Tâtonnement in matching markets[*]

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Abstract

I study tâtonnement processes in a matching market without transfers. In each period, schools set cutoffs, i.e. the preference ranks of the least preferred students they are willing to admit, and students accept their most preferred offers. Cutoffs are adjusted on the basis of demand-supply imbalances. A school's adjustment from one period to the next is moderate if it is bounded by the most recently observed imbalance at that school. I show that for any period in which all schools adjust moderately, the sum of demand-supply imbalances across all schools weakly decreases. Moreover, if all schools always adjust moderately and there is a unique stable matching, then adjustments converge to a market clearing cutoff vector. If there is more than one stable matching, moderate adjustments may cycle indefinitely but the supremum and the infimum of all cutoff vectors observed along a cycle are both market clearing.

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1 Introduction

Starting with Gale and Shapley (1962)'s seminal analysis of the marriage problem, matching theory has been successful in developing *centralized* mechanisms with appealing allocative and incentive properties. These mechanisms are not only theoretically appealing but also work well in practice. In fact, extensions of the Gale-Shapley deferred acceptance algorithm are now used to, for example, match children to public schools (see Abdulkadiroglu et al. (2006, 2009); Pathak (2011)) and medical graduates to their first professional position (see Roth and Peranson (1999)). However, many markets with very similar characteristics operate in a decentralized manner. An important example is university admissions, where centralized clearinghouses are the exception rather than the norm and full centralization often seems unlikely. Since the decentralized processing of admission offers, rejections and acceptances in these markets takes a non-negligible amount of time, there is often only room for relatively few rounds of offers. To avoid missing their enrolment targets, universities typically overbook, i.e. make more offers than they'd ideally like to be accepted. It is easy to find examples in which universities overbook too little or too much.[1] Motivated by these observations, I focus on the question of what happens in the long- and in the short-run when schools adjust their admission offers on the basis of observed discrepancies between target and realized enrolments. In particular, I am interested in finding simple conditions under which successive updating by schools from an arbitrary starting point eventually leads to market clearing or at least comes close to it.

At a theoretical level, my research speaks to the justification of stability as an equilibrium concept for many-to-one matching markets. One important justification for an equilibrium concept is that it can be learned by "myopic" agents via a distributed, iterative adjustment procedure.[2] A classic example of such a procedure for competitive equilibrium is tâtonnement

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[1]For example, US universities often admit a significant number of students from waitlists, indicating that not enough of their initial offers had been accepted to reach target enrolment. On the other hand, one can also find examples in which enrolment exceeded the target. A severe case of excess enrolment happened at the University of Bonn in 2009, when the university had 590 incoming law students - significantly exceeding its enrolment target of 350 (see here; accessed on January 23, 2025).

[2]To quote Kenneth J. Arrow (when talking about competitive equilibrium): *"How can equilibrium be established? The attainment of equilibrium requires a disequilibrium process. What does rational behavior mean in the presence of disequilibrium? Do individuals speculate on the equilibrating process?"* (Arrow (1986), p. S387)

where producers of goods in excess demand increase prices and producers of goods in excess supply decrease prices. In the context of many-to-one matching between universities and students, the natural analogy is a tâtonnement process in which universities make admission offers to students and each university adjusts its selectivity in the direction of its excess enrolment. If such a process was guaranteed to converge to a stable matching, it would provide us with a "learning foundation" for stability as an equilibrium concept. I show that convergence obtains whenever the underlying matching market has a unique stable matching but may fail otherwise.

I consider a standard many-to-one matching problem with responsive preferences and no monetary transfers. There are finite sets of students and schools. Each student has a strict preference ranking of available schools. Each school has a fixed enrolment target and responsive preferences with respect to some strict ranking of individual students. My analysis focuses on tâtonnement processes in which schools make binding admission offers, each student demands the most preferred school among those from which she received an offer, and schools then update offers on the basis of observed demand-supply imbalances. For the most part, I restrict attention to the case where schools' offers can be described by *cutoffs*, i.e. each school decides on the least preferred individual student it wants to admit and then makes an offer to all weakly preferred candidates. Cutoff adjustments from one period to the next are *moderate*, if they are bounded by the most recently observed demand-supply imbalances: Schools increase their cutoffs by no more than their most recent excess demand and decrease their cutoffs by no more than their most recent excess supply.[3]

My main results are as follows: First, moderate adjustments always bring the market weakly closer to clearing in the sense of weakly decreasing the *aggregate imbalance*, which I define as the sum of demand-supply imbalances across all schools (Theorem 1). Second, if schools adjust moderately and either (A) schools only decrease cutoffs when there is no school in excess demand, or (B) schools only increase cutoffs when there is no school in excess supply, then adjusted cutoffs are guaranteed to converge to a *market clearing cutoff vector*

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[3]Here, the "excess demand" for school s is taken to be the maximum of 0 and the difference between the demand for school s and its enrolment target. Similarly, the "excess supply" of school s is taken to be the maximum of 0 and the difference between the enrolment target of school s and its demand. One can actually allow a school in excess demand to increase its cutoff by more than the difference between demand and enrolment target - see Definition 1 and the discussion that follows it.

(Theorem 2).[4] Third, moderate adjustments may cycle indefinitely when the underlying matching market has more than one *stable matching* (Theorem 3). Finally, if moderate adjustments cycle indefinitely, then the supremum and the infimum of all cutoff vectors observed along the cycle are both market clearing (Theorem 4). As a corollary, I obtain that moderate adjustments are guaranteed to converge to market clearing if the underlying matching market has a unique stable matching (Corollary 3).

1.1 Related literature

Since I study a tâtonnement process in which schools use observed enrolments to update their admissions offers, one line of research that is closely connected to my paper is that on tâtonnement processes for markets with indivisible goods and continuous monetary transfers.[5] Demange et al. (1986) introduce a dynamic clock auction for markets with multiple heterogeneous goods and unit-demand bidders. This auction mechanism is guaranteed to converge to the smallest competitive equilibrium price vector, provided it starts from the vector of reservation prices. Gul and Stachetti (2000) extend the dynamic clock auction to the case where bidders' preferences satisfy the gross substitutes condition of Kelso and Crawford (1982). Contrary to the finding of Demange et al. (1986), Gul and Stachetti (2000) show that in general there is no dynamic clock auction mechanism with a single ascending price trajectory that implements the VCG outcome. Ausubel (2006) presents a different tâtonnement based auction process to compute competitive equilibrium prices. His process ensures that agents ultimately receive their VCG payments and converges from any initial price vector.[6] There are two crucial differences between this line of research and mine. First, cutoffs are different from prices since they determine who gets an offer from which school but otherwise do not affect students' utilities. Second, all of the above papers assume that there is *one* central auctioneer who controls price movements of all objects in the economy. In contrast, I explicitly want to allow for situations in which schools react independently to imbalances between realized and

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[4]A cutoff vector is market clearing, if there is no school in excess demand and all schools in excess supply make offers to all students.

[5]See Hahn (1982) for a survey of results on tâtonnement processes for markets with divisible goods. See Manea (2007) for an analysis of tâtonnement processes for coalitional games with transferable utility.

[6]For an extension of Ausubel's techniques to the class of preferences satisfying the gross substitutes and complements condition introduced by Sun and Yang (2006), see Sun and Yang (2009).

target enrolments.

Also related is a literature that has developed a linear programming approach to stable matchings. VandeVate (1989) and Rothblum (1992) have shown that the set of stable matchings in a one-to-one matching problem can be described by a linear programming problem whose extreme points are all integral.[7] That finding extends to many-to-one matching problems (see e.g. Baiou and Balinski (2000), Fleiner (2003), and Sethuraman et al. (2006)). Given the connection to linear programming, it is natural to try to leverage linear programming techniques to solve stable matching problems. For one-to-one matching problems, (both versions of) the Gale-Shapley algorithm can be interpreted as a dual ascent algorithm, see Abeledo and Rothblum (1995b) and Vohra (2012).[8] Dual ascent algorithms solve a linear programming problem through a series of relaxed programming problems that are connected via the updating of dual variables associated with constraints that are violated in the solutions of these relaxed problems. An important similarity between dual gradient methods and the tâtonnement processes that I study is that, for both, the individual adjustment at a specific school only depends on the extent to which feasibility and stability constraints involving that school are violated. A key difference is that dual gradient methods often rely on a carefully chosen starting point, whereas I study cutoff adjustments from arbitrary starting points. The focus on such fully distributed procedures - where there is no coordination on specific initial conditions or the adjustments of cutoffs across different schools - is motivated by the desire to understand the conditions under which completely decentralized matching markets arrive at a stable matching.

Several papers have studied different decentralized implementations of Gale and Shapley (1962)'s famous deferred acceptance algorithm (DAA). In these studies the main question typically is whether a stable matching is reached in (subgame perfect) equilibrium. For example, Niederle and Yariv (2009) consider a decentralized market game where in each period firms simultaneously make an offer to some worker and workers then simultaneously decide on received offers. Preferences of the agents are assumed to satisfy a strong alignment

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[7]Roth et al. (1993) show how to use complementary slackness conditions for the associated linear programming problem to derive several key results on stable matchings for the one-to-one matching problem.

[8]See Vries et al. (2007) for a primal-dual algorithm to find efficient allocations and supporting prices for a setting in which multiple indivisible and heterogenous objects are to be distributed among bidders who may depend multiple objects.

condition which, among other things, guarantees that there is a unique stable matching. Niederle and Yariv show that under complete information a stable matching is implemented with probability one in any equilibrium that survives iterative elimination of weakly dominated strategies. This finding does not extend to the case of incomplete information and it is possible that no (Bayes-Nash) equilibrium implements a stable matching with certainty. Other papers that have studied different decentralized implementations of the DAA under the assumption of complete information are Pais (2008) and Haeringer and Wooders (2011).

Also related to my work is the question of whether processes that, starting from an unstable matching, allow randomly selected blocking pairs to rematch converge to a stable matching. Roth and Vande Vate (1990) show that for two-sided matching markets in which no agent can be matched to more than one partner, such a random process converges to a stable matching with probability one.[9] This finding was extended to a certain class of many-to-many matching markets by Kojima and Ünver (2008), to (solvable) roommate markets by Diamantoudi et al. (2004), to (solvable) matching markets with couples by Klaus and Klijn (2007), and to matching markets with salaries by Chen et al. (2010).[10] While these papers are interesting studies of market equilibration in matching markets, they do rely on implicit assumptions about the coordination between and the information available to the agents. First, they assume that agents have sufficiently detailed information to identify a blocking opportunity. Second, it is taken for granted that agents coordinate in a way that guarantees that intended deviations are actually realized. I relate my findings to the random paths literature in more detail below.

Finally, Lauermann and Nöldeke (2014) embed a standard marriage market in a search model with random meetings and study the limit of steady-state equilibria when frictions vanish. Lauermann and Nöldeke show that equilibrium matchings are guaranteed to converge to stable matchings if and only if the underlying matching market has a unique stable matching. Although there does not seem to be a direct connection between sequences of steady-state equilibria and the adjustment processes that I study, the just described result mirrors my finding that moderate cutoff adjustments are guaranteed to converge to market

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[9]See Jackson and Watts (2002) for another type of dynamics that may lead to stable matchings.

[10]See also Pradelski (2015) and Leshno and Pradelski (2021) on the speed of convergence for random blocking dynamics in the assignment game.

clearing if and only if the core is a singleton.

2 Model

There is a finite set of students I with $|I| = N$ and a finite set of schools S with $|S| = M$. Each student i has a strict preference relation R_i over the set of available schools and strictly prefers all schools to being left unmatched. Each school $s \in S$ has a fixed enrolment target $q_s \in \{1, \ldots, N\}$ and a fixed strict ranking $R_s^{\#}$ of individual students in I.[11] In addition, each school s has a weak preference ranking[12] R_s on the set of all potential entering classes, 2^I, that is *responsive* (Roth (1985)) to $R_s^{\#}$ and q_s in the sense that for any $J \subseteq I$, $i, j \in I \setminus J$, $J \cup \{i\} P_s J \cup \{j\}$ if and only if $i P_s^{\#} j$, $J \cup \{i\} P_s J$ if $|J| < q_s$, and $J P_s J \cup \{i\}$ if $|J| \geq q_s$. For school $s \in S$ and student $i \in I$, i's *score at* s is given by $e_{is} = N + 1 - r_{is}$, where $r_{is} \in \{1, \ldots, N\}$ is the rank that i has in $R_s^{\#}$. Hence, the tth most preferred student according to $R_s^{\#}$ has a score of $N + 1 - t$ at school s. The *underlying matching market* $(I, S, (R_i)_{i \in I}, (R_s^{\#})_{s \in S}, (q_s)_{s \in S})$ is assumed fixed from here on out. To simplify notation, relevant market features, such as preferences or capacities, are typically suppressed in the following.

A *matching* is a mapping $\mu : I \cup S \to (S \cup I) \cup 2^I$ such that (i) $\mu(i) \in S \cup \{i\}$ for all $i \in I$,[13] (ii) $\mu(s) \in 2^I$ for all $s \in S$, (iii) $|\mu(s)| \leq q_s$, and (iv) $i \in \mu(s)$ if and only if $\mu(i) = s$. A matching μ is *stable*, if there is no student-school pair (i, s) such that $s P_i \mu(i)$ and either $|\mu(s)| < q_s$ or $e_{is} > e_{js}$ for some $j \in \mu(s)$. One of the central results in matching theory is that there always exist student- and school-optimal stable matchings (Gale and Shapley (1962); Roth (1984)). Denote the student-optimal stable matching by μ^I and the school-optimal stable matching by μ^S.

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[11]The formulation here assumes that all schools are acceptable to all students and all individual students are acceptable to all schools. These assumptions are without loss of generality since one can always introduce a "null school" that plays the role of being left unassigned and "null students" who represent the option of leaving places unfilled. The assumption that schools' rankings of individual students are strict is restrictive and may fail in some applications. For example, Abdulkadiroglu et al. (2006) empirically analyze the Boston school choice system where there are only four distinct classes of scores that are used to rank thousands of students. See Erdil and Ergin (2008), Abdulkadiroglu et al. (2009), Abdulkadiroglu et al. (2011), Abdulkadiroglu et al. (2015), and Ehlers and Westkamp (2018) for analyses of school choice problems with indifferences in priority orders.

