$$

$$\left\{(1-\beta^t)y^{1-\rho}+\beta^t\right\}^{\frac{1-\alpha}{1-\rho}} - \left\{1+\beta^{\tilde{\tau}}(1-\beta^t)\left(y^{1-\rho}-1\right)\right\}^{\frac{1-\alpha}{1-\rho}}$$

for all $x > y$ and all $t, \tilde{\tau}$. This holds iff

$$\frac{d}{dz}\left\{\left\{(1-\beta^t)z^{1-\rho}+\beta^t\right\}^{\frac{1-\alpha}{1-\rho}} - \left\{1+\beta^{\tilde{\tau}}(1-\beta^t)\left(z^{1-\rho}-1\right)\right\}^{\frac{1-\alpha}{1-\rho}}\right\} \leq 0$$

for all $z \in \mathbb{R}_+$.

Case 4: $\alpha < 1 < \rho$**.**

$$\left\{(1-\beta^t)x^{1-\rho}+\beta^t\right\}^{\frac{1-\alpha}{1-\rho}} - \left\{1+\beta^{\tilde{\tau}}(1-\beta^t)\left(x^{1-\rho}-1\right)\right\}^{\frac{1-\alpha}{1-\rho}}$$
$$\geq$$
$$\left\{(1-\beta^t)y^{1-\rho}+\beta^t\right\}^{\frac{1-\alpha}{1-\rho}} - \left\{1+\beta^{\tilde{\tau}}(1-\beta^t)\left(y^{1-\rho}-1\right)\right\}^{\frac{1-\alpha}{1-\rho}}$$

for all $x > y$ and all $t, \tilde{\tau}$. This holds iff

$$\frac{d}{dz}\left\{\left\{(1-\beta^t)z^{1-\rho}+\beta^t\right\}^{\frac{1-\alpha}{1-\rho}} - \left\{1+\beta^{\tilde{\tau}}(1-\beta^t)\left(z^{1-\rho}-1\right)\right\}^{\frac{1-\alpha}{1-\rho}}\right\} \geq 0$$

for all $z \in \mathbb{R}_+$.

To combine all cases, let

$$\Phi(z) \equiv \left\{(1-\beta^t)z^{1-\rho}+\beta^t\right\}^{\frac{1-\alpha}{1-\rho}} - \left\{1+\beta^{\tilde{\tau}}(1-\beta^t)\left(z^{1-\rho}-1\right)\right\}^{\frac{1-\alpha}{1-\rho}}.$$

We have shown that Stochastic Impatience requires $\Phi'(z) \geq 0$ if either $\alpha, \rho < 1$ or $\alpha < 1 < \rho$, and $\Phi'(z) \leq 0$ if either $\alpha, \rho > 1$ or $\alpha > 1 > \rho$. That is, Stochastic Impatience holds if and only if:

- $\Phi'(z) \geq 0$ for all z if $\alpha < 1$
- $\Phi'(z) \leq 0$ for all z if $\alpha > 1$

But note that

$$\Phi'(z) = (1-\alpha)(1-\beta^t)z^{-\rho}\left\{\begin{array}{l}\left\{(1-\beta^t)z^{1-\rho}+\beta^t\right\}^{\frac{\rho-\alpha}{1-\rho}} \\ -\left\{1+\beta^{\tilde{\tau}}(1-\beta^t)(z^{1-\rho}-1)\right\}^{\frac{\rho-\alpha}{1-\rho}}\beta^{\tilde{\tau}}\end{array}\right\}.$$

Moreover, $(1-\beta^t)z^{-\rho} > 0$ for all $z \in \mathbb{R}_+$. Combining the two cases for α, we find that Stochastic Impatience holds if and only if:

$$\left\{(1-\beta^t)z^{1-\rho}+\beta^t\right\}^{\frac{\rho-\alpha}{1-\rho}} - \left\{1+\beta^{\tilde{\tau}}(1-\beta^t)\left(z^{1-\rho}-1\right)\right\}^{\frac{\rho-\alpha}{1-\rho}}\beta^{\tilde{\tau}} \geq 0.$$

We have therefore shown the following lemma:

Lemma 7. *Stochastic Impatience holds if and only if*

$$\left\{(1-\beta^t)z^{1-\rho}+\beta^t\right\}^{\frac{\rho-\alpha}{1-\rho}} \geq \beta^{\tilde{\tau}}\left\{1+\beta^{\tilde{\tau}}(1-\beta^t)\left(z^{1-\rho}-1\right)\right\}^{\frac{\rho-\alpha}{1-\rho}}$$

for all $t, \tilde{\tau}$ *and all* $z \in \mathbb{R}_+$.

