Theoretical Economics 20 (2025), 911–939 1555-7561/20250911

Empirical welfare economics

Christopher P. Chambers
Department of Economics, Georgetown University

Federico Echenique
Department of Economics, UC Berkeley

Welfare economics relies on access to agents' utility functions: we revisit classical questions in welfare economics, assuming access to data on agents' past choices instead of their utilities. Our main result considers the existence of utilities that render a given allocation Pareto optimal. We show that a candidate allocation is efficient for some utilities consistent with the choice data if and only if it is efficient for an incomplete relation derived from the revealed preference relations and convexity. Similar ideas are used to make counterfactual choices for a single consumer, policy comparisons by the Kaldor criterion, and offer bounds on the degree of inefficiency in a Pareto suboptimal allocation.

Keywords. Revealed preference theory, Afriat's theorem.

JEL classification. D61.

1. Introduction

Consider a social planner facing a collection of agents in a neoclassical resource allocation problem. Pareto optimality is characterized by the equality of agents' marginal rates of substitution, but to use this characterization our planner needs access to agents' utility functions. Suppose instead that the planner has access to a data set consisting of a finite set of demand observations for each individual. The planner wants to know which allocations can be Pareto efficient for the collection of agents, given what she knows from the observed data set. As a minimal discipline, she asks that there are monotone and convex preferences that are consistent with the data, and for which a given allocation is Pareto efficient.

Our main result provides a complete characterization of the allocations that can be Pareto efficient for the observed data set. Our characterization parallels the definition of Pareto optimality, with an empirical domination relation standing in for unobservable

Christopher P. Chambers: Christopher.Chambers@georgetown.edu
Federico Echenique: fede@econ.berkeley.edu
This paper is dedicated to the memory of Kim Border. We are grateful to audiences at the CUHK-HKU-HKUST Joint Theory Seminar, UC Berkeley, the University of Chicago, the 2023 NSF-NBER Conference on Mathematical Economics at Caltech (which was celebrated in honor of Kim Border), McGill University, the 2022 Social Choice and Welfare meetings, Stanford University, the Workshop on Applications of Revealed Preferences, and Roy Allen for detailed comments. Three anonymous referees provided very useful feedback.

Available at https://econtheory.org. https://doi.org/10.3982/TE5845

utility comparisons. So, the characterization says that there should be no dominating alternative allocation, where the notion of domination captures what can be inferred about agents' utilities from the data set. In particular, the data set defines a revealed preference relation. The revealed preference is, in general, incomplete; it does not compare all alternatives. Given revealed preference, we can speak of making further comparisons based on monotonicity, transitivity, and convexity. For example, if it is known that both x and y are revealed preferred to z, then $\frac{1}{2}(x+y)$ should also be at least as good as z. Further, imposing monotonicity allows for additional comparisons: if x is revealed preferred to z, and $w \geq x$, then w should also be preferred to z. Each such comparison can be further combined with transitivity to impose additional comparisons. All the inferences that we can make recursively, using indirect revealed preference, convexity, and monotonicity, define what we call a domination relation for each individual agent. This domination relation is, in a sense, the "smallest" set of inferences we can make from the data by using rationality, convexity, and monotonicity alone.

The domination relation is typically highly incomplete. Incompleteness results from the limitations in the information contained in the data, even when augmented by the consequences of assuming monotone and convex preferences. This is in contrast with the normative statements about incomplete preferences, as in the work of Ok (2002), Dubra, Maccheroni, and Ok (2004), Eliaz and Ok (2006). Efficiency with respect to our relation is the same notion as is used in the matching literature, where the incomplete relation is typically the stochastic dominance relation on a set of lotteries induced by a linear order on the set of degenerate outcomes; see, for example, Bogomolnaia and Moulin (2001), McLennan (2002), Abdulkadiroğlu and Sönmez (2003), Manea (2008), Carroll (2010), Bogomolnaia and Heo (2012), Hashimoto, Hirata, Kesten, Kurino, and Ünver (2014), Aziz, Brandl, and Brandt (2015), Doğan and Yıldız (2016).

The paper actually uses the domination relation, and related concepts, to address a host of related questions in welfare economics. We start from individual welfare comparisons, and ask for counterfactual comparisons that may be inferred from individual-level consumption data. In particular, given data from one consumer, and two new bundles x and y, we ask when one can infer that the utility of x is greater than that of y, for all rationalizing concave utilities. The exercise follows Varian (1982), and is related to the literature on demand bounds; see, for example, Blundell, Browning, and Crawford (2007, 2008), Blundell, Browning, Crawford, De Rock, Vermeulen, and Cherchye (2015), Allen and Rehbeck (2020a,b). Our answer depends on a notion of empirical domination that is closely related to the notion behind our result on Pareto domination. There is, again, an empirically defined partial order among consumption bundles that captures all the comparisons that may be inferred from the data set and the hypotheses of monotonicity, transitivity, and concavity.

Next, we turn to collective welfare comparisons. Aside from our main result on Pareto optimal allocations, which we have already described, we consider the Kaldor criterion: whether an economic policy decision can be defended on the grounds that those who benefit from the policy could compensate those who lose (Kaldor (1939), Hicks (1939), Graaff and de Villiers (1967)). Again the idea of domination gives us an answer, and serves to rule out whether demand data validates a policy decision.

One approach to the problem could start from discrete choice. Imagine a single agent choosing from a finite set of alternatives. By observing choices from a set of possible finite menus, one could construct an incomplete revealed preference relation. Its transitive closure would in principle be incomplete, and a set of possible "completions" (or extensions) are possible. Now, with more than one consumer, and two alternative allocations x and y, we can decide if y might Pareto dominate x by checking if there are completions for each consumer so that y is ranked above x. This will occur as long as no agent's transitive closure ranks their consumption in x above the one in y. Notice that this gives a nice answer in the discrete case when we only have a single competing allocation, y. In testing for Pareto optimality of x, however, we need to account for *all* possible competing allocations.

Our approach deals with the (neoclassical) model of infinitely divisible consumption: Not a finite set of alternatives, and not discrete choice. The problem is handled by an appeal to the ideas behind the second welfare theorem. An allocation is Pareto optimal if and only if there is a common supporting price vector for each agent's consumption. So, we study a linear formulation of the problem of whether there exists utilities that are consistent with the observed data, and that render a candidate allocation Pareto optimal. Our theorem results from an application of linear programming duality. As a consequence, the question is computationally tractable, and our conditions can be checked in ways that are computationally efficient.

We focus on testing whether a given allocation could be Pareto optimal for some profile of utilities that are consistent with the data. We think of this as a natural and practical question that would come up in discussions of public policy. Consider a policy proposal that would result in an allocation $\bar{x}$. Can we say that the allocation $\bar{x}$, and by implication the underlying policy proposal, are possibly efficient? If the conditions we have laid out are violated, then there are not utilities for which the policy results in an efficient outcome for the economy.

A more general question takes as given multiple allocations, and wonders if there is a utility profile that is consistent with the data and for which all the allocations under consideration are efficient. This more general question is interesting, but somewhat harder to motivate because it is not obviously tied to a given policy proposal.

When considering multiple allocations, our approach falls short of a full characterization of efficiency, but still provides a practical and linear test. That is, if any of the allocations in a set of multiple allocations violates our single-allocation condition, we know that the set as a whole cannot be possibly efficient. While we cannot quantify how often it is that each member of a set passes the test while the set fails as a whole, clearly this is a nontrivial possibility (we demonstrate an example in Section 4). In the special case of quasilinear utility, we do offer a full characterization in the Appendix.

Related literature The paper starts with a discussion of the individual welfare comparisons that may be inferred from a consumption data set using revealed preference tools. Then the paper turns to collective choice. Our results on individual welfare extend the ideas of Varian (1982), who considered how two consumption bundles that are not observed in the data might be ranked by a utility that rationalizes the data. Varian provides an answer in terms of a system of linear inequalities. We show that the answers using his linear system is equivalent to checking a condition that is derived from the data.

Our results on collective choice fit into two strands of literature. First, the theory of efficiency in classical economic environments without completeness is studied in many works; a few of these include Shafer and Sonnenschein (1975), Gale and Mas-Colell (1975, 1977), Fon and Otani (1979), Weymark (1985), Rigotti and Shannon (2005), Bewley (2002); and Bewley et al. (1987). In our case, preference incompleteness arises because of limited data on agents' preferences, and gives rise to challenges that are not present in the previous literature.

Preference incompleteness goes away, and the results in our paper cease to be interesting, when agents' preferences can be recovered with high levels of precision from the observed data. The recovery question is, however, not straightforward; even when large consumption data sets are available. Mas-Colell (1977) discusses counterexamples, and conditions under which preferences may be recovered from a demand function, while Mas-Colell (1978) shows that the canonical Afriat rationalization may (under a Lipschitz condition on the underlying demand behavior) be used to recover agents' preferences; see also Chambers, Echenique, and Lambert (2021) and Ugarte (2022). Among other conditions, these results require that the data sample a rich enough subset of the possible budgets.

The second strand of literature concerns testing whether certain allocations can be equilibria of a given economy. Brown and Matzkin (1996) are the first to formulate the problem as a revealed preference exercise. In that paper, the authors check whether a collection of candidate objects could be equilibria of a given economy. Results in the revealed preference literature usually focus on establishing a list of polynomial inequalities that must be satisfied in order for the data to be rationalizable—these inequalities are analogous to the "Afriat inequalities" of rational consumer behavior. In showing that a particular rationalization problem reduces to one of verifying whether a solution exists to a list of polynomial inequalities establishes that these problems are decidable, in an algorithmic sense; see also Brown and Shannon (2000), Bossert and Sprumont (2002), Kubler (2003), Carvajal, Ray, and Snyder (2004), Carvajal (2004), Bachmann (2004, 2006b,a), Brown and Calsamiglia (2007), Brown and Kubler (2008), Carvajal (2010), Cherchye, Demuynck, and De Rock (2011), Carvajal and Song (2018) for testable implications of related environments. Some of these investigate efficiency directly: Bossert and Sprumont (2002) discuss how the core correspondence varies (for fixed preferences) as endowments vary. Their results characterize the testable implications of the core, but is restricted to the case of two agents and a fixed aggregate endowment; their "data" is generated by varying the distribution of a fixed aggregate endowment. Bachmann (2006b) considers an environment in which collections of endowments and consumption bundles (but not prices) are observed. His Proposition 5 establishes that Pareto efficiency has essentially no testable content in this environment, even if all preferences are represented by strictly concave and continuously differentiable utilities.[1]

[1] The idea is that a common linear preference renders every allocation efficient. Then perturb each agent's utility a bit to ensure strict concavity and smoothness.

Allen, Dziewulski, and Rehbeck (2019), Allen and Rehbeck (2020a,b) also consider notions of welfare or of group decision-making.

As mentioned, when it comes to welfare comparisons, what these papers primarily do is provide an analogue of the result of Afriat (1967), whereby rationalizability is equivalent to the satisfaction of a set of inequalities. In contrast, our work differs in two respects: first, we provide an economic characterization of whether a given bundle could possibly be efficient—our characterization is more analogous to the characterization of rationality via absence of cycles (also discussed by Afriat (1967), and termed "Generalized Axiom of Revealed Preference" by Varian (1982)). We take as the starting point of our proof a collection of "Afriat inequalities" that must be satisfied, and use these to uncover a dual system of linear inequalities that we can interpret—they have concrete economic meaning—and deliver a condition in terms of the domination relation.

2. The model

Basic definitions and notational conventions We use the following notational conventions: For vectors $x, y \in \mathbf{R}^n$, $x \leq y$ means that $x_i \leq y_i$ for all $i = 1, \dots, n$; $x < y$ means that $x \leq y$ and $x \neq y$; and $x \ll y$ means that $x_i < y_i$ for all $i = 1, \dots, n$. The set of nonnegative vectors in $\mathbf{R}^n$ is denoted $\mathbf{R}^n_+$, and the set of vectors that are strictly positive in all components is $\mathbf{R}^n_{++}$. When n is a nonnegative integer, we write the set $\{1, \dots, n\}$ as $[n]$; with $[0]$ denoting the empty set.

