Adoption epidemics and viral marketing

David McAdams and Yangbo Song *

September 2, 2024

Abstract

An innovation (e.g., new product or idea) spreads like a virus, transmitted by those who have previously adopted it. Agents update their beliefs about innovation quality based on private signals and when they hear about the innovation. We characterize equilibrium adoption dynamics and the resulting lifecycle of virally-spread innovations. Herding on adoption can occur but only early in the innovation lifecycle, and adoption eventually ceases for all virally-spread innovations. A producer capable of advertising directly to consumers finds it optimal to wait and allow awareness to grow virally initially after launch.

Keywords: Adoption epidemic, SIR model, innovation lifecycle, viral marketing

JEL Classification: C72, D62, D83

*McAdams: Fuqua School of Business and Economics Department, Duke University, david.mcadams@duke.edu. Song: School of Management and Economics, The Chinese University of Hong Kong (Shenzhen). Email: yangbosong@cuhk.edu.cn. We are grateful to seminar audiences at the 2018 ES Winter Meeting, 2019 ES North American Summer Meeting and Asian Meeting, HKUST IO Workshop, Duke, Fudan, Johns Hopkins Carey, Sun Yat-Sen U., Southwestern U. of Economics and Finance, Nanjing Audit U., Chinese U. of Hong Kong (Shenzhen), Bonn, and U. Connecticut for helpful comments and suggestions. We also thank Dihan Zou for excellent work as research assistant. A previous version of this paper circulated under the title "Viral Social Learning."

When a novel virus enters a population, infected hosts expose others who, if successfully infected, will start spreading the virus as well. In such an *infectious-disease epidemic*, virus strains that are more successful at causing infection spread more quickly through the population. In the same way, when a new product is launched, a new idea espoused, or a new method developed, an epidemic diffusion process ensues in which those who have purchased the product, accepted the idea, or adopted the method spread awareness and cause others to consider it as well. During such an *adoption epidemic*, consumers can make inferences about quality based on how long it took for them to be exposed, in addition to their private signals. For example, hearing about a movie long after it has been released is a sign that it is unlikely to be very good since, if it were, you would likely have heard about it sooner.

Our economic-epidemic model adapts the Susceptible-Infected (SI) model of viral epidemiology[1] to an economic context in which consumers receive informative private signals about quality and decide whether to adopt a new innovation. There is a unit-mass population of consumers and an "innovation" that is "good" with probability α and "bad" with probability $1-\alpha$. When first exposed to the innovation, each consumer i receives a conditionally independent private signal $s_i \in \{G, B\}$ that matches the true state with probability $\rho \in (1/2, 1)$. Consumer i then decides whether to adopt the innovation, preferring to adopt whenever she believes that the innovation is more likely to be good than bad. Those who adopt are "infected" and subsequently expose others, while those who choose not to adopt are "immune/removed" and do not expose anyone else to the innovation.

Our first main finding is that the adoption epidemic has a unique equilibrium epidemic trajectory, which depends on (i) consumers' ex ante belief $\alpha \in [0, 1]$ about the likelihood that the innovation is good, (ii) the precision $\rho \in (1/2, 1)$ of consumers' private signals, and (iii) the fraction L of the consumer population that learns about the innovation at "launch" at time $t = 0$. The case with $L = 1$ is relatively trivial since all consumers are exposed to the innovation at time $t = 0$ and simultaneously decide whether to adopt; we refer to this as a "traditional ad campaign." By contrast, when $L \approx 0$, almost all consumers encounter the innovation socially; we refer to this case as

[1] In the SI model, hosts progress from susceptible (state S) to infected (state I) once they are exposed to the virus, i.e., infectivity equals 100%. Our epidemiological model is a variation in which some exposed consumers do not become infected (because they choose not to adopt) and are henceforth immune.

a "viral campaign."

The qualitative features of the equilibrium trajectory of a viral campaign depend on whether or not the innovation is more likely to be good than bad, i.e., is $\alpha > 1/2$ or $\alpha < 1/2$? When $1/2 < \alpha < \rho$,[2] we show that consumers adopt regardless of their private signal ("herd on adoption") immediately after launch, but this herding phase eventually ends and is followed by subsequent phases in which newly-exposed consumers are less and less likely to adopt—until eventually all adoption ceases, an endogenous obsolescence. By contrast, when $1 - \rho < \alpha < 1/2$, consumers do not herd on adoption immediately after launch and newly-exposed consumers' belief about innovation quality initially rises over time. However, as when $1/2 < \alpha < \rho$, newly-exposed consumers eventually become sufficiently pessimistic about quality that all adoption ceases.

In an extension, we allow the producer of the innovation to launch it virally but then end the viral campaign at any time $T \in [0, \infty)$ with an ad that reaches all still-unexposed consumers. Our main finding in this extension is that a traditional ad campaign (corresponding to $T = 0$) leads to strictly less overall adoption than an optimal-length viral campaign.[3] On the other hand, we also show that it is never optimal in our model to run a viral campaign forever.

Relation to the literature. The idea that ideas can spread like a virus is widely appreciated[4] and well-studied, with some going even further to explore how ideas mutate as they circulate through a population; see e.g., Adamic et al. (2016) and Jackson et al. (2022). We abstract from the possibility of mutation, but push the literature forward by modeling becoming infected as an economic choice. In doing so, we characterize the equilibrium dynamics of the epidemic diffusion process and show how these dynamics change over time, passing through several phases with distinctive patterns of adoption.

[2] If $\alpha > \rho$ (or $\alpha < 1 - \rho$), then consumer behavior is trivial with everyone (or no one) adopting.

[3] The producer in our analysis seeks to maximize the mass of consumers who adopt the innovation. If quicker adoption is more valuable, such as when the innovation may become obsolete or when adoption corresponds to purchasing a new product and the producer is a firm that discounts profits, then the producer may prefer running a traditional ad campaign even though doing so leads to less overall adoption.

[4] See e.g., "The Age of the Viral Idea" by Bill Davidow, *The Atlantic*, Nov 17, 2011 and "The Internet Catches a Viral Epidemic" by Bill Wasik, *Wired*, April 16, 2013.

Most closely related is Banerjee (1993), who pioneered the study of adoption epidemics in the context of *rumors*, when only those exposed at launch have informative private signals about quality. Because those exposed after launch do not have any private information, their likelihood of adopting upon being exposed ("infectivity") at any given time is the same for low- and high-quality rumors. Moreover, the pattern of adoption is especially simple, with all socially-exposed consumers adopting the rumor until a critical moment after which no one adopts. By contrast, infectivity in our model depends on innovation quality and the epidemic transitions through up to four distinct phases.

Because awareness of the innovation spreads by word of mouth, this paper connects with the broader economic literature on diffusion; see e.g., Campbell (2013), Campbell et al. (2017), Leduc et al. (2017), and Sadler (2020). The main difference is that this literature mostly focuses on consumers' search technology and social network, whereas we focus on the impact of consumers' private information about quality. There is also a literature in marketing and consumer behavior on the diffusion of new products through influentials, e.g. Dodson and Muller (1978) and Van den Bulte and Joshi (2007). This literature also develops compartmental models where consumers transit between different states marking their awareness of the product and/or their adoption behavior. However, consumers in these models typically make decisions according to rules governed by exogenous parameters; see Watts and Dodds (2007) for a comprehensive survey. By contrast, the consumers in our analysis are Bayesian utility maximizers.

An extensive literature endogenizes the diffusion dynamics of an infectious pathogen; see e.g., Newman (2002) on disease spread over a social network, Laxminarayan and Brown (2001) and McAdams (2017) on when to switch to a new antibiotic in the face of rising resistance, and Farboodi et al. (2021) and McAdams et al. (2023) on the impact of social distancing during the outbreak and endemic phases of an epidemic. The basic difference with this literature is that agents in an infectious-disease epidemic prefer to avoid infection, whereas being "infected" in our model may or may not benefit consumers depending on whether the innovation is good or bad.

Finally, the paper relates indirectly to the literature on social learning. In the classic social learning model (Bikhchandani et al. (1992), Banerjee (1992)), infinitely-many agents are arrayed in a line and sequentially decide whether to adopt, based on their own private signal and all decisions made by those before them. By contrast, in our model, only those who have chosen to adopt expose others to the innovation and, when deciding whether to adopt, consumers do not know the length of the chain of exposures that led to their own exposure.[5]

The rest of the paper is organized as follows. Section 1 presents the model. Section 2 characterizes the equilibrium epidemic trajectory of innovation adoption over time. Section 3 extends the analysis to allow the producer to choose when to end the viral campaign with an ad that reaches all remaining consumers. Section 4 concludes by discussing the important assumptions of the model and a few interesting directions for future research. Formal proofs omitted in the main text are in the Appendix.

1 Model

There is an "innovation" which may be either "good" or "bad" and a continuum of consumers having unit mass. Each consumer i gets payoff $+1$ when adopting a good innovation, -1 when adopting a bad innovation, or zero when not adopting, and seeks to maximize their own expected payoff. Each consumer therefore strictly prefers to adopt if and only if they believe that the innovation's likelihood of being good exceeds $1/2$. Let $\alpha \in [0,1]$ be the ex ante probability that the innovation is good.

Epidemiological dynamics. Innovation awareness spreads through the consumer population following a variation of the classic Susceptible-Infected (SI) model of viral epidemiology (Kermack and McKendrick (1927)). At each point in time $t \geq 0$, each consumer is in one of three epidemiological states: *Susceptible* (state S), if not yet exposed to the innovation; *Infected* (state I), if previously exposed and chose to adopt; or *Immune/Removed* (state R), if previously exposed and chose not to adopt. We assume that mass $L > 0$ of consumers are exposed to the innovation at time $t = 0$ regardless of innovation quality. Each consumer who adopts becomes infected and spreads

[5] Classic social learning reemerges within a variation of our model if one instead assumes (i) all infected and immune consumers expose others at the same rate and (ii) each consumer is able to observe the history of decisions made along the entire chain of consumers leading to their exposure. In that context, consumers along each exposure chain behave exactly as in the classic model.

innovation awareness virally by exposing other randomly-selected consumers to the innovation; each infected consumer initiates such exposure events at rate normalized to one.[6] If an exposed consumer is susceptible, they receive a private signal and decide whether or not to adopt, transitioning immediately either to the infected state (if adopting) or to the immune state (if not adopting). If an exposed consumer is infected or immune, they remain in the same state; by assumption, adoption decisions are permanent.

Let $S_\omega(t)$, $I_\omega(t)$, and $R_\omega(t)$ denote the mass of susceptible, infected, and immune consumers at time t, conditional on the unobserved innovation-quality state $\omega \in \{g, b\}$. Since the population has unit mass, $R_\omega(t) = 1 - S_\omega(t) - I_\omega(t)$ and the overall epidemiological process is described by $(S_\omega(t), I_\omega(t) : t \geq 0, \omega = g, b)$. Let $q_\omega(t)$ denote time-t consumers' likelihood of adopting when the state is $\omega \in \{g, b\}$.

Epidemiological dynamics are characterized by the system of differential equations

$$S'_\omega(t) = -I_\omega(t)S_\omega(t) \tag{1}$$

$$I'_\omega(t) = q_\omega(t)I_\omega(t)S_\omega(t) \tag{2}$$

Equation (1) follows from the fact that each infected consumer meets another consumer at rate 1 and fraction $S_\omega(t)$ of others remain suspectible, generating a state-dependent flow $I_\omega(t)S_\omega(t)$ of newly-exposed consumers who are then no longer susceptible. Equation (2) follows from the fact that fraction $q_\omega(t)$ of these newly-exposed consumers choose to adopt. Note that epidemiological dynamics are completely determined by the adoption process $(q_\omega(t) : t \geq 0, \omega = g, b)$ and the mass L of consumers exposed at time $t = 0$.

Consumer belief formation. Let $p(t)$ be the probability that the innovation is good conditional on first encountering it socially at time t, what we refer to as the "interim belief" of consumers exposed socially at time t. Let $f(t|\omega)$ denote the endogenous[7] p.d.f. of consumers' time of exposure conditional on the state $\omega \in \{g, b\}$. By Bayes'

[6] The transmission rate being equal to one is without loss. Given any transmission rate $\beta \neq 1$, equilibrium epidemiological dynamics are exactly the same but happen β times faster than in our model.

[7] We characterize the equilibrium distribution of $t|\omega$, showing that $f(t|\omega)$ exists and is continuous in t at all but finitely-many points when the innovation lifecycle transitions from one phase to the next.

