Stable Matching in Large Markets with Occupational Choice*

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Abstract

We introduce a model of large many-to-one matching markets with occupational choice where each individual can choose which side of the market to belong to. We show that stable matchings exist under mild assumptions; in particular, both complementarities and externalities can be accommodated. Our model generalizes Greinecker & Kah (2021), which focuses on one-to-one matching and did not allow for occupational choice. Applications include the roommate problem with non-atomic participants, explaining the size and distribution of firms and wage inequality.

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1 Introduction

This paper establishes the existence of many-to-one stable matchings in large markets with complementarities, externalities and occupational choice. Stability in the presence of occupational choice differs from the standard stability notion for two-sided many-to-one matching markets. As individuals no longer have a fixed occupation stability requires someone being unable to find a better match even if this involves a change of occupation. Having all these features present simultaneously in the same model is important for at least the following reasons.

Labor markets match a large numbers of workers to managers in a many-to-one way. Unlike standard matching markets membership in one side or the other of the market is endogenous.

Complementarities and externalities are also an essential feature of labor markets. For example, firms typically want to hire workers with complementary skills and recent graduates may prefer to enter the same industry as their peers. In addition, knowledge spillovers may imply that the productivity of a manager depends on the aggregate quality of those who take managerial roles according to the matching.

Prior work, summarized in Section 2, has established the existence of stable matchings in models that contain a strict subset of these elements.

Our framework for large many-to-one matching markets with occupational choice subsumes several important special cases. It generalizes the two-sided one-to-one matching setting in distributional form of Greinecker & Kah (2021) by adding many-to-one matching and occupational choice; in particular, our existence result implies existence in Greinecker & Kah's (2021) one-to-one matching market.[1]

[1]In the working paper version, we establish formally that Greinecker & Kah's (2021) setting can be represented as a special case of our general framework and that, specialized to this setting, our stability notion coincides with theirs. We also introduce a new two-sided many-to-one matching model that generalizes Greinecker & Kah (2021) to allow for many-to-one matching (but not occupational choice). We show that this model is also a particular case of our framework and, specialized to this setting, our stability notion coincides with other stability concepts for two-sided markets where both sides are large, such as Azevedo & Hatfield's (2018).

In addition, we show how several classical models that feature occupational choice, many-to-one matching and a large number of participants, such as Lucas (1978), Rosen (1982), Garicano & Rossi-Hansberg (2004) and Garicano & Rossi-Hansberg (2006), can be seen as particular cases of our framework. These models also feature a continuum of types, which can be accomodated in our framework. To illustrate the flexibility of our setting and its technical advantages, we provide a detailed analysis of Rosen's (1982) model. We show that stable matchings exist and fully characterize them even though some of the assumptions of our general existence result do not hold.

Our model is not restricted to labor markets. We illustrate this by formalizing a non-atomic version of Gale & Shapley's (1962) roommate problem as a special case of our model – one in which individuals are indifferent between the two occupations. We show that our existence results imply the existence of stable matchings for the non-atomic roommate problem.

We present our model and stability notion in Section 4 after a brief literature review in Section 2 and a motivating example in Section 3.

Our existence results are in Section 5. In particular, we show that stable matchings exist in markets with occupational choice whenever preferences are rational and continuous and the set of feasible measures that managers can match with is bounded and rich.[2] Thus, we can accommodate externalities as long as preferences depend on the matching in a continuous way without any substitutability requirement – complementarities cause no problem for existence in our model. In addition, as is standard in models with a continuum of agents, preferences are not required to be convex.

Section 6 contains applications of our framework to the roommate problem (Section 6.1) and Rosen's (1982) model (Section 6.2), and a brief discussion of the settings of Lucas (1978), Garicano & Rossi-Hansberg (2004) and Garicano & Rossi-Hansberg (2006). Section 7 contains some concluding remarks. The proofs of our results are in

[2]Richness is a weak technical condition that implies that small perturbations of feasible measures are feasible.

the Appendix. Some omitted details are in the working paper version.[3]

2 Literature review

The study of large matching markets has commanded a great deal of recent attention, see, for example, Azevedo & Leshno (2016), Fisher & Hafalir (2016), Ashlagi, Kanoria & Leshno (2017), Eeckhout & Kircher (2018), Fuentes & Tohmé (2018), Nöldeke & Samuelson (2018) and Che & Tercieux (2019).[4] However, none of these papers allow for occupational choice.

Chiappori, Galichon & Salanié (2014), Pęski (2017) and Azevedo & Hatfield (2018) study large matching models with a restricted form of occupational choice and a large number of participants. See Section 6.1 for a more detailed discussion of these papers. Compared to these papers we accommodate many-to-one matching *and* more general forms of occupational choice.

Closest to this paper is Jagadeesan & Vocke (2021) which considers a many-to-many matching model where a continuum of agents of finitely many types can sign multiple contracts with each other. They do not require that the market be two-sided and hence their existence result holds in the presence of occupational choice. However, their assumption that the set of contracts available to each agent is finite makes it less convenient to capture settings such as Rosen (1982) which was part of our motivation. While our model cannot accomodate many-to-many matching, we allow for more general type and contract spaces and we allow preferences to depend on the matching. Wu (2021) also provides a general existence result for a broad class of finite-type many-to-many matching models under a convexity condition. However, Wu's (2021) result does not apply to our setting because we allow preferences to depend on the entire matching.

Externalities and complementarities cause problems for the existence of stable matchings in finite markets. Making the workers negligible allowed Che, Kim &

[3]The working paper version is available at https://klaohakunakorn.com/ocwp.pdf.

[4]See Greinecker & Kah (2021) for a survey.

Kojima (2019) to obtain the existence of stable matchings in two-sided many-to-one matching markets where managers' preferences exhibit complementarities. This result solved a longstanding problem in matching theory since, with finitely many workers and managers, Kelso & Crawford (1982), Hatfield & Milgrom (2005) and Hatfield & Kojima (2008) have shown that managers need to have substitutable preferences to guarantee the existence of stable matchings. In contrast to Che, Kim & Kojima (2019), we also allow for occupational choice and externalities. By assuming that *all* agents are negligible, we are able to show that a stable matching exists in the presence of complementarities, occupational choice and externalities.

Externalities raise some conceptual issues in finite markets. Indeed, when preferences depend on the matching, whether or not an individual gains by being part of a potential blocking coalition depends on the matching that results from such blocking. Thus, the definition of stability has to specify the (set of possible) matchings that result from each blocking coalition, and many such definitions have been proposed.[5] When there are finitely many managers but a continuum of workers and only workers' preferences depend on the matching, Cox, Fonseca & Pakzad-Hurson (2022), Leshno (2022) and Carmona & Laohakunakorn (2023) define stability and establish existence by specifying that each worker in a blocking coalition expects the matching to remain unchanged. In contrast to these papers, we consider the case where all agents are negligible and, thus, a blocking coalition of one (prospective) manager and a measure of (prospective) workers is negligible and, indeed, has no impact on the matching. Hence, externalities cause no conceptual issue in our framework and we can accommodate them on both sides of the market.