A function $f : A \subseteq \mathbf{R}^n \to \mathbf{R}$ is *weakly monotone increasing*, or *nondecreasing*, if $f(x) \leq f(y)$ when $x \leq y$; and *monotone increasing*, if it is weakly monotone increasing and $f(x) < f(y)$ when $x \ll y$. We often just write "increasing."

A function u is *explicitly quasiconcave* if it is quasiconcave and, for all $x, y \in \mathbf{R}^n_+$ and $\lambda \in (0, 1)$, $u(x) \neq u(y)$ implies that

$$u\big(\lambda x + (1 - \lambda)y\big) > \min\big\{u(x), u(y)\big\}.$$

Observe that explicit quasiconcavity of u is a behavioral property, meaning a property of the preference relation represented by u; and that it is weaker than concavity. Indeed, explicit quasiconcavity is only a minor strengthening of quasiconcavity; it is weaker than strict quasiconcavity ($u(\lambda x + (1 - \lambda)y) > \min\{u(x), u(y)\}$ for all $\lambda \in (0, 1)$), which corresponds to strict convexity of preferences. Strict quasiconcavity rules out that indifference curves contain any flat regions (i.e., contain any line segments), but flat regions are allowed by explicit quasiconcavity (some rather pathological examples with flat regions are ruled out). Perhaps explicit quasiconcavity is best known because it ensures that local maxima are global maxima, for which quasiconcavity alone does not suffice (see Theorem 192 in Border (2015)).

Definitions from welfare economics An agent is defined through a preference relation on $\mathbf{R}^m_+$, which we represent throughout by a utility function $u : \mathbf{R}^m_+ \to \mathbf{R}$.[2] The elements of $\mathbf{R}^m_+$ are called *consumption bundles*. Given a finite set of agents N, an *allocation* is a

[2]We restrict attention to continuous preference relations, but given that preferences are only constrained to rationalize a finite data set, continuity is without loss of generality.

vector $\bar{x} = (\bar{x}_i)_{i \in N} \in \mathbf{R}_+^{mN}$.[3] If each agent is endowed with a utility function u_i, an allocation $\bar{y}$ *Pareto dominates* the allocation $\bar{x}$ if $u_i(\bar{y}_i) \geq u_i(\bar{x}_i)$ for all i, with a strict inequality for at least one agent. An allocation $\bar{x}$ is *Pareto optimal* if there is no allocation satisfying

$$\sum_{i \in N} \bar{y}_i = \sum_{i \in N} \bar{x}_i$$

that Pareto dominates it.

Next, we turn to a criterion for comparing allocations based on the principle that winners may compensate the losers. The idea is that those who gain in moving from one allocation to the other may compensate those who lose with the move in allocations. Let $\bar{x}$ and $\bar{y}$ be two allocations. Say that $\bar{x}$ *weakly Kaldor dominates* $\bar{y}$ if there is no allocation $\bar{z}$ with $\sum_i \bar{z}_i \leq \sum_i \bar{y}_i$ that Pareto dominates $\bar{x}$. The idea is that if $\bar{x}$ does not weakly dominate $\bar{y}$, then there is a way of reassigning (whence losers are compensated by winners) the aggregate bundle $\sum_i \bar{y}_i$ in a way that Pareto dominates $\bar{x}$ (see Chapter 5 in Graaff and de Villiers (1967) for a discussion of the Kaldor criterion).

Data and rationalizability A pair $(p, x) \in \mathbf{R}_+^{m+m}$ is an *observation*, and should be interpreted as the datum that the consumption bundle $x \in \mathbf{R}_+^m$ was chosen from the budget set $\{y \in \mathbf{R}_+^m : p \cdot y \leq I\}$ in which the income or budget is $I = p \cdot x$. A (possibly empty) finite list of observations $\{(p^k, x^k)\}_{k \in [K]}$ is termed an *individual data set. N* is a finite set of individuals. A *group data set* is a collection of individual data sets, one for each $i \in N$. So, $D_i = \{(p_i^k, x_i^k)\}_{k \in [K_i]}$ denotes an individual data set for individual i, and $(D_i)_{i \in N}$ is a group data set.

An individual data set is *rationalizable* if there is an increasing utility function $u_i : \mathbf{R}_+^m \to \mathbf{R}$ for which for all k, $u_i(x) > u_i(x_i^k)$ implies $p_i^k \cdot x > p_i^k \cdot x_i^k$. In this case, we say that u_i *rationalizes* the individual data set (or that it is a *rationalizing* utility, when the data set is implied). Similarly, we say that a group data set is *rationalizable* if each individual data set is rationalizable.

In our paper, we insist that rationalizing utilities be monotone increasing. Clearly, some structure must be assumed on utilities, or any data becomes rationalizable by a constant utility. The most common approach is to impose local nonsatiation, and then resort to Afriat's theorem, which says that one may without loss of generality assume a rationalizing utility that is both increasing and concave. Thus monotonicity, but more importantly concavity, comes for free in the case of an individual agent's observed consumption behavior.

Revealed preferences involve the use of two binary relations. The *direct revealed preference* of agent i is denoted by $\succeq_i^R$, and defined by $x \succeq_i^R y$ if $x \geq x_i^k$ for some k that satisfies $p_i^k \cdot x_i^k \geq p_i^k \cdot y$ or if $x = y$. The *direct strict revealed preference* of agent i is denoted by $\succ_i^R$, and defined by $x \succ_i^R y$ if

$$x \gg x' \succeq_i^R y, \quad \text{or} \quad x \geq x_i^k, \quad \text{and} \quad p_i^k \cdot x_i^k > p_i^k \cdot y,$$

[3] One should think of an allocation $\bar{x}$ as "allocating" the aggregate bundle $\sum_{i \in N} \bar{x}_i$ among the agents in N.

for some x' or observation k. These definitions of revealed preferences are slightly unusual, in that they already incorporate the expectation of a monotone preference, and symmetry is built-in.[4] Observe that $\succ_i^R \subseteq \succeq_i^R$.

The *indirect revealed preference* $\succeq_i^I$ is defined as the transitive closure of $\succeq_i^R$. The *indirect revealed strict preference* $x \succ_i^I y$ obtains when there is a finite chain $x = z_1 \succeq_i^R \dots \succeq_i^R z_L = y$, where at least one instance of $\succeq_i^R$ is $\succ_i^R$.

An individual data set D_i satisfies the *Generalized Axiom of Revealed Preference (GARP)* if there is no $x, y \in \mathbf{R}_+^m$ such that $x \succeq_i^I y$ while $y \succ_i^I x$.

3. Results

We consider counterfactual welfare comparisons. Given data on individual consumption, we seek to characterize which counterfactual (i.e., unobserved) welfare conclusions may be drawn on the basis of what can be inferred about agents' preferences from the data. For individual agents, we want to evaluate unobserved bundles. For a group of agents, the welfare comparisons are about the possible Pareto optimality of some allocation, or consistency with the Kaldor criterion.

All proofs are relegated to Section 7.

3.1 *Individual welfare*

We begin by discussing individual welfare conclusions that may be drawn from a single agent's consumption data set. Aside from the intrinsic merit of these results, they serve to introduce some of the ideas we use later in our (main) results on collective welfare.

Our first result asks when we can say that one bundle is unambiguously better than another, given what the data tell us about the agent. Specifically, given an individual data set $\{(x^k, p^k) : 1 \leq k \leq K\}$, and two bundles $\bar{x}$ and $\bar{y}$, when is $\bar{x}$ ranked above $\bar{y}$ for all increasing and concave utility functions compatible with the data?

The answer turns out to depend on a binary relation that may be inferred from the consumer's choices. Varian (1982) also considers this question and offers an answer in the form of a linear program; what Varian calls Fact 4. Our binary relation essentially emerges from the dual program to Fact 4. Say that $\bar{x}$ *bests* $\bar{y}$ if $\bar{x}$ is a convex combination of some collection z^l of bundles, $1 \leq l \leq L$, such that, for each l, $z^l \succeq^I \bar{y}$. A bundle $\bar{x}$ *strictly bests* $\bar{y}$ for agent i if it weakly bests it and, moreover, if in the defining convex combination there is l with $z^l \succ^I \bar{y}$.[5]

It is easy to see that if $\bar{x}$ strictly bests $\bar{y}$, then it is ranked above $\bar{y}$ by any rationalizing concave and monotone increasing utility function u. Indeed, if $\bar{x} = \sum_l \lambda_l z^l$ is as above,

[4]See Chambers and Echenique (2009) and Nishimura, Ok, and Quah (2017) for such "compositions" of the revealed preference relation with the partial order on consumption bundles. It is easy to see that Afriat's theorem remains true under our definition of revealed preference.

[5]A bundle $\bar{x}$ strictly bests itself when it is incompatible as a choice with the existing data set. This means that there is no price $\bar{p}$ at which $\bar{x}$ could be demanded, and for which the resulting data set (obtained by adding $(\bar{x}, \bar{p})$ to the data set) is rationalizable. If the data set is rationalizable, however, we may choose $\bar{p}$ that supports the upper contour set of a (without loss, concave) rationalizing utility at $\bar{x}$. Adding the resulting observation to the data set preserves its rationalizability.

then

$$\begin{aligned} u(\bar{x}) &\geq \sum_{l=1}^{L} \lambda_l u(z^l) \\ &> \sum_{l=1}^{L} \lambda_l u(\bar{y}) = u(\bar{y}). \end{aligned}$$

The first inequality follows from concavity, and the second from u rationalizing the data and the requirements on z^l in the definition of besting. Our first result says that strict besting is not only sufficient for the counterfactual comparison of two bundles, but also necessary.

Theorem 1. *Let $(x^k, p^k)_{1\leq k\leq K}$ be an individual data set and $\bar{x}, \bar{y} \in \mathbf{R}^m_+$ be two bundles. Then $u(\bar{x}) > u(\bar{y})$ for all concave and monotone increasing u that rationalize the data set if and only if $\bar{x}$ strictly bests $\bar{y}$.*

In Theorem 1, we require that every concave, increasing, rationalizing utility satisfies a certain property. In our next result, Theorem 2, we asks about the existence of a rationalizing utility with a certain property. The latter sort of result is, of course, most conclusive when the condition fails, and thus certifies that the property is incompatible with any rationalizing utility. Our main results in Section 3.2 are of this nature.

Finally, observe that Theorem 2 only asks utilities to be explicitly quasiconcave. The same will be true of our main results.

Theorem 2. *Let $(x^k, p^k)_{1\leq k\leq K}$ be an individual data set and $\bar{x} \in \mathbf{R}^m_+$ a bundle. There exists a monotone increasing and explicitly quasiconcave rationalizing utility u for which $u(\bar{x}) \geq \max\{u(x^k) : 1 \leq k \leq K\}$ if and only if, once we add $\bar{x} \succeq^R x^k$ for all k to the revealed preference relation, as well as as well as $x^k \succeq^R \bar{x}$ when $p^k \cdot (\bar{x} - x^k) \leq 0$ and $x^k \succ^R \bar{x}$ when $p^k \cdot (\bar{x} - x^k) < 0$, we have:*

(i) GARP is satisfied.

(ii) There is no bundle $y \leq \bar{x}$ that strictly bests $\bar{x}$.

3.2 *Collective welfare*

Our main result characterizes the allocations that are efficient for some utility functions (with the requisite properties) that are consistent with a group data set.

An allocation $\bar{y}$ *empirically dominates* the allocation $\bar{x}$ if $\sum_i \bar{y}_i \leq \sum_i \bar{x}_i$ while $\bar{y}_i$ bests $\bar{x}_i$ for all i, and strictly bests it for at least one i. Observe the parallelism with the notion of Pareto domination: Given increasing utility functions $(u_i)_{i\in N}$, we may say that an allocation $\bar{y}$ Pareto dominates $\bar{x}$ if $\sum_i \bar{y}_i \leq \sum_i \bar{x}_i$, while $u_i(\bar{y}_i) \geq u_i(\bar{x}_i)$ for all i, and $u_i(\bar{y}_i) > u_i(\bar{x}_i)$ for at least one i. Theorem 3 implies that empirical domination really is the empirical counterpart to Pareto domination.

THEOREM 3. *Let $(D_i)_{i\in N}$ be a rationalizable group data set, and $\bar{x}$ an allocation. The following statements are equivalent:*

(i) There are monotone increasing, and explicitly quasiconcave, rationalizing utilities for which $\bar{x}$ is Pareto efficient.