Rule, $p(t) = \frac{\alpha f(t|\omega=g)}{\alpha f(t|\omega=g)+(1-\alpha)f(t|\omega=b)}$ or, equivalently,

$$\frac{p(t)}{1-p(t)} = \frac{\alpha}{1-\alpha} \times \frac{f(t|\omega=g)}{f(t|\omega=b)} \tag{3}$$

Once exposed to the innovation, each consumer i observes private signal $s_i \in \{G, B\}$. These signals are conditionally i.i.d. with $\Pr(s_i = G|\omega = g) = \Pr(s_i = B|\omega = b) = \rho \in (1/2, 1)$.[8]

A consumer i exposed at launch ($t_i = 0$) with signal $s_i \in \{G, B\}$ updates to "ex post belief" $p(0; s_i)$, where

$$\frac{p(0;G)}{1-p(0;G)} = \frac{\alpha}{1-\alpha} \times \frac{\rho}{1-\rho} \quad \text{and} \quad \frac{p(0;B)}{1-p(0;B)} = \frac{\alpha}{1-\alpha} \times \frac{1-\rho}{\rho}. \tag{4}$$

A consumer i exposed socially at time t_i updates her belief based on both her own private signal $s_i \in \{G, B\}$ and when she is exposed, forming "ex post belief" $p(t_i; s_i)$. Again by Bayes Rule,

$$\frac{p(t_i;G)}{1-p(t_i;G)} = \frac{p(t_i)}{1-p(t_i)} \times \frac{\rho}{1-\rho} \quad \text{and} \quad \frac{p(t_i;B)}{1-p(t_i;B)} = \frac{p(t_i)}{1-p(t_i)} \times \frac{1-\rho}{\rho}. \tag{5}$$

By assumption, all consumers receive equally-informative private signals, regardless of whether they encountered the innovation direction at launch or indirectly through a social interaction.

Belief dynamics. Since the consumer population has unit mass, the flow of newly-exposed consumers can be interpreted as the density of the time-until-exposure t, i.e., $f(t|\omega) = |S'_\omega(t)| = S_\omega(t) I_\omega(t)$, where $|S'_\omega(t)|$ is the flow of consumers exposed at time t ("time-t consumers") when the innovation is good ($\omega = g$) or bad ($\omega = b$). Thus, time-t consumers' interim belief is given by

$$\frac{p(t)}{1-p(t)} = \frac{\alpha}{1-\alpha} \times \frac{S_g(t) I_g(t)}{S_b(t) I_b(t)}. \tag{6}$$

[8] The fact that consumers receive binary private signals is not essential. In Appendix B of the working-paper version (McAdams and Song (2023)), we extend the analysis to a setting in which consumers receive continuous private signals satisfying the monotone likelihood ratio property.

Adoption dynamics. Let $a_{s_i}(t)$ denote the likelihood that each time-t consumer chooses to adopt given private signal $s_i \in \{G, B\}$. Time-t consumers are said to "herd on adoption" if $a_G(t) = a_B(t) = 1$ and to "herd on non-adoption" if $a_G(t) = a_B(t) = 0$. On the other hand, they are said to be "sensitive to signals" if $a_G(t) = 1$ but $a_B(t) = 0$. Note that time-t consumers find it optimal to herd on adoption whenever $p(t) > \rho$, to herd on non-adoption when $p(t) < 1 - \rho$, and to be sensitive to signals when $1 - \rho < p(t) < \rho$. Time-t consumers are indifferent whether to adopt after a bad private signal if $p(t) = \rho$ and indifferent whether to adopt after a good signal if $p(t) = 1 - \rho$.

Equilibrium. Our solution concept is Bayesian Nash equilibrium (or simply "equilibrium"). We will show by construction that an equilibrium exists and that generically this equilibrium is essentially unique, in the sense that all equilibria generate the same population-wide epidemiological dynamics $(S_\omega(t), I_\omega(t) : t \geq 0; \omega \in \{g, b\})$.

1.1 Discussion of modeling assumptions.

Two key features of our model are that (i) only those who have adopted spread awareness of the innovation, causing high-quality innovations to spread more rapidly,[9] and (ii) newly-exposed consumers can determine how long the innovation has been in circulation before deciding whether to adopt themselves. For example, word of mouth about a new movie spreads naturally from those who have chosen to go see it, causing people to hear about great movies more quickly than bad ones. Similarly, after a new scientific method is published, other scientists spread awareness by using it in their own published work. In each case, consumers (moviegoers, scientists) can determine when the innovation was launched (theatrical release, scientific publication) and update their own beliefs about its likely quality based on its recency before deciding whether to adopt themselves.

Other substantive economic assumptions play an important simplifying role in the analysis. In particular, (iii) consumers decide whether to adopt when they are first exposed to the innovation, (iv) adoption is irreversible and (v) those who adopt transmit awareness forever. Assumptions (iii-iv) dramatically simplify the analysis by allowing

[9]If those who have rejected the innovation spread awareness at the same rate as those who have adopted it, then consumers would learn nothing from the time at which they are exposed.

us to focus on consumers' beliefs only at the time of initial exposure, while (v) ensures that all consumers are eventually exposed to the innovation regardless of quality. Of course, in practice, consumers can often choose to wait before adopting and may only spread the word for a limited period of time. For instance, a moviegoer might wait until she hears about a movie from several people before seeing it and, even if she loves it, only gush to friends about it while it remains fresh in her mind.

The easiest of these assumptions to relax is (ii). Suppose that only fraction $1 - \eta$ of consumers are able to observe the time t since launch. Since all consumers are eventually exposed to the innovation, a consumer who is unable to observe the time since launch will not make any inference about innovation quality and so will decide whether to adopt *as if* encountering the innovation at launch. The overall likelihood that a consumer exposed at time $t > 0$ will adopt in innovation-quality state $\omega \in \{g, b\}$ is therefore $\tilde{q}_\omega(t) = \eta q_\omega(0) + (1 - \eta) q_\omega(t)$, where $q_\omega(0)$ and $q_\omega(t)$ are the likelihoods that consumers who *can* observe the time will adopt, respectively, at time 0 and time t. The rest of our analysis then carries over, with more complex formulas but little additional insight.

In the concluding remarks, we discuss how to modify the analysis to allow for temporary infectiousness, relaxing assumption (v). In the working-paper version McAdams and Song (2023), we also suggest some directions for future work in models that give consumers the option to wait or to "rent" the innovation, relaxing assumptions (iii) and (iv). Yet another valuable direction for future work would be to allow susceptible consumers to learn from more than just their meetings with infected individuals. For instance, in a standard random-meetings model, susceptible individuals would meet infected, immune, and other susceptible individuals (not just infected people, as in our model) and be able to learn from all of these meetings.[10]

2 Adoption Epidemic Dynamics

This section characterizes the unique equilibrium trajectory of the adoption epidemic throughout a viral campaign, from launch through endogenous obsolescence, in the

[10] As an example, suppose that a new movie is released on Thursday and you hear about it on Friday night from someone who saw it. What you infer about its quality on Friday night will depend on how many other people you have talked to since Thursday who did *not* mention the movie.

most interesting case with intermediate prior belief $\alpha \in (1-\rho, \rho)$.[11]

Consumer behavior at and immediately after launch. We begin by considering how consumers must behave at the very beginning of the viral campaign, at launch ($t = 0$) and shortly afterward. Since $1-\rho < \alpha < \rho$, we have $p(0; B) < 1/2 < p(0; G)$ and any consumer exposed at launch finds it optimal to adopt after getting a good signal but not after a bad signal, i.e., they are sensitive to signals. Since good signals are more likely for good innovations, more consumers adopt at launch and word of mouth spreads more rapidly for good innovations. Hearing quickly about an innovation is therefore good news about its quality. More precisely, $I_g(t) \approx \rho L$, $I_b(t) \approx (1-\rho)L$, and $S_g(t) \approx S_b(t) \approx 1 - L$ for all $t \approx 0$, where L is the mass of consumers exposed at launch. By equation (6), we conclude that

$$\frac{p(t)}{1-p(t)} \approx \frac{\alpha}{1-\alpha} \times \frac{\rho}{1-\rho} \text{ for all } t \approx 0, \tag{7}$$

regardless of L. That is, consumers' interim belief shortly after launch is the same as if they have gotten a good private signal of precision ρ. Because $\alpha > 1-\rho$, equation (7) implies that (i) $p(0+) \equiv \lim_{t \to 0} p(t) > 1/2$ and (ii) $p(0+) > \rho$ if and only if $\alpha > 1/2$. Consumers exposed immediately after launch will therefore herd on adoption if $\alpha > 1/2$ but remain sensitive to signals if $\alpha < 1/2$.

Interim belief dynamics after launch. Equation (6) characterizes consumers' interim belief $p(t)$ at time t, depending on the ex ante likelihood α that the innovation is good and the ratio $\frac{S_g(t) I_g(t)}{S_b(t) I_b(t)}$. Rather than focusing on $p(t)$ directly, we find it convenient to consider the percentage rate of change of the likelihood ratio $\frac{p(t)}{1-p(t)}$, gotten by taking the log of both sides of (6) and differentiating:

$$\begin{aligned} X(t) \equiv \frac{\mathrm{d}\log\left(\frac{p(t)}{1-p(t)}\right)}{\mathrm{d}t} &= \frac{S_g'(t)}{S_g(t)} - \frac{S_b'(t)}{S_b(t)} + \frac{I_g'(t)}{I_g(t)} - \frac{I_b'(t)}{I_b(t)} \\ &= -I_g(t) + I_b(t) + q_g(t) S_g(t) - q_b(t) S_b(t) \end{aligned} \tag{8}$$

[11]Agents herd on adoption forever if $\alpha > \rho$ and herd on non-adoption forever if $\alpha < 1-\rho$. The cases when $\alpha = \rho$ and $\alpha = 1-\rho$ are more complex because consumers are sometimes indifferent whether to adopt at launch, but this extra complexity does not lead to any additional insight.

where $\frac{S'_\omega(t)}{S_\omega(t)} = -I_\omega(t)$ and $\frac{I'_\omega(t)}{I_\omega(t)} = q_\omega(t)S_\omega(t)$ by equations (1-2). Since $\frac{p(t)}{1-p(t)}$ grows exponentially at rate $X(t)$, we have $p'(t) \gtreqless 0$ iff $X(t) \gtreqless 0$.

Lemma 1 summarizes some implications of equation (8), depending on whether consumers herd on adoption, are sensitive to signals, or herd on non-adoption. (All omitted proofs are provided in the Appendix.)

Lemma 1. *(i) Suppose that consumers herd on adoption at time t. $p'(t) < 0$ if $S_g(t) < S_b(t)$ and $I_g(t) > I_b(t)$. (ii) Suppose that consumers are sensitive to signals at time t. $p'(t) > 0$ if and only if the following inequality holds:*

$$\rho S_g(t) - (1-\rho)S_b(t) > I_g(t) - I_b(t). \tag{SS}$$

(We refer to this as "Condition SS," mnemonic for "sensitive to signal.") (iii) Suppose that consumers herd on non-adoption at time t. $p'(t) < 0$ if $I_g(t) > I_b(t)$.

Proof. $p'(t) \gtreqless 0$ iff $X(t) \gtreqless 0$ in equation (8). (i) Herding on adoption: When $q_g(t) = q_b(t) = 1$, $X(t) = -\left(S_b(t) - S_g(t)\right) - \left(I_g(t) - I_b(t)\right)$ which is negative so long as $S_g(t) < S_b(t)$ and $I_g(t) > I_b(t)$. (ii) Sensitive to signals: When $q_g(t) = \rho$ and $q_b(t) = 1-\rho$, $X(t) = \rho S_g(t) - (1-\rho)S_b(t) - \left(I_g(t) - I_b(t)\right)$ which is positive iff condition (SS) holds. (iii) Herding on non-adoption: When $q_g(t) = q_b(t) = 0$, $X(t) = -\left(I_g(t) - I_b(t)\right)$ which is negative so long as $I_g(t) > I_b(t)$. $\square$

Discussion of Lemma 1. The conditions $S_g(t) < S_b(t)$ and $I_g(t) > I_b(t)$ capture the idea that good innovations will reach more people and be adopted by more people by time t than bad innovations. As we show later in Proposition 4, these intuitive conditions must always hold in any equilibrium. Lemma 1(i) can therefore be restated more simply as "$p'(t) < 0$ whenever consumers herd on adoption" while Lemma 1(iii) is "$p'(t) < 0$ whenever consumers herd on non-adoption."