[12]That is, a complete and transitive binary relation on 2^I.

[13]Here, as usual in the literature, $\mu(i) = i$ means that i is left unmatched.

Cutoffs and market clearing   A *cutoff for school* $s \in S$ is an integer $c_s \in \{1, \ldots, N+1\}$ that represents the minimum score needed to get into school s. School s is called *selective* if $c_s > 1$. Let $C = \{1, \ldots, N+1\}^M$ be the set of all cutoff vectors. I say that *i can afford s at* $c \in C$ if $e_{is} \geq c_s$. Note that a cutoff vector acts like a price vector in that it determines exactly who can afford which school, but that cutoffs differ from prices in that students only care about whether a school is affordable to them or not. Given a vector of cutoffs $c \in C$, student i's *demand* is given by the most preferred school that she can afford (or by being unmatched if no school is affordable), that is,

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D_i(c) = \begin{cases} s^* & \text{, if } e_{is^*} \geq c_{s^*} \text{ and } c_{s'} > e_{is'} \text{ for all } s' \text{ such that } s' P_i s^* \\ i & \text{, if } c_s > e_{is} \text{ for all } s \in S. \end{cases}
$$

The *demand for school* $s \in S$ is the set of all students who demand s at c, that is, $D_s(c) = \{i \in I : D_i(c) = s\}$. I refer to the cardinality of s' entering class at cutoff vector c, $|D_s(c)|$, as the *enrolment* of s' at c. Note that students' scores lie in $\{1, \ldots, N\}$ and hence $|D_s(c)| = 0$ if $c_s = N + 1$. Allowing for cutoffs that exceed every student's score enables me to incorporate situations in which no offers are made. School s is *oversubscribed at* c if its enrolment strictly exceeds its target, i.e. $|D_s(c)| > q_s$, and *undersubscribed at* c if its enrolment falls strictly short of its target, i.e. $|D_s(c)| < q_s$. A cutoff vector $c \in C$ is *market clearing* if there are no oversubscribed schools and undersubscribed schools at c are not selective, that is, $|D_s(c)| \leq q_s$ for all $s \in S$ and $c_s = 1$ for all s such that $|D_s(c)| < q_s$. The requirement that undersubscribed schools are not selective translates the familiar notion that goods in excess supply should have a price of zero to the matching context.

Azevedo and Leshno (2016) have shown that market clearing and stability are essentially equivalent concepts.[14] To describe this insight more formally, first fix a matching μ and define an induced cutoff vector c^μ by setting $c_s^\mu = \min_{i \in \mu(s)} e_{is}$ if $|\mu(s)| \geq q_s$ and $c_s^\mu = 1$, otherwise. Next, given a cutoff vector c, define a mapping μ^c by setting $\mu^c(s) = D_s(c)$ and $\mu^c(i) = D_i(c)$. We have the following.

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[14]Balinski and Sönmez (1999) were the first to connect cutoff vectors to stability, but use a slightly different notion of market clearing.

Lemma 1 (Azevedo and Leshno (2016)). *If μ is stable, then c^μ is market clearing. If c is market clearing, then μ^c is stable. Moreover, there exists a uniformly lowest market clearing cutoff vector $\underline{\mathcal{C}}^*$ and a uniformly highest market clearing cutoff vector $\overline{\mathcal{C}}^*$.*

Given some stable matching μ, say that c *induces* μ if $\mu^c = \mu$. As discussed in Azevedo and Leshno (2016), $\underline{\mathcal{C}}^*$ induces μ^I and $\overline{\mathcal{C}}^*$ induces μ^S.

Tâtonnement processes   I consider simple tâtonnement processes in which schools set cutoffs, each student "demands" her most preferred school among those for which her score exceeds the cutoff, and schools then update cutoffs on the basis of observed discrepancies to their quotas. Tâtonnement can be interpreted as taking place either over multiple admission periods (where cutoffs represent binding admission offers), within a given admission period (where cutoffs represent tentative admission offers that only convert to binding ones when enrolment targets are met), or a mix between the two (for example, a few rounds of adjustments within a given period and then an initial round of offers in the next period that reflects students' reactions to the last round of offers in the previous period). In case of adjustments across multiple periods, my analysis assumes that the underlying matching problem is always the same, i.e. that a new and identical student population arrives in each period. That assumption is approximately satisfied if the admission environment is stable over time so that enrolments only change significantly when schools change their cutoffs.

Since the underlying matching market is fixed throughout, a cutoff vector $c \in C$ induces the demand vector $(D_s(c))_{s \in S}$ and the vector of demand-supply imbalances $(||D_s(c)| - q_s|)_{s \in S}$. Hence, while tâtonnement processes will typically be defined on the basis of some intermediary statistic, such as demand-supply imbalances, I can define them directly on the basis of cutoff vectors. For now, I restrict attention to time-invariant adjustment processes where adjustments only depend on the current cutoff vector. Formally, a *time-invariant adjustment process* is a mapping A from C to itself. Until Section 4.2 adjustments will always be assumed to be time-invariant and I will simply refer to adjustment processes. I assume throughout that $A(c) = c$ only if c is market-clearing or, equivalently, at least one school adjusts its cutoff when the current cutoff vector is not market-clearing. Finally, for any $c \in C$, let $A^t(c)$ denote the t-fold application of A to c.

To fix ideas, here are two examples.[15]

Example 1. Let $s \in S$ and $c \in C$ be arbitrary. The simple adjustment of s at c is given by

$$SIMP_s(c) = \begin{cases} c_s + 1 & \text{, if } |D_s(c)| > q_s \text{ and } c_s < N + 1 \\ c_s - 1 & \text{, if } |D_s(c)| < q_s \text{ and } c_s > 1 \\ c_s & \text{, in all other cases} \end{cases}$$

In words, the simple adjustment of s at c is to increase its cutoff by one if it is oversubscribed at c and to decrease its cutoff by one if it is undersubscribed at c.

Next, I focus on a generalization of Gale and Shapley (1962)'s famous deferred acceptance (DA) algorithm by Adachi (2000) that encompasses the student- and school proposing variants as special cases. The main idea is to characterize stable matchings as the set of fixed points of an operator from a finite lattice into itself. This idea figures heavily in the literature on generalized matching models following Hatfield and Milgrom (2005) and Ostrovsky (2008).[16]

Example 2. Let $s \in S$ and $c \in C$ be arbitrary. The generalized DA adjustment for school s is defined by

$$G_s(c) = \begin{cases} \Delta_s(c) & \text{, if } |D_s(c)| \geq q_s \\ c_s - \min\{q_s - |D_s(c)|, c_s - 1\} & \text{, otherwise,} \end{cases}$$

where for any school $s \in S$ such that $|D_s(c)| \geq q_s$, $\Delta_s(c)$ is the q_sth highest score of students in $D_s(c)$. Let $G(\cdot) = (G_s(\cdot))_{s \in S}$ be the collection of DA adjustments for the various schools. It is straightforward to see that the set of fixed points of G coincides with the set of market-clearing cutoff vectors. The student-proposing DA algorithm corresponds to the generalised Gale-Shapley adjustment process starting from $\underline{c}$ and the school-proposing DA algorithm to the generalised Gale-Shapley adjustment process starting from $\bar{c}$.

While the two DA variants have been central to matching theory and its applications, nothing is known about the behaviour of DA dynamics from arbitrary starting points. In

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[15]In the working paper version (Westkamp, 2025), I present another adjustment process that is related to the well known Boston, or Immediate Acceptance, mechanism.

[16]Azevedo and Leshno (2016) were the first to formulate a generalized Gale-Shapley algorithm on the basis of cutoff adjustments.

practice, even if the admissions process is centralized and relies on a DA algorithm, a school may be able to place bounds on its selectivity that have to be respected if it manages to hit its target enrolment. Hence, we might be faced with situations in which DA algorithms cannot be guaranteed to start from the extreme points for which their behavior is well understood. My results below show under which conditions DA algorithms are guaranteed to converge.

3   Main Results

The aim of my analysis is to derive simple conditions that guarantee adjustment processes to converge - or at least come closer - to market clearing. Given that everything else is fixed exogenously, the main determinant for the evolution of cutoffs is how schools adjust their cutoffs from one period to the next. As usual for tâtonnement processes, the main conditions require that cutoffs are adjusted in the direction of excess demand: An oversubscribed school becomes more selective, i.e. increases its cutoff, and an undersubscribed school becomes less selective, i.e. decreases its cutoff. Furthermore, the conditions also require that the size of cutoff adjustments is bounded from above by the most recently observed imbalance between demand and supply. In order to state the conditions formally, for any $c \in C$ and $s \in S$, let $\Delta_s(c)$ be the q_sth highest score of students in $D_s(c)$ if $|D_s(c)| \geq q_s$ and $\Delta_s(c) = c_s$ otherwise, that is,

$$\Delta_s(c) = \begin{cases} e_{i^*s} & \text{, if } D_{i^*}(c) = s \text{ and } |\{j \in D_s(c) : e_{js} \geq e_{i^*s}\}| = q_s, \\ c_s & \text{, if } |D_s(c)| < q_s. \end{cases}$$

With these preparations, we have the following.

Definition 1. An adjustment process A

(i) *increases moderately*, if, for all $c \in C$ and $s \in S$, $A_s(c) \leq \Delta_s(c)$;

(ii) *decreases moderately*, if, for all $c \in C$ and $s \in S$, $A_s(c) \geq c_s - \max\{q_s - |D_s(c)|, 0\}$;

(iii) is *moderate*, if it increases and decreases moderately.

Going back to the examples presented in the previous subsection, the simple adjustment process and the generalized Gale-Shapley process are both moderate.

Note that if a school s is oversubscribed at c and increases its cutoff moderately, then at

$A(c)$ it will remain affordable to at least q_s of the students who demanded it at c and can therefore only become undersubscribed if at least one other school has decreased its cutoff. Similarly, if a school s is undersubscribed at c and decreases its cutoff moderately, then at $A(c)$ it will become affordable to at most $q_s - |D_s(c)|$ additional students and can therefore only become oversubscribed if at least one other school has increased its cutoff. Note that both of these statements do not rely on information about students' preferences beyond what students demand at the cutoff vector c.[17]

In the first part of my analysis, I focus on the short-run properties of adjustment processes. The anecdotal evidence gathered in the introduction supports the view that demand-supply imbalances occur in practice and that these imbalances are costly - students have to cramp into crowded lecture halls for oversubscribed schools, while undersubscribed schools suffer significant revenue losses. Here is a simple aggregate measure for these costs.

Definition 2. The *aggregate imbalance* at $c \in C$ is given by

$$E(c) = \sum_{s \in S} ||D_s(c)| - q_s|.$$

Note that if $c \in C$ minimizes the aggregate imbalance, then either c is market clearing or there is a market clearing cutoff vector c' such that $c'_s = c_s$ for all s such that $|D_s(c)| \geq q_s$. Taking aggregate imbalance as a measure of distance to market clearing assumes that there is a constant cost associated to every student above or below a school's target enrolment. While it would certainly be interesting to study other cost measures, this is out of the scope of the current paper. Now remember that moderate adjustments ensure that the imbalance of a given school, defined as $||D_s(c)| - q_s|$, will weakly decrease if no other school changes its cutoff, that is, for any moderate adjustment process A, $||D_s(A_s(c), c_{-s})| - q_s| \leq ||D_s(c)| - q_s|$ for any cutoff vector c. However, this property may easily fail if multiple schools adjust their cutoffs simultaneously. Furthermore, even if only one school were to change its cutoff, it

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[17]Given the assumption of responsive preferences, schools always prefer enrolments that are closer to their target capacities. The preceding discussion then implies that $A_s(c)$ is a weakly better response to c_{-s} than c_s, if s' adjustment from c_s to $A_s(c)$ is moderate in the sense of Definition 1. Hence, moderate adjustment processes correspond to better reply dynamics in a cutoff setting game between schools. This connection and the relationship to the seminal work of Milgrom and Roberts (1990) is developed more fully in the working paper version (Westkamp, 2025).

may well be that imbalances at other schools increase. The first main result of this paper is that moderate adjustments *always* bring the market weakly closer to clearing in the sense of weakly decreasing the aggregate imbalance.

Theorem 1. *If A is moderate, then $E(A(c)) \leq E(c)$ for all $c \in C$.*

Theorem 1 relies on the fact that lowering a cutoff by x units makes the corresponding school affordable to exactly x additional students and increasing it by x units makes the school unaffordable to exactly x students. Hence, adjusting a cutoff is different from adjusting the (non-personalised) price of an object, since the latter affects the affordability and attractiveness of the object to all agents. Before discussing the intuition behind Theorem 1, I show via a small example that the aggregate imbalance may strictly increase if only one school does not adjust moderately.

Example 3. There are three students i, i', i'' and two schools s, s' with target enrolments $q_s = q_{s'} = 1$. Students' scores are given by $e_{is} = 3, e_{i's} = 2, e_{i''s} = 1$ and $e_{i's'} = 3, e_{is'} = 2, e_{i''s'} = 1$. Students' preferences are given by $s'P_i s, sP_{i'} s', sP_{i''} s'$.

Consider the cutoff vector $c = (c_s, c_{s'}) = (2, 3)$. At c, students i and i' both demand s, while i'' cannot afford any school. Hence, the aggregate imbalance at c is 2. Now consider the cutoff vector $c' = (3, 1)$. Note that the adjustment from c to c' is moderate for s but not for s'. At c', all students demand s' and the aggregate imbalance is 3.

To get some intuition for Theorem 1, assume there are only two schools, s^+ and s^-. Fix a moderate adjustment process A and some cutoff vector c such that s^+ is over- and s^- undersubscribed at c. Since A is moderate, s^+ cannot decrease and s^- cannot increase its cutoff from c to $A(c)$. Hence, no student can demand s^- at c but s^+ at $A(c)$. Therefore, we can restrict attention to students whose demand switches from s^+ to s^- when going from c to $A(c)$. First, let V^+ be the set of those students who switch their demands to s^- at $A(c)$ and who strictly prefer s^- over s^+. The total effect of demand changes by agents in V^+ is to decrease the aggregate imbalance by $2\min\{|V^+|, |D_{s^+}(c)| - q_{s^+}\}$. To see this, I distinguish two cases.