(ii) There are monotone increasing, and concave, rationalizing utilities for which $\bar{x}$ is Pareto efficient.

(iii) The allocation $\bar{x}$ is not empirically dominated by any other allocation.

The theorem provides a characterization of the allocations that could be efficient, for some monotone and convex preferences of the agents (with the minor strengthening of convexity implied by explicit quasiconcavity). The role of the unobserved utility functions in the definition of Pareto domination is taken by the observable empirical domination relations.[6]

Empirical domination ensures the existence of a common supporting price at the allocation $\bar{x}$, essentially the equality of marginal rates of substitution for a collection of rationalizing utilities. If we additionally require that this price supports the *Scitovsky contour* at $\bar{x}$, then the ideas behind Theorem 3 can be used to provide an empirical basis for the Kaldor criterion.[7]

COROLLARY 4. *Let $(D_i)_{i\in N}$, where $D_i = \{(x_i^k, p_i^k)\}_{k\in[K_i]}$ for $i \in N$ be a rationalizable group data set. Let $\bar{x}$ and $\bar{y}$ be allocations. There are increasing, concave, rationalizing utilities for which $\bar{x}$ weakly Kaldor dominates $\bar{y}$ if there is no allocation $(\bar{z}_i)$ that weakly dominates $\bar{x}_i$ for all i, and strictly dominates it for at least one i, and a scalar $\kappa \geq 0$, for which*

$$\sum_i \bar{z}_i \leq \sum_i \bar{x}_i + \kappa\left(\sum_i \bar{y}_i - \sum_i \bar{x}_i\right)$$

Observe that Corollary 4 only offers a sufficient condition for Kaldor domination. When the condition holds, then we may say that there are rationalizing utilities for which a switch from $\bar{x}$ to $\bar{y}$ could not be defended on the basis of the Kaldor criterion.

Our results assume that consumers' data sets are rationalizable. Empirical studies often document violations of this property, but there is (arguably) evidence of many environments where such violations are relatively small; see, for example, Echenique, Lee,

[6]The proof of Theorem 3 is based on an application of the theorem of the alternative. A different method of proof would be to construct the revealed preference relations of the candidate bundle $\overline{x}$, which is not empirically dominated, and then attempt to separate the implied Scitovsky set of this bundle from the set of bundles y for which $\sum_i \overline{x}_i \gg y$, resulting in a supporting price p. The idea would then be to show that adding, for each agent, the observation $(p, \overline{x}_i)$ results in a new data set for each agent, where each new data set satisfies GARP. Since the price p is common to all agents, there is a common marginal rate of substitution for any preference rationalization, and so we would have an efficient bundle. Though this method is certainly more intuitive than what we have done, we were unable to show in general that these new data sets generally satisfy GARP without reverting to the theorem of the alternative.

[7]Given utilities (u_i), the *Scitovsky contour* at $\bar{x}$ is the set $S(\bar{x}) = \{\sum_i z_i : u_i(z_i) \geq u_i(\bar{x}_i) \text{ for all } i \in N\}$. If a price q supports all individual upper contour sets at $\bar{x}$ and $q \cdot \sum_i \bar{y}_i < q \cdot \sum_i \bar{x}$, then $\sum_i \bar{y}_i \notin S(\bar{x})$.

and Shum (2011) and the summary of the empirical literature discussed in Chambers and Echenique (2016).

4. Multiple allocations

The results obtained in Section 3 exemplify the power of our approach, but there are also clear limits. Given a data set, one may ask a related question for a *collection* of allocations: whether there exists a single economy capable of generating all such allocations as Pareto efficient ones. It is natural to conjecture that there is such an economy if and only if each of the allocations is undominated. This conjecture turns out to be false, as shown by the following example.

Example 1. Let $N = \{1, 2\}$, and suppose there are two commodities, so that $m = 2$. Individual 1 has an empty individual data set. Individual 2 has four observations: $(p_2^1, x_2^1) = ((2, 1), (1, 2))$, $(p_2^2, x_2^2) = ((2, 1), (0, 4))$, $(p_2^3, x_2^3) = ((1, 2), (2, 1))$, and $(p_2^4, x_2^4) = ((1, 2), (4, 0))$.

Now, suppose we want to consider the allocations $\bar{x}_1^1 = (1, 0)$, $\bar{x}_2^1 = (0, 4)$, and $\bar{x}_1^2 = (0, 1)$, $\bar{x}_2^2 = (4, 0)$. Observe that because individual 1 has an empty individual data set, each of these allocations are possibly efficient by Theorem 3. On the other hand, they cannot both be efficient for the same economy.

To understand why, we argue by contradiction. First, observe that $(1, 3)$ strictly bests $(0, 4)$ for agent 2, as it is a convex combination of $(0, 4)$ and $(1.5, 2.5)$, where $(1.5, 2.5) \gg (1, 2) \succeq_2^R (0, 4)$ and $(0, 4) \succeq_2^R (0, 4)$. Therefore, we may infer by Theorem 1 that any utility function u_2 rationalizing agent 2's choices satisfies $u_2(1, 3) > u_2(0, 4)$. In particular, since allocation $\bar{x}_1$ is Pareto optimal, we may conclude that any rationalizing utility for agent 1 must feature $u_1(1, 0) > u_1(0, 1)$, as otherwise the allocation $((0, 1), (1, 3))$ would Pareto dominate $\bar{x}_1$. A symmetric argument establishes that any rationalizing utility for agent 1 must also feature $u_1(0, 1) > u_1(1, 0)$, which is a contradiction.[8] ♢

In the proof of Theorem 3, we reduced the problem of testing whether an allocation $\bar{x}$ could be efficient to the question of the existence of a supporting price q. Were we to ask that multiple allocations be efficient, we would need different supporting prices for each such allocation, but more to the point, the scale factors could differ across individuals, thus rendering the system nonlinear. In other words, we would need different λ for the different allocations, and the normalization used in the proof of Theorem 3 would no longer work.

The problem goes away when the value of λ is fixed, suggesting that we consider the special case of quasilinear utility: when $\lambda = 1$. We investigate this environment in the Appendix.

[8] We are grateful for this argument, which simplifies our previous argument.

5. Inefficient allocations

We now turn to an empirical evaluation of potentially inefficiency allocations. In particular, we present some results using the measure of Pareto inefficiency proposed by Debreu (1951): the *coefficient of resource utilization.*

To introduce the relevant concepts, consider an allocation $x = (x_i)_{i\in N}$ and fix a profile of utility functions $(u_i)_{i\in N}$ for the agents in N. Let $S^u(x_i) = \{z_i \in \mathbf{R}^m_+ : u(z_i) \geq u(x_i)\}$ denote the *upper contour set* for utility u at consumption vector x_i, and

$$S^{u_1,\dots,u_N}(x_1, \dots, x_N) = \sum_{i\in N} S^{u_i}(x_i)$$

the *Scitovsky contour* at x for the profile of utility functions $(u_1, \dots, u_N)$. In words, the Scitovsky contour of an allocation x is the set of aggregate bundles that may be decomposed into an allocation that guarantees each agent at least the utility that they obtain in x.

Debreu observes that, if the allocation x is not Pareto optimal, then as an aggregate consumption bundle, $\sum_{i=1}^N x_i$ will lie in the interior of the Scitovsky contour $S^{u_1,\dots,u_N}(x_1, \dots, x_N)$. Debreu proposes to measure the degree of inefficiency in x by the distance between $\sum_{i=1}^N x_i$ and the boundary of the Scitovsky contour: essentially his measure quantifies the degree to which agents' implied welfare in x can be reached with fewer resources than the aggregate $\sum_i x_i$.

Debreu's definition involves a price-dependent notion of welfare, but he shows that it reduces to a *coefficient of resource utilization* ρ defined by

$$\rho = \inf\left\{\rho' \in [0, 1] : \rho' \sum_{i=1}^N x_i \in S^{u_1,\dots,u_N}(x_1, \dots, x_N)\right\}.$$

We refer to Debreu (1951) for further details on his result (which requires convexity, continuity, and monotonicity on agents' preferences).

Now it should be clear that calculating the coefficient of resource utilization requires access to agents' utility functions. In our case, we use data on agents' consumption choices to obtain bounds on the possible values of the coefficient. In particular, consider a group data set: $(D_i)_{i\in N}$ where for each i, $D_i = \{(p^k_i, x^k_i)\}_{k\in[K_i]}$. Suppose, just to simplify our notation, that $K = K_i$ for all $i \in N$. We focus on the Kth allocation $x^K = (x^K_1, \dots, x^K_N)$, and want to measure its degree of inefficiency by means Debreu's coefficient.

A canonical utility rationalization in revealed preference theory is Afriat's construction. For each individual data set D_i, we may let the set $A_i \in \mathbf{R}^{2K}$ consist of all vectors $(V_i, \lambda_i) = ((V^1_i, \lambda^1_i), \dots, (V^K_i, \lambda^K_i))$ that solve the Afriat inequalities for i's data D_i, and that satisfy $V^K_i = 1$ and $\min\{V^k_i - \lambda^k_i p^k_i x^k_i : 1 \leq k \leq K\} = 0$. Now we may define the *Afriat rationalization* $u^{(V_i,\lambda_i)} : \mathbf{R}^m_+ \to \mathbf{R}$ by

$$u^{(V_i,\lambda_i)}(x_i) = \inf\{V^k_i + \lambda^k_i p^k_i \cdot (x_i - x^k_i) : 1 \leq k \leq K\}$$

for each $(V_i, \lambda_i) \in A_i$. One bound on the coefficient of resource utilization is obtained by

$$\underline{\rho} = \inf\left\{\rho' \in [0, 1] : \rho' \sum_{i=1}^N x^K_i \in \sum_{i=1}^N \bigcup_{(V,\lambda)\in A_i} S^{u^{(V,\lambda)}}(x^K_i)\right\}.$$

Another bound is found by means of the utility u_i^*:

$$u_i^*(x) = \inf\{u^{(V,\lambda)}(x) : (V, \lambda) \in A_i\},$$

and note that u_i^* is a rationalization of the data D_i. We may use these utilities to define a bound

$$\bar{\rho} = \inf\left\{\rho' \in [0, 1] : \rho' \sum_{i=1}^{N} x_i \in S^{u_1^*,\dots,u_N^*}\left(x_1^K, \dots, x_N^K\right)\right\}.$$

Proposition 5. *Consider a rationalizable group data set $(D_i)_{i\in N}$. The coefficient of resource utilization from any profile of Afriat rationalizations is bounded above by $\bar{\rho}$. The coefficient of resource utilization from any profile of concave and monotone rationalizations is bounded below by $\underline{\rho}$.*

6. Remarks

The key to our results is an observation based on Afriat's theorem, which says that an individual data set $\{(p_i^k, x_i^k) : 1 \leq i \leq K_i\}$ is rationalizable if and only if there is a solution U_i^k, $\lambda_i^k > 0$ to the following system of linear "Afriat inequalities:"[9]

$$U_i^l \leq U_i^k + \lambda_i^k p_i^k \cdot \left(x_i^l - x_i^k\right).$$

The observation is that we may normalize such a solution so that $\lambda_i^{k^*} = 1$ for some specific observation k^*. As a result, we obtain a system that remains linear, even if the prices $p_i^{k^*}$ at this particular observation were unknown.

With this observation in hand, we can now approach a problem like that in Theorem 3. For the allocation $\bar{x}$ to be Pareto optimal, agents' utilities would need to have a common supporting price q at $\bar{x}_i$. The existence of such a price q may be added to the above system of inequalities as if it were a new observation. Assuming that the corresponding value of λ has been normalized to 1, the system is still linear; see Bachmann (2004) or Bachmann (2006b) for related constructions. Now the work in proving the theorem amounts to interpreting the dual linear system.

We have discussed some obvious limits to our approach. Perhaps the main limitation is that the rationalizing utilities may not be unique, leading to an indeterminacy when the condition in our theorem is satisfied. But there are also additional applications that we have not exhausted. One of these is envy-freeness. Suppose given a group data set, and consider the existence of rationalizing utilities that render some proposed allocation $\bar{x}$ envy-free: meaning rationalizing utilities (u_i) with the property that $u_i(\bar{x}_i) \geq u_i(\bar{x}_j)$ for all $i, j \in N$. Our methods, based on working through the dual of *augmented* system of Afriat inequalities, provide an answer to this question.