Now we need to verify when this condition holds.

Case 1: $\alpha > 1 > \rho$**.**

Taking $t \to \infty$, Stochastic Impatience becomes

$$z^{\rho-\alpha} \geq \left\{1 + \beta^{\tilde{\tau}}\left(z^{1-\rho} - 1\right)\right\}^{\frac{\rho-\alpha}{1-\rho}} \beta^{\tilde{\tau}}$$

Since $\rho - \alpha < 0$, the condition becomes

$$z \leq \left\{1 + \beta^{\tilde{\tau}}\left(z^{1-\rho} - 1\right)\right\}^{\frac{1}{1-\rho}} \beta^{\frac{\tilde{\tau}}{\rho-\alpha}}$$

$$\iff \left[1 - \beta^{\left(\frac{1-\rho}{\rho-\alpha}+1\right)\tilde{\tau}}\right] \leq \beta^{\frac{1-\rho}{\rho-\alpha}\tilde{\tau}} \frac{1-\beta^{\tilde{\tau}}}{z^{1-\rho}}.$$

Note that the RHS converges to zero as $z \nearrow +\infty$ and the LHS is bounded away from zero since

$$1 > \beta^{\left(\frac{1-\rho}{\rho-\alpha}+1\right)\tilde{\tau}} \iff \frac{\alpha-1}{\alpha-\rho} > 0.$$

Therefore, Stochastic Impatience fails in this case.

Case 2: $\alpha > \rho > 1$**.**

Here, we can rearrange the Stochastic Impatience condition as:

$$\left\{(1-\beta^t)z^{1-\rho} + \beta^t\right\}^{\frac{\rho-\alpha}{1-\rho}} \geq \beta^{\tilde{\tau}} \left\{1 + \beta^{\tilde{\tau}}(1-\beta^t)\left(z^{1-\rho} - 1\right)\right\}^{\frac{\rho-\alpha}{1-\rho}}$$

for all $t, \tilde{\tau}$ and all $z \in \mathbb{R}_+$. Take $t \to \infty$, so the condition becomes

$$z^{\rho-\alpha} \geq \beta^{\tilde{\tau}} \left\{1 + \beta^{\tilde{\tau}}\left(z^{1-\rho} - 1\right)\right\}^{\frac{\rho-\alpha}{1-\rho}}$$

$$\iff 1 - \beta^{\tilde{\tau}\frac{1-\rho}{\rho-\alpha}+\tau} \geq \beta^{\tilde{\tau}\frac{1-\rho}{\rho-\alpha}}\left(1-\beta^{\tilde{\tau}}\right) z^{\rho-1}.$$

Taking $z \nearrow \infty$, we find that the RHS converges to $+\infty$, violating Stochastic Impatience.

Case 3: $1 > \alpha \geq \rho$**.**

Here, we can rearrange the Stochastic Impatience condition as:

$$(1-\beta^t)z^{1-\rho} + \beta^t \leq \beta^{\tilde{\tau}\frac{1-\rho}{\rho-\alpha}} \left\{1 + \beta^{\tilde{\tau}}(1-\beta^t)\left(z^{1-\rho} - 1\right)\right\}$$

for all $t, \tilde{\tau}$ and all $z \in \mathbb{R}_+$. Rearrange this condition as:

$$\left[1 - \beta^{\tilde{\tau}\left(\frac{1-\rho}{\rho-\alpha}+1\right)}\right] z^{1-\rho} \leq \frac{\beta^{\tilde{\tau}\frac{1-\rho}{\rho-\alpha}} - \beta^t}{1-\beta^t} - \beta^{\tilde{\tau}\left(\frac{1-\rho}{\rho-\alpha}+1\right)}$$

Note that

$$1-\beta^{\tilde{\tau}\left(\frac{1-\rho}{\rho-\alpha}+1\right)}<0 \iff \frac{1-\alpha}{\rho-\alpha}<0,$$

which is true.