Case 1: $|V^+| \leq |D_{s^+}(c)| - q_{s^+}$

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The demand change of each student in V^+ decreases the excess supply at s^- due to moderate decreases and the excess demand at s^+.

Case 2: $|V^+| > |D_{s^+}(c)| - q_{s^+}$

The demand changes of the first $|D_{s^+}(c)| - q_{s^+}$ students in V^+ decrease the excess supply at s^- and the excess demand at s^+. The demand changes of the remaining $|V^+| - (|D_{s^+}| - q_{s^+})$ students in V^+ do not affect the aggregate imbalance since they decrease the excess supply at s^- and increase the excess supply at s^+.

Second, let V^- be the set of those students who switch their demands to s^- at $A(c)$ and who strictly prefer s^+ to s^-. I will argue that the demand changes of students in V^- can at most increase the aggregate imbalance by $2\min\{|V^+|, |D_{s^+}(c)| - q_{s^+}\}$. I again distinguish two cases.

Case 1: $|V^+| \geq |D_{s^+}(c)| - q_{s^+}$

By moderate increases, we have that $|V^-| \leq |D_{s^+}(c)| - q_{s^+}$. Since the demand change of each individual student can increase the aggregate imbalance by at most two, we obtain that the demand changes of students in V^- increase the aggregate imbalance by at most $2(|D_{s^+}(c)| - q_{s^+})$.

Case 2: $|V^+| = k$ for some $k < |D_{s^+}(c)| - q_{s^+}$

The demand changes of at least $|D_{s^+}(c)| - q_{s^+} - k$ students in V^- cannot increase the aggregate imbalance since these demand changes decrease the excess demand at s^+. The demand changes of the remaining at most k students in V^- increase the aggregate imbalance by at most $2k$.

Hence, if each school limits its adjustments by the magnitude of its most recent demand-supply imbalance, then adjustments always at least weakly reduce the aggregate disparity between supply and demand. Since the aggregate imbalance is bounded from below, we immediately obtain the following important implication of Theorem 1.

Corollary 1. *If A is moderate, then $\lim_{t\to\infty} E(A^t(c))$ exists for all $c \in C$.*

From now on, I will focus on results about the long run behavior of adjustment processes. The first result shows that it is relatively easy to ensure convergence to market clearing from any initial cutoff vector if schools are willing to coordinate their cutoff adjustments and temporarily maintain cutoffs despite demand-supply imbalances.

Theorem 2. *Suppose A is moderate and either*

(i) *cutoffs are only increased when there is no undersubscribed selective school, i.e. for all $c \in C$, $A(c) \leq c$ if there is at least one school s such that $c_s > 1$ and $|D_s(c)| < q_s$, or*

(ii) *cutoffs are only decreased when there is no oversubscribed school, i.e. for all $c \in C$, $A(c) \geq c$ if there is at least one school s such that $|D_s(c)| > q_s$.*[18]

Then $\lim_{t \to \infty} A^t(c)$ exists for all $c \in C$.

The intuition for this result is straightforward: If A only increases cutoffs when there is no undersubscribed school with non-trivial cutoff, there must eventually come a point at which no school is undersubscribed if A satisfies moderate decreases. But from this point onwards, no school can become undersubscribed again if A satisfies moderate increases. As I show in Theorem 7 in Appendix B.2, both types of adjustment processes have a variety of other appealing features, such as independence of adjustment magnitudes (subject to being moderate) and convergence to market clearing cutoff vectors that are "close" to the initial cutoff vector.

Next, I focus on adjustment processes which randomly select a set of schools which are allowed to moderately adjust their cutoffs. Theorem 2 can be used to show that such processes are guaranteed to converge to market clearing if there is always a positive probability that only over- and a positive probability that only undersubscribed schools are selected. To formulate this corollary, some further notation and terminology is useful. Given a cutoff vector $c \in C$, let $S^+(c)$ be the set of all oversubscribed schools at c. Similarly, define $S^-(c)$ to be the set of undersubscribed schools with non-trivial cutoffs at c. With these preparations, we have the following.

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[18]If we start from a cutoff vector at which no school is oversubscribed, this class of dynamics is conceptually very similar to the vacancy chain dynamics analyzed in Blum et al. (1997). Both types of dynamics are related to the "gender consistent" rules for selecting blocking pairs of unstable matchings in marriage markets that were introduced by Abeledo and Rothblum (1995a).

Corollary 2. *Fix an arbitrary cutoff vector $c \in C$ and let $X(c) = (X^1(c), X^2(c), \ldots)$ be a random sequence of cutoff vectors such that for each $t \geq 1$, whenever $X^{t-1}(c)$ is not market-clearing, then $X^t(c)$ is obtained from $X^{t-1}(c)$[19] by*

(i) randomly selecting a set of schools S^t and

(ii) allowing schools in S^t to adjust their cutoffs moderately from $X^{t-1}(c)$ subject to the requirement that $X_s^t(c) \neq X_s^{t-1}(c)$ for at least one school in S^t.

If for any $t \geq 1$, there is a strictly positive probability that $S^t \subseteq S^+(X^{t-1}(c))$ and a strictly positive probability that $S^t \subseteq S^-(X^{t-1}(c))$, then $X(c)$ converges to a market clearing cutoff vector with probability one.

The corollary follows from Theorem 2 upon noting that this result implies that from any c, there are at least two finite paths of cutoff adjustments that lead to a market clearing cutoff vector. If the random sequence of cutoff vectors defined in the corollary were to cycle indefinitely, at least one cutoff vector must occur infinitely often and the random process would have to choose a non-convergent path each time it reaches that cutoff vector. Under the assumptions of the corollary, however, finite convergent paths always have a strictly positive probability of being chosen (since there is always a positive probability of only choosing oversubscribed schools until there are no more oversubscribed schools and always a positive probability of only choosing undersubscribed selective schools until there are no more undersubscribed selective schools) and we obtain a contradiction. The random blocking dynamics studied by Roth and Vande Vate (1990) are similar to a special case of the dynamic defined in Corollary 2 in which we require that only one school is selected at each point.[20]

Corollary 2 shows that if at each instance there is a positive probability that all schools who do adjust their cutoffs, adjust in the same direction, then convergence to market clearing is guaranteed. The just mentioned result implicitly relies on the existence of some coordinating entity that can identify which schools are over- and which are undersubscribed, and that can at least temporarily prevent some schools from adjusting their cutoffs. For the remainder of this section, I focus on adjustment processes that represent situations in which there is

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[19]Here, I set $X^0(c) \equiv c$ for all c.

[20]The correspondence is not completely exact, since moderate decreases precludes some blockings to be implemented in one step. To see this, consider a school s and a cutoff vector c such that $q_s = 1$ and $c_s \geq 3$. If $D_s(c) = \emptyset$ and the only student i who strictly prefers s over her offers at c has $e_{is} = 1$, the block between i and c cannot be implemented in one step.

no such coordinating entity and schools that do not achieve their enrollment target always adjust their cutoffs. The latter condition is the subject of the next definition.

Definition 3. An adjustment process A is *strict*, if for all $c \in C$ and all s, $A_s(c) = c_s$ only when either $|D_s(c)| = q_s$ or $|D_s(c)| < q_s$ and $c_s = 1$.

I also need the following more stringent notion of moderate adjustments.

Definition 4. An adjustment process A is *strongly moderate* if, for all $c \in C$,

$$A_s(c) \leq c_s + \max\{|D_s(c)| - q_s, 0\}.$$

Note that the simple adjustments in the sense of Example 1 are strongly moderate. On the other hand, the generalized DA adjustment process of Example 2 is moderate but not strongly so: A school that is oversubscribed at c increases its cutoff to the q_sth highest score at s among all students in $D_s(c)$ and that score may exceed $c_s + (|D_s(c)| - q_s)$.

The next result shows that strict adjustment processes can only be guaranteed to converge if the underlying matching market has a unique stable matching.

Theorem 3. *Suppose A is strict and strongly moderate. If $\mu^I \neq \mu^S$, then there exists a cutoff vector c such that $\lim_{t \to \infty} A^t(c)$ does not exist.*

The intuition for this result is that if there are two stable matchings, we can always find a cutoff vector for which some students can afford their more preferred stable match, while others cannot. Furthermore, the cutoff vector can be constructed so that schools, which are affordable to all students who match to them in the more preferred stable outcome for the students, are over- and all other schools are undersubscribed. Given the restrictions on adjustment magnitudes, a strict adjustment process cycles forever. However, as the next result shows, the lowest and the highest cutoffs observed along any cycle of an eventually moderate adjustment process are both market clearing.

Theorem 4. *Let A be a strict adjustment process, $c \in C$ be an arbitrary cutoff vector, and T be such that $A^t(c) = A^{T+(t \mod T)}(c)$ for all $t \geq T$.[21] Define cutoff vectors c^{max} and c^{min} by*

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[21]Note that such a T must always exist since the set of possible cutoff vectors is finite and since the adjustment in each period only depends on the current cutoff vector.

setting, for each school s, $c_s^{max} = \max_{t \geq T} A_s^t(c)$ and $c_s^{min} = \min_{t \geq T} A_s^t(c)$. If A is moderate, then c^{max} and c^{min} are market-clearing.

Theorem 4 implies immediately that any strict and moderate adjustment process will eventually enter the "market-clearing corridor" between $\underline{\mathcal{C}}^*$ and $\overline{\mathcal{C}}^*$, that is, for all $c \in C$ there exists a $\hat{T}$ such that $\underline{\mathcal{C}}^* \leq A^t(c) \leq \overline{\mathcal{C}}^*$ for all $t \geq \hat{T}$. Hence, if the difference between the two extreme market-clearing cutoff vectors is very small, strict and moderate adjustment procedures will become almost constant in the long-run.

Theorem 4 is also readily seen to imply that moderate adjustment processes always converge if the underlying market has a unique stable matching.[22]

Corollary 3. *If A is moderate and strict, and $\mu^I = \mu^S$, then, $\lim_{t \to \infty} A^t(c)$ exists for all $c \in C$.*

Corollary 3 is particularly useful when the market under consideration is large. For such markets, there is often a unique stable matching - see e.g. Kojima and Pathak (2009) and Azevedo and Leshno (2016) - and thus moderate, strict adjustment procedures are guaranteed to converge to market-clearing.

4 Extensions

The analysis up to this point has assumed that schools use cutoff strategies and that adjustments are time-invariant. However, in decentralized and congested matching markets, schools may sometimes prefer to make offers that are not representable by cutoffs, for example because they also care about the likelihood with which an offer is accepted.[23] On the other hand, a school may condition adjustments on the whole history of cutoffs, for example because it is initially uncertain about the fraction of its offers that will be accepted and then tries to revise its estimated "yield" on basis of all enrolments observed so far. I now briefly sketch two extensions that allow for more general offers by schools and time-dependent adjustments, respectively.

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[22]In the working paper version (Westkamp, 2025), I show that the adjustment process related to the Boston mechanism is guaranteed to converge if and only if all schools essentially have the same preferences over students, which is a stronger requirement than that there is a unique stable matching.

[23]For a classic example of such "strategic targeting" see Roth and Xing (1997)'s study of the entry level labor market for clinical psychologists in the United States.

4.1 Beyond cutoffs

An *offer for school* $s \in S$ is a subset $O_s \subseteq I$ of students. Note that the analysis so far focused on the special case where offers have a cutoff structure, i.e. where for any school s, $i \in O_s$ implies $j \in O_s$ for all j such that $e_{js} \geq e_{is}$. Now let $\mathcal{O}_s = 2^I$ be the set of all possible offers by school s and $\mathcal{O} = \times_{s \in S} \mathcal{O}_s$ be the set of all possible offer vectors. Say that school s is *selective at* $O \in \mathcal{O}$ if $O_s \subsetneq I$. Given a vector of offers $O \in \mathcal{O}$, student i demands her most preferred offer, i.e.

$$\hat{D}_i(O) = \{s \in S : i \in O_s \text{ and } i \notin O_{s'} \text{ for all } s' \text{ s.t. } s'P_is\}$$

and $\hat{D}_i(O) = \{i\}$ if $i \notin \cup_{s \in S} O_s$. The *demand for school* $s \in S$ *at* O is the set of all students who demand s at O, that is, $\hat{D}_s(O) = \{i \in I : \hat{D}_i(O) = \{s\}\}$. I refer to the cardinality of the demand for s at O, $|\hat{D}_s(O)|$, as the *enrolment* of s at O. School s is *oversubscribed at* O if its enrolment strictly exceeds its target, i.e. $|\hat{D}_s(O)| > q_s$, and *undersubscribed at* O if its enrolment falls strictly short of its target, i.e. $|\hat{D}_s(O)| < q_s$. An offer vector $O \in \mathcal{O}$ is *market clearing* if there are no oversubscribed schools and undersubscribed schools at O are not selective, that is, $|\hat{D}_s(O)| \leq q_s$ for all $s \in S$ and $O_s = I$ for all s such that $|\hat{D}_s(O)| < q_s$. While market-clearing offer vectors are desirable since they prevent the excess supply of or demand for schools, they may lack the strong normative foundation of market clearing cutoff vectors. In particular, market clearing offer vectors can support unstable and even inefficient matchings.[24]

One can now define generalized adjustment processes starting at some exogenously given vector of offer sets O. As for adjustment processes, I assume that $A(O) = O$ only if O is market-clearing. For generalized adjustment processes, Definition 6 in Appendix A introduces a natural generalization of Definition 1: A generalized adjustment process $\hat{A}$ increases moderately if undersubscribed schools expand their set of admission offers and if each oversubscribed school s continues to offer admission to at least q_s of the students who demanded it at O; a generalized adjustment process $\hat{A}$ decreases moderately at O, if

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[24]For an easy example, assume that $|I| \geq \sum_{s \in S} q_s$. In this case, any profile of offer vectors O such that $|O_s| \geq q_s$ for all s and $O_s \cap O_{s'} = \emptyset$ whenever $s \neq s'$ is market clearing.

oversubscribed schools shrink their set of admission offers and if each undersubscribed school s makes at most as many new admission offers as it had vacant seats at O. In Appendix A, I show that Theorems 1 and 2 both extend to moderate generalized adjustment processes. By contrast, Theorem 4 does not directly extend unless offer sets are "very close" to having a cutoff structure (details are available upon request).