3 Motivating example

This example is a particular case of the model in Rosen (1982). There are two types of individuals, 1 and 2. Individuals have preferences that are fully described by their

[5] See, for example, Sasaki & Toda (1996), Dutta & Massó (1997), Echenique & Yenmez (2007), Hafalir (2008), Mumcu & Saglam (2010), Bando (2012) and Fisher & Hafalir (2016).

types and their population is described by a measure ν over the type space $Z = \{1, 2\}$. Let $\nu(1) = \nu(2) = 1/2$.

Each individual can be a manager, a worker or self-employed (i.e. remain unmatched). For each type $z \in \{1, 2\}$, some individuals of type z can be managers and some others can be workers; furthermore, those who are managers (if any) can be matched with workers of type z or of type $z' \neq z$. Those who are managers can hire a workforce, which we represent as a measure over worker types and contracts, from the set X, where each $\delta \in X$ is a measure over $Z \times C$ with C being the set of contracts. For this example, let $C = \mathbb{R}_+$ and $X = \{n1_{(z,c)} : z \in Z, n, c \in \mathbb{R}_+\}$.[6] Specifically, each manager can be matched with a measure $n1_{(z,c)}$, where $z \in Z$ denotes the type of workers he employs, $n \in \mathbb{R}_+$ denotes their number and $c \in \mathbb{R}_+$ denotes the wage paid to them.

The preferences of each individual depend on her type, her occupation and on her match. In this example, we specify that if someone of type $z \in \{1, 2\}$ chooses to be a manager and is matched with $n1_{(z',c)}$, then her payoff is $U_z(m, n1_{(z',c)}) = z^{1+\alpha}n^{1-\alpha} - cn$ for each $z' \in Z$, where $\alpha \in (0, 1)$. If she chooses to be a worker and is matched with manager z' at wage c, then her payoff is the wage: $U_z(w, 1_{(z',c)}) = c$. An individual can also choose to be unmatched, in which case she receives a payoff of zero.

The managers' rents are obtained via a production function of the form $g(z)z^{\alpha}n^{1-\alpha}$, with $g(z) = z$, which has labor and managers' type as inputs, the latter being interpreted as the managers' quality.

In the context of this example, a matching is a measure μ over $Z \times X$ with $\mu(z, n1_{(z',c)})$ describing the measure of type z who are managers and hire n workers of type z' at wage c.

Consider first the case where each individual's occupation is fixed, with type 1 individuals being managers and type 2 individuals being workers. There is a unique stable matching in this example without occupational choice: $\mu(1, 1_{(2,1-\alpha)}) = 1/2$. In such matching, all workers (i.e. type 2 individuals) are matched with a manager (i.e. a type 1 individual), each manager hires a workforce consisting of a measure $n = 1$

[6]If Y is a metric space and $y \in Y$, 1_y denotes the probability measure degenerate on y.

of workers at wage $c = 1 - \alpha$. Since both managers and workers obtain a strictly positive utility in this matching and zero if they were unmatched, such matching is individually rational. Furthermore, no manager and group of workers can block this matching since hiring a measure one of workers is optimal given the wage; hence, the manager cannot gain by changing his workforce since at least the newly hired workers would require a wage higher than $1 - \alpha$.

In the example without occupational choice, type 1 individuals can only be managers and type 2 individuals can only be workers; these restrictions are now removed by the introduction of occupational choice. The specification of our example implies that individuals of type 2 are better managers than those of type 1 since they have higher quality. This then means that the stable matching μ for the setting without occupational choice is intuitively not stable when occupational choice is allowed. For instance, any type 2 individual could choose to be a manager and attract, for example, a measure one of workers of type 2 by paying them $1 - \alpha + \varepsilon$ to obtain a rent of $2^{1+\alpha} - (1 - \alpha) - \varepsilon$; for sufficiently small $\varepsilon > 0$, such workers are willing to work for her and her payoff is higher than $1 - \alpha$ which is her payoff in the matching μ.

Thus, stability in the presence of occupational choice is more demanding than the stability notion for two-sided many-to-one matching markets. The latter roughly requires that no manager can improve his well being by changing the number of workers who work for him or by employing (an optimal number of) workers that he can target, which are those who would prefer to work for him at the proposed wage rather than for the manager with whom they are currently matched.[7] With occupational choice, since anyone can choose to be a manager, this condition must hold not just for those who are managers in the current match but also for those who are workers and unmatched. Similarly, since anyone can be a worker, the targets of a prospective manager are no longer restricted to be the current workers but rather can include current managers and unmatched individuals.

When $\alpha = 1/2$, the unique stable matching in the above example is for all type 2 individuals to be managers, each of them being matched with a measure one of

[7]Stability also requires individual rationality for the workers.

type 1 individuals at wage $w \simeq 1.41$.[8] At this wage, the firm size is optimal for type 2 managers. Their rent is equal to w, so that type 2 individuals are actually indifferent between being a manager or a worker. Type 1 individuals would get a rent approximately equal to 0.18 if they were to hire an optimal number of workers at wage w and, thus, they strictly prefer to be workers rather than managers. It follows from these properties that this matching is indeed stable.[9]

4 Matching with occupational choice

The setting we introduce in this paper is that of a matching market featuring occupational choice, many-to-one matching and a large number of participants. We frame this problem in the context of a labor market for simplicity, so that individuals have a choice of being a manager, a worker or self-employed.

4.1 Environment and matching

Individuals are (potentially) heterogenous in e.g. their talent or knowledge. This is captured by a (nonempty, Polish) *set Z of types*. The population of individuals is described by a nonzero, finite, Borel measure ν on Z; ν is the *type distribution*. A *dummy type* $\emptyset \notin Z$ is used to represent unmatched i.e. self-employed individuals, and we let $Z_\emptyset = Z \cup \{\emptyset\}$, with the assumption that $\emptyset$ is an isolated point in $Z_\emptyset$.

A manager of type z may be matched with a worker of type z' under some contract c. In particular, there is a (nonempty, Polish) *set C of contracts* and a *contract correspondence* $\mathbb{C} : Z \times Z_\emptyset \rightrightarrows C$ describing the set $\mathbb{C}(z, z')$ of contracts that are feasible for a manager of type z and a worker of type z' (when $z' = \emptyset$ the manager is,

[8] In the working paper version, we fully characterize the stable matchings in this example for each $\alpha \in (0, 1)$; in fact, there is a unique stable matching for each α.

[9] Our general framework allows for externalities and their presence is often natural. In the context of the above example, it might be that the production function depends on the aggregate managerial quality in an analogous way to Romer (1986), so that the rent of a manager with quality z is e.g. $\left(\int_{Z\times X} \hat{z}\mathrm{d}\mu(\hat{z}, \delta)\right) z^{1+\alpha} n^{1-\alpha} - cn$ when the matching is μ.

in fact, self-employed and $\mathbb{C}(z, \emptyset)$ describes the feasible contracts for a self-employed individual of type z).