To gain intuition, recall from equation (6) that $p(t)$ co-moves with the ratio $\frac{S_g(t)I_g(t)}{S_b(t)I_b(t)}$.

Suppose that consumers herd on non-adoption. With no new infections, $\frac{I_g(t)}{I_b(t)}$ is constant. But since $I_g(t) > I_b(t)$, each susceptible agent is exposed at a faster rate when the innovation is good; so, $S_g(t)$ falls at a faster percentage rate than $S_b(t)$ and $\frac{S_g(t)}{S_b(t)}$ decreases over time. Thus, the ratio $\frac{S_g(t)I_g(t)}{S_b(t)I_b(t)}$ and hence $p(t)$ must fall.

Next, suppose that consumers herd on adoption. Since $S_g(t) < S_b(t)$, each infected agent exposes others at a slower rate when the innovation is good; so, $I_g(t)$ rises at a slower percentage rate than $I_b(t)$ and the ratio $\frac{I_g(t)}{I_b(t)}$ decreases over time. Since the ratio $\frac{S_g(t)}{S_b(t)}$ also declines (for the same reason as before), the overall effect is that $\frac{S_g(t)I_g(t)}{S_b(t)I_b(t)}$ and hence $p(t)$ must fall even more quickly.[12]

Finally, suppose that consumers are sensitive to signals. Although infected agents encounter susceptible agents less frequently when the innovation is good, each of these exposure events is more likely to convert into an infection because newly-exposed agents are more likely to get a positive signal. In particular, each infected agent causes a new infection at rate $\rho S_g(t)$ when the innovation is good, compared to rate $(1-\rho)S_b(t)$ when it is bad. So long as $\rho S_g(t) > (1-\rho)S_b(t)$, the ratio $\frac{I_g(t)}{I_b(t)}$ rises over time. And so long as condition (SS) holds, the resulting "upward pressure" on beliefs overwhelms the "downward pressure" due to the ratio $\frac{S_g(t)}{S_b(t)}$ falling over time, and $p(t)$ will rise. However, as soon as condition (SS) fails, the downward pressure dominates and $p(t)$ must fall.

2.1 Equilibrium Lifecycle of an Innovation

This section characterizes equilibrium economic-epidemiological dynamics, focusing on the case of a very small launch ($L \approx 0$) so that essentially all consumers are exposed socially.[13] Our main finding is that consumer behavior transitions over time through up to four distinct phases, what we refer to collectively as the "innovation lifecycle"; see Figure 1. Behavior immediately after launch (Phase I) depends on whether the innovation is more likely to be good ($\alpha > 1/2$) or bad ($\alpha < 1/2$). Subsequent behavior then passes through a period of partial herding (Phase II), a period in which consumers are sensitive to signals (Phase III), and a final period with zero adoption (Phase IV).

[12]By this reasoning, consumers' interim belief $p(t)$ must fall whenever $q_g(t) = q_b(t)$, i.e., whenever good and bad innovations have equal infectivity. This is true at all times $t > 0$ in Banerjee (1993), since socially-exposed consumers in his model of virally-spread rumors do not receive private signals. This explains why the interim belief is monotone decreasing in Banerjee (1993), but may be increasing in our model during periods when consumers are sensitive to signals.

[13]Some qualitative features of the equilibrium epidemic trajectory only hold when L is sufficiently small. In particular, condition (SS) fails immediately after launch whenever $L > \frac{1}{2}$, causing consumer beliefs to fall even if consumers are sensitive to signals.

Theorem 1. *Suppose that $\alpha \in (1-\rho, \rho)$ and $L \approx 0$. Equilibrium epidemiological dynamics $(S_\omega(t), I_\omega(t) : t \geq 0; \omega \in \{g, b\})$ are uniquely determined, with consumers' post-launch equilibrium behavior transitioning through four phases at times $0 \leq t_1 < t_2 < t_3 < \infty$.*

Phase I: *(i) If $\alpha \in (1/2, \rho)$, then consumers herd on adoption after launch and interim belief $p(t) > \rho$ decreases until time $t_1 \in (0, \infty)$ at which $p(t_1) = \rho$. (ii) If $\alpha \in (1-\rho, 1/2)$, then consumers are sensitive to signals after launch and $p(t) \in (1/2, \rho)$ increases until time $t_1 \in (0, \infty)$ at which $p(t_1) = \rho$. (iii) If $\alpha = 1/2$, then $t_1 = 0$ and $p(0+) \equiv \lim_{\epsilon \to 0} p(\epsilon) = \rho$.*

Phase II: *After time t_1, consumers partially herd on adoption, adopting always after a good signal and with probability $a_B(t) \in (0, 1)$ after a bad signal, where $a_B(t)$ is decreasing in t, until time $t_2 \in (t_1, \infty)$ at which $a_B(t_2) = 0$. Consumers' interim belief $p(t) = \rho$ for all $t \in [t_1, t_2]$.*

Phase III: *After time t_2, consumers are sensitive to signals and interim belief $p(t) \in (1-\rho, \rho)$ is decreasing in t until time $t_3 \in (t_2, \infty)$ is reached at which $p(t_3) = 1 - \rho$.*

Phase IV: *After time t_3, consumers herd on non-adoption, what we refer to as "viral obsolescence," and consumers' interim belief $p(t) < 1 - \rho$ continues to decline with $\lim_{t \to \infty} p(t) = 0$.*

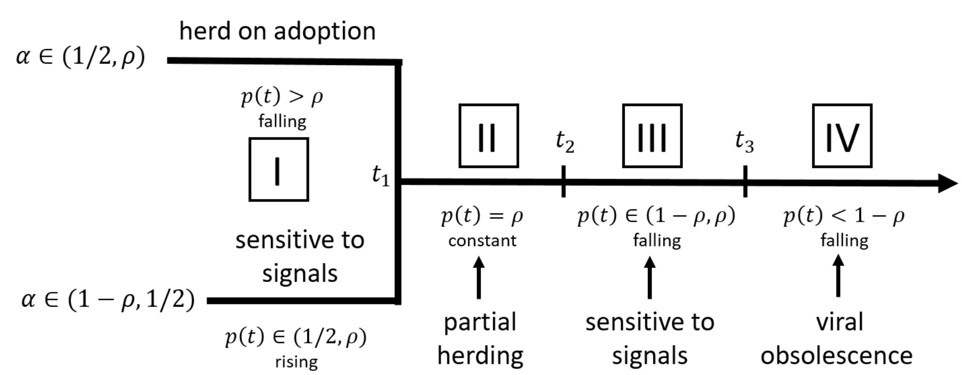
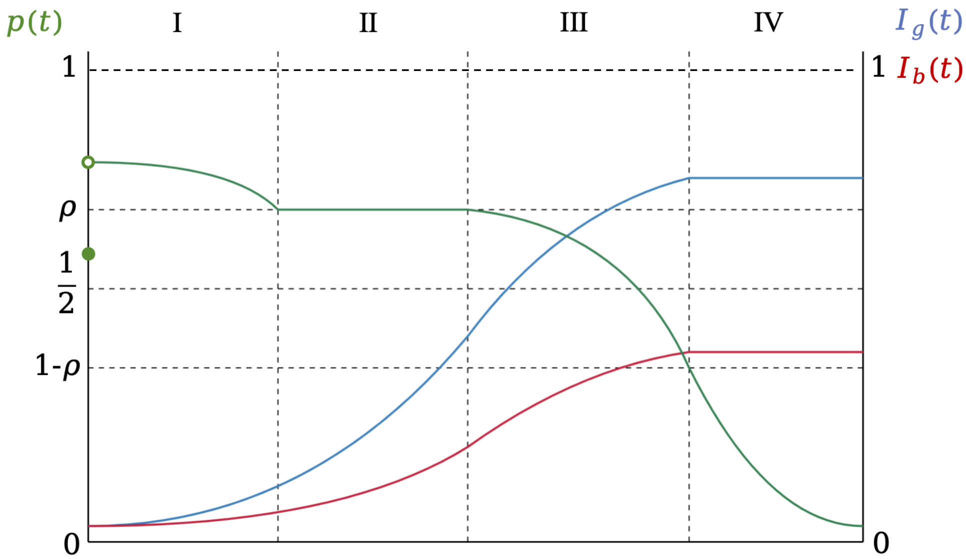
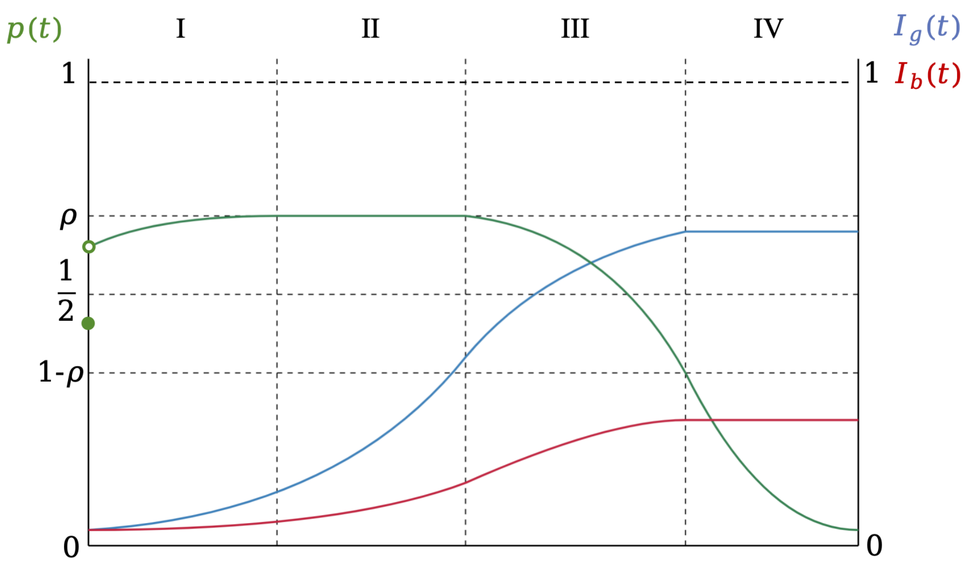


Figure 1: Visual summary of equilibrium adoption behavior and interim beliefs over the innovation lifecycle, when consumers' ex ante belief $\alpha \in (1-\rho, \rho)$.

The rest of this section establishes Theorem 1 through a series of five propositions.



(a) $\alpha = 0.55$ and $\rho = 0.65$



(b) $\alpha = 0.45$ and $\rho = 0.65$

Figure 2: Dynamics of interim beliefs ($p(t)$) and innovation adoption ($I_g(t)$, $I_b(t)$) in the equilibrium adoption epidemic in two examples with (a) $\alpha > \frac{1}{2}$ and (b) $\alpha < \frac{1}{2}$.

Phase I: herding on adoption case. Suppose first that $\alpha \in (1/2, \rho)$, as in the numerical example illustrated in Figure 2(a). By previous analysis around equation (7), $p(0+) > \rho$ and consumers must initially herd on adoption.

Proposition 1 (Phase I: herding on adoption)**.** *Suppose that $\alpha \in (1/2, \rho)$. There exists $t_1 \in (0, \infty)$ such that, in any equilibrium trajectory, (i) consumers herd on adoption for all $t \in (0, t_1)$, (ii) $p(t)$ is strictly decreasing over $t \in (0, t_1)$, and (iii) $p(t_1) = \rho$.*

The fact that consumers' interim belief must fall follows immediately from Lemma 1(i) due to herding on adoption. The critical time t_1 is the first moment after launch at which newly-exposed consumers no longer strictly prefer to herd on adoption.

Phase I: sensitive to signals case. Suppose next that $\alpha \in (1 - \rho, 1/2)$, as in the numerical example illustrated in Figure 2(b). By previous analysis, $p(0+) \in (1/2, \rho)$ and consumers are sensitive to signals after launch.

Proposition 2 (Phase I: sensitive to signals)**.** *Suppose that $\alpha \in (1 - \rho, 1/2)$ and $L \approx 0$. There exists $t_1 \in (0, \infty)$ such that, in any equilibrium trajectory, (i) consumers are sensitive to signals for all $t \in (0, t_1)$, (ii) $p(t)$ is strictly increasing over $t \in (0, t_1)$, and (iii) $p(t_1) = \rho$.*

Consumers being sensitive to signals and our small-launch assumption ($L \approx 0$) ensure that condition (SS) is initially satisfied. Consumers' interim belief $p(t)$ must therefore rise initially. The proof in the Appendix shows that, in fact, $p(t)$ continues to rise until a critical time t_1 at which $p(t_1) = \rho$. We refer to the period up to time t_1 as "Phase I."