A sketch of the solution follows: the trick is to add supporting prices for each agent at the proposed consumption of other agents in the allocation $\bar{x}$. The normalization idea keeps the system linear, and we just need to include utility values $u_{i,j}$ for i's utility at the bundle intended for j:

[9] See Chambers and Echenique (2016) for a discussion of Afriat's theorem and this system of linear inequalities.

(i) For all $i \in N$ and all $k, l \in [K_i]$ for which $p_i^l \cdot (x_i^k - x_i^l) \leq 0$, we have $u_i^k \leq u_i^l + \lambda_i^l p_i^l \cdot (x_i^k - x_i^l)$.

(ii) For all $i, j \in N$ and all $k \in [K_i]$ for which $p_i^k \cdot (\overline{x}_j - x_i^k) \leq 0$, we have $u_{i,j} \leq u_i^k + \lambda_i^k p_i^k \cdot (\overline{x}_j - x_k^i)$.

(iii) For all $i, j \in N$ and all $k \in [K_i]$, $u_i^k \leq u_{i,j} + p_{i,j} \cdot (x_k^i - \overline{x}_j)$.

(iv) For all $i, j, h \in N$, $u_{i,j} \leq u_{i,h} + p_{i,h} \cdot (\overline{x}_j - \overline{x}_h)$.

(v) For all $i, j \in N$, $u_{i,i} \geq u_{i,j}$.

We omit the details, but hope that it is clear how to proceed on the basis of this system.

7. Proofs

7.1 *Proof of Theorem 1*

In proving Theorem 1, we shall make use of an auxiliary "besting" definition: Say that $\bar{x}$ *bests'* $\bar{y}$ if $\bar{x}$ can be written as a convex combination of bundles z^l, where for each l $z^l \succeq^I \bar{x}$, or $z^l \succeq^I \bar{y}$, with at least one occurrence of the latter. Say that $\bar{x}$ *strictly bests'* $\bar{y}$ if it weakly bests it, and one of the revealed-preference comparisons is strict ($\succ^I$ for $\succeq^I$).

Lemma 6. *If $\bar{x}$ bests' $\bar{y}$, then $\bar{x}$ bests $\bar{y}$. And if $\bar{x}$ strictly bests' $\bar{y}$, then $\bar{x}$ strictly bests $\bar{y}$.*

Proof. Suppose that $\bar{x}$ bests' $\bar{y}$ and by means of contradiction that $\bar{x}$ does not best $\bar{y}$.

We can express $\bar{x} = \sum_k \mu^k w^k + \sum_l \lambda^l z^l$, where $\mu^k, \lambda^l \geq 0$, $\sum_k \mu^k + \sum_l \lambda^l = 1$, each $w^k \succeq^I \bar{x}$ but not $w^k \succeq^I \bar{y}$, and each $z^l \succeq^I \bar{y}$. By definition of bests', there is some l for which $\lambda^l > 0$ and there must also be some $\mu^k > 0$ since $\bar{x}$ does not best $\bar{y}$. So, we may write $\bar{x} = \sum_{k=1}^{K_0} \mu^k w^k + \sum_{l=1}^{L_0} \lambda^l z^l$, μ^k and $\lambda^l > 0$ for $k = 1, \ldots, K_0$ and $l = 1, \ldots, L_0$, and $\sum_{k=1}^{K_0} \mu^k + \sum_{l=1}^{L_0} \lambda^l = 1$.

Consider first any w^k for which the indirect preference is merely a consequence of $w^k \geq \bar{x}$. In other words, there is no observed data point w' with $w^k \geq w'$ and $w' \succeq^I \bar{x}$. Without loss, suppose that this is $w^k = w^1$. Then note that $w^1 > \bar{x}$ and thus $\mu^1 < 1$, so we may consider $\Delta = w^1 - \bar{x} > 0$ and represent $\bar{x}$ as

$$\bar{x} = \mu^1[\bar{x} + \Delta] + \sum_{k=2}^{K_0} \mu^k w^k + \sum_{l=1}^{L_0} \lambda^l z^l = \frac{1}{1-\mu^1}\left(\sum_{k=2}^{K_0} \mu^k w^k + \lambda^1\left[z^1 + \frac{\mu^1 \Delta}{\lambda^1}\right] + \sum_{l=2}^{L_0} \lambda^l z^l\right)$$

Then $z^1 \succeq^I \bar{y}$ and $\Delta \geq 0$ implies that $z^1 + \frac{\mu^1 \Delta}{\lambda^1} \succeq^I \bar{y}$, and we have reduced the number of $w^k \succeq^I \bar{x}$ by one. We may then assume that for each w^k there exists some sequence $w', \ldots, w^*$ in the data so that $w^k \geq w' \succeq^R \ldots w^* \succeq^R \bar{x}$.

Consider now the set $\mathcal{W}$ consisting of the bundles that are (1) revealed indirectly preferred to $\bar{x}$, in the sense that there is an observed w' with $w^k \geq w' \succeq^I \bar{x}$, and (2) not revealed indirectly preferred to $\bar{y}$. The bundles in $\mathcal{W}$ may not be in the support of $\bar{x}$, but $\mathcal{W}$ includes $w^1, \ldots, w^{K_0}$.

We claim that for any $w^k \in \mathcal{W}$ in the support of $\bar{x}$, there exists $w^{k'} \in \mathcal{W}$, also in the support of $\bar{x}$, for which $w^k \succ^I w^{k'}$ (in particular if k is unique then $w^k \succ^I w^k$). The claim provides a contradiction because it implies the existence of a strict $\succ^I$ cycle among the elements w^k, contradicting that the original data were rational.

To prove the claim, let $w^k \in \mathcal{W}$ in the support of $\bar{x}$ be arbitrary. Note that, if w' is in the data, then $w' \geq z$ implies that $w' \succeq^R z$. So, we may assume the existence of $w', \dots, w^*$ with $w^k \geq w' \succeq^R \dots \succeq^R w^* \succeq^R \bar{x}$, where all members of the chain are members of $\mathcal{W}$ (as otherwise $w^k \succeq^I \bar{y}$, which we assumed false by the definition of w^k). Note that the observed bundle w^* is part of an observation (p^*, w^*), so that $p^* \cdot w^* \geq p^* \cdot \bar{x}$.

Recall that there is at least one z^l, and that, for all z^l, $p^* \cdot w^* < p^* \cdot z_l$ (the latter as otherwise we would have $w^* \succeq^I z^l$, implying $w^* \succeq^I \bar{y}$, and hence $w^k \succeq^I \bar{y}$, again contradicting the definition of w^k).

So, we have $p^* \cdot w^* \geq p^* \cdot (\sum_k \mu^k w^k + \sum_l \lambda^l z^l)$, and $p^* \cdot \lambda^l z^l > p^* \cdot \lambda^l w^*$ for all l, so that there must be k' for which $p^* \cdot w^* > p^* \cdot w^{k'}$. Conclude $w^* \succ^R w^{k'}$, and hence $w^k \succ^I w^{k'}$. This then implies that there is a $\succ^I$ cycle of length at least two, contradicting the fact that GARP is satisfied.

Finally, we show that strict besting implies strict besting. Suppose then that $\bar{x}$ strictly bests' $\bar{y}$. We may write $\bar{x} = \sum_k \mu^k w^k + \sum_l \lambda^l z^l$, with $z^l \succeq^I \bar{y}$ for all l, and $w^k \succ^I \bar{x}$ for all k. By the previous proof, we also have $w^k \succeq^I \bar{y}$. In fact, since $\bar{x}$ bests' $\bar{y}$ we can write $\bar{x}$ as a convex combination $\bar{x} = \sum_h \eta^h r^h$ with each $r^h \succeq^I \bar{y}$.

Now consider w^1. First, if $w^1 \gg \bar{x}$ then $\mu^1 < 1$ and we may proceed as above to eliminate w^1 from the representation of $\bar{x}$. Second, if $w^1 \succ^I \bar{x}$ but it is not the case that $w^1 \gg \bar{x}$ then by definition of $\succ^I$ there exists w^* with $w^k \succ^I w^*$ and (p^*, w^*) is part of the data, with

$$p^* \cdot w^* \geq p^* \cdot \bar{x} = p^* \cdot \left(\sum_h \eta^h r^h\right).$$

The latter implies that $p^* \cdot w^* \geq p^* \cdot r^h$ for some r^h, and hence that $w^1 \succ^I r^h \succeq^I \bar{y}$. Thus $\bar{x}$ strictly bests $\bar{y}$. □

We may now proceed with the proof of Theorem 1. The starting point is the system of linear inequalities introduced by Varian (1982) for this problem. Indeed, these are essentially Varian's Fact 4 (Varian (1982)). In Varian's terminology, $\bar{y}$ is revealed worse than $\bar{x}$ if and only if there is no solution $q > 0$ to the system of linear inequalities comprised by the following collection of inequalities:

(i) $q \cdot \bar{x} \leq q \cdot x^k$ for all k with $x^k \succeq^I \bar{x}$

(ii) $q \cdot \bar{x} \leq q \cdot x^k$ for all k with $x^k \succeq^I \bar{y}$

(iii) $q \cdot \bar{x} < q \cdot x^k$ for all k with $x^k \succ^I \bar{x}$

(iv) $q \cdot \bar{x} < q \cdot x^k$ for all k with $x^k \succ^I \bar{y}$

(v) $q \cdot \bar{x} \leq q \cdot \bar{y}$

Note that each of the first four listed inequalities really describes multiple linear inequalities. For example, there is one inequality $q \cdot \bar{x} \leq q \cdot x^k$ for each observation (p^k, x^k) that satisfies $x^k \succeq^I \bar{x}$.

The first and third inequalities require that no revealed-preference cycle arises if we add the hypothesized price q to support $\bar{x}$, meaning that we add the observation $(q, \bar{x})$ to the data. The remaining inequalities require that with this hypothesized price, $\bar{x}$ is not revealed strictly preferred, either directly or indirectly, to $\bar{y}$. If these inequalities are satisfied, then there is a price q that supports $\bar{x}$ for which $\bar{x}$ is not revealed strictly preferred to $\bar{y}$. No matter which price we choose to support $\bar{y}$, it will then never be the case that $\bar{x}$ is revealed strictly preferred to $\bar{y}$. It is known that Afriat's theorem then allows the flexibility to choose a rationalization where $u(y) \geq u(x)$ (see Fact 16 in Varian (1982)).

Let us set up a matrix to capture this system, with one row for each of the inequalities that are collected in (i)-(v) above. These rows are of the form $x^k - \bar{x} \in \mathbf{R}^m$ or $\bar{y} - \bar{x} \in \mathbf{R}^m$. We want $q > 0$ so there is also one row for each $q_h \geq 0$ inequality, and one row for the inequality that $\sum_h q_h > 0$. Consider a dual solution with weights $\theta^k \geq 0$ for each of the inequalities involving $\bar{x}$, $\eta^k \geq 0$ for the inequalities that involve $\bar{y}$, and $\eta^{\bar{y}}$ for the 5th inequality.

We use a prime to distinguish revealed preference from strict revealed preference. Let $\xi^h \geq 0$ be the dual variable for the $q_h \geq 0$ inequalities and $\xi^M \geq 0$ for the last $\sum_h q_h > 0$ inequality. The dual then says, for each h,

$$\sum_{\{k:x^k \succeq^I \bar{x}\}} \theta^k \big(x_h^k - \bar{x}_h\big) + \sum_{\{k:x^k \succ^I \bar{x}\}} \theta'^k \big(x_h^k - \bar{x}_h\big) + \sum_{\{k:x^k \succeq^I \bar{y}\}} \eta^k \big(x_h^k - \bar{x}_h\big)$$
$$+ \sum_{\{k:x^k \succ^I \bar{y}\}} \eta'^k \big(x_h^k - \bar{x}_h\big) + \underbrace{\eta^{\bar{y}}(\bar{y}_h - \bar{x}_h)}_{\bigstar} + \xi^M = 0$$

In an abuse of notation, we shall not distinguish between variables with and without prime. The term indicated by $\bigstar$, with dual variable $\eta^{\bar{y}}$, corresponds to equation 5. For ease of exposition, label $x^{K+1} = \bar{y}$ and $\eta^{K+1} = \eta^{\bar{y}}$, so that inequality 5 becomes an inequality of type 2, and we write $\eta^{\bar{y}}(\bar{y}_h - \bar{x}_h) = \eta^{K+1}(x_h^{K+1} - \bar{x}_h)$.