A manager is allowed to hire as many workers as he likes; to capture the many-to-one aspect of matching, a manager is matched with a measure of workers and contracts $\delta \in \mathcal{M}(Z \times C)$.[10] The definition of a matching below will impose feasibility constraints on δ via the contract correspondence $\mathbb{C}$ and, thus, constrain the contracts that the manager can offer to each of his employees. These constraints are of the form $c \in \mathbb{C}(z, z')$ and are, therefore, independent across workers. To capture interdependent and other feasibility constraints, we let X be a subset of $\mathcal{M}(Z \times C)$ and require that managers be matched with $\delta \in X$.

Self-employed (or unmatched) managers are those matched with the dummy type $\emptyset$. To specify his contract (e.g. the number of hours worked as self-employed), we use matches of the form $(z, 1_{(\emptyset,c)})$ to describe a self-employed individual of type z with contract c. To unify the two cases, we let $X_\emptyset = X \cup \{1_{(\emptyset,c)} : c \in C\}$ be the *set of possible matches of managers and self-employed individuals.*

The *set of occupations* is $A = \{w, s, m\}$, where w stands for worker, s for self-employed and m for manager. The choice set of each individual depends on his occupation; namely, a worker chooses among managers' types and contracts, a self-employed individual among contracts, and a manager among measures $\delta \in X$ describing whom to hire and the contracts offered. To capture these differences, let $X_m = X$, $X_s = \{1_{(\emptyset,c)} : c \in C\}$, $X_w = \{1_{(z,c)} : (z, c) \in Z \times C\}$ and $\Delta = \{(a, \delta) : \delta \in X_a\}$.[11] The set Δ is the *choice set of each individual* as she can choose her occupation and a match feasible for the chosen occupation.

We allow for externalities and, thus, preferences are allowed to depend on the matching. Matchings with occupational choice are elements of $\mathcal{M}(Z \times X_\emptyset)$ satisfying

[10] Whenever Y is a metric space, $\mathcal{M}(Y)$ denotes the set of finite, Borel measures on Y endowed with the weak (narrow) topology (see Varadarajan (1958) for details). We often focus on $\mathcal{M}_R(Y)$ where, for each $R > 0$, $\mathcal{M}_R(Y) = \{\delta \in \mathcal{M}(Y) : \delta(Y) \leq R\}$.

[11] We do not distinguish between (z, c) and $1_{(z,c)}$ for each $(z, c) \in Z_\emptyset \times C$, hence it would be simpler to replace the latter with the former in the definition of X_s and X_w. The formalization we use above provides an unified notation which simplifies the exposition elsewhere.

certain properties described below. The *preferences* of an individual of type z are then described by a relation $\succ_z$ defined on $\Delta \times \mathcal{M}(Z \times X_\emptyset)$ for each $z \in Z$.

In summary, a *matching market with occupational choice* (a market, henceforth) is $E = (Z, \nu, C, \mathbb{C}, X, (\succ_z)_{z\in Z})$.

A *matching with occupational choice* (a matching, henceforth) is a Borel measure $\mu \in \mathcal{M}(Z \times X_\emptyset)$ such that

1. $\{z\} \times \operatorname{supp}(\delta) \subseteq \operatorname{graph}(\mathbb{C})$ for each $(z, \delta) \in \operatorname{supp}(\mu)$, and

2. $\nu_M + \nu_S + \nu_W = \nu$

where, for each Borel subset B of Z, $\nu_M(B) = \mu(B \times X)$, $\nu_S(B) = \mu(B \times (X_\emptyset \setminus X))$ and $\nu_W(B) = \int_{Z\times X} \delta(B \times C)\mathrm{d}\mu(z, \delta)$.

The interpretation of μ is as follows. First, μ describes the occupational choices by the place in the match (z, δ), namely, the first coordinate refers to managers and the second to workers (as part of a firm) when $\delta \in X$ and, when $\delta \in X_\emptyset \setminus X$, the first coordinate refers to a self-employed individual and the second, which is equal to $1_{(\emptyset,c)}$ for some $c \in C$, describes the individual's contract. Condition 1 requires that the contract is feasible according to the contract correspondence. Condition 2 requires that everyone in the market is accounted for, as follows: For each Borel subset B of Z, $\mu(B \times X)$ is the measure of managers whose type belongs to B and we call it $\nu_M(B)$. Similarly, $\mu(B \times (X_\emptyset \setminus X))$ is the measure of self-employed individuals whose type belongs to B and we call it $\nu_S(B)$. Finally, $\int_{Z\times X} \delta(B \times C)\mathrm{d}\mu(z, \delta)$ is the measure of workers whose type belongs to B and, thus, we call it $\nu_W(B)$.[12] Since an individual must be either a manager, or a worker or self-employed, condition 2 must hold if everyone in the market is accounted for.

[12]For each Borel subset E of a metric space Y, the function $\delta \mapsto \delta(E) : \mathcal{M}(Y) \to \mathbb{R}$ is Borel measurable. This follows by the argument in Aliprantis & Border (2006, Theorem 15.13, p. 514) together with Varadarajan (1958, Theorem 3.1).

4.2 Stability

Heading towards the definition of stable matchings, we start by defining the targets of individuals at a given matching and then define the stability set of a matching.

Targets at a given matching μ depend on the type z and on the occupational choice a, and are denoted by $T_z^a(\mu)$. Because one's occupation is a choice and not a fixed characteristic, these targets are for someone planning to choose occupation a, i.e. if someone chooses occupation a, then his targets are $T_z^a(\mu)$. The targets for the prospective self-employed are simply the contracts that are feasible when someone is unmatched: For each $z \in Z$, let $T_z^s(\mu) = \{\emptyset\} \times \mathbb{C}(z, \emptyset)$.

The targets of prospective managers and workers are more complicated as they consist of contracts and types of people on the other side of the market that managers or workers can attract. But with occupational choice, there is not a fixed "other side of the market" since anyone can change his occupation. In more detail, even if all individuals of type z^* are managers in the matching μ, any type z^* person can choose to became a worker. In particular, if such z^* person gains by becoming a worker and by working for a manager of type z at some contract c, then (z^*, c) is a target for those of type z planning to be a manager, i.e. it belongs to $T_z^m(\mu)$. We then let, for each $z \in Z$, $T_z^m(\mu)$ be the set of $(z^*, c) \in Z \times C$ such that $c \in \mathbb{C}(z, z^*)$ and there exists

(a) $(z', c', \delta') \in Z \times C \times X$ such that $(z', \delta') \in \text{supp}(\mu)$, $(z^*, c') \in \text{supp}(\delta')$ and $(w, 1_{(z,c)}, \mu) \succ_{z^*} (w, 1_{(z',c')}, \mu)$, or

(b) $\delta' \in X_\emptyset \setminus X$ such that $(z^*, \delta') \in \text{supp}(\mu)$ and $(w, 1_{(z,c)}, \mu) \succ_{z^*} (s, \delta', \mu)$, or

(c) $\delta' \in X$ such that $(z^*, \delta') \in \text{supp}(\mu)$ and $(w, 1_{(z,c)}, \mu) \succ_{z^*} (m, \delta', \mu)$.