Note that $\frac{\beta^{\tilde{\tau}\frac{1-\rho}{\rho-\alpha}}-\beta^t}{1-\beta^t}$ is decreasing in t whenever $\beta^{\tilde{\tau}\frac{1-\rho}{\rho-\alpha}}>1$, which is true since $\frac{1-\rho}{\rho-\alpha}<0$. Thus, Stochastic Impatience holds if and only if the condition above holds for $t=\infty$. Take $t\to+\infty$, so it becomes:

$$\left[1-\beta^{\tilde{\tau}\left(\frac{1-\rho}{\rho-\alpha}+1\right)}\right]z^{1-\rho}\leq \beta^{\tilde{\tau}\frac{1-\rho}{\rho-\alpha}}-\beta^{\tilde{\tau}\left(\frac{1-\rho}{\rho-\alpha}+1\right)}.$$

This is true if and only if

$$\beta^{\tau\left(\frac{1-\rho}{\rho-\alpha}+1\right)}-\beta^{\tau\frac{1-\rho}{\rho-\alpha}}\leq 0 \iff \beta\leq 1,$$

verifying that Stochastic Impatience holds.

Case 4: $1<\alpha\leq\rho$**.**

Recall the condition for Stochastic Impatience to hold:

$$\left\{(1-\beta^t)z^{1-\rho}+\beta^t\right\}^{\frac{\rho-\alpha}{1-\rho}}\geq\beta^{\tilde{\tau}}\left\{1+\beta^{\tilde{\tau}}(1-\beta^t)\left(z^{1-\rho}-1\right)\right\}^{\frac{\rho-\alpha}{1-\rho}}$$

for all $t,\tilde{\tau}$ and all $z\in\mathbb{R}_+$.

Since $\frac{\rho-\alpha}{1-\rho}<0$, we can rewrite this condition as

$$(1-\beta^t)z^{1-\rho}\left(1-\beta^{\tilde{\tau}\frac{1-\rho}{\rho-\alpha}+\tilde{\tau}}\right)\leq\beta^{\tilde{\tau}\frac{1-\rho}{\rho-\alpha}}-\beta^{\tilde{\tau}\frac{1-\rho}{\rho-\alpha}+\tilde{\tau}}(1-\beta^t)-\beta^t.$$

Note that the LHS is negative since $(1-\beta^t)z^{1-\rho}>0$ and

$$1-\beta^{\tilde{\tau}\frac{1-\rho}{\rho-\alpha}+\tilde{\tau}}<0 \iff \frac{1-\alpha}{\rho-\alpha}<0,$$

which is true. Note also that the RHS is positive:

$$\beta^{\tilde{\tau}\frac{1-\rho}{\rho-\alpha}}-\beta^{\tilde{\tau}\frac{1-\rho}{\rho-\alpha}+\tilde{\tau}}(1-\beta^t)-\beta^t>0$$

$$\iff \beta^{\tilde{\tau}\frac{1-\rho}{\rho-\alpha}}\left(1-\beta^{\tilde{\tau}}\right)>\beta^t\left(1-\beta^{\tilde{\tau}\frac{1-\rho}{\rho-\alpha}+\tilde{\tau}}\right),$$

but

$$\beta^{\tilde{\tau}\frac{1-\rho}{\rho-\alpha}}\left(1-\beta^{\tilde{\tau}}\right)>0>\beta^t\overbrace{\left(1-\beta^{\tilde{\tau}\frac{1-\rho}{\rho-\alpha}+\tilde{\tau}}\right)}^{<0\text{ by our previous calculations}}.$$

This establishes that Stochastic Impatience holds.

Case 5: $\alpha < \rho < 1$**.**

Recall the condition for Stochastic Impatience to hold:

$$\left\{(1-\beta^t)z^{1-\rho}+\beta^t\right\}^{\frac{\rho-\alpha}{1-\rho}} \geq \beta^{\tilde{\tau}}\left\{1+\beta^{\tilde{\tau}}(1-\beta^t)\left(z^{1-\rho}-1\right)\right\}^{\frac{\rho-\alpha}{1-\rho}}$$

for all $t,\tilde{\tau}$ and all $z \in \mathbb{R}_+$. Since $\frac{\rho-\alpha}{1-\rho} > 0$, we can rewrite this condition as

$$(1-\beta^t)\left(1-\beta^{\tilde{\tau}\frac{1-\rho}{\rho-\alpha}+\tilde{\tau}}\right)z^{1-\rho} \geq \beta^{\tilde{\tau}\frac{1-\rho}{\rho-\alpha}} - \beta^{\tilde{\tau}\frac{1-\rho}{\rho-\alpha}+\tilde{\tau}}(1-\beta^t)-\beta^t.$$