Suppose first that $\xi^M > 0$. Then we get that $\sum_k (\theta^k + \eta^k) x^k \ll \bar{x} \sum_k (\theta^k + \eta^k)$, which means that $\sum_k (\theta^k + \eta^k) > 0$ and that we may normalize so that $\sum_k \theta^k + \eta^k = 1$. Set $z^{k^*} \gg x^{k^*}$ for some $\theta^{k^*} + \eta^{k^*} > 0$, and $z^k = x^k$ for all other $k \neq k^*$, so that $\bar{x} = \sum_k (\theta^k + \eta^k) z^k$ with $z^k \succeq^I \bar{x}$ or $z^k \succeq^I \bar{y}$ for each k, and where the comparison becomes $\succ^I$ for $k = k^*$. Notice that we can choose k^* so that $\eta^{k^*} > 0$ because if all the η variables were zero we would have a certificate for the inequalities in (i) and (iii) being infeasible; we know, however, that these are feasible.[10] We conclude then that $\bar{x}$ strictly best' $\bar{y}$.

If instead $\xi^M = 0$, then we must have $\theta^k + \eta^k > 0$ for some k with either $x^k \succ^I \bar{x}$ or $x^k \succ^I \bar{y}$. Again this allows us to assume that $\sum_k \theta^k + \eta^k = 1$ and we get that $\sum_k (\theta^k + \eta^k) x^k \leq \bar{x}$. Again we obtain that $\bar{x}$ strictly best' $\bar{y}$. By Lemma 6, the theorem follows.

[10] Indeed, if we consider only the inequalities (i) and (iii), and if the data set is rationalizable, then we may choose $q > 0$ to support a rationalizing utility at $\bar{x}$. The resulting data set, adding the observation $(q, \bar{x})$, must be rationalizable.

7.2 *Proof of Theorem 3*

We begin with the following lemma, which is stated in Chambers and Echenique (2016), Remark 3.6.

Lemma 7. *Let $i \in N$. Suppose that for all $k \in [K_i]$, there are $u_i^k \in \mathbf{R}$ and $\lambda_i^k > 0$ for which for all $k, l \in [K_i]$ satisfying $p_i^k \cdot (x_i^l - x_i^k) \leq 0$, we have*

$$u_i^l \leq u_i^k + \lambda_i^k p_i^k \cdot (x_i^l - x_i^k).$$

Then the individual data set $\{(p_i^k, x_i^k)\}_{k \in [K_i]}$ is rationalizable.

Proof. Suppose that the condition in the statement of the lemma is satisfied. Define the pair of binary relations $x_i^k \succeq_i^R x_i^l$ if $p_i^k \cdot (x_i^l - x_i^k) \leq 0$ and $x_i^k \succ_i^R x_i^l$ if $p_i^k \cdot (x_i^l - x_i^k) < 0$.

A *cycle* is a finite list $x_i^{l_1} \succeq_i^R x_i^{l_2} \succeq_i^R \ldots x_i^{l_a} \succ_i^R x_i^{l_1}$. We claim that there can be no cycle. For, if there were, then we would have

$$u_i^{l_{j+1}} - u_i^{l_j} \leq \lambda_i^{l_j} p_i^{l_j} \cdot (x_i^{l_{j+1}} - x_i^{l_j}),$$

for all $j = 1, \ldots, a-1$ and

$$u_i^{l_1} - u_i^{l_a} \leq \lambda_i^{l_a} p_i^{l_a} \cdot (x_i^{l_1} - x_i^{l_a}).$$

Reading addition of indices as modulo a, observe that

$$0 = \sum_{j=1}^{a} (u_i^{l_{j+1}} - u_i^{l_j}) \leq \sum_{j=1}^{a} \lambda_i^{l_j} p_i^{l_j} \cdot (x_i^{l_{j+1}} - x_i^{l_j}) < 0.$$

The first equality is by telescoping, the weak inequality by summing the original inequalities, and the strict inequality because of the right-hand sides of the original inequalities are nonpositive (and at least one strictly negative). So, we arrive at a contradiction and there can be no cycle. Conclude by Afriat's theorem (Afriat (1967), Chambers and Echenique (2016)) that the individual data set is rationalizable. □

Now we proceed with the proof of the theorem.

First, that (i) implies (iii) follows because if u_i are rationalizing monotone and explicitly quasiconcave utilities, then $z_i \succeq_i^I \bar{x}_i$ implies $u_i(z_i) \geq u_i(\bar{x}_i)$, and $z_i \succ_i^I \bar{x}_i$ implies $u_i(z_i) > u_i(\bar{x}_i)$. So, when y_i is a convex combination of bundles $z_i^l \succeq_i^I \bar{x}_i$ we must have that $u_i(y_i) \geq u_i(\bar{x}_i)$ by quasiconcavity of utility. Moreover, if $z_i^l \succ_i^I \bar{x}_i$ for some l, then we obtain $u_i(y_i) > u_i(\bar{x}_i)$ by explicit quasiconcavity. In all, then when y_i bests $\bar{x}_i$ for all agents, and strictly bests for at least one agent, we have that $\bar{x}$ is Pareto dominated for the rationalizing utilities.

Second, it is obvious that (ii) implies (i). So, we focus our attention on showing that (iii) implies (ii). (Indeed our argument shows that (ii) and (iii) are equivalent.) Suppose then that (iii) is satisfied. We will demonstrate that there exists some $q \in \mathbf{R}^m_{++}$ so that, for all $i \in N$, the individual data set given by $\{(p_i^k, x_i^k)\}_{k \in [K_i]} \cup \{(\bar{x}_i, q)\}$ is rationalizable. This then implies (by Afriat's theorem) the existence of a concave, increasing utility function for which for all $y \in \mathbf{R}^m_+$ satisfying $q \cdot y \leq q \cdot \overline{x}_i$, we have $u_i(y) \leq u_i(\overline{x}_i)$, and consequently, that $u_i(y) > u_i(\overline{x}_i)$ implies $q \cdot y > q \cdot \overline{x}_i$. Consequently, it also follows that $u_i(y) \geq u_i(\overline{x}_i)$ implies $q \cdot y \geq q \cdot \overline{x}_i$, by continuity and monotonicity of u_i. It then follows that $\overline{x}$ is efficient for these utility indices.[11]

The proof relies on a homogeneous theorem of the alternative; see Border (2020).

The content of Afriat's theorem is that for each $i \in N$ and $k \in [K_i]$, there is u_i^k and $\lambda_i^k > 0$ for which for all $k, l \in [K_i]$,

$$u_i^k \leq u_i^l + \lambda_i^l p_i^l \cdot \left(x_i^k - x_i^l\right).$$

What we would now like to find are additional unknown parameters. Namely, for each $i \in N$, a scalar $\overline{u}_i \in \mathbf{R}$ and $q \in \mathbf{R}^m$. The vector q is required to be common to all individuals and will reflect the common prices supporting the hypothesized efficient allocation $\overline{x}$.

Our task is then to find $q \in \mathbf{R}^m$, and for each $i \in N$, a real number $\overline{u}_i \in \mathbf{R}$, and for each $i \in N$ and $k \in [K_i]$, $u_i^k \in \mathbf{R}$ and $\lambda_i^k \in \mathbf{R}$ for which the following linear inequalities are satisfied:

(i) For all $i \in N$ and all $k, l \in [K_i]$ for which $p_i^k \cdot (x_i^l - x_i^k) \leq 0$, we have $u_i^l \leq u_i^k + \lambda_i^k p_i^k \cdot (x_i^l - x_i^k)$.

(ii) For all $i \in N$ and all $k \in [K_i]$, $u_i^k \leq \overline{u}_i + q \cdot (x_i^k - \overline{x}_i)$.

(iii) For all $i \in N$ and all $k \in [K_i]$, for which $p_i^k \cdot (\overline{x}_i - x_i^k) \leq 0$, we have $\overline{u}_i \leq u_i^k + \lambda_i^k p_i^k \cdot (\overline{x}_i - x_i^k)$.

(iv) For all $i \in N$ and all $k \in [K_i]$, $\lambda_i^k > 0$.

(v) $q \geq 0$ and $q \neq 0$.

The inequalities can be represented in matrix notation. We display part of the matrix below, as the matrix itself is quite large. The matrix below displays four horizontal blocks. The first two correspond to vectors corresponding to weak inequalities, the latter two to strict. This matrix has, for each agent i, $2(K_i + 1)$ columns, and an additional m columns; in total the number of columns is $m + \sum_i (2K_i + 1)$. Observe that, in the matrix written below, the column labeled by q actually represents m columns; for example, $\mathbf{1}_{m'}$ is an indicator function of the dimension $m' \in \{1, \dots, m\}$.

As to rows, the matrix has, for each agent i, one row for each ordered pair (l, k) where $l, k \in [K_i]$, $k \neq l$, and $p_i^k \cdot (x_i^l - x_i^k) \leq 0$. When agent i is understood, the row is labeled (l, k), as in the displayed matrix below. Continuing with the rows for agent i, there are also three rows for each k: one labeled by $(k, *)$, one by $(*, k)$, and one by k. The row labeled (k, l) for agent i is meant to capture inequality (i): there is a 1 in the column k for agent i, a -1 in column l, and $p_i^k \cdot (x_i^l - x_i^k)$ in the column for k among the second set of K_i columns. The rest of the entries in that row are zero. In a similar vein, the rows labeled by $(k, *)$ and $(*, k)$ are there to encode the inequalities in (ii) and in (iii). The

[11] If not, then there is $\overline{y}$ for which $\sum_i \overline{y}_i = \sum_i \overline{x}_i$ and for all $i \in N$, we have $u_i(\overline{y}_i) \geq u_i(\overline{x}_i)$, with inequality strict for some $j \in N$, implying $\sum_i q \cdot \overline{y}_i > \sum_i q \cdot \overline{x}_i$, a contradiction.

row labeled k is meant to capture the basic positivity constraint (iv), and has a one in column k, among the second collection of K_i columns.

Finally, the matrix has a collection of rows $m+1$ that are not specific to any agent and seek to capture (v). There is then one column for each $m' \in \{1, \dots, m\}$ (labeled $(*, m)$), expressing the nonnegativity of q, and a row asserting that $\sum_{m'=1}^{m} q(m') > 0$, the row labeled M.

Because this matrix is large, we only show certain portions of it. The rows listed in the matrix have zeroes everywhere for every remaining column.

$$
\begin{array}{c|ccccccccc|ccccc|c}
 & 1 & \cdots & k & \cdots & l & \cdots & K_i & \cdots & * & 1' & \cdots & k' & \cdots & K_i' & q \\
(l,k) & 0 & \cdots & 1 & \cdots & -1 & \cdots & 0 & \cdots & 0 & 0 & \cdots & p_i^k \cdot \left(x_i^l - x_i^k\right) & \cdots & 0 & 0 \\
\vdots & \vdots & & \vdots & & \vdots & & \vdots & & \vdots & \vdots & & \vdots & & \vdots & 0 \\
(*,k) & 0 & \cdots & 1 & \cdots & 0 & \cdots & 0 & \cdots & -1 & 0 & \cdots & p_i^k \cdot \left(\overline{x}_i - x_i^k\right) & \cdots & 0 & 0 \\
\vdots & \vdots & & \vdots & & \vdots & & \vdots & & \vdots & \vdots & & \vdots & & \vdots & 0 \\
(k,*) & 0 & \cdots & -1 & \cdots & 0 & \cdots & 0 & \cdots & 1 & 0 & \cdots & 0 & \cdots & 0 & x_i^k - \overline{x}_i \\
\hline
\vdots & \vdots & & \vdots & & \vdots & & \vdots & & \vdots & \vdots & & \vdots & & \vdots & 0 \\
(*,m') & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 & \mathbf{1}_{m'} \\
\vdots & \vdots & & \vdots & & \vdots & & \vdots & & \vdots & \vdots & & \vdots & & \vdots & 0 \\
\hline
M & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 & \mathbf{1}_{\{1,\dots,m\}} \\
\hline
\vdots & \vdots & & \vdots & & \vdots & & \vdots & & \vdots & \vdots & & \vdots & & \vdots & 0 \\
k & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 & \vdots & 0 & 0 & \cdots & 1 & \cdots & 0 & 0 \\
\vdots & \vdots & & \vdots & & \vdots & & \vdots & & \vdots & \vdots & & \vdots & & \vdots & 0
\end{array}
$$

We are searching for a vector in $m + \sum_i (2K_i + 1)$-dimensional real space, which when multiplied with this matrix to yield a linear combination of its columns, results in a vector whose coordinates in the first two horizontal blocks are nonnegative, and in the last two are strictly positive. Such a vector would represent a solution to the system of inequalities (i)–(v). This is the system to which we will apply a duality result.