Anyone of type z can be a manager if he finds workers, here of type z^*, who prefer to work for him than to be in their current occupation. Each of these workers can be someone who was already a worker in μ as described in condition (a), or self-employed as described by condition (b), or even a manager as described by condition (c).

The targets of prospective workers are defined analogously. Thus, for each $z \in Z$, let $T^w_z(\mu)$ be the set of $(z^*, c) \in Z \times C$ such that $c \in \mathbb{C}(z^*, z)$ and there is $\delta \in X$ such that $(z, c) \in \text{supp}(\delta)$ and

(a) $\text{supp}(\delta) \setminus \{(z, c)\} \subseteq T^m_{z^*}(\mu)$ and there is $(z', c', \delta') \in Z \times C \times X$ such that $(z', \delta') \in \text{supp}(\mu)$, $(z^*, c') \in \text{supp}(\delta')$ and $(m, \delta, \mu) \succ_{z^*} (w, 1_{(z',c')}, \mu)$, or

(b) $\text{supp}(\delta) \setminus \{(z, c)\} \subseteq T^m_{z^*}(\mu)$ and there is $\delta' \in X_\emptyset \setminus X$ such that $(z^*, \delta') \in \text{supp}(\mu)$ and $(m, \delta, \mu) \succ_{z^*} (s, \delta', \mu)$, or

(c) there is $\delta' \in X$ such that $\text{supp}(\delta) \setminus \{(z, c)\} \subseteq T^m_{z^*}(\mu) \cup \text{supp}(\delta')$, $(z^*, \delta') \in \text{supp}(\mu)$ and $(m, \delta, \mu) \succ_{z^*} (m, \delta', \mu)$.

As above, anyone of type z can be a worker if she finds a manager, here of type z^*, that hires her, possibly alongside other workers as described by $\delta \in X$, and both agree on a feasible contract $c \in \mathbb{C}(z^*, z)$. This manager can be someone who was already a manager in μ as described in condition (c), or self-employed as described by condition (b), or even a worker as described by condition (a).

The stability set $S(\mu)$ of matching μ is the set of $(z, \delta) \in Z \times X_\emptyset$ such that, if $\delta \in X$, then

(i) there does not exist $(a, \delta') \in \Delta$ such that $\text{supp}(\delta') \subseteq T^a_z(\mu) \cup \text{supp}(\delta)$ if $a = m$, $\text{supp}(\delta') \subseteq T^a_z(\mu)$ if $a \neq m$, and $(a, \delta', \mu) \succ_z (m, \delta, \mu)$,

(ii) for each $(z', c) \in \text{supp}(\delta)$, there does not exist $(a, \delta') \in \Delta$ such that $\text{supp}(\delta') \subseteq T^a_{z'}(\mu)$ and $(a, \delta', \mu) \succ_{z'} (w, 1_{(z,c)}, \mu)$,

and, if $\delta \in X_\emptyset \setminus X$, then

(iii) there does not exist $(a, \delta') \in \Delta$ such that $\text{supp}(\delta') \subseteq T^a_z(\mu)$ and $(a, \delta', \mu) \succ_z (s, \delta, \mu)$.

The set $S(\mu)$ describes matches (z, δ) that do not suffer from instability. Instability could come from those who are managers in μ if a manager of type z can find a match δ' that is better than his current one δ by employing workers of the types currently employed or those of his targets. In addition, he could instead be better off by changing his occupation and matching with some of his targets for the alternative occupation. Condition (i) rules out instability arising from the current managers, whereas condition (ii) does the same for current workers and (iii) for self-employed. A matching μ is *stable* if $\text{supp}(\mu) \subseteq S(\mu)$.

Theorem 1 provides a characterization of stable matchings that is simpler to use. Let $S_M(\mu)$ be defined as $S(\mu)$ but with "$(a, \delta') \in \Delta$" being replaced with "$(a, \delta') \in \Delta$ such that $a = m$" and, analogously, $IR(\mu)$ be defined as $S(\mu)$ but with "$(a, \delta') \in \Delta$" being replaced with "$(a, \delta') \in \Delta$ such that $a = s$".

Theorem 1 *A matching μ is stable if and only if* $\text{supp}(\mu) \subseteq S_M(\mu) \cap IR(\mu)$.

4.3 Discussion

We conclude this section with some comments on our definition of stability. First, note that it focuses on the support of the matching. In some cases, however, not all elements of $\text{supp}(\delta)$ in a match (z, δ) are pairs of worker types and contracts that are matched with a manager of type z. This may happen, for example, if $\delta = \sum_{k=1}^{\infty} 2^{-k} 1_{(z_k, c_k)}$ for some countable subset $D = \{(z_k, c_k)\}_{k=1}^{\infty}$ of $Z \times C$. In this case, it would seem more appropriate to require only that $\{z\} \times D \subseteq \text{graph}(\mathbb{C})$ instead of $\{z\} \times \text{supp}(\delta) \subseteq \text{graph}(\mathbb{C})$ in the definition of a matching. When the correspondence $\mathbb{C}$ is continuous, this issue does not arise since then the two requirements are equivalent. Similar considerations apply to the definition of stability when preferences are also continuous. For instance, when a market E also satisfies a richness condition, we have that a matching μ is stable if and only if $S(\mu)$ has full μ-measure.[13]

A more important issue concerns what we require for a manager of type z, currently matched with δ, and a potential workforce δ' to qualify as a blocking coalition.[14] We require that $(m, \delta') \succ_z (m, \delta)$ and $\text{supp}(\delta') \subseteq T_z^m(\mu) \cup \text{supp}(\delta)$. This requirement is unusual in that it is between weak and strong domination – but as we now argue,

[13]See Section 5 for the notion of continuity and richness we use and the working paper version for a proof of this claim.

[14]I.e. what condition (i) of the definition of $S(\mu)$ for $a = m$ rules out.

it is the weakest requirement for blocking (and hence associated with the strongest stability notion) such that stable matchings exist under general conditions.