4.2 Dynamic adjustments

If adjustments are allowed to depend on the whole history of cutoffs, we can define an *adjustment procedure for school s* as a mapping $\mathcal{A}_s : \cup_{t=1}^{\infty} C^{t-1} \to \{1, \ldots, N+1\}$, where, for any $t \geq 1$, C^{t-1} is the set of all sequences of cutoff vectors of length t. Here, for any $\mathbf{c}^{t-1} = (c^0, \ldots, c^{t-1}) \in C^{t-1}$, $\mathcal{A}_s(\mathbf{c}^{t-1})$ is the cutoff that s sets for period t if cutoffs in periods 0 to $t-1$ are given by the vectors c^0 to c^{t-1}. The adjustment of school s at $\mathbf{c}^{t-1} = (c^1, \ldots, c^{t-1})$ may depend on any information directly or indirectly (for example, via demand-supply imbalances) conveyed by $(c^1, \ldots, c^{t-1})$. As in the analysis of the preceding sections, I focus on the evolution of cutoffs induced by a profile of adjustment procedures from an arbitrary initial condition as defined in the following.

Definition 5. Let $\mathcal{A} = (\mathcal{A}_s)_{s \in S}$ be a profile of adjustment procedures and c be an arbitrary cutoff vector. The *adjustment process induced by* $\mathcal{A}$ *and* c is defined inductively by setting $A^0(\mathcal{A}, c) = c$ and $A^t(\mathcal{A}, c) = \mathcal{A}(A^0(\mathcal{A}, c), \ldots, A^{t-1}(\mathcal{A}, c))$ for each $t \geq 1$.

For the remainder of this discussion, I take a profile of adjustment procedures $\mathcal{A}$ to be fixed and focus directly on properties of the induced adjustment processes. For each $c \in C$ and t, let $A^t(c) \equiv A^t(\mathcal{A}, c)$. Instead of assuming directly that adjustment processes can only come to rest at market-clearing cutoff vectors, I make the weaker assumption that adjustment processes are *responsive to demand* in the sense that no school ignores upward or downward pressure on its cutoff forever.

Assumption 1. There do not exist a cutoff vector $c \in C$, a school $s \in S$, and a time T, such that either

(i) $A_s^t(c) = A_s^T(c)$ and $|D_s(A^t(c))| > q_s$, for all $t \geq T$, or

(ii) $A_s^t(c) = A_s^T(c)$, $|D_s(A^t(c))| < q_s$, and $c_s > 1$, for all $t \geq T$.

Thus, from any point onwards, a school that is continuously oversubscribed eventually increases its cutoff and thus makes fewer offers. Similarly, a selective school that is continuously undersubscribed eventually decreases its cutoff and thus makes more offers. It is straightforward to show that Assumption 1 ensures that adjustment processes can only come to rest at market clearing cutoff vectors.

Observation 1. If an adjustment process A satisfies Assumption 1, then for all $c \in C$ such that $\lim_{t \to \infty} A^t(c)$ exists, $\lim_{t \to \infty} A^t(c)$ is market clearing.

To illustrate, here is an example in which adjustments are not time-invariant.

Example 4. Suppose each school s has some prior belief on its *yield* γ_s^0, that is, a belief about the fraction of offers that will be accepted by students. Given its prior belief, each school sets its cutoff so that expected enrolment equals its target. Hence, a school s with prior γ_s^0 initially sets a cutoff of $c_s^0 = N + 1 - \lceil \frac{q_s}{\gamma_s^0} \rceil$.[25] If all schools follow this routine, we get an initial cutoff vector $c^0 = (c_s^0)_{s \in S}$ that induces observed enrolments $|D_s(c^0)|$ and yields $\hat{\gamma}_s(c^0) = \frac{|D_s(c^0)|}{n+1-c_s^0}$. Given observations about past enrolments, school s updates its yield to $\gamma_s^1 = (1 - \omega^1)\gamma_s^0 + \omega^1 \hat{\gamma}_s^0(c^0)$, where $\omega^1 \in (0, 1)$ is the weight s places on the observed yield in period 1, and then updates its cutoff to $Y_s(c^0) = N + 1 - \lceil \frac{q_s}{\gamma_s^1} \rceil$, again ensuring that expected equals target enrolment.

More generally, in each non-initial period $t \geq 1$ schools similarly use past enrolments in previous periods to update their yield estimates. I assume that at the beginning of period t, school s estimates its yield to be $\gamma_s^t = (1 - \omega^t)\gamma_s^{t-1} + \omega^t \hat{\gamma}_s(Y^{t-1}(c^0))$, where γ_s^{t-1} is school s' yield estimate at the beginning of period $t - 1$ and $\omega^t \in (0, 1)$ is the weight school s attaches to the realized yield in period $t - 1$, $\hat{\gamma}_s(Y^{t-1}(c^0))$. For period $t + 1$, school s then sets a cutoff of $Y_s^t(c^0) = N + 1 - \lceil \frac{q_s}{\gamma_s^t} \rceil$, so that it meets its enrolment target in expectation given its revised estimate of the yield.

Note that the adjustment process Y satisfies Assumption 1. In contrast to the generalized DA adjustment process in Example 2, this process uses information about all previous enrolments and not just the most recent one.

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[25]Note that this corresponds to being in the budget set of $\lceil \frac{q_s}{\gamma_s} \rceil$ students.

Depending on schools' priors, the average yields adjustment process may initially require large downward or upward adjustments and therefore fail to be moderate. However, if $\omega^t > 0$ for all t and $\lim_{t \to \infty} \omega^t = 0$, adjustments become moderate for large enough t. More formally, I say that an adjustment process is *eventually moderate*, if, for any c there exists some $T \geq 0$, such that $A^t(c)$ is a moderate adjustment from $A^{t-1}(c)$ for all $t \geq T$.

In terms of results, Theorem 1 extends straightforwardly: whenever we arrive at a period where adjustments to the next period are moderate, the aggregate imbalance will weakly decrease. Hence, if adjustments are moderate for large enough t, the aggregate imbalance will converge. Similarly, if adjustments satisfy the conditions in Theorem 2 or Corollary 2 for large enough t, we obtain convergence. The negative result of Theorem 3 obviously extends to the more general adjustment processes I consider in this subsection. Finally, for Theorem 4 note that for a non time-invariant adjustment process A, $A^T(c) = A^t(c)$ for some cutoff vector c does not imply that adjustments cycle from period t onwards. However, it is not difficult to see that the proof of Theorem 4 implies that for any eventually moderate A and any c, the supremum and infimum of all accumulation points of the sequence $\{A^t(c)\}_{t \geq 0}$ both define market-clearing cutoff vectors - see Theorem 8 in Appendix B.6 for a formal statement and proof.

5 Conclusion

I have studied tâtonnement processes for matching markets without transfers. Schools have fixed enrolment targets and use their observations about realized enrolments to adjust their cutoffs. It was shown that whenever schools' cutoff adjustments are bounded by the most recently observed imbalance between actual and target enrolment, the market moves weakly closer to clearing. If at each point in time only over- or only undersubscribed schools adjust their cutoffs, convergence to market clearing was seen to be guaranteed from any initial situation. However, whenever the underlying matching market has more than one stable matching and all schools instantaneously react to enrolment imbalances, the existence of initial conditions from which adjustment processes cycle indefinitely was established. For the case where schools adjust moderately, it was shown that market clearing cutoff vectors

can be constructed from cycles in the adjustment process as the supremum and the infimum of all cutoff vectors observed along a cycle are both always market clearing. In particular, convergence is guaranteed when the underlying matching market has a unique stable matching.

There are many avenues for future research on the topics presented in this paper. Most importantly, if one takes tâtonnement processes as a model of decentralized matching markets, there are a number of assumptions that one might want to relax. First, the analysis implicitly assumed that application costs are negligible so that students apply to all schools. If this assumption is not satisfied, students face a difficult portfolio allocation problem. In this case, one should expect students to also rely on their observations about past market conditions to decide where to apply. Second, the environment was assumed to be completely stationary and it is important to study the effects of substantial preference shocks on adjustment processes. Third, it would be useful to derive optimal learning rules as a theoretical benchmark. This and most of the other open questions will most likely require additional assumptions about (the evolution of) schools' and students' preferences.

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Appendices

A Beyond cutoffs

In this section, I consider the more general version of my model introduced in Section 4.1. In the process, I present and prove more general versions of several of the results in the main body of the paper. I start by generalizing the notion of moderate adjustments from the main body of the paper.

Definition 6. A generalized adjustment process $\hat{A}$

(i) *increases moderately at $O \in \mathcal{O}$ if for all $s \in S$,*

$$\hat{A}_s(O) \supseteq O_s \text{ if } q_s > |\hat{D}_s(O)|$$

and

$$|\hat{A}_s^t(O) \cap \hat{D}_s(O)| \geq \min\{q_s, |\hat{D}_s(O)|\};$$

(ii) *decreases moderately at $O \in \mathcal{O}$ if for all $s \in S$,*

$$\hat{A}_s^t(O) \subseteq O_s \text{ if } |\hat{D}_s(O)| > q_s$$

and

$$|\hat{A}_s^t(O) \setminus O_s| \leq \max\{q_s - |\hat{D}_s(O)|, 0\};$$

(iii) *is moderate at $O \in \mathcal{O}$, if it increases and decreases moderately at O.*

I now relate cutoffs to offer sets. First, note that a cutoff vector c induces the offer vector O^c, where $O_s^c = \{i \in I : e_{is} \geq c_s\}$. Furthermore, say that $O \in \mathcal{O}$ has a cutoff structure, if there exists a cutoff vector c such that $O = O^c$. Second, given an adjustment process A, we can define an associated generalized adjustment process $\tilde{A}$ as follows: Let $\hat{O} \in \mathcal{O}$ be such that $\hat{O} = O^c$ for some cutoff vector c and then set $\tilde{A}(\hat{O}) = O^{A(c)}$. Note that an adjustment process A converges if and only if the associated generalized adjustment process $\tilde{A}$ converges. Note also that, for any c, it holds that $D_s(c) = \hat{D}_s(O^c)$ for all $s \in S$. The next lemma describes the basic relationships between cutoff adjustment processes and their generalized counterparts that apply for any vector of offer sets.

Lemma 2. *If adjustment process A*

(i) *increases moderately in the sense of Definition 1, then $\tilde{A}$ increases moderately at all $\hat{O} \in \mathcal{O}$ that have a cutoff structure;*

(ii) *decreases moderately in the sense of Definition 1, then $\tilde{A}$ decreases moderately at all $\hat{O} \in \mathcal{O}$ that have a cutoff structure;*

(iii) *is moderate in the sense of Definition 1, then $\tilde{A}$ is moderate at all $\hat{O} \in \mathcal{O}$ that have a cutoff structure.*

Proof of Lemma 2. (i) Fix some $\hat{O} \in \mathcal{O}$ with cutoff structure and let c be such that $O^c = \hat{O}$. If $|\hat{D}_s(O^c)| = |D_s(c)| < q_s$, then $\Delta_s(c) = c_s$ and thus $A_s(c) \leq c_s$. The last inequality

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immediately implies $O_s^{A(c)} \supseteq O_s^c$. If $|\hat{D}_s(O^c)| = |D_s(c)| \geq q_s$, then $\Delta_s(c)$ is equal to the q_sth highest score of students in $D_s(c)$. Hence, we obtain that $|O_s^{A(c)} \cap \hat{D}_s(O^c)| \geq q_s$.

(ii) Fix some $\hat{O} \in \mathcal{O}$ with cutoff structure and let c be such that $O^c = \hat{O}$. By the second part of Definition 1, we have that $A_s(c) \geq c_s - \max\{q_s - |D_s(c)|, 0\}$. The last inequality immediately implies $|O_s^{A(c)} \setminus O_s^c| \leq \max\{q_s - |\hat{D}_s(O^c)|, 0\}$. Moreover, I also obtain that $O_s^{A(c)} \subseteq O_s^c$ if $|\hat{D}_s(O^c)| > q_s$.

(iii) Follows immediately from the first two parts.

$\square$

The next definition generalizes the notion of aggregate imbalance.

Definition 7. The *aggregate imbalance* at $O \in \mathcal{O}$ is given by

$$\hat{E}(O) = \sum_{s \in S} ||\hat{D}_s(O)| - q_s|.$$

With these preparations, I now state and prove a generalized version of Theorem 1.

Theorem 5. *Let $O \in \mathcal{O}$ be arbitrary. If the generalized adjustment process $\hat{A}$ is moderate at O, then $\hat{E}(\hat{A}(O)) \leq \hat{E}(O)$.*

Proof of Theorem 5. Suppose the generalized adjustment process $\hat{A}$ is moderate at $O \in \mathcal{O}$ and let $O' = \hat{A}(O)$. Define $S^+(O)$ to be the set of over- and $S^-(O)$ to be the set of undersubscribed schools at O. I now consider the aggregate effects of changes in individual demands caused by the move from O to O'.