By Motzkin's transposition theorem (a version of the theorem of the alternative, see Theorem 47 in Border (2020)), there is no solution to the set of inequalities (and consequently to the enumerated list of inequalities above) if and only if there is, for each row of the matrix, a nonnegative weight, where for some row corresponding to a strict inequality (either in the third or fourth horizontal block), one of the weights is strict, for which the weighted sum of rows is the zero vector.

So, let us suppose by means of contradiction that there is no solution to the linear system. Therefore, there exists a solution to the dual system. Interpret the solution as a collection of weights on the rows of the matrix. For the rows corresponding to agent $i \in N$ (any row except the one labeled M), we let $\xi_i^A \geq 0$ denote the weight for the row labeled by A. For example, in the row of the above matrix labeled (l, k), $\xi_i^{(l,k)}$ is the associated weight. We let $\xi^M \geq 0$ be the weight associated with row M (which is common to all $i \in N$), and we let $\xi^{(*,m')} \geq 0$ be the weight associated with row $(*, m')$.

The matrix has a special structure. Observe that, restricted to the first $\sum_i (K_i + 1)$ block of columns on the left, and the rows labeled (k, l), $(k, *)$, or $(k, *)$ for some agent (and some k, l), the matrix becomes the incidence matrix of a graph with vertexes that can be identified with these $\sum_i (K_i + 1)$ columns. So, each vertex is identified with a pair (i, k), of an agent and an observation $k \in [K_i]$, or with a pair $(i, *)$ for the hypothesized efficient bundle. An edge goes from a node (i, k) to (i, l) when $p_i^k \cdot (x_i^l - x_i^k) \leq 0$. An edge goes from $(i, *)$ to (i, k) when $p_i^k \cdot (\overline{x}_i - x_i^k) \leq 0$. An edge always goes from (i, k) to $(i, *)$.

Now, the solution to the dual, when restricted to the incidence submatrix, provides a nonnegative linear combination of rows that equals the null vector. The Poincaré–Veblen–Alexander theorem (Berge (2001)) claims that for any nonnegative weighted sum of incidence vectors of a directed graph, which is zero, there is a collection of positively oriented cycles in the graph, each cycle being associated with a weight, and the total weight ascribed to an incidence vector is the sum of all weights associated to cycles in which the incidence vector appears. Here, a cycle includes no repetitions of nodes.

Because the individual data set $\{(p_i^k, x_i^k)\}_{k \in [K_i]}$ is rationalizable, we may assume without loss of generality that every such cycle involves an edge of the type connecting (i, k) to $(i, *)$. This is because otherwise, along all elements of the cycle, rationalizability implies that $p_i^{k_j} \cdot (x_i^{k_{j+1}} - x_i^{k_j}) = 0$, and thus the weighted sum of vectors across that cycle is zero. Removing them does not affect the total weighted sum of rows.

Let us now represent the cycles associated with agent $i \in N$ by $\mathcal{C}_i$, as described, each of them comes with a weight $\mu(c) \geq 0$. What we just claimed is that for each $c \in \mathcal{C}_i$, there is some $k \in [K_i]$ and an edge connecting (i, k) to $(i, *)$. This implies, in particular, that $x_i^k \succeq_i^I \bar{x}_i$. To see why, let the cycle be written via a sequence of nodes: $(i, *), (i, k_1), \ldots, (i, k_l = k), (i, *)$. Because $(i, *)$ is connected to (i, k_1) by an edge, it means that $p_i^{k_1} \cdot (\bar{x}_i - x_i^{k_1}) \leq 0$, so that $x_i^{k_1} \succeq_i^R \bar{x}_i$. Similarly, $x_i^{k_{j+1}} \succeq_i^R x_i^{k_j}$ for all $j = 1, \ldots, l-1$. Consequently, by definition, $x_i^k \succeq_i^* \bar{x}_i$.

What we have just claimed is that if $\xi_i^{(k,*)} > 0$, it must be that $x_i^k \succeq_i^I \bar{x}_i$.

Now, again by Motzkin's transposition theorem, one of the following must be true: either $\xi^M > 0$, or there is $i \in N$ and $k \in [K_i]$ for which $\xi_i^k > 0$.

Let us consider each of the two cases in turn.

Case 1: There is a dual solution with $\xi^M > 0$.

The only columns for which row M are nonzero are the last m columns. Rows of type $(*, m')$ add (potentially) nonnegative terms to these last m columns. Since the weighted sum of rows equals zero, it follows that

$$\sum_i \sum_{k \in [K_i]} \xi_i^{(*,k)} \big(x_i^k - \overline{x}_i\big) = -\sum_{m'=1}^{m} \xi^{*,m'} \mathbf{1}_{m'} - \xi^M \mathbf{1}_{1,\ldots,m} \ll 0. \tag{1}$$

In other words, for each $i \in N$ and each $k \in [K_i]$, there is a number $\theta_i^k \geq 0$ for which

$$\sum_i \sum_{k \in [K_i]} \theta_i^k \big(x_i^k - \overline{x}_i\big) \ll 0,$$

where by the preceding discussion, $\theta_i^k > 0$ implies $x_i^k \succeq_i^I \overline{x}_i$. Furthermore, there is $i \in N$ and $k \in [K_i]$ for which $\theta_i^k > 0$, since equation (1) is strictly negative in every coordinate.

Without loss of generality (since the system is homogeneous), we may assume that $\sup_{i\in N}\sum_{k\in[K_i]}\theta_i^k = 1$.

For each $i \in N$, let $\theta_i^0 = 1 - \sum_{k\in[K_i]}\theta_i^k$. Then

$$\sum_i\left(\theta_i^0\bar{x}_i + \sum_k \theta_i^k x_i^k\right) = \sum_i\left(\bar{x}_i + \sum_k \theta_i^k\left(x_i^k - \bar{x}_i\right)\right) \ll \sum_i \bar{x}_i.$$

So, we can define

$$\bar{y}_i = \theta_i^0\bar{x}_i + \sum_{k\in[K_i]}\theta_i^k x_i^k.$$

for all $i \neq 1$. Observe that $\bar{y}_i$ is a convex combination of $\bar{x}_i \succeq_i^I \bar{x}_i$ (by definition), and $x_i^k \succeq_i^I \bar{x}_i$. If $\theta_1^0 > 0$, choose $y_1' \gg \bar{x}_1$ so that $\bar{y}_1 = \theta_1^0 y_1' + \sum_{k=1}^{K_1}\theta_1^k x_1^k$ and $y_1' \succ_1^I \bar{x}_1$; otherwise choose $y_1^{k^*} \gg x_1^{k^*}$ so that $\bar{y}_1 = \theta_1^0\bar{x}_1 + \sum_{k=1}^{K_1}\theta_1^k x_1^k + \theta_1^{k^*}(y_1^{k^*} - x_1^{k^*})$ and $y_1^{k^*} \succ_1^I x_1^{k^*}$. Either way the allocation $\bar{y}_i$ bests $\bar{x}_i$ for all agents, and strictly bests it for agent 1.

Case 2: There is a dual solution with $\xi_i^k > 0$.

This means that there is $i \in N$ and $k \in [K_i]$ for which $\xi_i^k > 0$. Fix such an $i^* \in N$ and a $k^* \in [K_i]$. Because $\xi_M = 0$ is possible, we may only conclude in this case that $\sum_i\sum_{k\in[K_i]}\xi_i^{(*,k)}(x_i^k - \overline{x}_i) \leq 0$.

On the other hand, we may conclude, since $\xi_{i^*}^{k^*} > 0$, that there is also $l \in \{1, \dots, K_{i^*}\}$ with $\xi_{i^*}^{(l,k^*)} > 0$ and $p_{i^*}^{k^*}\cdot(x_{i^*}^l - x_{i^*}^{k^*}) < 0$; or in other words, $x_{i^*}^{k^*} \succ_i^R x_{i^*}^l$. In particular, the edge (i^*, k^*) to (i^*, l) belongs to some $c \in \mathcal{C}_i$, which has a corresponding $\xi_{i^*}^{(*,k)} > 0$; we may conclude then that $x_{i^*}^k \succ_{i^*}^I \bar{x}_{i^*}$.

Now $\sum_i\sum_{k\in[K_i]}\xi_i^{(*,k)}(x_i^k - \overline{x}_i) \leq 0$ implies that we can again as in Case 1 set $\theta_i^k = \xi_i^{(*,k)}$, assume without loss that $\sum_k \theta_i^k \leq 1$, and define $\theta_i^0 = 1 - \sum_k \theta_i^k$. Then we may set $z_i^0 = \bar{x}_i$ when $\theta_i^0 > 0$ and $z_i^k = x_i^k$ when $\theta_i^k > 0$ and then we have (ignoring terms where $\theta_i^k = 0$)

$$\sum_i\sum_{k=0}^{K_i}\theta_i^k z_i^k \leq \sum_i \bar{x}_i$$

so that if we define an allocation by $y_i = \sum_{k=0}^{K_i}\theta_i^k z_i^k$, and recall that $x_{i^*}^k \succ_{i^*}^I \bar{x}_{i^*}$, we conclude that the allocation (y_i) empirically dominates $(\bar{x}_i)$.

7.3 *Proof of Theorem 2*

For this proof, we start by constructing the same matrix as in the proof of Theorem 3 but with $N = 1$, and where we now add a row $\mathbf{1}_* - \mathbf{1}_k$ for each k to capture the inequality $u^k \leq \bar{u}$. The idea is to consider the same collection of linear inequalities as before, but where we in addition require that the level of utility in the new observation exceeds that of any existing observation in the data. Consider a solution to the dual. Again when restricted to the incidence matrix there is a collection of oriented cycles in the graph, each cycle being associated with a weight, and the total weight ascribed to an incidence vector is the sum of all weights associated to cycles in which the incidence vector appears. A cycle includes no repetitions of nodes.

Because the individual data set $\{(p_i^k, x_i^k)\}_{k\in[K_i]}$ is rationalizable, we may assume without loss of generality that every such cycle involves an edge of the type connecting (i, k) to $(i, *)$. This is because otherwise, along all elements of the cycle, rationalizability implies that $p_i^{k_j} \cdot (x_i^{k_{j+1}} - x_i^{k_j}) = 0$, and thus the weighted sum of vectors across that cycle is zero. Removing them does not affect the total weighted sum of rows.

By the same argument as in Theorem 3, if $\mathcal{C}$ denotes the set of cycles, each of them with weight $\mu(c)$, we know that a cycle has an edge connecting (say) (k) to $(*)$, where $\xi^{(k,*)} > 0$ and that in consequence $x^k \succeq^I \bar{x}$. What is different from the proof of Theorem 3 is that now the cycle may involve an edge going from (say) (l) to $(*)$, which was added from a row $\mathbf{1}_* - \mathbf{1}_l$ due to the inequality $u^l \leq \bar{u}$.

Now as before there are two cases to contend with. First, when $\xi^M > 0$ we obtain as before that $\sum_k \xi^{(k,*)}(x^k - \bar{x}) \ll 0$. This means that there is a convex combination $\theta^- \bar{x} + \sum_k \theta^k x^k \ll \bar{x}$ with support in $\bar{x}$ and the $x^k \succeq^I \bar{x}$ (as $\theta^k = \xi^{(k,*)} > 0$ means that the argument in previous paragraph applies). Second, when $\xi^M = 0$ then we must have $\xi^k > 0$ for some k. This may again lead to the same case as in Theorem 3, or it may be the case that $\xi^{(k,*)} = 0$ for all k and we have a strict cycle involving the new $\bar{x} \succeq^R x^l$ edges. This would be a violation of GARP.