Phase II: Partial herding. After time t_1, consumers randomize whether to adopt after a bad private signal (and always adopt after a good signal), what we call "partial herding." Consumers' interim belief $p(t)$ remains constant ρ and the likelihood $a_B(t)$ that consumers adopt after a bad signal declines continuously until, at a critical time t_2, $a_B(t) = 0$ and consumers become sensitive to signals. We refer to the period from t_1 until t_2 as "Phase II".

Proposition 3 (Phase II)**.** *Suppose that $\alpha \in (1 - \rho, \rho)$ and $L \approx 0$. There exists $t_2 \in (t_1, \infty)$ such that, in any equilibrium trajectory, (i) consumers partially herd with adoption probability*

$a_B(t) \in (0, 1)$ *after a bad signal for all* $t \in (t_1, t_2)$*, where*

$$a_B(t) = \frac{\rho S_g(t) - (1-\rho) S_b(t) - \big(I_g(t) - I_b(t)\big)}{\rho S_b(t) - (1-\rho) S_g(t)} \tag{9}$$

and (ii) $p(t) = \rho$ *for all* $t \in (t_1, t_2)$*. Moreover,* $a_B(t)$ *is continuously decreasing over* $t \in (t_1, t_2)$ *with* $a_B(t_1) < 1$ *and* $a_B(t_2) = 0$*.*

The intuition for why there must be partial herding after time t_1 is that, if consumers were to herd on adoption, then the interim belief $p(t)$ would fall below ρ and they would strictly prefer to be sensitive to signals, a contradiction. On the other hand, if consumers were sensitive to signals, then $p(t)$ would rise above ρ and they would strictly prefer to herd on adoption, another contradiction. Mixing after a bad signal balances the upward and downward pressure on interim beliefs so that $p(t)$ is able to remain constant over time.[14] The time t_2 at which Phase II ends is the first time at which $\rho S_g(t) - (1-\rho) S_b(t) = I_g(t) - I_b(t)$, so that Condition SS is satisfied with equality. After that point, there is overall downward pressure on consumer beliefs *even if* they become sensitive to signals. And indeed, that is what happens next.

Phases III and IV: End of the innovation lifecycle. After time t_2, consumers are sensitive to signals and interim belief $p(t)$ falls until a critical time t_3 at which $p(t_3) = 1 - \rho$ (Proposition 4). Consumers then herd on non-adoption after time t_3, what we refer to as "viral obsolescence" (Proposition 5). We refer to the period from t_2 to t_3 as "Phase III" and the obsolescent period after t_3 as "Phase IV".

The fact that consumers suddenly stop adopting at time t_3 is a consequence of our assumption of binary private signals. In the working-paper version McAdams and Song (2023), we extend the analysis to a richer setting with continuous private signals. In that context, there is never full herding on adoption or full herding on non-adoption, and newly-exposed consumers' likelihood of adopting falls continuously to zero during the last part of the epidemic.

Proposition 4 (Phase III)**.** *Suppose that* $\alpha \in (1-\rho, \rho)$ *and* $L \approx 0$*. There exists* $t_3 \in (t_2, \infty)$ *such that, in any equilibrium trajectory, (i) consumers are sensitive to signals for all*

[14] The equilibrium mixed strategies here can be "purified" by augmenting the model so that consumers' private signals have differing precision. See the working-paper version for details.

$t \in (t_2, t_3)$, *(ii)* $p(t)$ *is strictly decreasing over* $t \in (t_2, t_3)$, *and (iii)* $p(t_3) = 1 - \rho$. *Moreover,* $S_g(t) < S_b(t)$ *and* $I_g(t) > I_b(t)$ *for all* $t \in [0, t_3]$.

Proposition 5 (Phase IV)**.** *Suppose that* $\alpha \in (1-\rho, \rho)$ *and* $L \approx 0$. *In any equilibrium trajectory, consumers herd on non-adoption after time* t_3 *and* $p(t)$ *is strictly decreasing with* $\lim_{t\to\infty} p(t) = 0$.

The proofs of Propositions 4-5 are the most technically challenging in the paper, but the intuition underlying these results is easy to explain. After Phase II, the epidemic is sufficiently mature that the downward pressure on consumer beliefs is so large that $p(t)$ must fall over time no matter what newly-exposed consumers do. Phase III is the period of time while $p(t)$ is falling from ρ to $1-\rho$, causing consumers to be sensitive to signals, while Phase IV is the final period when $p(t)$ is below $1-\rho$.

3 Stopping the Viral Campaign

Here we extend the analysis to allow the producer to decide *how long* to continue the viral campaign. Suppose that, at any time $T \geq 0$, the producer can stop the viral campaign by running a "broadcast advertisement" (or simply "broadcast") that reaches all still-unexposed consumers. $T = \infty$ corresponds to a purely-viral campaign as analyzed in Section 2, while $T = 0$ corresponds to a "traditional ad campaign" in which all consumers are exposed non-socially and must decide independently whether to adopt.

In this section, we characterize the optimal time at which to run the broadcast. To keep the analysis as simple as possible, we assume that the producer must choose the broadcast time $T \in [0, \infty]$ before launch and before knowing whether its innovation will be good or bad; running the broadcast is costless; and the producer's objective is to maximize the expected mass of consumers who adopt the innovation.[15]

Consumers who encounter the innovation socially before the broadcast make the same inference and the same adoption decision as in a purely-viral campaign. The difference is that consumers who would encounter the innovation socially *after T* now see the broadcast. Depending on how broadcast-exposed consumers update their beliefs, this may increase or decrease overall adoption relative to a purely-viral campaign.

[15] For simplicity, we assume that the producer does not care about the timing of adoption. Introducing discounting complicates the analysis but does not generate any additional insight.

Broadcast-updated beliefs. Consumers who see the broadcast at time T update their belief about innovation quality based on the fact that they did not encounter the innovation during the preceding viral campaign. Let $p_{BR}(T)$ denote consumers' updated belief after seeing the broadcast at time T. Conditional on the innovation being good or bad, each consumer will encounter the innovation via broadcast with ex ante probability $S_g(T)$ or $S_b(T)$, respectively. By Bayes' Rule:

$$\frac{p_{BR}(T)}{1-p_{BR}(T)} = \frac{\alpha}{1-\alpha} \times \frac{S_g(T)}{S_b(T)}. \tag{10}$$

Lemma 2 establishes several useful facts about broadcast-updated beliefs.

Lemma 2. *Suppose that $\alpha \in (1-\rho, \rho)$ and $L \approx 0$. (i) $p_{BR}(T) < p(T)$ for all $T > 0$. (ii) $p_{BR}(0+) = \alpha$ and $\frac{p_{BR}(T)}{1-p_{BR}(T)}$ falls exponentially at rate $I_g(T) - I_b(T) > 0$ for all T. Define $\overline{T}$ implicitly by $p_{BR}(\overline{T}) = 1-\rho$. (iii) $\overline{T} \in (t_1, t_3)$. (iv) If $\alpha \in (1/2, \rho)$, then $\overline{T} \in (t_2, t_3)$.*

Discussion of Lemma 2: Since awareness spreads more widely during the viral campaign when the innovation is good, seeing the broadcast is bad news about innovation quality. Moreover, broadcast-exposed consumers' negative inference gets worse as time goes on (Lemma 2(ii)) and is worse than the inference they would make if encountering the innovation socially at the same time (Lemma 2(i)).

The threshold time $\overline{T}$ is the moment at which broadcast-exposed consumers are indifferent whether to adopt with a good private signal. We refer to this moment as "broadcast obsolescence" since any broadcast after time $\overline{T}$ will generate zero adoption. Lemma 2(iii) states that broadcast obsolescence always occurs during Phase II or Phase III, after partial herding has begun but before viral obsolescence. When $\alpha \in (1/2, \rho)$ so that the epidemic begins in a herding phase, Lemma 2(iv) implies further that broadcast obsolescence must occur during Phase III, after partial herding has ended.

Optimal-length viral campaigns. We are now ready to characterize the optimal stopping time for the viral campaign, in terms of the threshold times t_1, t_2, and t_3 derived in the proof of Theorem 1. Note that, by definition, $\overline{T} \gtrless t_2$ iff $p_{BR}(t_2) \gtrless 1-\rho$.

Theorem 2. *Suppose that $\alpha \in (1-\rho, \rho)$ and $L \approx 0$, and let $\mathcal{T}^*$ denote the set of optimal stopping times. (i) If $\overline{T} \geq t_2$, then $\mathcal{T}^* = [t_2, \overline{T}]$. (ii) If $\overline{T} < t_2$, then* either $\mathcal{T}^* = \overline{T}$ or

$\mathcal{T}^* = [t_3, \infty]$. *Moreover,* $\mathcal{T}^* = [t_3, \infty]$ *if and only if* $p_{BR}(t_2) < 1 - \rho$ *and*

$$\alpha \left(\int_{\overline{T}}^{t_2} a_B(t)(1-\rho)|S_g'(t)|\mathrm{d}t - \rho S_g(t_3) \right) + (1-\alpha) \left(\int_{\overline{T}}^{t_2} a_B(t)\rho|S_b'(t)|\mathrm{d}t - (1-\rho)S_b(t_3) \right) \geq 0 \tag{11}$$

where $\left(S_g(t), S_b(t), a_B(t) : t \geq 0\right)$ *were derived in the proof of Theorem 1.*

Theorem 2 lays out three basic possibilities, depending on whether broadcast obsolescence $\overline{T}$ occurs in Phase II or Phase III and on whether inequality (11) holds:

(a) If $\overline{T}$ is in Phase III (always true when $\alpha \in (1/2, \rho)$ by Lemma 3(iv)), then $\overline{T}$ is optimal and stopping prior to Phase III or after $\overline{T}$ is suboptimal.

(b) If $\overline{T}$ is in Phase II and (11) holds, then a purely-viral campaign is optimal and stopping prior to Phase IV is suboptimal.

(c) If $\overline{T}$ is in Phase II and (11) fails, then $\overline{T}$ is the unique optimal stopping time.

The proof of Theorem 2 is provided below, after some discussion.

Intuition for Theorem 2. Focus first on the case when most innovations are good, i.e., $\alpha \in (1/2, \rho)$ and compare three options: $T = 0$, a traditional ad campaign; $T = \overline{T}$; and $T = \infty$, a purely-viral campaign. When $T = 0$, all consumers are sensitive to signals; so, the producer gets adoption from all consumers with a good private signal and none with a bad signal. When $T = \infty$, consumers herd on adoption during Phase I, partially herd on adoption during Phase II, are sensitive to signals during Phase III, and herd on non-adoption during Phase IV. Compared to $T = 0$, the producer is more likely to get consumers who are exposed during Phases I-II, equally likely to get those exposed during Phase III, and less likely to get those exposed during Phase IV.

Whether $T = 0$ or $T = \infty$ is better is unclear, as it depends on how the extra adoptions from consumers exposed during Phases I-II compares to the lost adoptions from those exposed during Phase IV. But waiting until $\overline{T}$ is better than both of these options, as it allows the producer to get all the extra adoptions associated with Phases I-II of a purely-viral campaign while *also* still inducing consumers who would have been exposed during Phase IV to adopt after a good signal. Indeed, $T = \overline{T}$ is always an optimal stopping time in the case when $\alpha \in (1/2, \rho)$.

What about the case when $\alpha \in (1-\rho, 1/2)$? Waiting until $T = \overline{T}$ remains superior to a traditional ad campaign, but now the comparison between $T = \overline{T}$ and a purely-viral campaign is unclear. The reason is that, if broadcast obsolescence occurs during Phase II, ending the viral campaign at $\overline{T}$ forces the producer to forgo some extra adoptions that otherwise would occur due to partial herding during the rest of Phase II. The two terms in (11) capture this new tradeoff between lost adoptions from those who would be exposed during Phase II after $\overline{T}$ versus the gain from those who would be exposed during Phase IV.