We illustrate the above with the following example, where for simplicity contracts are omitted and preferences do not depend on the matching. Let $Z = \{1, 2\}$, $\nu(1) = \nu(2) = 1/2$ and $X = \{n1_z : n \leq 1, z \in Z\}$. Let preferences be represented by: $u_z(m, n1_{z'}) = 2nz'$, $u_z(w, 1_{z'}) = z'$, and $u_z(s, 1_\emptyset) = 0$. It is easy to see that μ such that $\mu(2, 1_1) = 1/2$ is a stable matching. Here, every individual gets payoff 2 (thus the matching is individually rational and $\text{supp}(\mu) \subseteq IR(\mu)$), and since being a worker yields payoff at most 2, $T_1^m(\mu) = T_2^m(\mu) = \emptyset$. Since $\text{supp}(\mu) = \{(2, 1_1)\}$, $(w, 1_2) \succeq_1 (m, \delta)$ for all $\delta \in X$ such that $\text{supp}(\delta) \subseteq \emptyset$ and $(m, 1_1) \succeq_2 (m, \delta)$ for all $\delta \in X$ such that $\text{supp}(\delta) \subseteq \{1\}$, it follows that $\text{supp}(\mu) \subseteq S_M(\mu)$.

The strongest notion of stability is the one that defines a blocking coalition via *weak domination*, i.e. to require that every individual in the coalition is weakly better off with at least one individual being strictly better off. Let $\nu_W^{\preceq z}(\mu)$ be the measure of types who would *weakly prefer* to work for type z than remain in their current match, given μ. Under weak domination, our requirement that $\text{supp}(\delta') \subseteq T_z^m(\mu) \cup \text{supp}(\delta)$ would be replaced with $\text{supp}(\delta') \subseteq \text{supp}(\nu_W^{\preceq z}(\mu))$.[15] Note that $T_z^m(\mu) \cup \text{supp}(\delta) \subseteq \text{supp}(\nu_W^{\preceq z}(\mu))$ since $T_z^m(\mu)$ is the set of types[16] that would *strictly prefer* to work for type z given matching μ and those in $\text{supp}(\delta)$ are currently working for type z and hence indifferent; thus, the resulting notion of stability is stronger.

However, this yields a notion of stability for which there are no stable matchings in the current example. In the matching of the previous paragraph, we now have

[15] To see this in the context of the current example, suppose that $(z, \delta) \in \text{supp}(\mu)$ and there exists δ' such that $(m, \delta') \succ_z (m, \delta)$ and $\text{supp}(\delta') \subseteq \text{supp}(\nu_W^{\preceq z}(\mu))$. Then there is a nonnull coalition S of individuals, described by a measure $\nu^S = \nu_M^S + \nu_W^S$, and a matching μ^S for the coalition such that $\text{supp}(\nu_M^S) = \{z\}$, $\mu^S(z, \delta') = \nu_M^S(z)$, $\mu^S(z, \delta')\delta'(z') = \nu_W^S(z')$ for each $z' \in Z$, each manager in ν_M^S is strictly better off and each worker in ν_W^S is weakly better off. Indeed, let $\nu_M^S = \varepsilon 1_z$ and $\nu_W^S(z') = \varepsilon\delta'(z')$ for each $z' \in Z$. For each $z' \in \text{supp}(\nu_W^S) = \text{supp}(\delta')$, we have that $z' \in \text{supp}(\nu_W^{\preceq z}(\mu))$; thus for ε sufficiently small, $\nu_W^S(z') = \varepsilon\delta'(z') \leq \nu_W^{\preceq z}(\mu)(z')$ for each $z' \in \text{supp}(\nu_W^S)$ and so the coalition can be chosen such that each worker is weakly better off. In addition, for ε sufficiently small, $\nu_M^S(z) = \varepsilon \leq \mu(z, \delta)$ and so the coalition can be chosen such that each manager is strictly better off.

[16] Recall that we are omitting contracts for simplicity.

$\text{supp}(\nu_W^{\preceq 2}(\mu)) = \{1, 2\}$, $\text{supp}(1_2) \subseteq \text{supp}(\nu_W^{\preceq 2}(\mu))$ and $(m, 1_2) \succ_2 (m, 1_1)$. It is easy to see that there are no other stable matchings; in any stable matching all type 2 individuals must be managers and employ type 2 individuals but this is impossible.

We could alternatively use *strong domination* to define a blocking coalition, i.e. to require that every individual in the coalition is strictly better off. Then for type z, currently a manager and matched with δ, to form a blocking coalition with potential workforce δ', we would need $(m, \delta') \succ_z (m, \delta)$ and $\text{supp}(\delta') \subseteq T_z^m(\mu)$. Note that $T_z^m(\mu) \subseteq T_z^m(\mu) \cup \text{supp}(\delta) \subseteq \text{supp}(\nu_W^{\preceq z}(\mu))$; hence stability defined via weak domination is the strongest notion, followed by ours, followed by the one defined via strong domination.

Our existence result, Theorem 2, shows generally that, when managers can only hire a bounded number of workers as in the above example, a stable matching exists when blocking coalitions are defined using our requirement $\text{supp}(\delta') \subseteq T_z^m(\mu) \cup \text{supp}(\delta)$ (and hence when they are defined via strong domination). Our reason for adopting our stability notion is that it is a refinement of the stability notion defined via strong domination but its existence is nevertheless guaranteed under general conditions. We prefer our notion to the one defined via strong domination because our notion implies existing stability notions in special cases (see Section 6).

5 Existence of stable matchings

In this section we establish the existence of stable matchings and discuss the conditions needed to prove this result.

One requirement in our existence result is that preferences are rational. We say that a market is *rational* if $\succ_z$ is asymmetric and negative transitive for each $z \in Z$.[17] Note that $\succ_z$ is asymmetric and negative transitive if and only if $\succeq_z$ is complete and transitive (i.e. rational).[18] Rational preferences can be represented by an utility

[17] A relation $\succ$ on a set Y is asymmetric if, for each $x, y \in Y$, if $x \succ y$ then $\neg(y \succ x)$. It is negative transitive if, for each $x, y, z \in Y$, if $\neg(x \succ y)$ and $\neg(y \succ z)$, then $\neg(x \succ z)$.

[18] The relation $\succeq_z$ is defined as usual by setting, for each $(a, \delta, \mu), (a', \delta', \mu') \in \Delta \times \mathcal{M}(Z \times X_\emptyset)$, $(a, \delta, \mu) \succeq_z (a', \delta', \mu')$ if and only if $(a, \delta, \mu) \succ_z (a', \delta', \mu')$ or $\neg\big((a', \delta', \mu') \succ_z (a, \delta, \mu)\big)$.

function and this plays an important role in our proof.

Another basic requirement in our existence results is some form of continuity. We say that a market E is *continuous* if $\{(a, \delta, \mu, a', \delta', \mu', z) \in (\Delta \times \mathcal{M}(Z \times X_\emptyset))^2 \times Z : (a, \delta, \mu) \succ_z (a', \delta', \mu')\}$ is open,[19] $\mathbb{C}$ is continuous with nonempty and compact values, and X is closed.