7.4 *Proof of Proposition 5*

The result follows from two simple lemmas.

Lemma 8.

$$S^{u_i^*}(x_i^K) = \bigcap_{(V_i,\lambda_i)\in A_i} S^{u^{(V_i,\lambda_i)}}(x_i^K)$$

Proof. Suppose that $u^{(V,\lambda)}(x_i) \geq u^{(V,\lambda)}(x_i^K)$ for all $(V, \lambda) \in A_i$. Then since $u^{(V,\lambda)}(x_i^K) = V_i^K = 1$ for all $(V, \lambda) \in A_i$, it follows that $u_i^*(x_i) = \inf\{u^{(V,\lambda)}(x_i) : (V, \lambda) \in A_i\} \geq u_i^*(x_i^K)$.

Conversely, suppose that $u_i^*(x) \geq u_i^*(x^K)$. Then again, since $u_i^*(x^K) = 1 = u^{(V_i,\lambda_i)}(x^K)$ for any $(V_i, \lambda_i) \in A_i$, we conclude that $u^{(V_i,\lambda_i)}(x_i) \geq u^{(V_i,\lambda_i)}(x_i^K)$ for all $(V_i, \lambda_i) \in A_i$. □

Lemma 9. *Let u_i be concave and monotone rationalization of the data with $u(x_i^K) = 1$. Then there is $(V_i, \lambda_i) \in A_i$ such that $S^{u_i}(x_i^K) \subseteq S^{u^{(V_i,\lambda_i)}}(x_i^K)$.*

Proof. For each $x_i \in \mathbf{R}_+^m$, let $V_i^{x_i} = u_i(x_i)$ and $q_i^{x_i} \in \partial u_i(x_i)$. Then we have, for any x_i and y_i that $V_i^{y_i} \leq V_i^{x_i} + q_i^{x_i} \cdot (y_i - x_i)$. We also have that

$$u_i(x_i) = \inf\{V_i^{y_i} + q_i^{y_i} \cdot (x_i - y_i) : y_i \in \mathbf{R}_+^m\}.$$

Let (p_i^k, x_i^k), $k = 1, \dots, K$ be a data set.

If u_i rationalizes the data, then we can identify $V_i^k = V_i^{x^k}$ and choose λ_i^k so that $q_i^{x^k} = \lambda_i^k p_i^k$. Because u is a rationalization, then $(V_i, \lambda_i) \in A_i$. The resulting Afriat utility satisfies that, for any $x_i \in S_{u_i}(x_i^K)$,

$$\begin{aligned} u^{(V_i,\lambda_i)}\big(x_i^K\big) = V_i^K = u_i\big(x_i^K\big) \le u_i(x_i) \\ = \inf\big\{V_i^y + q_i^y \cdot (x_i - y_i) : y_i \in \mathbf{R}_+^m\big\} \\ = \inf\big\{V_i^k + q_i^k \cdot \big(x_i - x_i^k\big) : k = 1, \dots, K\big\} \end{aligned}$$

Hence, $S_{u_i}(x_i^K) \subseteq S_{u^{(V_i,\lambda_i)}}(x_i^K)$. □

Appendix: Quasilinear rationalization with multiple allocations

The paper focuses on the single-allocation case, as discussed in Section 4. If we restrict attention to quasi-linear preferences, then we can accommodate multiple allocations. In particular, we can characterize the collections of allocations that may jointly be Pareto efficient for some rationalizing quasilinear utility.

We first revisit the rationalizability question for a single consumer, a question first analyzed by Brown and Calsamiglia (2007), and then turn to the problem of a collection of allocations.

A.1 *Individual data*

Consider an individual data set $(x^k, p^k)_{1\le k\le K}$. We say that it is *quasilinear rationalizable* if there exists a utility function $U : \mathbf{R}_+^m \to \mathbf{R}$ so that, for all $k \in [K]$,

$$U\big(x^k\big) - p^k \cdot x^k \ge U(x) - p^k \cdot x$$

for all $x \in \mathbf{R}^n$.

A matrix $\eta \in \mathbb{R}_+^{K\times K}$ is *bistochastic* if for every $t \in [K]$, $\sum_{s\in[K]} \eta(t, s) = \sum_{s\in[K]} \eta(s, t) = 1$.

The following result is a form of the theorem in Brown and Calsamiglia (2007), essentially an economic analogue of the notion of *cyclic monotonicity* due to Rockafellar (1966).[12] We state it without proof (its proof is implicit in the proof of the following theorem as well).

Proposition 10. *An individual data set $(x^k, p^k)_{1\le k\le K}$ is quasilinear rationalizable by a concave and monotonic utility if and only if, for any bistochastic matrix $\eta \in \mathbb{R}_+^{K\times K}$, we have $\sum_k \sum_t \eta(k, t) p^t \cdot (x^k - x^t) \ge 0$.*

To interpret Proposition 10, think of a bistochastic matrix as a probability distribution over pairs (k, t), after a normalization. If U is a rationalization of the data, then the sum $\frac{1}{\sum_{(k,t)} \eta(k,t)} \sum_{(k,t)} \eta(k, t)[U(x^k) - U(x^t)]$ is the expected change in utility when going from the consumption x^t to x^k. If the matrix is bistochastic, this expected change is zero. On the other hand, since U rationalizes the data, for each k and t, $U(x^t) - p^t \cdot x^t \ge U(x^k) - p^t \cdot x^k$. Thus then change in utility $U(x^k) - U(x^t)$ is bounded above by $p^t \cdot (x^k - x^t)$ and, therefore, the expected value of $p^s \cdot (x^k - x^t)$, $\frac{1}{\sum_{(k,t)} \eta(k,t)} \sum_{(k,t)} \eta(k, t) p^t \cdot (x^k - x^t)$ must be nonnegative.

[12] See also Browning (1989).

A.2 *Multiple allocations*

Here, we will show that the quasilinear model allows a natural linear test of the hypothesis that multiple allocations could potentially be Pareto efficient. Example 1 shows that, in general, an allocation-by-allocation approach does not capture all of the implications imposed by hypothesizing that multiple allocations are Pareto efficient. In the general setting, there is no linear test that we could perform. But in the quasilinear setting, it becomes quite simple.

Let $x^s = (x_1^s, \dots, x_N^s) \in \mathbf{R}_+^{LN}$ for $s \in [L]$ be a collection of L allocations. In a notational abuse, we regard the elements of $[K_i]$ and $[L]$ as distinct, even if they are the same number.

In the following, a matrix η is constant row-column sum if there is some number c such that, for every k and l, $\sum_t \eta(t,k) = c = \sum_t \eta(l,t)$. That is, if it is a scaled version of a bistochastic matrix.

Theorem 11. *There exist concave utilities U_i that quasilinear rationalize the data, and for which the allocations x^s are Pareto optimal if and only if there are no constant row–column sum matrices $\eta_i \in \mathbb{R}_+^{([K_i]\cup[L])\times([K_i]\cup[L])}$ for which*

$$\sum_i \sum_{t\in K_i} \sum_{k\in[K_i]\cup[L]} \eta_i(k,t)\, p_i^t \cdot \left(x_i^t - x_i^k\right) > 0$$

and for all $s \in L$,

$$\sum_i \sum_{k\in[K_i]\cup[L]} \eta_i(k,s)\left(x_i^s - x_i^k\right) \geq 0.$$

The idea follows as in Proposition 10. Suppose the data are rationalizable, and suppose η_i satisfies the conditions in the theorem, and that each x^s could be efficient. We will argue that a contradiction entails.

Hypothesizing that for each s, x^s is an efficient allocation means that there are prices q^s at which agent i demands x_i^s. Let us for now set $p_i^s = q^s$ for each $i \in N$ and $s \in L$. Then, owing to Theorem 11, we must have

$$\sum_i \sum_{t\in[K_i]\cup[L]} \sum_{k\in[K_i]\cup[L]} \eta_i(k,t)\, p_i^t \cdot \left(x_i^t - x_i^k\right) \leq 0. \tag{2}$$

Now, since for any $s \in L$, $\sum_i \sum_{k\in[K_i]\cup[L]} \eta(k,s)(x_i^k - x_i^s) \leq 0$ and since $p_i^s = q^s \geq 0$, using linearity we have

$$\sum_i \sum_{k\in[K_i]\cup[L]} \eta(k,s)\, p_i^s \cdot \left(x_i^s - x_i^k\right) \geq 0 \tag{3}$$

for any $s \in [L]$. For each s, subtracting equation (3) from (2), we get that $\sum_i \sum_{t\in K_i} \sum_{k\in K_i \cup Ly} \eta(k,t)\, p_i^t \cdot (x_i^t - x_i^k) \leq 0$, contradicting the first equation in the statement of the theorem. We offer a formal proof in Section A.3.

To interpret the conditions in Theorem 11, we may assume that each η_i is actually a bistochastic matrix, and, by renormalizing, that there is some probability distribution $\alpha \in \Delta(N)$ such that the equations in the theorem may be rewritten as

$$\sum_i \sum_{t\in[K_i]} \sum_{k\in[K_i]\cup[L]} \alpha_i \eta_i(k,t) p_i^t \cdot \left(x_i^k - x_i^t\right) < 0$$

and for each $s \in [L]$,

$$\sum_i \sum_{k\in[K_i]\cup[L]} \alpha_i \eta_i(k,s)\left(x_i^k - x_i^s\right) \leq 0.$$

We shall see that the conditions in the theorem are, roughly speaking, multiagent analogues of the conditions for quasilinear rationalizability in Proposition 10, once we hypothesize common supporting prices for the allocations x^s, $s \in [L]$.

Interpret the product $\alpha_i \eta_i(k,t)$ as a probability: draw an agent at random according to α, and then a pair (k,s) using the bistochastic matrix. Just like in Proposition 10, the change in utility from x_i^t to x_i^k is upper bounded by $p^t \cdot (x_i^k - x_i^t)$. In a bistochastic matrix, the expected utility change must be zero and, therefore, Proposition 10 results as the expectation of $p^t \cdot (x_i^k - x_i^t)$ cannot be negative.

Theorem 11 is, however, about efficiency, which demands that we find supporting prices q^s for each allocation $s \in [L]$. We may set individual prices $p_i^s = q^s$, because efficiency requires that the same prices support each individual agent's consumption (a generalization of the equalization of marginal rates of substitution). Now, since $q^s > 0$, if $\sum_i \sum_{k\in[K_i]\cup[L]} \alpha_i \eta_i(k,s)(x_i^k - x_i^s) < 0$ holds for each $s \in [L]$, then we obtain

$$\sum_{s\in[L]} q^s \cdot \sum_i \sum_{k\in[K_i]\cup[L]} \alpha_i \eta_i(k,s)\left(x_i^k - x_i^s\right) = \sum_i \sum_{s\in[L]} \sum_{k\in[K_i]\cup[L]} \alpha_i \eta_i(k,s) p_i^s \cdot \left(x_i^k - x_i^s\right) < 0.$$

Using the upper bound on utility changes that we used in Proposition 10, this means that the expected change in utility is negative (when drawing an agent at random, and a pair of allocations from $[L]$ and $[K_i] \cup [L]$). But the overall expected change must be zero, so the inequality in the formula is inconsistent with efficiency.

A.3 *Proof of Theorem 11*

We offer only a sketch, as the details are similar to our other results.

For all $i \in N$ and all $k \in [K_i] \cup [L]$ and $t \in [K_i]$, consider the Afriat inequalities:

$$U_i^k \leq U_i^t + p_i^t \cdot \left(x_i^k - x_i^t\right).$$

In these inequalities, U_i^k is unknown.

For all $k \in [K_i] \cup [L]$ and all $s \in L$, consider the Afriat inequalities for the L proposed allocations,

$$U_i^k \leq U_i^s + q^s \cdot \left(x_i^k - x_i^s\right).$$

In these inequalities, U_i^k is unknown for $k \in [K_i] \cup [L]$ and $q^s \in \mathbb{R}_+^n$ is unknown for $s \in [L]$.

Consider three matrices, A, B, C. These matrices have one row for each triple (i, k, t) with $i \in N$, $k, t \in [K_i] \cup [L]$, and $k \neq t$.