First, let $I^- = \cup_{s \in S \setminus S^+(O)} \hat{D}_s(O)$ denote the set of all students who demand a school in $S \setminus S^+(O)$ at O. By the weak axiom of revealed preference we must have $\hat{D}_i(O') \in S \setminus S^+(O)$ for all $i \in I^-$ since $O' = \hat{A}(O)$ and, given that $\hat{A}$ is moderate at t, all schools in $S^+(O)$ make less (in a subset sense) offers at O' than at O. Furthermore, if $s' := \hat{D}_i(O') \neq \hat{D}_i(O)$ for some $i \in I^-$, then $s' \in S^-(O)$ and $i \in O'_{s'} \setminus O_{s'}$. Now let $s \in S \setminus S^+(O)$ be arbitrary. Since $\hat{A}$ decreases moderately at t and O, there are at most $q_s - |\hat{D}_s(O)|$ students who get an offer from s at $O' = \hat{A}^t(O)$ but not at O. Hence, if we momentarily disregard potential demand changes of agents in $I \setminus I^-$, then no school in $S \setminus S^+(O)$ can be oversubscribed at O'. This implies that the aggregate imbalance is exactly the same after we have accounted for the demand changes of agents I^-: The demand change of an agent $i \in I^-$ from $s = \hat{D}_i(O)$ to $s' = \hat{D}_i(O')$ increases the excess supply of seats at s by one and decreases the excess

supply of seats at s' by one. Let $D_s^1 = \hat{D}_s(O') \cap I^-$ for all schools $s \in S \setminus S^+(O)$, $D_s^1 = \hat{D}_s(O)$ for all $s \in S^+(O)$, and $E^1 (= \hat{E}(O))$ be the associated aggregate imbalance.

Next, I account for the demand changes of agents in $I^+ := I \setminus I^- = \cup_{s \in S^+(O)} \hat{D}_s(O)$. Call an agent $i \in I^+$ a *voluntary leaver* of $s \in S^+(O)$, if $\hat{D}_i(O) = s$ and $\hat{D}_i(O') P_i s$. Let V_s denote the set of all voluntary leavers of s and $V = \cup_{s \in S^+(O)} V_s$. Note that for all $i \in V$, $\hat{D}_i(O') \in S^-(O)$. I will first consider the aggregate effects caused by demand changes of agents in V. For this analysis, I neglect demand changes of agents in $I \setminus (I^- \cup V)$, as these will be considered in the last step of the proof. Set $D_s^2 = \hat{D}_s(O) \setminus V_s$ for all $s \in S^+(O)$, $D_s^2 = D_s^1 \cup \{i \in V : \hat{D}_i(O') = s\}$ for all $s \in S \setminus S^+(O)$,[26] and E^2 be the associated aggregate imbalance.

Consider first a school $s \in S^+(O)$ such that $|D_s^2| \geq q_s$. In this case, the aggregate imbalance at O' is $2|V_s|$ units lower than the aggregate imbalance at O after we have accounted for the demand changes of agents in V_s. To see this, note first that $|D_s^2| - q_s = |\hat{D}_s(O)| - |V_s| - q_s \geq 0$, so that after accounting for demand changes of agents in V_s the excess demand for places at s is $|V_s|$ units lower at O' than at O. Furthermore, for all $s' \in S^-(O)$, $|D_{s'}^2| \leq q_{s'}$ since from O to O', s' has become affordable to at most $q_{s'} - |\hat{D}_{s'}(O)|$ additional agents given that $\hat{A}$ decreases moderately at t and $\tilde{O}$. This implies in particular that $\sum_{s' \in S^-(O)} (|q_{s'} - (|D_{s'}^1| + |\{i \in V_s : \hat{D}_i(O') = s'\}|)|) = \sum_{s' \in S^-(O)} (q_{s'} - |D_{s'}^1|) - |V_s|$, so that after accounting for demand changes of agents in V_s aggregate excess supply of schools in $S^-(O)$ is $|V_s|$ units lower at O' than at O.

Next, consider a school $s \in S^+(O)$ such that $|D_s^2| < q_s$. In this case, the aggregate imbalance at O' is $2(|\hat{D}_s(O)| - q_s)$ units lower than at O after we have accounted for demand changes of agents in V_s. To see this, note first that $||D_s^2| - q_s| = q_s - |\hat{D}_s(O)| + |V_s|$, so that from O to O' the absolute value of excess enrolment at school s changes by $q_s - |\hat{D}_s(O)| + |V_s| - (|\hat{D}_s(O)| - q_s) = |V_s| - 2(|\hat{D}_s(O)| - q_s)$. On the other hand, as in the previous case, after accounting for demand changes of agents in V_s aggregate excess supply of schools in $S^-(O)$ is $|V_s|$ units lower at O' than at O. Summing up these changes we see that after we have accounted for demand changes of agents in V_s the aggregate imbalance is $2(|\hat{D}_s(O)| - q_s)$ units lower at O' than at O.

Summing up, after we have accounted for demand changes of agents in V_s for all $s \in S$, the aggregate imbalance is $E^2 = \hat{E}(O) - 2 \sum_{s \in S^+(O)} \min\{|V_s|, |\hat{D}_s(O)| - q_s\}$.

To complete the proof, I now consider the effect of demand changes of students $i \in I \setminus (I^- \cup V)$. For the following, fix a school $s \in S^+(O)$ and let $F_s = \{i \in \hat{D}_s(O) : s P_i \hat{D}_i(O')\}$ be the set of students who are forced to leave s going from O to O'. Note that since $\hat{A}$ increases moderately

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[26]Note that $D_s^2 = D_s^1$ for all $s \notin S^+(O) \cup S^-(O)$ by the weak axiom of revealed preference.

at t and $\tilde{O}$, $|F_s| \leq |\hat{D}_s(O)| - q_s$. I now consider the change in aggregate imbalance caused by the demand changes of agents in F_s starting at E^2. I distinguish three cases.

Suppose first that $|\hat{D}_s(O)| - |V_s| - q_s \geq 0 > |\hat{D}_s(O)| - |F_s| - q_s - |V_s|$. In this case, the change in the imbalance at s due to demand changes of agents in F_s starting at E^2 is $|F_s| + |V_s| - (|\hat{D}_s(O)| - q_s) - (|\hat{D}_s(O)| - |V_s| - q_s) = |F_s| + 2|V_s| - 2(|\hat{D}_s(O)| - q_s)$. Furthermore, in the worst case each student in F_s increases the excess enrolment at some school $s' \in S \setminus \{s\}$ by one unit. Hence, the total increase in the aggregate imbalance caused by demand changes in F_s starting at E^2 is at most $\max\{2|F_s| - 2(|\hat{D}_s(O)| - q_s) + 2|V_s|, 0\} \leq 2|V_s|$ given moderate decreases. Since $|V_s| \leq |\hat{D}_s(O)| - q_s$ in the case I consider here, $E^2 = \hat{E}(O) - 2|V_s| - 2\sum_{\tilde{s} \in S^+(O) \setminus \{s\}} \min\{|V_{\tilde{s}}|, |\hat{D}_{\tilde{s}}(O)| - q_{\tilde{s}}\}$ and the aggregate imbalance demand after accounting for demand changes of agents in F_s is at most $\hat{E}(O) - 2\sum_{\tilde{s} \in S^+(O) \setminus \{s\}} \min\{|V_{\tilde{s}}|, |\hat{D}_{\tilde{s}}(O)| - q_{\tilde{s}}\}$.

Next, suppose that $0 > |\hat{D}_s(O)| - |V_s| - q_s$. In this case, the increase in the excess supply of seats at s due to demand changes of agents in F_s starting at E^2 is $|F_s|$. Furthermore, in the worst case each agent in F_s increases the excess enrolment at some school $s' \in S \setminus \{s\}$ by one. Since $|V_s| > |\hat{D}_s(O)| - q_s$ in the case I consider here, $E^2 = \hat{E}(O) - 2(|\hat{D}_s(O)| - q_s) - 2\sum_{\tilde{s} \in S^+(O) \setminus \{s\}} \min\{|V_{\tilde{s}}|, |\hat{D}_{\tilde{s}}(O)| - q_{\tilde{s}}\}$. After accounting for demand changes of agents in F_s, the aggregate imbalance is at most

$$\hat{E}(O) + 2|F_s| - 2(|\hat{D}_s(O)| - q_s) - 2\sum_{\tilde{s} \in S^+(O) \setminus \{s\}} \min\{|V_{\tilde{s}}|, |\hat{D}_{\tilde{s}}(O)| - q_{\tilde{s}}\}$$

$$\leq \hat{E}(O) - 2\sum_{\tilde{s} \in S^+(O) \setminus \{s\}} \min\{|V_{\tilde{s}}|, |\hat{D}_{\tilde{s}}(O)| - q_{\tilde{s}}\},$$

where the inequality follows since $|F_s| \leq |\hat{D}_s(O)| - q_s$ given that $\hat{A}$ is moderate at t and $\tilde{O}$.

Finally, when $|\hat{D}_s(O)| - q_s - |F_s| - |V_s| \geq 0$ demand changes of agents in F_s cannot increase the aggregate imbalance relative to E^2.

An iterative application of this argument for all schools in $S^+(O)$ shows that $E(O') \leq E(O)$ and thus completes the proof.

$\square$

In the following, I refer to a moderate generalized adjustment process $\hat{A}$ that satisfies $\hat{A}_s(O) \subseteq O_s$ for all $s \in S$ as long as there is at least one oversubscribed school as an *upward-downward generalized adjustment process*. Similarly, I refer to a generalized adjustment process $\hat{A}$ that satisfies $\hat{A}_s(O) \supseteq O_s$ for all $s \in S$ whenever there is at least one undersubscribed school as a *downward-upward generalized adjustment process*. The next result summarizes a key property of the two processes just described.

Theorem 6. *If $\hat{A}$ is either a moderate generalized upward-downward adjustment process or a moderate generalized downward-upward adjustment process, then $\lim_{t\to\infty}\hat{A}^t(O)$ exists for all $O \in \mathcal{O}$.*

Proof of Theorem 6. The proof relies on the following lemma.

Lemma 3. (i) *If $O \in \mathcal{O}$ is such that for all $s \in S$, either $|\hat{D}_s(O)| \geq q_s$ or $O_s = I$, then $\lim_{t\to\infty}\hat{A}^t(O)$ exists for any moderate generalized adjustment process $\hat{A}$.*

 (ii) *If $O \in \mathcal{O}$ is such that for all $s \in S$, $|\hat{D}_s(O)| \leq q_s$, then $\lim_{t\to\infty}\hat{A}^t(O)$ exists for any moderate generalized adjustment process $\hat{A}$.*

Proof of Lemma 3.   (i) Let $\hat{A}$ be an arbitrary moderate generalized adjustment process.

I show first that if no selective school is undersubscribed at O, then all students are weakly worse off under $O' = \hat{A}(O)$ than under O and no selective school is undersubscribed at O'. The first property follows immediately since all selective schools are undersubscribed at O and $\hat{A}$ is moderate. For the second property, note first that the weak axiom of revealed preference implies that, for all $s \in S$, if $i \in \hat{D}_s(O) \cap O'_s$, then $i \in \hat{D}_s(O')$. Since $\hat{A}$ is moderate, $|\hat{D}_s(O)| \geq q_s$ therefore implies that $|\hat{D}_s(O')| \geq q_s$. By the contrapositive of the last observation, we have that $q_s > |D_s(O')|$ implies $q_s > |D_s(O)|$ and thus, by our assumptions about O, $O_s = I$. Since $\hat{A}$ is moderate, $O'_s = I$ as well.

Iterating the above reasoning, for all t such that $\hat{A}^t(O)$ is not market clearing, we must have $\hat{A}^{t+1}_s(O) \subsetneq \hat{A}^t_s(O)$ for at least one s. Hence, there must exist a $t' \geq t+1$ such that $D_i(\hat{A}^t(O))P_iD_i(\hat{A}^{t'}(O))$ for some i. Since student welfare is bounded from below, $\lim_{t\to\infty}\hat{A}^t(O)$ exists.

 (ii) Let $\hat{A}$ be an arbitrary moderate generalized adjustment process.

I show first that if no school is oversubscribed at O, then all students are weakly better off under $O' = \hat{A}(O)$ and no school is oversubscribed at O'. The first property follows immediately since no school is oversubscribed at O and $\hat{A}$ is moderate. For the second property, note first that the weak axiom of revealed preference implies that, for all $s \in S$, if $i \in O_s$ and $\hat{D}_i(O) \neq s$, then $\hat{D}_i(O') \neq s$. Hence, $\hat{D}_s(O') \setminus \hat{D}_s(O) \subseteq O'_s \setminus O_s$ for all $s \in S$. Since $\hat{A}$ is moderate, $|O'_s \setminus O_s| \leq q_s - |\hat{D}_s(O)|$ and, combined with the previous finding, I obtain $|\hat{D}_s(O')| \leq |\hat{D}_s(O)| + q_s - |\hat{D}_s(O)| = q_s$. Hence, no school is oversubscribed at O' if no school is oversubscribed at O.

Iterating the above reasoning, for all t such that $\hat{A}^t(O)$ is not market clearing, we must have $\hat{A}^{t+1}_s(O) \supsetneq \hat{A}^t_s(O)$ for at least one s. Hence, there must exist a $t' \geq t+1$ such

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that $D_i(\hat{A}^{t'}(O))P_i D_i(\hat{A}^t(O))$ for some i. Since student welfare is bounded from above, $\lim_{t\to\infty} \hat{A}^t(O)$ exists.

<div style="text-align: right">□</div>

Let $\hat{A}$ be any moderate upward-downward generalized adjustment process and $O \in \mathcal{O}$ be arbitrary. Note that there must be a T_1 such that $|\hat{D}_s(\hat{A}^{T_1}(O))| \le q_s$ for all $s \in S$: For any t such that there is a school s for which $|\hat{D}_s(A^t(O))| > q_s$, then the assumption that $\hat{A}$ is upward-downward jointly implies $\hat{A}^{t+1}(O) \subsetneq \hat{A}^t(O)$; since the set of offers for each school is bounded from below by $\emptyset$, I obtain a contradiction. Let $O' = \hat{A}^{T_1}(O)$ and note that (i) of Lemma 3 implies that $\lim_{t\to\infty} \hat{A}^t(O')$ exists. This completes the proof of convergence for the case of moderate upward-downward generalized adjustment processes. To establish the convergence of moderate downward-upward generalized adjustment processes one first establishes that such a generalized adjustment process must eventually reach an offer vector at which no selective school is undersubscribed and then applies (ii) of Lemma 3. I omit the details.