Stable matchings may fail to exist in the absence of a bound on the measure of workers a manager can hire. This existence problem arises because each manager is negligible and, therefore, is effectively unconstrained by the size of the market. In Section A.6, we provide an example showing that, without any boundedness assumptions on X, a stable matching fails to exist.[20] Thus, we focus on bounded markets, defined as follows: We say that a market E is *bounded* if there exists $R > 0$ such that $\delta(Z \times C) \leq R$ for each $\delta \in X$. More succinctly, E is bounded if $X \subseteq \mathcal{M}_R(Z \times C)$ for some $R > 0$.

Note that boundedness is essentially a uniform satiation condition. Indeed, suppose that there exists $R > 0$ such that, for each $z \in Z$ and $\mu \in \mathcal{M}(Z \times X_\emptyset)$, there exists $\delta \in X$ such that $\delta(Z \times C) \leq R$ and $(m, \delta, \mu) \succeq_z (m, \delta', \mu)$ for each $\delta' \in X$. In this case, as far as existence of stable matchings is concerned, we may focus on $\delta \in \mathcal{M}_R(Z \times C)$ and, thus, assume that the market is bounded.

We will also focus on rich markets. The reason is that our approach to the existence problem consists in first addressing discrete markets where Z, C and X are finite. In such markets, managers are matched with measures of workers that are finitely supported and richness will then allow us to extend our existence results from discrete to general markets. We say that a market E is *rich* if the correspondences $\Lambda : Z \times X \times \mathcal{M}(Z \times X_\emptyset) \rightrightarrows X$ and $\Lambda_0 : Z \times \mathcal{M}(Z \times X_\emptyset) \rightrightarrows X$ defined by setting, for each $(z, \delta, \mu) \in Z \times X \times \mathcal{M}(Z \times X_\emptyset)$, $\Lambda(z, \delta, \mu) = \{\delta' \in X : \text{supp}(\delta') \subseteq \text{supp}(\delta) \cup T_z^m(\mu)\}$ and $\Lambda_0(z, \mu) = \{\delta' \in X : \text{supp}(\delta') \subseteq T_z^m(\mu)\}$ are lower hemicontinuous.

The richness assumption is a mild requirement which is satisfied in several special

[19] The set A of occupations is endowed with the discrete topology.

[20] A stable matching would fail to exist even under the weakest form of stability we discuss in Section 4 which is defined via strong domination.

cases, including those of Che, Kim & Kojima (2019) and Greinecker & Kah (2021) where, respectively, $X = \mathcal{M}_1(Z \times C)$ and $X = \{1_{(z,c)} : (z,c) \in Z \times C\}$ (the boundedness assumption is clearly also satisfied in these two cases). This can be seen by noting that, for a market to be rich, it is sufficient that the set of finitely supported measures on $Z \times C$ is dense in X (this is (β) below) and that measures δ obtained via a small perturbation to the support of a finitely supported measure in X remain in X (this is (α) below). More formally, the following conditions are sufficient for richness:[21]

(α) For each $\delta \in X$ such that $\delta = \sum_{j=1}^{J} a_j 1_{(z_j,c_j)}$ for some $J \in \mathbb{N}$, $a_j \in \mathbb{R}_{++}$ and $(z_j, c_j) \in Z \times C$ for each $j = 1, \ldots, J$ and each open neighborhood V_δ of δ in X, there exist open neighborhoods $V_{(z_j,c_j)}$ of (z_j, c_j) for each $j = 1, \ldots, J$ such that, whenever $(\hat{z}_j, \hat{c}_j) \in V_{(z_j,c_j)}$ for each $j = 1, \ldots, J$, there exists $\hat{a} = (\hat{a}_1, \ldots, \hat{a}_J) \in \mathbb{R}_+^J$ such that $\sum_{j=1}^{J} \hat{a}_j 1_{(\hat{z}_j,\hat{c}_j)} \in V_\delta$.

(β) For each $\delta \in X$ and open neighborhood V_δ of δ in X, there exists $\hat{\delta} \in V_\delta$ such that $\mathrm{supp}(\hat{\delta})$ is a finite subset of $\mathrm{supp}(\delta)$.

The following is our main existence result. As Greinecker & Kah's (2021) framework is a special case of ours, it has Greinecker & Kah's (2021) Theorem 5 as a special case.

Theorem 2 *Every rational, continuous, bounded and rich market has a stable matching.*

When there are no externalities, the rationality of E can be replaced with the requirement that preferences are acyclic. This is because when Z, C and X are finite, acyclic preferences defined on Δ (as opposed to $\Delta \times \mathcal{M}(Z \times X_\emptyset)$) can be extended to linear orders, which are rational. We say that E is a *market without externalities* if for each $z \in Z$, $(a, \delta), (a', \delta') \in \Delta$, if $(a, \delta, \hat{\mu}) \succ_z (a', \delta', \hat{\mu})$ for some $\hat{\mu} \in \mathcal{M}(Z \times X_\emptyset)$, then $(a, \delta, \mu) \succ_z (a', \delta', \mu)$ for all $\mu \in \mathcal{M}(Z \times X_\emptyset)$. Moreover, we say that E is *acyclic*

[21] See the working paper version for a proof of this claim.

if $\succ_z$ is acyclic for each $z \in Z$.[22] We then obtain the following corollary which has Greinecker & Kah's (2021) Theorem 1 as a special case.

Corollary 1 *Every acyclic, continuous, bounded and rich market without externalities has a stable matching.*

6 Applications

6.1 Roommate market

Gale & Shapley (1962) considered a roommate problem in which an "even number of boys wish to divide up into pairs of roommates." This is an example of matching with occupational choice since there aren't two exogenously given sets of individuals to match; it has also the particular feature that individuals are indifferent between the different occupations.

In this section we formulate a general version of the roommate problem with a continuum of individuals in distributional form, a roommate market, as a special case of our framework. We show that, in contrast to the case of finitely many individuals of Gale & Shapley (1962), a stable matching exists in roommate markets that are acyclic and continuous when there are no externalities in preferences; when there are externalities, our existence result requires preferences to be rational and continuous. In particular, roommate markets are always bounded and rich.

The importance of large markets for the existence of stable matchings in the roommate problem has been established by Chiappori, Galichon & Salanié (2014), Pęski (2017), Azevedo & Hatfield (2018), Wu (2021) and Jagadeesan & Vocke (2021). Both Chiappori, Galichon & Salanié (2014) and Pęski (2017) show the existence of approximately stable matchings in roommate problems with a large finite number of individuals, respectively, with and without transferable utility, and Azevedo & Hatfield (2018) establish the existence of (exact) stable matchings with a continuum

[22]A relation $\succ$ on a set Y is acyclic if there is no finite sequence $y_1, y_2, \ldots, y_n$ in Y such that $y_1 \succ y_2 \succ \cdots \succ y_n \succ y_1$.

of individuals and with transferable utility. Our existence result for the roommate problem, like Wu's (2021) and Jagadeesan & Vocke's (2021), dispenses with the requirement of transferable utility and allows us to cover the continuum version of Gale & Shapley (1962); in contrast to Jagadeesan & Vocke (2021) and Wu (2021), we allow preferences to depend on the entire matching.