Numerical exploration of the case $\alpha \in (1-\rho, 1/2)$. Given the theoretical ambiguity in this case, we conducted an exhaustive numerical exploration to determine when $T = \overline{T}$ is optimal and when $T = \infty$ is optimal, given every possible $\rho \in (1/2, 1)$ and every possible $\alpha \in (1-\rho, 1/2)$. For each such (α, ρ) pair, we computed the equilibrium epidemic trajectory and compared the overall mass of consumers who adopt when $T = \overline{T}$ versus a purely-viral campaign. We found that stopping the campaign at time $\overline{T}$ is strictly better across the entire parameter space, increasing adoption by as much as 72% for some parameter values; see Figure 3. Thus, a purely-viral campaign is never optimal (and setting $T = \overline{T}$ is always optimal) given any model parameters.

Consumer welfare implications. Being exposed virally to an innovation provides an informative "social signal" about its quality. Thus, consumers' ex ante expected payoff is higher when an innovation is marketed virally (for any $T > 0$) than in a traditional ad campaign in which they get no social signal at all. That said, consumers exposed to the broadcast at time $\overline{T}$ get zero expected payoff, since they are indifferent whether to adopt even after a positive private signal. By contrast, in a purely-viral campaign, all those exposed socially after $\overline{T}$ but before viral obsolescence at time t_3 get positive expected payoff. Thus, a purely-viral campaign is better for consumers than an optimal-length campaign.[16]

[16] The seller's choice of when to run the broadcast can be viewed as a limited Bayesian-persuasion problem (Kamenica (2019)). Let t_i be the time that consumer i encounters the innovation in a purely-viral campaign, which serves as i's social signal about quality. Running the broadcast at time T changes the distribution of this signal, revealing only "$t_i \geq T$" to all those with $t_i \geq T$. Viewed in this light, the fact that consumers are worse off under the optimal-length campaign is unsurprising.

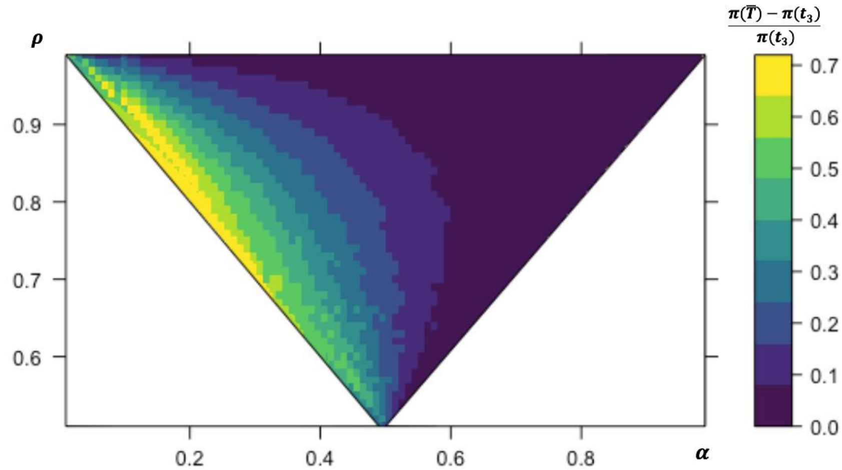


Figure 3: Percentage increase in the producer's expected measure of adopting consumers, denoted $\pi(T)$, when stopping the viral campaign at broadcast obsolescence ($T = \overline{T}$) versus a purely-viral campaign ($T = \infty$). For all combinations of (α, ρ), stopping at $T = \overline{T}$ is more profitable.

Proof of Theorem 2. Should the viral campaign continue until time $\overline{T}$, the producer must decide whether to run the broadcast right at that moment, so that still-unexposed consumers are willing to adopt after a good signal ("go"), or never run the broadcast at all, allowing the campaign to continue until viral obsolescence ("no-go").

Case #1: when $\overline{T} \geq t_2$, always "go". Suppose first that $\overline{T} \geq t_2$. In this case, the producer unambiguously prefers to run the broadcast at time $\overline{T}$ rather than allowing the viral campaign to continue. Why? Consumers who are socialy exposed after time $\overline{T}$ are either sensitive to signals (if exposed in Phase III) or herd on non-adoption (if exposed during Phase IV). By comparison, if the producer runs the broadcast at (or infinitesimally before) time $\overline{T}$, all of these consumers are be sensitive to signals—leading to strictly more adoption, whether the innovation is good or bad.

Case #2: when $\overline{T} < t_2$, "no go" if and only if inequality (11) holds. Suppose next that $\overline{T} < t_2$. Running the broadcast at time $\overline{T}$ still ensures that all remaining consumers will be sensitive to signals, avoiding the downside that consumers exposed in Phase IV never adopt. However, there is also a benefit associated with continuing to run the viral campaign, that consumers exposed in the remainder of Phase II (at times $t \in (\overline{T}, t_2)$) will sometimes adopt after getting a negative private signal as well as after a positive signal.[17] Whether the producer prefers to continue the viral campaign past time $\overline{T}$ depends on the magnitudes of these countervailing effects.

The downside of continuing the viral campaign is that all consumers who get a positive signal and would have been exposed during Phase IV choose to adopt under the time-$\overline{T}$ broadcast but not under the continued viral campaign. These consumers have mass $\rho S_g(t_3)$ when the innovation is good and mass $(1-\rho)S_b(t_3)$ when it is bad. Overall, then, the "viral downside" equals $\alpha\rho S_g(t_3) + (1-\alpha)(1-\rho)S_b(t_3)$.

The upside of continuing the viral campaign is that some consumers who get a negative signal and would have been exposed during the remainder of Phase II choose to adopt under the continued viral campaign but not under the time-$\overline{T}$ broadcast. These consumers have mass $\int_{\overline{T}}^{t_2} a_B(t)(1-\rho)|S_g'(t)|\mathrm{d}t$ when the innovation is good and mass $\int_{\overline{T}}^{t_2} a_B(t)\rho|S_b'(t)|\mathrm{d}t$ when it is bad, where $a_B(t)$ is consumers' equilibrium likelihood of adopting after a bad signal during Phase II. Overall, then, the "viral upside" equals $\alpha \int_{\overline{T}}^{t_2} a_B(t)(1-\rho)|S_g'(t)|\mathrm{d}t + (1-\alpha)\int_{\overline{T}}^{t_2} a_B(t)\rho|S_b'(t)|\mathrm{d}t$, and the upside exceeds the downside if and only if inequality (11) holds.

The analysis thus far has shown: (a) any stopping time $T \in (\overline{T}, t_3)$ is always worse than $T = \overline{T}$ and all stopping times $T \in [t_3, \infty]$ generate identical adoption since no one exposed after t_3 ever adopts; (b) when $\overline{T} \geq t_2$, $\overline{T}$ is optimal and all stopping times $T \in [t_2, \overline{T}]$ generate identical adoption; and (c) when $\overline{T} \geq t_2$, a purely-viral campaign is better than stopping at $\overline{T}$ if and only if inequality (11) holds.

Next, we show that all stopping times prior to $\min\{t_2, \overline{T}\}$ are strictly worse than $\overline{T}$. The reason is simple: stopping at some time $T' < \min\{t_2, \overline{T}\}$ causes consumers who would have otherwise encountered the innovation between $\max\{T', t_1\}$ and $\min\{t_2, \overline{T}\}$ (the portion of Phase II that is after T' and before $\overline{T}$) to adopt less often—they are sensitive to signals rather than partially herding on adoption—without inducing any other consumer to adopt more often. Thus, lengthening the viral campaign from T' until time $\min\{t_2, \overline{T}\}$ unambiguously increases overall adoption.

Putting these pieces together allows us to complete the proof. First, when $\overline{T} \geq t_2$, we have shown that $\overline{T}$ is strictly better than all stopping times before t_2 or after $\overline{T}$. Since all stopping times in $[t_2, \overline{T}]$ generate identical adoption, $\mathcal{T} = [t_2, \overline{T}]$. This completes the

[17]We can ignore the consumers exposed in Phase III, since they are sensitive to signals and hence adopt exactly as they would have under a time-$\overline{T}$ broadcast.

proof of Theorem 2(i). Next, when $\overline{T} < t_2$, we have shown that $\overline{T}$ is strictly better than all stopping times before $\overline{T}$ and strictly better than all those between $\overline{T}$ and t_3. Moreover, stopping at $\overline{T}$ is better than a purely-viral campaign ($T = \infty$) if and only if inequality (11) fails. Since all stopping times after viral obsolescence generate identical adoption, we conclude that $\mathcal{T} = \overline{T}$ when inequality (11) fails and $\mathcal{T} = [t_3, \infty]$ when inequality (11) holds.[18] This completes the proof of Theorem 2(ii). $\square$

4 Concluding remarks

This paper introduces and analyzes an economic-epidemiological model of innovation diffusion and adoption, in which awareness of an innovation (e.g., new product or practice, scientific finding, etc.) spreads by word of mouth from those who have already adopted it. The paper follows Banerjee (1993) in bridging the economic literature on social learning and the epidemiological literature on social transmission, combining ideas and methods from both fields. Because agents choose whether to adopt in our model, we endogenize the *infectivity* of a virally-spread innovation and show how infectivity changes over the course of the adoption epidemic.

In future work, our methodology could be extended in several directions to endogenize other key parameters of the innovation diffusion process, including the transmission rate (if agents choose how actively to meet others) and the informativeness of agents' private signals (if they choose how intensively to examine the innovation). Interesting future work could also seek to relax some of our simplifying assumptions, to build more detailed and realistic models of adoption epidemics. Here we highlight one such extension, relaxing the assumption that adopters remain permanently infectious.

Temporary infectiousness. In practice, consumers who adopt an innovation may only remain infectious for a limited period of time. For example, people may eventually get bored of a new game and stop telling others about it as they stop playing themselves. To model this possibility, suppose that each adopter "recovers" from the transmissive infectious state I to a quiescent state Q at rate $\gamma \geq 0$. The differential equation (2)

[18] When we say "inequality (11) fails," we mean that it holds in the opposite direction. If inequality (11) holds with equality, then stopping at time $\overline{T}$ generates equal expected adoption as a purely-viral campaign and $\mathcal{T} = \overline{T} \cup [t_3, \infty]$.

governing the dynamics of infection changes to

$$I'_\omega(t) = q_\omega(t) I_\omega(t) S_\omega(t) - \gamma I_\omega(t), \tag{2'}$$

with $Q'(t) = \gamma I(t)$ and $\lim_{t\to\infty} Q(t)$ being the mass of consumers who eventually adopt. An important difference in this variation of our model is that the innovation will only reach a fraction of the population, with good innovations reaching more people than bad ones. Thus, even for consumers who cannot observe the time since launch, simply being exposed to the innovation is a positive signal about its quality.

References

ADAMIC, L. A., T. M. LENTO, E. ADAR, AND P. C. NG (2016): "Information evolution in social networks," *Proceedings of the ninth ACM international conference on web search and data mining*, 473–482.

BANERJEE, A. V. (1992): "A Simple Model of Herd Behavior," *The Quarterly Journal of Economics*, 107, 797–817.

——— (1993): "The Economics of Rumours," *The Review of Economic Studies*, 60, 309–327.

BIKHCHANDANI, S., D. HIRSHLEIFER, AND I. WELCH (1992): "A Theory of Fads, Fashion, Custom, and Cultural Change as Informational Cascades," *Journal of Political Economy*, 100, 992–1026.

CAMPBELL, A. (2013): "Word-of-Mouth Communication and Percolation in Social Networks," *American Economic Review*, 103, 2466–2498.

CAMPBELL, A., D. MAYZLIN, AND J. SHIN (2017): "Managing buzz," *RAND Journal of Economics*, 48, 203–229.

DODSON, J. A. AND E. MULLER (1978): "Models of new product diffusion through advertising and word-of-mouth," *Management Science*, 24, 1568–1578.

FARBOODI, M., G. JAROSCH, AND R. SHIMER (2021): "Internal and external effects of social distancing in a pandemic," *Journal of Economic Theory*, 196, 105293.

JACKSON, M. O., S. MALLADI, AND D. MCADAMS (2022): "Learning through the grapevine: the impact of noise and the breadth and depth of social networks," *Proceedings of the National Academy of Sciences*, 119, e2205549119.

KAMENICA, E. (2019): "Bayesian persuasion and information design," *Annual Review of Economics*, 11, 249–272.

KERMACK, W. O. AND A. G. MCKENDRICK (1927): "A contribution to the mathematical theory of epidemics," *Proceedings of the Royal Society London A*, 115, 700–721.