<div style="text-align: right">□</div>

B   Proofs

The following additional lemma describes some important structural properties of cutoffs that induce a given stable matching.

Lemma 4. *Let μ be some stable matching.*

1. *There exist cutoff vectors $\bar{c}(\mu)$ and $\underline{c}(\mu)$ such that $\bar{c}(\mu)$ and $\underline{c}(\mu)$ both induce μ and, for any cutoff vector c that induces μ, $\bar{c}(\mu) \ge c \ge \underline{c}(\mu)$.*

2. *For any c such that $\bar{c}(\mu) \ge c \ge \underline{c}(\mu)$, $D_s(c) = \mu(s)$ for all $s \in S$.*

Proof.   1. For each school s, let $\bar{c}_s(\mu) = \min_{i \in \mu(s)} e_{is}$ if $\mu(s) \ne \emptyset$ and $\bar{c}_s(\mu) = N + 1$, otherwise. Since μ is stable, it is straightforward that $\bar{c}(\mu)$ induces μ. Now consider some cutoff vector c such that $c_{s^*} > \bar{c}_{s^*}(\mu)$ for some s^*. I claim that c does not induce μ. Given the construction of $\bar{c}(\mu)$ and the assumption that $c_{s^*} > \bar{c}_{s^*}(\mu)$, we must have $\mu(s^*) \ne \emptyset$. But then, $c_{s^*} > \min_{i \in \mu(s^*)} e_{is^*}$ and c cannot induce μ since there is at least one student in $\mu(s^*)$ who cannot afford s^* at c.

<div style="text-align: center">33</div>

Next, I construct $\underline{c}(\mu)$. For any school s, let $D_s^*(\mu) = \{i \in I : sP_i\mu(i)\}$ be the set of students who desire s at μ. For any school s such that $D_s^*(\mu) = \emptyset$, let $\underline{c}_s(\mu) = 1$. For any school s such that $D_s^*(\mu) \neq \emptyset$, let $\underline{c}_s(\mu) = \max\{e_{is} : i \in D_s^*(\mu)\} + 1$. Stability of μ again immediately implies that $\underline{c}(\mu)$ induces μ. Now consider some cutoff vector c such that $c_s < \underline{c}_{s^*}(\mu)$ for some s^*. I claim that c does not induce μ. Given the construction of $\underline{c}(\mu)$, it is immediate that $D_s^*(\mu) \neq \emptyset$. Let i^* be such that $e_{i^*s^*} = \underline{c}_{s^*}(\mu)$. By the definitions of $\underline{c}_{s^*}(\mu)$ and $D_{s^*}^*(\mu)$, we must have $s^*P_{i^*}\mu(i^*)$. Since i^* can afford s^* at c, we have $D_{i^*}(c) \neq \mu(i^*)$ and thus c does not induce μ.

2. Let c be an arbitrary cutoff vector such that $\bar{c}(\mu) \geq c \geq \underline{c}(\mu)$. Note first that since $c_s \leq \min_{i \in \mu(s)} e_{is}$ for all s such that $\mu(s) \neq \emptyset$, all students can afford the schools that they are matched to under μ. Next, let i be an arbitrary student for whom there exists a school s such that $sP_i\mu(i)$. Since $D_s^*(\mu) \neq \emptyset$, we have $\underline{c}_s(\mu) = \min\{e_{is} : i \in D_s^*(\mu)\} + 1$ and thus $c_s > e_{is}$. Hence, i cannot afford s at c and $D_i(c) = \mu(i)$.

$\square$

B.1   Proof of Theorem 1

The proof follows immediately from Theorem 5 and Lemma 2.

B.2   Proof of Theorem 2

Similar to Appendix A, I refer to an adjustment process A that satisfies $A(c) \geq c$ if there is at least one oversubscribed school at c as an *upward-downward adjustment process*. Furthermore, I refer to an adjustment process A that satisfies $A(c) \leq c$ if there is at least one undersubscribed selective school at c as a *downward-upward adjustment process*. I start by introducing a stronger version of Theorem 2.

Theorem 7.   *(i) If A is a moderate upward-downward adjustment process, then for all c, $\lim_{t\to\infty} A^t(c)$ exists. Moreover, for any c and any moderate upward-downward adjustment process B, $D_s(\lim_{t\to\infty} B^t(c)) = D_s(\lim_{t\to\infty} A^t(c))$ for all $s \in S$.*

(ii) If A is a moderate downward-upward adjustment process, then for all c, $\lim_{t\to\infty} A^t(c)$ exists. Moreover, for any c and any moderate downward-upward adjustment process B, $D_s(\lim_{t\to\infty} B^t(c)) = D_s(\lim_{t\to\infty} A^t(c))$ for all $s \in S$.

(iii) If A is a moderate upward-downward adjustment process and $c \leq \overline{\mathcal{C}}^*$, then $\lim_{t\to\infty} A^t(c) \geq c$ and there is no market clearing cutoff vector c' such that $c \lneq c' \lneq \lim_{t\to\infty} A^t(c)$ and $D_s(c') \neq D_s(\lim_{t\to\infty} A^t(c))$ for some $s \in S$.

(iv) If A is a strongly moderate downward-upward adjustment process[27] and $c \geq \underline{\mathcal{C}}^*$, then $\lim_{t\to\infty} A^t(c) \leq c$ and there is no market clearing cutoff vector c' such that $c \gneq c' \gneq \lim_{t\to\infty} A^t(c)$ and $D_s(c') \neq D_s(\lim_{t\to\infty} A^t(c))$ for some $s \in S$.

Proof of Theorem 7. The existence of the limits in Parts (i) and (ii) follows immediately from Theorem 6 and Lemma 2. For the remaining parts of Theorem 7, I now introduce and prove two additional lemmata.

Lemma 5. *(i) If c and c' are two cutoff vectors such that $c' \geq c$, then $\Delta(c') \geq \Delta(c)$.*

(ii) Given a cutoff vector c and a school s, define

$$
\mathcal{Y}_s(c) = \begin{cases} c_s - \min\{q_s - |D_s(c)|, c_s - 1\} & , \ \textit{if } q_s \geq |D_s(c)| \\ c_s & , \ \textit{otherwise} \end{cases}
$$

and let $\mathcal{Y}(c) = (\mathcal{Y}_s(c))_{s\in S}$. If c and c' are two cutoff vectors such that $c' \leq c$, then $\mathcal{Y}(c') \leq \mathcal{Y}(c)$.

Proof of Lemma 5. (i) I only need to establish that $\Delta_s(c') \geq \Delta_s(c)$ for any s that is oversubscribed at c since the statement is trivially true otherwise. If $|\{i \in I : e_{is} \geq c'_s \text{ and } i \in D_s(c)\}| < q_s$, the definition of Δ implies that $\Delta_s(c) \leq c'_s \leq \Delta_s(c')$. Otherwise, note that since $c' \geq c$, we must have $\{i \in I : e_{is} \geq c'_s \text{ and } i \in D_s(c)\} \subseteq D_s(c')$. In particular, the q_sth highest score of agents in $\{i \in I : e_{is} \geq c'_s \text{ and } i \in D_s(c)\}$ must be weakly lower than the q_sth highest score of agents in $D_s(c')$. This establishes the statement.

(ii) Consider some school s that is undersubscribed at c and let $\delta = \min\{q_s - |D_s(c)|, c_s - 1\}$. If $c_s - \delta \geq c'_s$, the statement follows since $c' \geq \mathcal{Y}(c')$. So suppose that $c_s \geq c'_s > c_s - \delta$ and let $l > 0$ be such that $c'_s = c_s - (\delta - l)$. Note that for any i such that $e_{is} \geq c_s$ and $i \notin D_s(c)$, we must have $i \notin D_s(c')$ since $c' \leq c$. Hence, it has to be the case that $D_s(c') \subseteq D_s(c) \cup \{i \in I : c_s > e_{is} \geq c'_s\}$. This in turn implies $|D_s(c')| \leq |D_s(c)| + |\{i \in I : c_s > e_{is} \geq c_s - (\delta - l)\}|$. Since $|D_s(c)| \leq q_s - \delta$ and $|\{i \in I : c_s > e_{is} \geq c_s - (\delta - l)\}| = \delta - l$, we obtain $|D_s(c')| \leq q_s - l$. The definition of $\mathcal{Y}$ implies that $\mathcal{Y}_s(c') \leq c'_s - l = c_s - (\delta - l) - l = c_s - \delta = \mathcal{Y}_s(c)$ and this completes the proof. $\qquad\square$

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[27]Recall that the difference between a moderate and a strongly moderate adjustment process is that for the latter $A_s(c) \leq c_s + \max\{|D_s(c)| - q_s, 0\}$ and, in particular, $A_s(c) \leq c_s$ if $|D_s(c)| = q_s$.

Lemma 6. *Let c and c' be such that $D(c) = D(c')$ and one of the following two conditions is satisfied:*

1. For all $s \in S$, either $|D_s(c)| \geq q_s$ or $c_s = c'_s = 1$.

2. For all $s \in S$, $|D_s(c)| \leq q_s$.

For any pair of moderate adjustment processes A and B we have that $D_s(A^{T_1}(c)) = D_s(B^{T_2}(c'))$ for all $s \in S$, where T_1 is the smallest integer such that $A^{T_1}(c)$ is market-clearing and T_2 is the smallest integer such that $B^{T_2}(c')$ is market-clearing.

Proof.   1. Assume that c and c' are such that $D(c) = D(c')$ and, for all $s \in S$, either $|D_s(c)| \geq q_s$ or $c_s = c'_s = 1$.

I show first that for all $T \leq T_2$ there exists a $T' \leq T_1$ such that $D_i(B^T(c'))R_i D_i(A^t(c))$ for all $i \in I$ and all $t \in \{T', \ldots, T_1\}$.

Note first that the statement is trivially true for $T = 0$ since (A) $B^0(c') = c'$, (B) $D_s(c) = D_s(c')$ for all $s \in S$, and (C) $A^t(c) \geq c$ for all $t \leq T_1$ given that A is moderate and given that, for all $s \in S$, either $|D_s(c)| \geq q_s$ or $c_s = c'_s = 1$.

So suppose the statement is true up to some $T < T_2$ and let $T' \leq T_1$ be such that $D_i(B^T(c'))R_i D_i(A^t(c))$ for all $i \in I$ and all $t \in \{T', \ldots, T_1\}$. I need to show that there is some $T'' \leq T_1$ such that $D_i(B^{T+1}(c'))R_i D_i(A^t(c))$ for all $i \in I$ and all $t \in \{T'', \ldots, T_1\}$. Suppose not. Then there has to exist some i^* such that $D_{i^*}(A^t(c))P_{i^*} D_{i^*}(B^{T+1}(c'))$ for all $t \leq T_1$. Let $s^* = D_{i^*}(A^{T_1}(c))$. Note that we must have

$$\{i \in D_{s^*}(A^{T'}(c)) : e_{is^*} \geq A^{T_1}_{s^*}(c)\} \subseteq D_{s^*}(A^{T_1}(c))$$

since $A^{T_1}(c) \geq A^{T'}(c)$. Furthermore, given the inductive assumption that $D_i(B^T(c'))R_i D_i(A^t(c))$ for all $i \in I$ and all $t \in \{T', \ldots, T_1\}$, we also have

$$\{i \in D_{s^*}(B^T(c')) : e_{is^*} \geq A^{T_1}_{s^*}(c)\} \subseteq \{i \in D_{s^*}(A^{T'}(c)) : e_{is^*} \geq A^{T_1}_{s^*}(c)\}.$$

Since $s^* = D_{i^*}(A^{T_1}(c))P_{i^*} D_{i^*}(B^{T+1}(c'))$, we have that $B^{T+1}_{s^*}(c') > A^{T_1}_{s^*}(c)$ and hence

$$\{i \in D_{s^*}(B^T(c')) : e_{is^*} \geq B^{T+1}_{s^*}(c')\} \subseteq \{i \in D_{s^*}(B^T(c')) : e_{is^*} \geq A^{T_1}_{s^*}(c)\}.$$

Combining the last insights, I find that

$$\{i \in D_{s^*}(B^T(c')) : e_{is^*} \geq B^{T+1}_{s^*}(c')\} \subseteq D_{s^*}(A^{T_1}(c)).$$

Next, we must have $B_{s^*}^{T+1}(c') > 1$ given that $B_{s^*}^{T+1}(c') > A_{s^*}^{T_1}(c)$. Since B is moderate and, by the conditions on demands at c', either $|D_{s^*}(c')| \geq q_{s^*}$ or $c'_{s^*} = 1$, we must have $|D_{s^*}(B^T(c'))| \geq q_{s^*}$ and $|\{i \in D_{s^*}(B^T(c')) : e_{is^*} \geq B_{s^*}^{T+1}(c')\}| \geq q_{s^*}$. Since $\{i \in D_{s^*}(B^T(c')) : e_{is^*} \geq B_{s^*}^{T+1}(c')\} \subseteq D_{s^*}(A^{T_1}(c))$ and $i^* \in D_{s^*}(A^{T_1}(c)) \setminus D_{s^*}(B^{T+1}(c'))$, we obtain $|D_{s^*}(A^{T_1}(c))| > q_{s^*}$, which is a contradiction to the definition of T_1.

By the statement that I have just established, I obtain that $D_i(B^{T_2}(c'))R_iD_i(A^{T_1}(c))$ for all $i \in I$. A symmetric argument yields the converse. Since preferences are strict, I obtain $D_s(A^{T_1}(c)) = D_s(B^{T_2}(c'))$ for all $s \in S$.

2. Assume that c and c' are such that $D(c) = D(c')$ and, for all $s \in S$, $|D_s(c)| \leq q_s$.

I show first that for all $T \leq T_2$ there exists a $T' \leq T_1$ such that $D_i(A^t(c))R_iD_i(B^T(c'))$ for all $i \in I$ and all $t \in \{T', \ldots, T_1\}$.

Note first that the statement is trivially true for $T = 0$ since (A) $B^0(c') = c'$, (B) $D_s(c) = D_s(c')$ for all $s \in S$, and (C) $A^t(c) \leq c$ for all $t \leq T_1$ given that A is moderate and given that, for all $s \in S$, $|D_s(c)| \leq q_s$.