A roommate market can be defined as a market where matching is restricted to be one-to-one and preferences and the contract correspondence satisfy certain restrictions that reflect the fact that the roles of worker and manager have no meaning in the roommate setup. In particular, we define a roommate market as a market $E = (Z, \nu, C, \mathbb{C}, X, (\succ_z)_{z\in Z})$ satisfying the following restrictions:

(R1) $X = \{1_{(z,c)} : (z, c) \in Z \times C\}$,

(R2) $(m, 1_{(z',c)}, \mu) \sim_z (w, 1_{(z',c)}, \mu)$ for each $z, z' \in Z$, $c \in C$ and $\mu \in \mathcal{M}(Z \times X_\emptyset)$,

(R3) $\mathbb{C}(z, z') = \mathbb{C}(z', z)$ for each $z, z' \in Z$, and

(R4) $(m, 1_{(z',c)}, \mu) \sim_z (m, 1_{(z',c)}, \mu \circ f^{-1})$ for each $z, z' \in Z$, $c \in C$ and measurable $f \in F$,

where $F = \{f : f(z, z') = (z, z')$ or $f(z, z') = (z', z)$ for each $z, z' \in Z,$ and $f(z, \emptyset) = (z, \emptyset)$ for each $z \in Z\}$. (R1) requires that matching in a roommate market is one-to-one. (R2) requires that each type cares only about who he is matched with (and not the role he occupies in the match). (R3) requires that switching the roles of two types in a match does not affect the set of feasible contracts, and (R4) requires that matchings that differ only according to who occupies which role in a match are treated the same way.

The particular setting of a roommate market allows for some simplification in its description. In fact, we can identify $1_{(z,c)}$ with (z, c) for each $(z, c) \in Z_\emptyset \times C$ and, thus, we can write (R1) as requiring $X = Z \times C$ and $X_\emptyset = Z_\emptyset \times C$. In particular, a matching is $\mu \in \mathcal{M}(Z \times Z_\emptyset \times C)$.

(R2) implies that individual preferences can be defined on $Z_\emptyset \times C \times \mathcal{M}(Z \times$

$Z_\emptyset \times C$).[23] In light of this comment and the one in the previous paragraph, we can equivalently define a *roommate market* as $E = (Z, \nu, C, \mathbb{C}, (\succ_z)_{z \in Z})$ such that $(Z, \nu, C, \mathbb{C})$ are as in the general framework of Section 4, $\mathbb{C}$ satisfies $\mathbb{C}(z, z') = \mathbb{C}(z', z)$ for each $z, z' \in Z$, and $\succ_z$ is defined on $Z_\emptyset \times C \times \mathcal{M}(Z \times Z_\emptyset \times C)$ and satisfies $(z', c, \mu) \sim_z (z', c, \mu \circ f^{-1})$ for each $z \in Z$, $(z', c) \in Z_\emptyset \times C$ and measurable $f \in F$.

A matching, which we refer to as a *roommate matching*, is then a Borel measure $\mu \in \mathcal{M}(Z \times Z_\emptyset \times C)$ such that $\text{supp}(\mu) \subseteq \text{graph}(\mathbb{C})$ and $\nu_W + \nu_S + \nu_M = \nu$ where, for each Borel subset B of Z, $\nu_M(B) = \mu(B \times Z \times C)$, $\nu_W(B) = \mu(Z \times B \times C)$ and $\nu_S(B) = \mu(B \times \{\emptyset\} \times C)$.

The targets become

$$\begin{aligned} T^m_z(\mu) = T^w_z(\mu) = \{&(z^*, c) \in Z \times C : c \in \mathbb{C}(z, z^*) \text{ and } \exists (z', c') \in Z \times C \text{ such that} \\ &\text{supp}(\mu) \cap \{(z^*, z', c'), (z', z^*, c')\} \neq \emptyset \text{ and } (z, c, \mu) \succ_{z^*} (z', c', \mu)\} \end{aligned}$$

and $T^s_z(\mu) = \{\emptyset\} \times \mathbb{C}(z, \emptyset)$. Let $T_z(\mu) = T^m_z(\mu) \cup T^s_z(\mu)$. Then $S(\mu)$ becomes the set of $(z, z', c) \in Z \times Z_\emptyset \times C$ such that

(i) there does not exist $(\hat{z}, \hat{c}) \in T_z(\mu)$ such that $(\hat{z}, \hat{c}, \mu) \succ_z (z', c, \mu)$, and

(ii) if $z' \neq \emptyset$, there does not exist $(\hat{z}, \hat{c}) \in T_{z'}(\mu)$ such that $(\hat{z}, \hat{c}, \mu) \succ_{z'} (z, c, \mu)$.

Since a roommate market is a particular case of the setting of Section 4, the existence of a stable matching for each roommate market follows from Theorem 2.

Corollary 2 *If E is a rational and continuous roommate market or an acyclic and continuous roommate market without externalities, then E has a stable roommate matching.*

[23] Indeed, given $\hat{\succ}_z$ defined on $\Delta \times \mathcal{M}(Z \times Z_\emptyset \times C)$, define $\succ_z$ on $Z_\emptyset \times C \times \mathcal{M}(Z \times Z_\emptyset \times C)$ by setting, for each $z', z'' \in Z$, $c', c'' \in C$, and $\mu', \mu'' \in \mathcal{M}(Z \times Z_\emptyset \times C)$, (i) $(z', c', \mu') \succ_z (z'', c'', \mu'')$ if and only if $(m, z', c', \mu') \hat{\succ}_z (m, z'', c'', \mu'')$, (ii) $(z', c', \mu') \succ_z (\emptyset, c'', \mu'')$ if and only if $(m, z', c', \mu') \hat{\succ}_z (s, \emptyset, c'', \mu'')$, (iii) $(\emptyset, c', \mu') \succ_z (z'', c'', \mu'')$ if and only if $(s, \emptyset, c', \mu') \hat{\succ}_z (m, z'', c'', \mu'')$, and (iv) $(\emptyset, c', \mu') \succ_z (\emptyset, c'', \mu'')$ if and only if $(s, \emptyset, c', \mu') \hat{\succ}_z (s, \emptyset, c'', \mu'')$. These four conditions, together with (R2), also define $\hat{\succ}_z$ on $\Delta \times \mathcal{M}(Z \times Z_\emptyset \times C)$ from $\succ_z$ on $Z_\emptyset \times C \times \mathcal{M}(Z \times Z_\emptyset \times C)$.