LAXMINARAYAN, R. AND G. M. BROWN (2001): "Economics of antibiotic resistance: a theory of optimal use," *Journal of Environmental Economics and Management*, 42, 183–206.

LEDUC, M. V., M. O. JACKSON, AND R. JOHARI (2017): "Pricing and referrals in diffusion on networks," *Games and Economic Behavior*, 104, 568–594.

MCADAMS, D. (2017): "Resistance diagnosis and the changing epidemiology of antibiotic resistance," *Antimicrobial Therapeutics Reviews*, 1388, 5–17.

MCADAMS, D. AND Y. SONG (2023): "Adoption Epidemics and Viral Marketing," working paper.

MCADAMS, D., Y. SONG, AND D. ZOU (2023): "Equilibrium social activity during an epidemic," *Journal of Economic Theory*, 207, 105591.

NEWMAN, M. E. J. (2002): "Spread of epidemic disease on networks," *Physical Review E*, 66, 016128.

SADLER, E. (2020): "Diffusion games," *American Economic Review*, 110, 225–270.

VAN DEN BULTE, C. AND Y. V. JOSHI (2007): "New product diffusion with influentials and imitators," *Marketing Science*, 26, 400–421.

WATTS, D. J. AND P. S. DODDS (2007): "Influentials, networks, and public opinion formation," *Journal of Consumer Research*, 34, 441–458.

Appendix: Omitted Proofs

The following lemma is useful in several of the proofs that follow.

Lemma 3. *Fix any $\alpha \in (1-\rho, \rho)$. Suppose that $p(t) > \alpha$ for all $t \in (0, \hat{t}]$ for some $\hat{t}$ along some equilibrium epidemic trajectory. Then $I_g(t) > I_b(t)$, $I_g'(t) > I_b'(t)$, $S_g(t) < S_b(t)$, and $S_g'(t) < S_b'(t)$ for all $t \in (0, \hat{t}]$.*

Proof. By equation (6), $\frac{p(t)}{1-p(t)} = \frac{\alpha}{1-\alpha} \times \frac{I_g(t)S_g(t)}{I_b(t)S_b(t)} = \frac{\alpha}{1-\alpha} \times \frac{|S_g'(t)|}{|S_b'(t)|}$. Since $p(t) > \alpha$ by assumption, $|S_g'(t)| > |S_b'(t)|$ and hence $S_g'(t) < S_b'(t) < 0$ for all $t \in (0, \hat{t}]$. By equations (1-2), $I_\omega'(t) = -q_\omega(t)S_\omega'(t)$. Since $q_g(t) \geq q_b(t)$ at all times, we conclude that $I_g'(t) > I_b'(t) > 0$ for all $t \in (0, \hat{t}]$. Finally, because launch-exposed consumers are sensitive to signals (due to $\alpha \in (1-\rho, \rho)$), we have $S_g(0) = S_b(0) = 1-L$ and $I_g(0) > I_b(0)$. Since $S_g'(t) < S_b'(t)$ and $I_g'(t) > I_b'(t)$, we conclude as desired that $S_g(t) < S_b(t)$ and $I_g(t) > I_b(t)$ for all $t \in (0, \hat{t}]$. □

Proof of Proposition 1. Because $\alpha \in (1/2, \rho)$, $p(0+) > \rho$ and socially-exposed consumers herd on adoption until the first time t_1 at which $p(t_1) = \rho$. Prior to t_1, $X(t) = -\left(I_g(t) - I_b(t)\right) - \left(S_b(t) - S_g(t)\right)$ and $p(t) > \rho > \alpha$. By Lemma 3, $I_g(t) - I_b(t)$ and $S_b(t) - S_g(t)$ are each strictly increasing from time 0 (when $X(0) = L(2\rho - 1) > 0$) until t_1. The likelihood ratio $\frac{p(t)}{1-p(t)}$ therefore falls exponentially at an increasing rate, implying that $p(t)$ reaches the threshold ρ in finite time. □

Proof of Proposition 2. Because $\alpha \in (1-\rho, 1/2)$, $p(0+) \in (1/2, \rho)$ and socially-exposed consumers are sensitive to signals until the first time t_1 at which either $p(t_1) = \rho$ or $p(t_1) = 1-\rho$. Since launch-exposed consumers are also sensitive to signals, $I_g(t) = \rho(1 - S_g(t))$ and $I_b(t) = \rho(1 - S_b(t))$. Prior to time t_1, condition (SS) in the main text can now be simplified to

$$2\left(I_g(t) - I_b(t)\right) < 2\rho - 1 \tag{SS'}$$

If the launch size $L > \frac{1}{2}$, then $I_g(0) - I_b(0) = L(2\rho - 1)$ and condition (SS') would fail and $p(t)$ would fall after launch. However, because of our small-launch assumption, $I_g(t) - I_b(t) \approx 0$ and $p(t)$ must rise after launch. In particular, $\frac{p(t)}{1-p(t)}$ rises exponentially at rate $X(t) = 2\rho - 1 - \left(I_g(t) - I_b(t)\right)$, which equals $(2\rho - 1)(1 - L) \approx 2\rho - 1$ at $t = 0$.

Let $\hat{t}$ be the time at which $p(t)$ would reach the threshold ρ in a hypothetical situation in which $X(t) = \hat{X} \equiv \rho - \frac{1}{2} > 0$ at all times. We have shown that $X(0) \approx 2\hat{X}$ and $X(t) > \hat{X}$ so long as $I_g(t) - I_b(t) < \hat{X}$. But $I_g(t) - I_b(t) < I_g(t) < \rho L e^{\rho t}$.[19] Thus, for all L small enough that $\rho L e^{\rho \hat{t}} < \hat{X}$, $X(t)$ remains strictly above $\hat{X}$ and $p(t)$ continues to increase until reaching the threshold ρ in finite time, i.e., $p(t_1) = \rho$, as desired. □

Proof of Proposition 3. We begin by showing that $p(t) = \rho$ for some period of time after t_1. If $p(t)$ were to rise above ρ after t_1, then consumers would herd on adoption and $p(t)$ must fall by Lemma 1(i), a contradiction. On the other hand, if $p(t)$ were to fall below ρ, consumers would then be sensitive to signals. As discussed in the proof of Proposition 2, our assumption of a small launch ($L \approx 0$) guarantees that only a small mass of consumers are exposed to the innovation during Phase I;[20] in particular, $S_g(t_1), S_b(t_1) \in (1-\epsilon, 1)$ and $I_g(t_1), I_b(t_1) \in (0, \epsilon)$ for some small ϵ. Consequently, Condition (SS) holds and $p(t)$ must rise after time t_1 by Lemma 1(ii), a contradiction.

By equation (6), interim belief $p(t) = \rho$ requires that $\frac{\rho}{1-\rho} = \frac{\alpha I_g(t) S_g(t)}{(1-\alpha) I_b(t) S_b(t)}$ or, equivalently, $\frac{I_g(t)S_g(t)}{I_b(t)S_b(t)} = \frac{(1-\alpha)\rho}{\alpha(1-\rho)}$. In order for this ratio not to change over time, the ratio of derivatives $\frac{(I_g(t)S_g(t))'}{(I_b(t)S_b(t))'}$ must also equal $\frac{(1-\alpha)\rho}{\alpha(1-\rho)}$. Taking derivatives, using equations (1-2), and re-arranging yields

$$\begin{aligned}\frac{(1-\alpha)\rho}{\alpha(1-\rho)} &= \frac{I_g'(t)S_g(t) + I_g(t)S_g'(t)}{I_b'(t)S_b(t) + I_b(t)S_b'(t)} = \frac{I_g(t)S_g^2(t)q_g(t) - I_g^2(t)S_g(t)}{I_b(t)S_b^2(t)q_b(t) - I_b^2(t)S_b(t)} \\ &= \frac{I_g(t)S_g(t)(S_g(t)q_g(t) - I_g(t))}{I_b(t)S_b(t)(S_b(t)q_b(t) - I_b(t))}\end{aligned}$$

Since $\frac{I_g(t)S_g(t)}{I_b(t)S_b(t)} = \frac{(1-\alpha)\rho}{\alpha(1-\rho)}$, this condition holds iff

$$S_g(t)q_g(t) - I_g(t) = S_b(t)q_b(t) - I_b(t). \tag{12}$$

Let $a_B(t)$ denote the likelihood that consumers exposed at time t adopt after a bad

[19] $\rho L e^{\rho t}$ is the mass of consumers who would be infected if (i) each infected agent encounters a susceptible agent at rate one and (ii) newly-exposed consumers are sensitive to signals. In fact, each infected agent encounters *some* agent at rate one, but fraction $I_g(t)$ of these encounters are with someone already infected.

[20] If L is not sufficiently small, condition (SS) may not hold at time t_1. In that case, Phase II has zero length and the epidemic progresses directly to Phase III, with consumers sensitive to signals and $p(t)$ falling immediately after t_1.

signal, resulting in overall adoption likelihoods $q_g(t) = \rho + (1-\rho)a_B(t)$ and $q_b(t) = 1-\rho+\rho a_B(t)$ that good and bad innovations, respectively. Equation (12) now becomes

$$\left(\rho S_g(t) - (1-\rho)S_b(t)\right) - \left(I_g(t) - I_b(t)\right) + a_B(t)\left((1-\rho)S_g(t) - \rho S_b(t)\right) = 0 \quad (13)$$

or, equivalently,

$$a_B(t) = \frac{\rho S_g(t) - (1-\rho)S_b(t) - \left(I_g(t) - I_b(t)\right)}{\rho S_b(t) - (1-\rho)S_g(t)}. \quad (14)$$

Equation (14) uniquely determines $a_B(t_1+)$. Note that so long as $p(t) = \rho$, Lemma 3 implies that $I_g(t_1) > I_b(t_1)$ and $S_b(t_1) > S_g(t_1)$; we conclude by equation (14), $a_B(t) < 1$ so long as $p(t)$ remains at ρ. Moreover, because Condition SS holds at time t_1 (discussed earlier), the numerator in (14) is positive; so, $a_B(t_1+) > 0$.

Equations (1,2,14) now uniquely determine the path of $\left(a_B(t), S_g(t), S_b(t), I_g(t), I_b(t)\right)$, starting at time t_1 and so long as $a_B(t) \in [0,1]$. Let t_2 be the first time after t_1 at which $a_B(t) = 0$, or $t_2 = \infty$ if $a_B(t)$ remains forever between zero and one. To complete the proof, we need to show that $a_B(t)$ is strictly decreasing after t_1 and reaches zero in finite time.

Let t_2 denote the first time after t_1 at which $a_B(t_2) = 0$, or $t_2 = \infty$ if consumers partially herd forever. Since $p(t) > \alpha$ throughout Phase I and $\rho > \alpha$, Lemma 3 implies that $S_b(t) > S_g(t)$, ensuring that the denominator of (14) remains positive. Moreover, t_2 is the first time as which the numerator of (14) equals zero, i.e., when Condition SS holds with equality.