Matrix A has one column for each element of $(\bigcup_{i \in N} [K_i]) \cup ([n] \times [L])$: identify each column with the unknown U_i^s. In the row for (i, k, s), A is equal to zero everywhere except for a -1 in the column for U_i^k and 1 in the column for U_i^s.

Matrix B has $n \times L$ columns: identify each with the unknown q_ℓ^s. In the row for (i, k, s) in which $s \in [L]$ matrix B has zero in all entries except for a $x_{i,\ell}^k - x_{i,\ell}^s$ in the column for q_ℓ^s.

Matrix C has a single column. In the row for (i, k, s) with $s \in [K_i]$, this column equals $p_i^s \cdot (x_i^k - x_i^s)$. It equals zero in any row (i, k, s) with $s \in [L]$.

For each row $r = (i, k, s)$ and matrix $a \in \{A, B, C\}$, we write r_a for row r in matrix a. The system is infeasible if and only if there exists weights $\theta(r) \geq 0$ for each row $r = (i, k, s)$ such that:

(i) $\sum_r \theta(r) r_A = 0$

(ii) $\sum_r \theta(r) r_B \leq 0$

(iii) $\sum_r \theta(r) r_C < 0$

Note that, for each i and k, $\sum_r \theta(r) r_A = 0$ implies that

$$\sum_{s \in [K_i] \cup [L], s \neq k} \theta(i, k, s) - \sum_{s \in [K_i] \cup [L], s \neq k} \theta(i, s, k) = 0.$$

Let $\eta_i(k, s) = \theta(i, k, s)$ and define $\eta_i(k, k)$ for each k so that the matrix η has constant row–column sum.

Since $\sum_r \theta(r) r_B \leq 0$ and η_i is independent of ℓ we obtain that, for each $s \in L$, $\sum_{i \in N} \sum_{k \in [K_i] \cup [L]} \eta_i(k, s)(x_i^k - x_i^s) \leq 0$. Finally, $\sum_r \theta(r) r_C < 0$ implies that

$$\sum_i \sum_{t \in K_i} \sum_{k \in [K_i] \cup [L]} \eta_i(k, t) p_i^t \cdot (x_i^t - x_i^k) > 0.$$

References

Abdulkadiroğlu, Atila and Tayfun Sönmez (2003), “Ordinal efficiency and dominated sets of assignments.” *Journal of Economic Theory*, 112, 157–172. [0912]

Afriat, Sydney N. (1967), “The construction of utility functions from expenditure data.” *International economic review*, 8, 67–77. [0915, 0926]

Allen, Roy, Paweł Dziewulski, and John Rehbeck (2019), “Revealed statistical consumer theory.” Available at SSRN 3474472. [0914, 0915]

Allen, Roy and John Rehbeck (2020a), “Counterfactual and welfare analysis with an approximate model.” arXiv:2009.03379. arXiv preprint. [0912, 0915]

Allen, Roy and John Rehbeck (2020b), “Satisficing, aggregation, and quasilinear utility.” Available at SSRN 3180302. [0912, 0915]

Aziz, Haris, Florian Brandl, and Felix Brandt (2015), "Universal Pareto dominance and welfare for plausible utility functions." *Journal of Mathematical Economics*, 60, 123–133. [0912]

Bachmann, Ruediger (2004), "Rationalizing allocation data: A nonparametric Walrasian theory when prices are absent or non-Walrasian." *Journal of Mathematical Economics*, 40, 271–295. [0914, 0922]

Bachmann, Ruediger (2006a), "Testable implications of coalitional rationality." *Economics Letters*, 93, 101–105. [0914]

Bachmann, Ruediger (2006b), "Testable implications of Pareto efficiency and individual rationality." *Economic Theory*, 29, 489–504. [0914, 0922]

Berge, Claude (2001), *The Theory of Graphs*. Dover Publications. [0929]

Bewley, Truman F. et al. (1987), "Knightian decision theory, part ii: Intertemporal problems." Cowles Foundation Discussion Papers, 835. [0914]

Bewley, Truman F. (2002), "Knightian decision theory. Part I." *Decisions in economics and finance*, 25, 79–110. [0914]

Blundell, Richard, Martin Browning, and Ian Crawford (2007), "Improving revealed preference bounds on demand responses." *International Economic Review*, 48, 1227–1244. [0912]

Blundell, Richard, Martin Browning, and Ian Crawford (2008), "Best nonparametric bounds on demand responses." *Econometrica*, 76, 1227–1262. [0912]

Blundell, Richard W., Martin Browning, Ian Crawford, Bram De Rock, Frederic Vermeulen, and Laurens Cherchye (2015), "Sharp for SARP: Nonparametric bounds on the behavioural and welfare effects of price changes." AEJ Microeconomics, 7. [0912]

Bogomolnaia, Anna and Eun Jeong Heo (2012), "Probabilistic assignment of objects: Characterizing the serial rule." *Journal of Economic Theory*, 147, 2072–2082. [0912]

Bogomolnaia, Anna and Hervé Moulin (2001), "A new solution to the random assignment problem." *Journal of Economic theory*, 100, 295–328. [0912]

Border, Kim C. (2015), "Miscellaneous notes on optimization theory and related topics." Report, Caltech. [0915]

Border, Kim C. (2020), "Alternative linear inequalities.". Available at http://www.its.caltech.edu/~kcborder/Notes/Alternative.pdf. Accessed: 2020-1-25. [0927, 0928]

Bossert, Walter and Yves Sprumont (2002), "Core rationalizability in two-agent exchange economies." *Economic theory*, 20, 777–791. [0914]

Brown, Donald J. and Caterina Calsamiglia (2007), "The nonparametric approach to applied welfare analysis." *Economic Theory*, 31, 183–188. [0914, 0932]

Brown, Donald J. and Felix Kubler (2008), "Refutable theories of value." In *Computational Aspects of General Equilibrium Theory*, 1–10, Springer. [0914]

Brown, Donald J. and Rosa L. Matzkin (1996), "Testable restrictions on the equilibrium manifold." *Econometrica*, 64, 1249–1262. [0914]

Brown, Donald J. and Chris Shannon (2000), "Uniqueness, stability, and comparative statics in rationalizable Walrasian markets." *Econometrica*, 68, 1529–1539. [0914]

Browning, Martin (1989), "A nonparametric test of the life-cycle rational expections hypothesis." International Economic Review, 979–992. [0932]

Carroll, Gabriel (2010), "An efficiency theorem for incompletely known preferences." *Journal of Economic Theory*, 145, 2463–2470. [0912]

Carvajal, Andrés (2004), "Testable restrictions on the equilibrium manifold under random preferences." *Journal of Mathematical Economics*, 40, 121–143. [0914]

Carvajal, Andrés (2010), "The testable implications of competitive equilibrium in economies with externalities." *Economic theory*, 45, 349–378. [0914]

Carvajal, Andrés, Indrajit Ray, and Susan Snyder (2004), "Equilibrium behavior in markets and games: Testable restrictions and identification." *Journal of Mathematical Economics*, 40, 1–40. [0914]

Carvajal, Andrés and Xinxi Song (2018), "Testing Pareto efficiency and competitive equilibrium in economies with public goods." *Journal of Mathematical Economics*, 75, 19–30. [0914]

Chambers, Christopher P. and Federico Echenique (2009), "Supermodularity and preferences." *Journal of Economic Theory*, 144, 1004–1014. [0917]

Chambers, Christopher P. and Federico Echenique (2016), *Revealed Preference Theory*, volume 56. Cambridge University Press. [0920, 0922, 0926]

Chambers, Christopher P., Federico Echenique, and Nicolas S. Lambert (2021), "Recovering preferences from finite data." *Econometrica*, 89, 1633–1664. https://onlinelibrary.wiley.com/doi/abs/10.3982/ECTA17845. [0914]

Cherchye, Laurens, Thomas Demuynck, and Bram De Rock (2011), "Testable implications of general equilibrium models: An integer programming approach." *Journal of Mathematical Economics*, 47, 564–575. [0914]

Debreu, Gerard (1951), "The coefficient of resource utilization." Econometrica. 273–292. [0921]

Doğan, Battal and Kemal Yıldız (2016), "Efficiency and stability of probabilistic assignments in marriage problems." *Games and Economic Behavior*, 95, 47–58. [0912]

Dubra, Juan, Fabio Maccheroni, and Efe A. Ok (2004), "Expected utility theory without the completeness axiom." *Journal of Economic Theory*, 115, 118–133. [0912]

Echenique, Federico, Sangmok Lee, and Matthew Shum (2011), "The money pump as a measure of revealed preference violations." *Journal of Political Economy*, 119, 1201–1223. [0919, 0920]

Eliaz, Kfir and Efe A. Ok (2006), "Indifference or indecisiveness? Choice-theoretic foundations of incomplete preferences." *Games and Economic Behavior*, 56, 61–86. [0912]

Fon, Vincy and Yoshihiko Otani (1979), "Classical welfare theorems with non-transitive and non-complete preferences." *Journal of Economic Theory*, 20, 409–418. [0914]

Gale, David and Andreu Mas-Colell (1975), "An equilibrium existence theorem for a general model without ordered preferences." *Journal of Mathematical Economics*, 2, 9–15. [0914]

Gale, David and Andreu Mas-Colell (1977), "On the role of complete, transitive preferences in equilibrium theory." In *Equilibrium and Disequilibrium in Economic Theory*, 7–14, Springer. [0914]

Graaff, Johannes and de Villiers (1967), *Theoretical Welfare Economics*. Cambridge University Press. [0912, 0916]

Hashimoto, Tadashi, Daisuke Hirata, Onur Kesten, Morimitsu Kurino, and Utku Ünver (2014), "Two axiomatic approaches to the probabilistic serial mechanism." *Theoretical Economics*, 9, 253–277. [0912]

Hicks, John R. (1939), "The foundations of welfare economics." *The economic journal*, 49, 696–712. [0912]

Kaldor, Nicholas (1939), "Welfare propositions of economics and interpersonal comparisons of utility." *The Economic Journal*, 49, 549–552. [0912]

Kubler, Felix (2003), "Observable restrictions of general equilibrium models with financial markets." *Journal of Economic Theory*, 110, 137–153. [0914]

Manea, Mihai (2008), "A constructive proof of the ordinal efficiency welfare theorem." *Journal of Economic Theory*, 141, 276–281. [0912]

Mas-Colell, Andreu (1977), "The recoverability of consumers' preferences from market demand behavior." *Econometrica*, 45, 1409–1430. http://www.jstor.org/stable/1912308. [0914]

Mas-Colell, Andreu (1978), "On revealed preference analysis." *The Review of Economic Studies*, 45, 121–131. [0914]

McLennan, Andrew (2002), "Ordinal efficiency and the polyhedral separating hyperplane theorem." *Journal of Economic Theory*, 105, 435–449. [0912]

Nishimura, Hiroki, Efe A. Ok, and John K.-H. Quah (2017), "A comprehensive approach to revealed preference theory." *American Economic Review*, 107, 1239–1263. [0917]

Ok, Efe A. (2002), "Utility representation of an incomplete preference relation." *Journal of Economic Theory*, 104, 429–449. [0912]

Rigotti, Luca and Chris Shannon (2005), "Uncertainty and risk in financial markets." *Econometrica*, 73, 203–243. [0914]

Rockafellar, Ralph (1966), "Characterization of the subdifferentials of convex functions." *Pacific Journal of Mathematics*, 17, 497–510. [0932]

Shafer, Wayne and Hugo Sonnenschein (1975), "Equilibrium in abstract economies without ordered preferences." *Journal of Mathematical Economics*, 2, 345–348. [0914]

Ugarte, Cristián (2022), "Preference recoverability from inconsistent choices." UC Berkeley Working Paper. [0914]

Varian, Hal R. (1982), "The nonparametric approach to demand analysis." *Econometrica*, 50, 945–973. [0912, 0913, 0915, 0917, 0924, 0925]

Weymark, John A. (1985), "Remarks on the first welfare theorem with nonordered preferences." *Journal of Economic Theory*, 36, 156–159. [0914]

Co-editor Todd D. Sarver handled this manuscript.

Manuscript received 18 August, 2023; final version accepted 7 November, 2024; available online 20 November, 2024.