Recall that

$$1-\beta^{\tilde{\tau}\frac{1-\rho}{\rho-\alpha}+\tilde{\tau}} > 0 \iff \frac{1-\alpha}{\rho-\alpha} > 0,$$

which is true here. Therefore, the LHS is positive. Because $\rho < 1$, the condition holds if and only if it holds as $z \searrow 0$. Since

$$\lim_{z\searrow 0}(1-\beta^t)z^{1-\rho}\left(1-\beta^{\tilde{\tau}\frac{1-\rho}{\rho-\alpha}+\tilde{\tau}}\right) = 0,$$

Stochastic Impatience holds if and only if

$$\beta^{\tilde{\tau}\frac{1-\rho}{\rho-\alpha}} - \beta^{\tilde{\tau}\frac{1-\rho}{\rho-\alpha}+\tilde{\tau}}(1-\beta^t)-\beta^t \leq 0$$

for all $t,\tilde{\tau}$. Rearrange this inequality as

$$\beta^{\tilde{\tau}\frac{1-\rho}{\rho-\alpha}} - \beta^{\tilde{\tau}\frac{1-\rho}{\rho-\alpha}+\tilde{\tau}} \leq \beta^t \underbrace{\left(1-\beta^{\tilde{\tau}\frac{1-\rho}{\rho-\alpha}+\tilde{\tau}}\right)}_{+}.$$

Since the RHS is decreasing in t, it holds for all t if and only if it holds as $t \nearrow +\infty$. Thus, Stochastic Impatience holds if and only if

$$\beta^{\tilde{\tau}\frac{1-\rho}{\rho-\alpha}} - \beta^{\tilde{\tau}\frac{1-\rho}{\rho-\alpha}+\tilde{\tau}} \leq 0 \iff \tilde{\tau}\frac{1-\rho}{\rho-\alpha} \geq \tilde{\tau}\frac{1-\rho}{\rho-\alpha}+\tilde{\tau},$$

which is false. Therefore, Stochastic Impatience fails.

Case 6: $\alpha < 1 < \rho$**.**

Since $\frac{\rho-\alpha}{1-\rho} < 0$, the condition for Stochastic Impatience to hold becomes

$$(1-\beta^t)z^{1-\rho}+\beta^t \leq \beta^{\tilde{\tau}\frac{1-\rho}{\rho-\alpha}} + \beta^{\tilde{\tau}\frac{1-\rho}{\rho-\alpha}+\tilde{\tau}}(1-\beta^t)\left(z^{1-\rho}-1\right)$$

for all $t,\tilde{\tau}$ and all $z \in \mathbb{R}_+$. Rearrange it as

$$(1-\beta^t)\left(1-\beta^{\tilde{\tau}\frac{1-\rho}{\rho-\alpha}+\tilde{\tau}}\right)z^{1-\rho} \leq \beta^{\tilde{\tau}\frac{1-\rho}{\rho-\alpha}} - \beta^{\tilde{\tau}\frac{1-\rho}{\rho-\alpha}+\tilde{\tau}}(1-\beta^t)-\beta^t.$$

Recall that

$$1-\beta^{\tilde{\tau}\frac{1-\rho}{\rho-\alpha}+\tilde{\tau}}>0 \iff \frac{1-\alpha}{\rho-\alpha}>0,$$

which is true here. Therefore, the LHS is positive and decreasing in z. It follows that Stochastic Impatience holds if and only if the condition holds as $z \searrow 0$. Since

$$\lim_{z\searrow 0}(1-\beta^{t})\left(1-\beta^{\tilde{\tau}\frac{1-\rho}{\rho-\alpha}+\tilde{\tau}}\right)z^{1-\rho}=+\infty,$$

Stochastic Impatience fails in this case.

Combining all cases, Stochastic Impatience holds if and only if either $1>\alpha\geq\rho$ or $1<\alpha\leq\rho$.