Next, note that

$$a_B'(t) = \frac{\begin{array}{c}(\rho S_g'(t) - (1-\rho)S_b'(t) - (I_g'(t) - I_b'(t))(\rho S_b(t) - (1-\rho)S_g(t)) \\ -(\rho S_g(t) - (1-\rho)S_b(t) - (I_g(t) - I_b(t)))(\rho S_b'(t) - (1-\rho)S_g'(t))\end{array}}{(\rho S_b(t) - (1-\rho)S_g(t))^2}.$$

Rearranging and simplifying the numerator, we have

$$\begin{aligned}\text{numerator } &= (\rho^2 - (1-\rho)^2)(S_g'(t)S_b(t) - S_b'(t)S_g(t)) \\ &\quad - (I_g'(t) - I_b'(t))(\rho S_b(t) - (1-\rho)S_g(t)) \\ &\quad + (I_g(t) - I_b(t))(\rho S_b'(t) - (1-\rho)S_g'(t)).\end{aligned}$$

By (1-2), the second term above can be re-written as

$$\begin{aligned}&- (I_g'(t) - I_b'(t))(\rho S_b(t) - (1-\rho)S_g(t)) \\ =& - (I_g(t)S_g(t)(\rho + (1-\rho)a_B(t)) - I_b(t)S_b(t)(1-\rho+\rho a_B(t)))(\rho S_b(t) - (1-\rho)S_g(t)) \\ =& - I_b(t)(I_g(t) - I_b(t))(\rho S_b(t) - (1-\rho)S_g(t)) \\ &- (I_g(t) - I_b(t))S_g(t)(\rho + (1-\rho)a_B(t))(\rho S_b(t) - (1-\rho)S_g(t)) \end{aligned} \tag{15}$$

Similarly, the third term above can be re-written as

$$\begin{aligned}&(I_g(t) - I_b(t))(\rho S_b'(t) - (1-\rho)S_g'(t)) \\ =& - (I_g(t) - I_b(t))(\rho I_b(t)S_b(t) - (1-\rho)I_g(t)S_g(t)) \\ =& - I_b(t)(I_g(t) - I_b(t))(\rho S_b(t) - (1-\rho)S_g(t)) \\ &+ (I_g(t) - I_b(t))S_g(t)(1-\rho)(I_g(t) - I_b(t)) \end{aligned} \tag{16}$$

To establish that the entire numerator is negative, we will show that the first term is negative and that the sum of the second term (15) and third term (16) is negative. To that end, recall that $I_g(t) > I_b(t)$, $I_g'(t) > I_b'(t)$, $S_g(t) < S_b(t)$, and $S_g'(t) < S_b'(t)$ at all times $t < t_2$ (Lemma 3). The fact that the first term is negative now follows immediately from (1-2), since $S_g'(t)S_b(t) - S_b'(t)S_g(t) = -S_g(t)S_b(t)(I_g(t) - I_b(t)) < 0$. Moreover, $\rho S_b(t) > (1-\rho)S_g(t)$ because $S_b(t) > S_g(t)$ and $\rho > 1/2$; so, the first part of (15) and the first part of (16) are negative. To show that the sum of (15) and (16) is negative, it therefore suffices to show that $(\rho + (1-\rho)a_B(t))(\rho S_b(t) - (1-\rho)S_g(t)) > (1-\rho)(I_g(t) - I_b(t))$. But this follows immediately from the fact that $\rho S_b(t) - (1-\rho)S_g(t) > I_g(t) - I_b(t)$ (since Condition SS remains satisfied) and $\rho + (1-\rho)a_B(t) > 1-\rho$ (since $\rho > 1/2$ and $a_B(t) \geq 0$).

Overall, we conclude that $a_B(t) > 0$ but that $a_B'(t) < 0$ so long as the numerator of equation (14) continues to be positive, i.e., so long as Condition SS continues to be satisfied. Moreover, there is a finite time t_2 at which partial herding ceases. To see why, suppose for the sake of contradiction that consumers were to partially herd forever. Because all consumers are eventually exposed to the innovation, $\lim_{t\to\infty} S_g(t) = \lim_{t\to\infty} S_b(t) = 0$. On the other hand, because $I_g'(t) > I_b'(t)$ so long as $a_B(t) > 0$, $\lim_{t\to\infty}(I_g(t) - I_b(t)) > I_g(t_1) - I_g(t_1) > 0$. All together, then, the numerator of (14) must eventually become negative, a contradiction. □

Proofs of Propositions 4-5. We prove Propositions 4-5 together, dividing the proof into four main steps.

Step 1: After time t_2, $\frac{p(t)}{1-p(t)}$ declines exponentially at an increasing rate until some time $\tilde{t}$ at which $p(\tilde{t}) = \max\{1-\rho, \underline{\alpha}\}$, where $\underline{\alpha} \equiv \frac{(1-\rho)\alpha}{(1-\rho)\alpha+\rho(1-\alpha)} \in \left(\frac{(1-\rho)^2}{(1-\rho)^2+\rho^2}, \frac{1}{2}\right)$.

By Lemma 1, $\frac{p(t)}{1-p(t)}$ declines exponentially at rate $X(t)$. So, it suffices to show that $X(t) < 0$ and $X'(t) < 0$ at all times after t_2 until a time $\tilde{t}$ is reached at which $p(\tilde{t}) = \max\{1-\rho, \underline{\alpha}\}$. By the proof of Proposition 3: $p(t_2) = \rho$; consumers are sensitive to signal at time t_2 (because $a_B(t_2) = 0$); and $X(t_2) = \left(\rho S_g(t_2) - I_g(t_2)\right) - ((1-\rho)S_b(t_2) - I_b(t_2)) = 0$. It suffices to show that $X'(t) < 0$ at all times $t \in [t_2, \tilde{t})$, since then it must also be that $X(t) < 0$ at all times $t \in (t_2, \tilde{t})$.

According to the proof of Lemma 1(ii), $X'(t) = -2\left(\rho S_g(t) I_g(t) - (1-\rho) S_b(t) I_b(t)\right)$ while consumers are sensitive to signals. Thus, $X'(t) < 0$ so long as $\frac{S_g(t_2)I_g(t_2)}{S_b(t_2)I_b(t_2)} > \frac{1-\rho}{\rho}$. By equation (6), $\frac{p(t)}{1-p(t)} = \frac{\alpha S_g(t_2)I_g(t_2)}{(1-\alpha)S_b(t_2)I_b(t_2)}$; so, $\frac{S_g(t_2)I_g(t_2)}{S_b(t_2)I_b(t_2)} > \frac{1-\rho}{\rho}$ if and only if $p(t) > \underline{\alpha}$ or, equivalently, $\frac{p(t)}{1-p(t)} > \frac{\alpha(1-\rho)}{(1-\alpha)\rho} = \frac{\underline{\alpha}}{1-\underline{\alpha}}$. In other words:

$$\text{when consumers are sensitive to signal, } X'(t) \gtrless 0 \text{ iff } p(t) \gtrless \underline{\alpha} \tag{17}$$

At time t_2, consumers are sensitive to signal and $p(t_2) = \rho > \underline{\alpha}$; so, $X'(t_2) < 0$. Moreover, $X'(t) < 0$ at times $t \in (t_2, \tilde{t})$ since (i) consumers remain sensitive to signal (because $p(t) \in (1-\rho, \rho)$) and (ii) $p(t) > \underline{\alpha}$. We conclude that $\frac{p(t)}{1-p(t)}$ decreases exponentially at an increasing rate from time t_2 until time $\tilde{t}$.

What about after time $\tilde{t}$? There are two relevant cases. First, suppose that $\alpha \in (1-\rho, 1/2]$, so that $\underline{\alpha} \leq 1-\rho$. In this case, $p(\tilde{t}) = 1-\rho$ and Phase III ends at time $\tilde{t}$, i.e., $t_3 = \tilde{t}$. Second, suppose that $\alpha \in (1/2, \rho)$. In this more challenging case, $\underline{\alpha} \in$

$(1-\rho, 1/2)$ and the argument so far shows that $\frac{p(t)}{1-p(t)}$ declines at an increasing rate until time $\tilde{t}$, when consumers' interim belief hits $\underline{\alpha}$. However, we still need to show that consumers' interim belief *continues* falling long enough after time $\tilde{t}$ to reach $1-\rho$.

Step 2: In the case when $\alpha \in (1/2, \rho)$, $\frac{p(t)}{1-p(t)}$ declines exponentially at a decreasing rate from time $\tilde{t}$ until time t_3 at which $p(t_3) = 1-\rho$.

The argument in Step 1 established that $p(\tilde{t}) = \underline{\alpha} \in (1-\rho, 1/2)$ and $X(\tilde{t}) < 0$; thus, consumers' interim belief continues to fall below $\underline{\alpha}$ right after time $\tilde{t}$. By condition (17), we conclude that $X'(t) > 0$ right after $\tilde{t}$ and at all times $t > \tilde{t}$ so long as consumers' interim belief remains between $1-\rho$ and $\underline{\alpha}$.

This leaves three possibilities for what happens after time $\tilde{t}$: (i) $p(t)$ decreases until a time t_3 at which point $p(t_3) = 1-\rho$ and Phase III ends; (ii) $p(t)$ decreases forever but never reaches $1-\rho$; or (iii) $p(t)$ stops decreasing (and starts increasing) at some time $\hat{t}$ before reaching $1-\rho$.

We will prove that possibility (i) always occurs, by ruling out (ii) and (iii).

As shorthand, define $X(\infty) = \lim_{t\to\infty} X(t)$, $I_g(\infty) = \lim_{t\to\infty} I_g(t)$, and so on.

"Possibility (ii)" cannot occur.

Suppose for the sake of contradiction that consumers' interim belief continues falling forever after time t_2 but never reaches $1-\rho$. This is only possible if $X(\infty) = 0$, which in turn requires that $I_g(\infty) - \rho S_g(\infty) = I_b(\infty) - (1-\rho)S_b(\infty)$. Since all consumers eventually encounter the innovation, $S_g(\infty) = S_b(\infty) = 0$. Thus, it must be that $I_g(\infty) = I_b(\infty)$. We will reach a contradiction by showing that $I_g(\infty) > I_b(\infty)$.

Recall that we are focusing here on the case in which $\alpha \in (1/2, \rho)$. We have shown: consumers are sensitive to signals at launch ($t = 0$), adopting good innovations with probability ρ and bad ones with probability $1-\rho$; consumers herd on adoption in Phase I ($t \in (0, t_1)$), adopting all innovations with probability one; and consumers partially herd on adoption in Phase II ($t \in (t_1, t_2)$), adopting good innovations with probability $\rho + a_B(t)(1-\rho)$ and bad ones with probability $1-\rho + a_B(t)\rho$. Moreover, given the presumption that possibility (ii) is occurring, consumers are again sensitive to signals at all times $t > t_2$. Overall, the mass of consumers who adopt a good innovation therefore takes the form:

$$\begin{aligned} I_g(\infty) &= \rho L + \int_0^{t_1} |S_g'(t)|dt + \int_{t_1}^{t_2} (\rho + (1-\rho)a_B(t))|S_g'(t)|dt + \int_{t_2}^{\infty} \rho |S_g'(t)|dt \\ &= \rho + \int_0^{t_1} (1-\rho)|S_g'(t)|dt + \int_{t_1}^{t_2} (1-\rho)a_B(t)|S_g'(t)|dt \end{aligned} \tag{18}$$

where $|S_g'(t)|$ is the flow of consumers being exposed at time t and $L + \int_0^\infty |S_g'(t)|dt = 1$ because the consumer population has unit mass. Similarly, the overall share of consumers who adopt a bad innovation takes the form:

$$\begin{aligned} I_b(\infty) &= (1-\rho)L + \int_0^{t_1} |S_b'(t)|dt + \int_{t_1}^{t_2} (1-\rho+\rho a_B(t))|S_b'(t)|dt + \int_{t_2}^{\infty} (1-\rho)|S_b'(t)|dt \\ &= (1-\rho) + \int_0^{t_1} \rho |S_b'(t)|dt + \int_{t_1}^{t_2} \rho a_B(t)|S_b'(t)|dt \end{aligned} \tag{19}$$

Since consumers' interim belief exceeds ρ throughout Phase I and equals ρ throughout Phase II, $|S_g'(t)| > |S_b'(t)|$ for all $t \in (0, t_2)$ by Lemma 3. Thus,

$$I_b(\infty) < (1-\rho) + \int_0^{t_1} \rho |S_g'(t)|dt + \int_{t_1}^{t_2} \rho a_B(t)|S_g'(t)|dt \tag{20}$$

(18, 20) together imply

$$I_g(\infty) - I_b(\infty) > (2\rho - 1)\left(1 - \int_0^{t_1} |S_g'(t)|dt - \int_{t_1}^{t_2} a_B(t)|S_g'(t)|dt\right). \tag{21}$$

Finally, note that $\int_0^{t_1} |S_g'(t)|dt = (1-L) - S(t_1)$ and, since $a_B(t) < 1$ for all $t \in (t_1, t_2)$, $\int_{t_1}^{t_2} a_B(t)|S_g'(t)|dt < S(t_1) - S(t_2)$. We conclude that $I_g(\infty) - I_b(\infty) > (2\rho-1)(L + S(t_2)) > 0$; so, $I_g(\infty) > I_b(\infty)$, completing the desired contradiction.

"Possibility (iii)" cannot occur.