So suppose the statement is true up to some $T < T_2$ and let $T' \leq T_1$ be such that $D_i(A^t(c))R_iD_i(B^T(c'))$ for all $i \in I$ and all $t \in \{T', \ldots, T_1\}$. I need to show that there is some $T'' \leq T_1$ such that $D_i(A^t(c))R_iD_i(B^{T+1}(c'))$ for all $i \in I$ and all $t \in \{T'', \ldots, T_1\}$. Suppose not. Then there has to exist some i^* such that $D_{i^*}(B^{T+1}(c'))P_{i^*}D_{i^*}(A^t(c))$ for all $t \leq T_1$. In particular, $D_{i^*}(B^{T+1}(c'))P_{i^*}D_{i^*}(A^{T_1}(c))$.

Let $s^* = D_{i^*}(B^{T+1}(c'))$ and note that we must have $\min\{A_{s^*}^{T_1}(c), B_{s^*}^T(c')\} > e_{i^*s^*} \geq B_{s^*}^{T+1}(c')$. Next, observe that

$$D_{s^*}(A^{T_1}(c)) \subseteq D_{s^*}(A^{T'}(c)) \cup \{i \in I : A_{s^*}^{T'}(c) > e_{is^*} \geq A_{s^*}^{T_1}(c)\}$$

since $A^{T_1}(c) \leq A^{T'}(c)$. Furthermore, given the inductive assumption that $D_i(A^t(c))R_iD_i(B^T(c'))$ for all $i \in I$ and all $t \in \{T', \ldots, T_1\}$, if $B_{s^*}^T(c') > A_{s^*}^{T'}(c)$, then

$$D_{s^*}(A^{T'}(c)) \subseteq D_{s^*}(B^T(c')) \cup \{i \in I : B_{s^*}^T(c') > e_{is^*} \geq A_{s^*}^{T'}(c)\},$$

and if $A_{s^*}^{T'}(c) \geq B_{s^*}^T(c')$, then $D_{s^*}(A^{T'}(c)) \subseteq D_{s^*}(B^T(c'))$.

Now consider first the case where $B_{s^*}^T(c') > A_{s^*}^{T'}(c)$. Combining the above insights, we have

$$D_{s^*}(A^{T_1}(c)) \subseteq D_{s^*}(B^T(c')) \cup \{i \in I : B_{s^*}^T(c') > e_{is^*} \geq A_{s^*}^{T_1}(c)\}.$$

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Since B is moderate, I obtain

$$|\{i \in I : B^T_{s*}(c') > e_{is*} \geq B^{T+1}_{s*}(c)\}| \leq q_{s*} - |D_{s*}(B^T(c'))|.$$

Since $s^* = D_{i*}(B^{T+1}(c'))P_{i*}D_{i*}(A^{T_1}(c))$, we have $i^* \notin \{i \in I : B^T_{s*}(c') > e_{is*} \geq A^{T_1}_{s*}(c)\}$ and therefore $|D_{s*}(A^{T_1}(c))| < q_{s*}$, which is a contradiction to the definition of T_1.

Next, assume that $A^{T'}_{s*}(c) \geq B^T_{s*}(c')$. By the inductive assumption, there cannot be an agent i such that $A^{T'}_{s*}(c) > e_{is*} \geq B^T_{s*}(c')$, $D_i(A^{T_1}(c)) = s^*$, and $D_i(B^T(c')) \neq s^*$ (because such an agent would strictly prefer $D_i(B^T(c'))$ over $D_i(A^{T_1}(c))$). Hence,

$$D_{s*}(A^{T_1}(c)) \subseteq D_{s*}(B^T(c')) \cup \{i \in I : B^T_{s*}(c') > e_{is*} \geq A^{T_1}_{s*}(c)\}$$

and we obtain a contradiction as before.

$\square$

With the help of Lemmas 5 and 6, I now complete the proof of Theorem 7.

(i) Let A and B be two moderate upward-downward adjustment processes and c be an arbitrary cutoff vector. For any integer t, set $a^t := A^t(c)$ and $b^t := B^t(c)$.

Let T_1 be the smallest integer such that no school is oversubscribed at a^{T_1} and T_2 be the smallest integer such that no school is oversubscribed at b^{T_2}. Define a second sequence of cutoffs δ^t by setting $\delta^0 = c$ and $\delta^t = \Delta(\delta^{t-1})$ for all $t \geq 1$. Let T be the smallest integer such that no school is oversubscribed at δ^T.

Claim 1: $D_s(a^{T_1}) = D_s(b^{T_2}) = D_s(\delta^T)$ for all $s \in S$.

Proof. I show the statement for the sequence a^t. The proof for the sequence b^t is completely analogous. Note first that $a^t \leq \delta^t$ for all t: The statement is trivially true for $t = 0$; suppose we had already shown that $\delta^{t-1} \geq a^{t-1}$. Then, by Lemma 5 (i) and since A is moderate, $\delta^t = \Delta(\delta^{t-1}) \geq \Delta(a^{t-1}) \geq a^t$.

Next, I argue that $a^{T_1} \leq \delta^T$. By definition of T and what was just established, I immediately obtain $a^T \leq \delta^T$. If $T = T_1$, I immediately obtain the desired inequality. So assume that $T < T_1$.[28] By the first part of Lemma 5, I obtain $\Delta(a^T) \leq \Delta(\delta^T)$. Since no school is

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[28]Note that $T > T_1$ is impossible given that A is moderate.

oversubscribed at δ^T, we have that $\Delta(\delta^T) = \delta^T$. Since A is moderate, $a^{T+1} \leq \Delta(a^T)$. Hence, $a^{T+1} \leq \delta^T$ and iterating these arguments, I obtain the desired result.

To complete the proof of Claim 1, I now show by induction that for all $l \in \{0, \ldots, T'\}$, there exists a $t_l \leq T_1$ such that for all $t \in \{t_l, \ldots, T_1\}$ and all $i \in I$, $D_i(\delta^l) R_i D_i(a^t)$. In particular, $D_i(\delta^T) R_i D_i(a^{T_1})$ for all $i \in I$ and, given the already established fact that $\delta^T \geq a^{T_1}$, we must have $D_s(\delta^T) = D_s(a^{T_1})$ for all $s \in S$.

Now the desired statement is trivially true for $l = 0$ since $a^t \geq c$ and therefore $D_i(c) R_i D_i(a^t)$ for all $t \leq T_1$. Now suppose the statement is true for all $l' \leq l < T$. I show that it is also true for $l + 1$. Fix $t \in \{t_l, \ldots, T_1\}$ and i so that $s' := D_i(a^t) P_i D_i(\delta^{l+1})$. We must have $\delta_{s'}^{l+1} > e_{is'} \geq a_{s'}^t \geq c_{s'}$. By definition of $\Delta(\cdot)$, there must be at least $q_{s'}$ students $j \neq i$ such that $e_{js'} \geq \delta_s^{l+1}$ and $j \in D_{s'}(\delta^l)$. By the inductive assumption, for all $t \geq t_l$, none of these students can afford a more preferred school s' at a^t and consequently all enrol at s' at a^t. Hence, for any $t \geq t_l$ such that $D_i(a^t) P_i D_i(\delta^{l+1})$ for at least one $i \in I$, at least one school must be oversubscribed. Since $A(c) = c$ only if c is market-clearing, there must exist a $t_{l+1} \geq t_l$ such that for all $t \in \{t_{l+1}, \ldots, T_1\}$, each student weakly prefers her demand at δ^{l+1} over her demand at a^t. $\square$

To complete the proof, I can now apply the second part of Lemma 6 to a^{T_1} and b^{T_2}.

(ii) Let A and B be two moderate downward-upward adjustment processes and c be an arbitrary cutoff vector. For any integer t, set $a^t := A^t(c)$ and $b^t := B^t(c)$.

Let T_1 be the smallest integer such that no selective school is undersubscribed at a^{T_1} and T_2 be the smallest integer such that no selective school is undersubscribed at b^{T_2}. Define a second sequence of cutoffs y^t by setting $y^0 = c$ and $y^t = \mathcal{Y}(y^{t-1})$ for all $t \geq 1$. Let T be the smallest integer such that no selective school is undersubscribed at y^T.

Claim 2: $D_s(a^{T_1}) = D_s(b^{T_2}) = D_s(y^T)$ for all $s \in S$.

Proof. I show the statement for the sequence a^t. The proof for the sequence b^t is completely analogous. Note first that $y^t \leq a^t$ for all t. This is trivially true for $t = 0$. So suppose we had already established that $y^{t-1} \leq a^{t-1}$ for some $t \geq 1$. By the second part of Lemma 5, I obtain $y^t = \mathcal{Y}(y^{t-1}) \leq \mathcal{Y}(a^{t-1}) \leq a^t$.

Next, I argue that $a^{T_1} \geq y^T$. By definition of T and what was just established, I immediately obtain $a^T \geq y^T$. If $T = T_1$, I immediately obtain the desired inequality. So assume that $T < T_1$. By the second part of Lemma 5, I obtain $\mathcal{Y}(a^T) \geq \mathcal{Y}(y^T)$. Since no selective school

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is undersubscribed at y^T, we have that $\mathcal{Y}(y^T) = y^T$. Since A is moderate, $a^{T+1} \geq \mathcal{Y}(a^T)$. Hence, $a^{T+1} \geq y^T$ and iterating these arguments, I obtain the desired result.

To complete the proof of Claim 2, I now show by induction that for all $l \in \{0, \ldots, T\}$, there exists a $t_l \leq T_2$ such that for all $t \in \{t_l, \ldots, T_2\}$ and all $i \in I$, $D_i(a^t)R_iD_i(y^l)$. In particular, $D_i(a^{T_1})R_iD_i(y^T)$ for all $i \in I$ and, given the already established fact that $a^{T_1} \geq y^T$, we must have $D_s(a^{T_1}) = D_s(y^T)$ for all $s \in S$.

The desired statement is trivially true for $l = 0$ since $a^t \leq c$ and therefore $D_i(a^t)R_iD_i(c)$ for all $t \leq T_1$. Now suppose the statement is true for all $l' \leq l < T$. I show that it is also true for $l + 1$. Fix $t \in \{t_l, \ldots, T_1\}$ and i so that $s' := D_i(y^{l+1})P_iD_i(a^t)$. Since $t \geq t_l$ the inductive assumption yields $D_j(a^t)R_jD_j(y^l)$ for all $j \in I$. Furthermore, $s' = D_i(y^{l+1})P_iD_i(a^t)R_iD_i(y^l)$ so that s' must have been undersubscribed at y^l and $y^{l+1}_{s'} \leq e_{is'} < \min\{y^l_{s'}, a^t_{s'}\}$. The inductive assumption also implies that for all j such that $s' \neq D_j(y^l)$ and $e_{js'} \geq y^l_{s'}$, $s' \neq D_j(a^t)$. Hence,

$$D_{s'}(a^t) \quad \subseteq D_{s'}(y^l) \cup \{i \in I : y^l > e_{is'} \geq a^t_{s'}\}$$
$$\subsetneq D_{s'}(y^l) \cup \{i \in I : y^l > e_{is'} \geq y^{l+1}_{s'}\}.$$

By definition of $\mathcal{Y}(\cdot)$, $y^{l+1}_{s'} = \max\{c_{s'} - (q_{s'} - |D_{s'}(y^l)|), 1\}$. Hence, we must have

$$|\{i \in I : y^l_{s'} > e_{is'} \geq y^{l+1}_{s'}\}| \leq q_{s'} - |D_{s'}(y^l)|$$

and consequently $|D_{s'}(a^t)| < q_{s'}$. In particular, for any t such that $D_i(y^{l+1})P_iD_i(a^t)$ for at least one $i \in I$, at least one school must be undersubscribed. Since $A(c) = c$ only if c is market-clearing, there is a $t_{l+1} \in \{t_l, \ldots, T_2\}$ such that, for all $t \geq t_{l+1}$, $D_i(a^t)R_iD_i(y^{l+1})$ for all $i \in I$.

□

To complete the proof, I can now apply the first part of Lemma 6 to a^{T_1} and b^{T_2}.

(iii) Since A is upward-downward, there exists a T_1 such that $A^{T_1}(c) \geq c$ and $|D_s(A^{T_1}(c))| \leq q_s$ for all $s \in S$. Furthermore, $\Delta(\overline{\mathcal{C}}^*) = \overline{\mathcal{C}}^*$ and Lemma 5 is easily seen to imply $A^{T_1}(c) \leq \overline{\mathcal{C}}^*$. Let $S^* = \{s \in S : |D_s(\overline{\mathcal{C}}^*)| = q_s\}$ and $I^* = \{i \in I : D_i(\overline{\mathcal{C}}^*) \in S^*\}$. Note that we must have $|I^*| = \sum_{s \in S^*} q_s$. Since $A^{T_1}(c) \leq \overline{\mathcal{C}}^*$ and $\overline{\mathcal{C}}^*_s = 1$ for all $s \in S \setminus S^*$, we have that $A^{T_1}_s(c) = \overline{\mathcal{C}}^*_s$ for all $s \in S \setminus S^*$. By the weak axiom of revealed preference, I then obtain $D_i(A^{T_1}(c)) \in S^*$ for all $i \in I^*$ and therefore $I^* \subseteq \cup_{s \in S^*} D_s(A^{T_1}(c))$. Since $|D_s(A^{T_1}(c))| \leq q_s$ for all $s \in S$ and

$|I^*| = \sum_{s \in S^*} q_s$, I obtain that $|D_s(A^{T_1}(c))| = q_s$ for all $s \in S^*$. Since $A_s^{T_1} = 1$ for all $s \in S \backslash S^*$, $A^{T_1}(c)$ is market clearing and given that A is moderate, we have that $\lim_{t \to \infty} A^t(c) \geq c$.