Suppose for the sake of contradiction that there exists $t' > t_2$ such that $X(t) < 0$ for all $t \in (t_2, t')$, $X(t') = 0$, and $p(t') > 1-\rho$. For future reference, note that $X(t') = 0$ requires that $\rho S_g(t') - I_g(t') = (1-\rho)S_b(t') - I_b(t')$. Also recall that, since $X(t_1) = 0$ and $p(t_1) = \rho > \underline{\alpha}$, condition (17) implies that $X'(t_1) < 0$ and that $X(t)$ grows more negative until time $\tilde{t}$ at which $p(\tilde{t}) = \underline{\alpha}$. Thus, it must be that $t' > \tilde{t}$ and that $p(t') \in (1-\rho, \underline{\alpha})$ or equivalently, given equation (6), $\frac{(1-\rho)(1-\alpha)}{\rho\alpha} < \frac{I_g(t')S_g(t')}{I_b(t')S_b(t')} < \frac{1-\rho}{\rho}$.

Several equations that follow are quite complex, so we introduce the following shorthand: $a = S_g(t_2)$; $b = S_b(t_2)$; $c = \rho S_g(t_2) - I_g(t_2) = (1-\rho)S_b(t_2) - I_b(t_2)$; and $d = -(\rho S_g(t') - I_g(t')) = -((1-\rho)S_b(t') - I_b(t'))$.

We know that

$$\begin{aligned} c + d &= (\rho S_g(t_2) - I_g(t_2)) - (\rho S_g(t') - I_g(t')) \\ &= \int_{t_2}^{t'} 2\rho I_g(t) S_g(t) \mathrm{d}t = 2(I_g(t') - I_g(t_2)) = -2\rho(S_g(t') - S_g(t_2)) \\ &= \int_{t_2}^{t'} 2(1-\rho) I_b(t) S_b(t) \mathrm{d}t = 2(I_b(t') - I_b(t_2)) = -2(1-\rho)(S_b(t') - S_b(t_2)), \end{aligned} \tag{22}$$

which implies that

$$\begin{aligned} I_g(t') - I_g(t_2) = I_b(t') - I_b(t_2) &= \frac{c+d}{2} \\ S_g(t') - S_g(t_2) &= -\frac{c+d}{2\rho} \\ S_b(t') - S_b(t_2) &= -\frac{c+d}{2(1-\rho)}. \end{aligned}$$

Therefore,

$$\begin{aligned} \frac{I_g(t')S_g(t')}{I_b(t')S_b(t')} &= \frac{(I_g(t_2) + I_g(t') - I_g(t_2))(S_g(t_2) + S_g(t') - S_g(t_2))}{(I_b(t_2) + I_b(t') - I_b(t_2))(S_b(t_2) + S_b(t') - S_b(t_2))} \\ &= \frac{(a - \frac{c+d}{2\rho})((a\rho - c) + \frac{c+d}{2})}{(b - \frac{c+d}{2(1-\rho)})((b(1-\rho) - c) + \frac{c+d}{2})} \\ &= \frac{a(a\rho - c) + \frac{c^2-d^2}{4\rho}}{b(b(1-\rho) - c) + \frac{c^2-d^2}{4(1-\rho)}} \end{aligned}$$

We already know that $\frac{a(a\rho-c)}{b(b(1-\rho)-c)} = \frac{(1-\alpha)\rho}{\alpha(1-\rho)} > 1$. Hence, no matter whether $c^2 - d^2 \geq 0$ or $c^2 - d^2 < 0$, $\frac{I_g(t')S_g(t')}{I_b(t')S_b(t')} > \frac{1-\rho}{\rho}$, a contradiction.

Step 3: At all times $t \leq t_3$, $S_g(t) < S_b(t)$ and $I_g(t) > I_b(t)$.

Let $LS(t) \equiv S_b(t) - S_g(t)$ denote the "exposure gap," the extra share of consumers who have been exposed to good innovations by time t, and let $LI(t) \equiv I_g(t) - I_b(t)$ denote the "adoption gap," the extra share who have adopted. At launch, $S_g(0) =$

$S_b(0) = L$, $I_g(t) = \rho L$, and $I_b(t) = (1-\rho)L$; so, $LS(0) = 0$ and $LI(0) = (2\rho - 1)L > 0$. Here we will show that $LS(t) > 0$ and $LI(t) > 0$ at all times $t \in (0, t_3)$.

$LS(t) > 0$ and $LI(t) > 0$ for all $t \leq t_2$.

By Steps 1-2, consumers' interim belief $p(t)$ declines throughout Phase III, from ρ at time t_2 to $1-\rho$ at time t_3; so, there is a unique time $\hat{t} \in (t_2, t_3)$ at which $p(\hat{t}) = \alpha$. Note that $p(t)$ exceeds consumers' ex ante belief α at all times $t \in (0, t_2]$ by Propositions 1-3 and that $p(t) > \alpha$ for all $t \in (t_2, \hat{t})$ by definition of $\hat{t}$. Lemma 3 therefore implies that $LS'(t) = S_b'(t) - S_g'(t) > 0$ and $LI'(t) = I_g'(t) - I_b'(t) > 0$ for all $t \in (0, \hat{t})$. Since $LS(0) = 0$ and $LI(0) > 0$, we conclude that $LS(t) > 0$ and $LI(t) > 0$ for all $t \in (0, \hat{t})$, and thus for all $t \leq t_2$.

$LS(t) > 0$ and $LI(t) > 0$ for all $t \in [t_2, t_3]$.

We begin by showing that the "adoption gap" $LI(t)$ exceeds $LI(t_2)$ during all of Phase III. Fix any $t' \in (t_2, t_3)$. Recall that $X(t_2) = 0$ (shown in the proof of Proposition 3), $X(t') < 0$ (proven in Step Two), and $X(t) = (\rho S_g(t) - I_g(t)) - ((1-\rho)S_b(t) - I_b(t))$ for all $t \in [t_2, t_3)$ (by Lemma 1, because consumers are sensitive to signals). Thus,

$$(\rho S_g(t') - I_g(t')) - ((1-\rho)S_b(t') - I_b(t')) < (\rho S_g(t_2) - I_g(t_2)) - ((1-\rho)S_b(t_2) - I_b(t_2)).$$

Rearranging and reformulating terms as in equation (22) yields

$$\int_{t_2}^{t'} -2\rho I_g(t) S_g(t) \mathrm{d}t < \int_{t_2}^{t'} -2(1-\rho) I_b(t) S_b(t) \mathrm{d}t. \tag{23}$$

Since $I_g'(t) = \rho I_g(t) S_g(t)$ and $I_b'(t) = (1-\rho) I_b(t) S_b(t)$, inequality (23) implies that $I_g(t') - I_g(t_2) > I_b(t') - I_b(t_2)$, which in turn implies that $LI(t') > LI(t_2)$. Since $LI(t_2) > 0$, we conclude that $LI(t') > 0$ for all $t' \in (t_2, t_3]$, as desired.

The "exposure gap" $LS(t) = S_b(t) - S_g(t)$ is non-monotone during Phase III, but we can show that $LS(t) > 0$ for all $t \in (t_2, t_3]$. Recall that $p(t) > \alpha$ for all $t \in [t_2, \hat{t})$ and $p(t) < \alpha$ for all $t \in (\hat{t}, t_3]$, where $\hat{t} \in (t_2, t_3)$ is the unique time during Phase III at which consumers' interim belief $p(t)$ equals their ex ante belief α. Also recall that, by equation (6),

$$p(t) \gtrless \alpha \text{ iff } S_g(t) I_g(t) \gtrless S_b(t) I_b(t) \text{ iff } -S_g'(t) \gtrless -S_b'(t). \tag{24}$$

Prior to time $\hat{t}$, $p(t) > \alpha$ and condition (24) implies that $LS'(t) > 0$, i.e., the exposure gap is increasing and hence obviously still positive. After time $\hat{t}$, $p(t) < \alpha$ and condition (24) implies that $S_g(t)I_g(t) < S_b(t)I_b(t)$; since $I_g(t) > I_b(t)$, this is only possible if $S_b(t) > S_g(t)$. Thus, even though the exposure gap tightens after time $\hat{t}$, it must remain positive throughout Phase III.

Step 4: After time t_3, $\frac{p(t)}{1-p(t)}$ declines exponentially at a constant rate, $LI(t)$ is constant, and $LS(t)$ is decreasing but positive.

Consumers' interim belief at time t_3 equals $1-\rho$, making them indifferent whether to adopt after a good private signal. Let $a_G(t_3) \in [0,1]$ be the probability with which consumers exposed to the innovation at time t_3 adopt after a good signal. Note that

$$X(t_3) = a_G(t_3)\left(\rho S_g(t_3) - (1-\rho)S_b(t_3)\right) - \left(I_g(t_3) - I_b(t_3)\right).$$

To establish that consumers' interim belief continues declining below $1-\rho$, it suffices to show that $X(t_3) < 0$. However, this follows immediately from the facts that $X(t_3-) < 0$ (proven in Step 2), $I_g(t_3) > I_b(t_3)$ (proven in Step 3), and $a_B(t_3) \in [0,1]$.

Once consumers' interim belief falls below $1-\rho$, immediately after time t_3, consumers herd on non-adoption; so, $X(t_3+) = -\left(I_g(t_3) - I_b(t_3)\right) < 0$ by Step 3 and beliefs continue to fall. Consumers therefore still herd on non-adoption, meaning that $I_g(t) = I_g(t_3)$, $I_b(t) = I_b(t_3)$, and hence $X(t) = X(t_3)$ and $LI(t) = LI(t_3)$ for all $t > t_3$. We conclude that all adoption ceases after time t_3 and that $\frac{p(t)}{1-p(t)}$ forevermore declines exponentially at the constant rate $|X(t_3)|$. In particular, $\lim_{t\to\infty} p(t) = 0$.

Finally, as discussed in Step 3, the fact that $p(t) < \alpha$ implies that $S_b(t)I_b(t) > S_g(t)I_g(t)$; hence, the exposure gap must shrink during obsolescence, i.e., $LS'(t) < 0$ for all $t > t_3$. At the same time, because $I_g(t) > I_b(t)$, the condition $S_b(t)I_b(t) > S_g(t)I_g(t)$ is only possible if $S_b(t) > S_g(t)$; thus, $LS(t) > 0$ for all $t > t_3$. □

Proof of Lemma 2. (i) In the proof of Theorem 1, we showed that $I_g(t) > I_b(t)$ at all times $t \geq 0$ during a purely-viral campaign. Comparing equations (6,10), this implies $p_{BR}(T) < p(T)$ for all $T \geq 0$.

(ii) $p_{BR}(0+) = \alpha$ by (10) and $S_g(0+) = S_b(0+) = 1-L$. $\frac{\mathrm{d}\log\left(S_g(T)/S_b(T)\right)}{\mathrm{d}T} = \frac{S_g'(T)}{S_g(T)} - \frac{S_b'(T)}{S_b(T)} = -(I_g(T) - I_b(T))$ by(1); thus $\frac{p_{BR}(T)}{1-p_{BR}(T)}$ falls exponentially at rate $I_g(T) - I_b(T)$.

(iii) By part (i) and Proposition 5, $\lim_{T\to\infty} p_{BR}(T) \leq \lim_{T\to\infty} p(T) = 0$. By part (ii), $p_{BR}(0+) = \alpha > 1-\rho$ and $p_{BR}(T)$ is strictly decreasing and continuous. $\overline{T}$ is therefore well-defined as the unique time at which $p_{BR}(\overline{T}) = 1-\rho$. Moreover, $\overline{T} > t_1$ because $p_{BR}(t_1) \approx \alpha > 1-\rho$ and $\overline{T} < t_3$ because $p_{BR}(t_3) < p(t_3) = 1-\rho$.

(iv) So far, we have shown that $\overline{T}$ must occur during Phase II or Phase III. To complete the proof, we need to show that $\overline{T}$ occurs during Phase III when $\alpha \in (\frac{1}{2}, \rho)$. In Section 2.1, we showed that condition SS holds throughout Phase II (corresponding to the intuition that there is "upward pressure" on beliefs when consumers are sensitive to signals) but fails to hold throughout Phase III. When $\alpha \in (\frac{1}{2}, \rho)$, the fact that $\frac{\alpha S_g(\overline{T})}{(1-\alpha)S_b(\overline{T})} = \frac{1-\rho}{\rho}$ (by definition of $\overline{T}$) implies $\frac{S_g(\overline{T})}{S_b(\overline{T})} < \frac{1-\rho}{\rho}$ (because $\alpha > 1/2$) and hence $\rho S_g(\overline{T}) - (1-\rho)S_b(\overline{T}) < 0$. Because $I_g(\overline{T}) - I_b(\overline{T}) > 0$, we conclude that condition SS must fail at time $\overline{T}$ and hence $\overline{T} \in (t_2, t_3)$. $\square$