Now take any market clearing cutoff vector c' such that $c' \geq c$. By Lemma 5, we have that $\Delta(c) \leq \Delta(c')$. Furthermore, the definitions of Δ and $\bar{c}(\mu^{c'})$ immediately imply $\Delta(c') \leq \bar{c}(\mu^{c'})$. Since A is moderate, I obtain $A(c) \leq \bar{c}(\mu^{c'})$. Iterating this argument, I find that $\lim_{t \to \infty} A^t(c) \leq \bar{c}(\mu^{c'})$, which completes the proof.

(iv) Since A is downward-upward, there exists a T_1 such that $A^{T_1}(c) \leq c$ and for all $s \in S$, either $|D_s(A^{T_1}(c))| \geq q_s$ or $A_s^{T_1}(c) = 1$. Furthermore, $\mathcal{Y}(\underline{\mathcal{C}}^*) = \underline{\mathcal{C}}^*$ and Lemma 5 is easily seen to imply $A^{T_1}(c) \geq \underline{\mathcal{C}}^*$. Let $S^* = \{s \in S : |D_s(\underline{\mathcal{C}}^*)| = q_s\}$ and $I^* = \{i \in I : D_i(\underline{\mathcal{C}}^*) \in S^*\}$. Note that we must have $|I^*| = \sum_{s \in S^*} q_s$. Now for any $i \in I \backslash I^*$, $A^{T_1}(c) \geq \underline{\mathcal{C}}^*$ and the weak axiom of revealed preference imply $D_i(A^{T_1}(c)) \notin S^*$. Hence, $\cup_{s \in S^*} D_s(A^{T_1}(c)) \subseteq I^*$. Furthermore, for any $s \in S^*$ such that $A_s^{T_1}(c) = \underline{\mathcal{C}}_s^* = 1$, the weak axiom of revealed preference implies $|D_s(A^{T_1}(c)| \geq |D_s(\underline{\mathcal{C}}^*)| = q_s$. So $|D_s(A^{T_1}(c))| \geq q_s$ for all $s \in S^*$. Since $\cup_{s \in S^*} D_s(A^{T_1}(c)) \subseteq I^*$ and $|I^*| = \sum_{s \in S^*} q_s$, I thus obtain $|D_s(A^{T_1}(c))| = q_s$ for all $s \in S^*$. Given that A is strongly moderate, it is then straightforward to show $A_s^t(c) = A_s^{T_1}(c)$ for all $s \in S^*$ and all $t \geq T_1$. By (ii), $A^t(c)$ converges to a market clearing cutoff vector $\lim_{t \to \infty} A^t(c) =: c^*$. By the rural hospitals theorem (Roth, 1986), we have that $|D_s(c^*)| < q_s$ for all $s \in S \backslash S^*$ and therefore $c_s^* = 1$ for all such s. Combining the previous two insights, I obtain that $c^* \leq c$.

Now take any market clearing cutoff vector c' such that $c' \leq c$. By Lemma 5, I obtain $y(c) \geq y(c') = c'$. Hence, $\lim_{t \to \infty} A^t(c) \geq c'$ and the desired result follows. $\square$

B.3   Proof of Theorem 3

Let A be any strongly moderate and strict adjustment process. Take any stable matching $\mu \neq \mu^I$. By Erdil and Ergin (2008), there exists a stable improvement cycle $i_1, \dots, i_L$ of μ, that is, for all l, $s_{l+1} := \mu(i_{l+1}) P_{i_l} \mu(i_l) =: s_l$ and $e_{i_l, s_{l+1}} > e_{j, s_{l+1}}$ for all $j \neq i_l$ such that $s_{l+1} P_j \mu(j)$ (with $L+1 := 1$). Consider the following cutoff vector:

$$c_{s_1} = e_{i_L, s_1} + 1, c_{s_l} = e_{i_{l-1}, s_l}, \text{ for all } l \in \{2, \dots, L\}$$

$$c_s = \min_{j \in \mu(s)} e_{j,s}, \text{ for all } s \in S \backslash \{s_1, \dots, s_L\} \text{ such that } |\mu(s)| = q_s$$

$$c_s = 1, \text{ for all } s \text{ such that } |\mu(s)| < q_s.$$

I start by deriving the result of the first round of adjustments.

Consider first an arbitrary school $s \in S \setminus \{s_1, \ldots, s_L\}$. If there is an agent $j \in I$ such that $j \notin \mu(s)$ but $s = D_j(c)$, we obtain a contradiction to the stability of μ since (1) $c_{\mu(j)} \leq e_{j,\mu(j)}$, so that $s = D_j(c)$ implies $sP_j\mu(j)$, and (2) $e_{j,s} > c_s = \min_{j' \in \mu(s)} e_{j',s}$, since scores are strict. The same arguments show that if there is an agent $j \in \mu(s)$ such that $s \neq D_j(c)$, we must have $D_j(c) = s_l$ for some $l \in \{1, \ldots, L\}$. This is impossible unless $s_l P_j s$ and $e_{j,s} \geq e_{i_{l-1},s_l} + 1$, which would contradict the definition of a stable improvement cycle. Hence, we must have $\mu(s) = D_s(c)$ so that $A_s(c) = c_s$ since either $|\mu(s)| = q_s$ or $c_s = 1$.

Next, I consider the schools involved in the stable improvement cycle. First, take some $l \in \{1, \ldots, L-1\}$ and agent i_l. Since $s_{l+1}P_{i_l}s_l$ and $c_{s_{l+1}} = e_{i_l,s_{l+1}}$, $D_{i_l}(c) \neq s_l$. If $D_{i_l}(c) = s_{l'}$ for some $l' \neq l+1$, we obtain a contradiction to the definition of a stable improvement cycle since $s_{l'}P_{i_l}\mu(i_l)$ and $e_{i_l,s_{l'}} \geq e_{i_{l'-1},s_{l'}}$. Hence, we must have $D_{i_l}(c) = s_{l+1}$. A completely analogous argument shows that $\mu(s_l) \setminus \{i_l\} \subseteq D_{s_l}(c)$ for all $l \in \{1, \ldots, L\}$. Now consider agent i_L. By the definition of a stable improvement cycle and $c_{s_L} < e_{i_L,s_L}$, we must have $D_{i_L}(c) \in \{s_1, s_L\}$. Since $c_{s_1} = e_{i_L,s_1} + 1$, $D_{i_L}(c) = s_L$. Together with the above this implies $|D_{s_1}(c)| = q_{s_1} - 1$, $|D_{s_L}(c)| = q_{s_L} + 1$, and $|D_{s_l}(c)| = q_{s_l}$ for all $l \in \{2, \ldots, L-1\}$. Since A is strongly moderate and strict, I obtain $A_{s_1}(c) = e_{i_L,s_1}$, $A_{s_L}(c) = e_{i_{L-1},s_L} + 1$, and $A_{s_l}(c) = e_{i_{l-1},s_l}$ for all $l \in \{2, \ldots, L-1\}$.

An iterative application of these arguments is easily seen to imply that, for all $t \leq L$, $A^t_{s_{L-t+1}}(c) = e_{i_{L-t},s_{L-t+1}} + 1$ (where $i_0 := i_L$), $A^t_{s_l}(c) = e_{i_{l-1},s_l}$ for all $l \neq L-t+1$, and $A^t_s(c) = c_s$ for all $s \in S \setminus \{s_1, \ldots, s_L\}$. In particular, $A^L(c) = c$ and this completes the proof.

$\square$

B.4   Proof of Theorem 4

To simplify the notation in the proof, set $c^t := A^t(c)$ for all t.

- I first establish that no school is oversubscribed at c^{min}. Suppose to the contrary that s is such that $|D_s(c^{min})| > q_s$ and let $I_s = D_s(c^{min})$. Note that by the definition of c^{min} and the weak axiom of revealed preference, we must have, for all $i \in I_s$, $D_i(c^t) = s$ whenever t is such that $e_{is} \geq c^t_s$. Since the adjustment process is strict, there must thus exist a smallest T' such that $c^{T'}_s > c^{min}_s$.

  Let $l_s = |\{i \in I_s : e_{is} \geq c^{T'}_s\}|$. By the above arguments, we must have $|D_s(c^{T'})| \geq l_s$. Hence, if $l_s \geq q_s$, then the above arguments imply that s is not undersubscribed at $c^{T'}$ and

$c_s^{T'+1} \geq c_s^{T'} > c_s^{min}$ since A decreases moderately. If $l_s < q_s$, then since A decreases moderately, we obtain $c_s^{T'+1} \geq c_s^{T'} - (q_s - l_s)$. Furthermore, $c_s^{T'} \geq c_s^{min} + (|I_s| - l_s)$. Combining the last two observations, I find that $c_s^{T'+1} \geq c_s^{min} + (|I_s| - l_s) - (q_s - l_s)$. Hence, $c_s^{T'+1} > c_s^{min}$ given that $|I_s| > q_s$. I have shown that if $c_s^{T'} > c_s^{min}$, then $c_s^{T'+1} > c_s^{min}$ as well. Iterating this argument, I find that $c_s^t > c_s^{min}$ for all $t \geq T'$. But the last relationship contradicts the definition of c_s^{min} and this contradiction completes the proof.

- Next, I show that no selective school is undersubscribed at c^{max}. Suppose to the contrary that s is such that $|D_s(c^{max})| < q_s$ as well as $c_s^{max} > 1$. Note that for all t and i, $D_i(c^t)R_iD_i(c^{max})$. The last observation implies that for any student i such that $e_{is} \geq c_s^{max}$ but $D_i(c^{max}) \neq s$, we have $D_i(c^t)P_is$ for all t. Hence, s must be undersubscribed whenever $c_s^t = c_s^{max}$. Since the adjustment process is strict, there must thus exist a smallest T' such that $c_s^{T'} < c_s^{max}$.

  If $|D_s(c^{T'})| < q_s$, we must have $c_s^{T'+1} \leq c_s^{T'}$ since A increases moderately. Next, consider the case of $|D_s(c^{T'})| \geq q_s$. Since A increases moderately, we must have $c_s^{T'+1} \leq \Delta_s(c^{T'})$. Since $|D_s(c^{max})| < q_s$, we must therefore have that $c_s^{T'+1} < c_s^{max}$. The preceding arguments show that whenever $c_s^{T'} < c_s^{max}$, $c_s^{T'+1} < c_s^{max}$ if A increases moderately. Iterating this argument, we find that $c_s^t < c_s^{max}$ for all $t \geq T'$. This contradicts the definition of c_s^{max}.

Using the just established findings, I now show that c^{min} and c^{max} are both market-clearing. Note first that $c^{min} \geq \underline{\mathcal{C}}^*$ given that no school is oversubscribed at c^{min} and $c^{max} \leq \overline{\mathcal{C}}^*$ given that no selective school is undersubscribed at c^{max}. Now let s' be a school such that $|\mu(s')| < q_{s'}$ for some stable matching μ. By the rural hospitals theorem (Roth, 1986), we have $\mu(s') = \mu'(s')$ for all stable matchings μ'. Hence, $\underline{\mathcal{C}}^*_{s'} = \overline{\mathcal{C}}^*_{s'} = 1$ and $c^{min}_{s'} = c^{max}_{s'} = 1$. Furthermore, it is straightforward to verify that $D_{s'}(c^{min}) = D_{s'}(c^{max}) = D_{s'}(\underline{\mathcal{C}}^*) = D_{s'}(\overline{\mathcal{C}}^*)$. Next, let $S^* = \{s \in S : |D_s(\underline{\mathcal{C}}^*)| = q_s\}$ be the set of all schools that fill their capacities in any stable matching. By the previous observations, we have that $\cup_{s \in S^*} D_s(c^{min}) = \cup_{s \in S^*} D_s(c^{max}) = \cup_{s \in S^*} D_s(\underline{\mathcal{C}}^*)$, that is, between c^{min} and c^{max} demand can only flow between schools that fill their capacities in all stable matchings. Furthermore, by definition of S^*, $\sum_{s \in S^*} |D_s(c^{min})| = \sum_{s \in S^*} |D_s(c^{min})| = \sum_{s \in S^*} q_s$. But then, if there was an undersubscribed selective school at c^{min}, there would also have to be an oversubscribed school at c^{min}. Hence, c^{min} is market clearing. Finally, if there was an oversubscribed school s^+ at c^{max}, the previous findings indicate that there would also have to be a school $s^- \in S^*$ that is undersubscribed at c^{max}. Since there are no undersubscribed selective schools at c^{max}, we must have $c_{s^-}^{max} = 1$ and thus also $c_{s^-}^{min} = 1$. But by revealed preference and $c^{max} \geq c^{min}$, we then obtain $D_{s^-}(c^{min}) \subseteq D_{s^-}(c^{max})$ and

thus $|D_{s^-}(c^{min})| < q_{s^-}$. Since we have already shown that c^{min} is market clearing, we obtain a contradiction to $s^- \in S^*$.

$\square$

B.5   Proof of Corollary 3

By Theorem 4 (iii) there is a T such that $\overline{\mathcal{C}}^* \geq A^t(c) \geq \underline{\mathcal{C}}^*$ for all $t \geq T$. It is straightforward to show that in the case I consider here, any cutoff vector that lies between the lowest and highest market clearing cutoff vector must be market clearing.

$\square$

B.6   Time-varying adjustments

In this subsection, I first formulate a more general version of Theorem 4 and then discuss the necessary adjustments to the proof.

Theorem 8. *Let A be an arbitrary adjustment process, $c \in C$ be an arbitrary cutoff vector, and $\mathcal{C}^{acc}$ be the set of accumulation points of the sequence $\{A^t(c)\}_{t \geq 0}$. Let c^{max} be the supremum of $\mathcal{C}^{acc}$ and c^{min} be the infimum of $\mathcal{C}^{acc}$. If A is moderate, then c^{max} and c^{min} are market-clearing.*

Proof. Note that there exists T such that $A^t(c) \in \mathcal{C}^{acc}$ for all $t \geq T$ by the definition of $\mathcal{C}^{acc}$. Conditional on this observation, the remainder of the proof proceeds almost exactly as the proof of Theorem 4. The only remaining difference is that we now use Assumption 1 to infer that an oversubscribed school at c^{min} and an undersubscribed selective school at c^{max} must both eventually adjust their cutoffs. $\square$