To illustrate our stability condition for the roommate market, first consider the example from Gale & Shapley (1962) with four individuals α, β, γ and δ. Preferences are given by: $\beta \succ_\alpha \gamma \succ_\alpha \delta \succ_\alpha \emptyset$, $\gamma \succ_\beta \alpha \succ_\beta \delta \succ_\beta \emptyset$, and $\alpha \succ_\gamma \beta \succ_\gamma \delta \succ_\gamma \emptyset$. As Gale & Shapley (1962) argue, a stable matching does not exist regardless of δ's preferences. In a finite market, someone must be matched with δ or unmatched. But whoever is matched with δ or unmatched would prefer to be matched with either of the other two individuals, one of whom must also prefer to be matched with him.

Suppose instead that there is a continuum of individuals with four types of agents α, β, γ and δ, where each type of agent has the same preference as the single individual of that type given above,[24] and let the measure of each type of agent be $\nu(z) = 1$ for $z \in Z = \{\alpha, \beta, \gamma, \delta\}$. We will now argue that $\mu(\alpha, \beta) = \mu(\beta, \gamma) = \mu(\gamma, \alpha) = 1/2$ and $\mu(\delta, \emptyset) = 1$ is a stable matching in our model.[25]

First, note that $\mu(\{z\} \times Z) + \mu(Z \times \{z\}) + \mu(\{z\} \times \{\emptyset\}) = 1$ for each $z \in Z$, so μ is a roommate matching. The targets are: $T_\alpha(\mu) = \{\gamma, \emptyset\}$, $T_\beta(\mu) = \{\alpha, \emptyset\}$, $T_\gamma(\mu) = \{\beta, \emptyset\}$, and $T_\delta(\mu) = \{\emptyset\}$. Note that $\operatorname{supp}(\mu) = \{(\alpha, \beta), (\beta, \gamma), (\gamma, \alpha), (\delta, \emptyset)\}$. To see that $(\alpha, \beta) \in S(\mu)$, note that type α likes β the most so there is no $\hat{z}$ such that $\hat{z} \succ_\alpha \beta$; thus condition (i) is satisfied. Type β prefers γ to α but $\gamma \notin T_\beta(\mu)$; thus condition (ii) is satisfied. Analogous arguments establish that $\operatorname{supp}(\mu) \subseteq S(\mu)$, and hence μ is stable.

A stable matching exists in this example with a continuum of individuals because it is possible for individuals of type α, β and γ all to be matched with each other, leaving individuals of type δ unmatched. More generally, our results imply that the large market version of the roommate problem admits a stable solution with or without transfers and even in the presence of externalities as long as the market is rational and continuous.

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[24] With a continuum of individuals, it is possible for a given type to match with itself. To ensure that this does not happen in a stable matching, we specify for this example that $z' \succ_z z$ for each $(z, z') \in Z \times Z_\emptyset$ with $z' \neq z$.

[25] Formally, a model without contracts can be modelled in our framework by letting C be singleton and $\mathbb{C}(z, z') = C$ for each $z, z' \in Z$, but here we omit contracts altogether for simplicity.

6.2 Rosen (1982)

In this section we consider the setting in Rosen (1982, Section 3).

Individuals are characterized by their general ability, with $Z \subseteq \mathbb{R}$ denoting the set of possible abilities and ν denoting its (nonzero, finite) distribution. Individuals can be workers, managers or self-employed (here more correctly interpreted as unemployed as it will be clear from the individuals' payoffs) and their productivity is determined both by this choice and their ability, with $q = q(z)$ denoting the productivity of someone of ability z who chooses to be a worker and $r = r(z)$ his productivity if he chooses to be a manager; both r and q are non-decreasing functions of the ability z.

A firm consists of one manager and several workers of the same type, i.e. there is many-to-one matching. Managers have one unit of time and need to supervise workers: The output produced by a worker with productivity q in a firm with a manager with productivity r is $g(r)f(tr, q)$, where t is the time spent by the manager supervising the worker, $g(r)$ represents the quality of management decisions of a manager of productivity r, $g : \mathbb{R}_+ \to \mathbb{R}_+$ is increasing and $f : \mathbb{R}^2_+ \to \mathbb{R}_+$ is continuously differentiable, homogeneous of degree 1, strictly increasing and strictly concave in each coordinate in the interior of its domain[26] and satisfies $f(0, y) = f(x, 0) = 0$ for each $x, y \in \mathbb{R}_+$. For convenience, we define $\theta : \mathbb{R}_+ \to \mathbb{R}_+$ as $\theta(x) = f(x, 1)$ for each $x \in \mathbb{R}_+$; note that θ is strictly increasing and strictly concave. The output of a firm with a manager of ability r and a measure n of workers with productivity q is

$$ng(r)f\left(\frac{r}{n}, q\right) = g(r)f(r, nq) = g(r)nq\theta\left(\frac{r}{nq}\right)$$

since the time spent in each worker is $t = 1/n$.[27] The manager's rent is

$$g(r)f(r, nq) - cn = g(r)nq\theta\left(\frac{r}{nq}\right) - cn,$$

[26] Meaning that for $(x, y) \in \mathbb{R}^2_{++}$, $\partial f(x, y)/\partial x > 0$, $\partial f(x, y)/\partial y > 0$, and $x \mapsto \partial f(x, y)/\partial x$ and $y \mapsto \partial f(x, y)/\partial y$ are strictly decreasing over $\mathbb{R}_{++}$.

[27] This claim follows from the Jensen's integral inequality as follows. Let $\mu \in \mathcal{M}([0, 1])$ be a probability distribution of time spent on workers so that $\mu(B)$ is the fraction of workers who get supervision time in B, for each Borel subset B of $[0, 1]$, and $1_{1/n} \in \mathcal{M}([0, 1])$ be the probability distribution degenerate on $1/n$. Then $n \int t\mathrm{d}\mu(t) = 1$ and $\int g(r)nq\theta(rt/q)\,\mathrm{d}\mu(t) = nqg(r)\int \theta(rt/q)\,\mathrm{d}\mu(t) \leq nqg(r)\theta\left(r\int t\mathrm{d}\mu(t)/q\right) = nqg(r)q\theta(r/nq) = \int g(r)nq\theta(rt/q)\,\mathrm{d}1_{1/n}(t)$.

where c is the wage paid by the manager to the workers.

To represent the above setting in the general framework of Section 4, let, in addition to Z and ν as above, the set of contracts be $C = \mathbb{R}_+$, interpreted as the set of possible wages, and the contract correspondence be $\mathbb{C} \equiv C$. The set of feasible matches for managers is $X = \{n1_{(z,c)} : (z,c) \in Z \times C \text{ and } n \in \mathbb{R}_+\}$ since managers can hire several workers all of the same type. Occupations are the same as in the general framework: $A = \{w, s, m\}$. Finally, preferences are defined by specifying payoff functions as follows:

$$
\begin{aligned}
&U_z(w, 1_{(z',c)}) = c \text{ for each } 1_{(z',c)} \in X_w, \\
&U_z(s, 1_{(\emptyset,c)}) = 0 \text{ for each } 1_{(\emptyset,c)} \in X_s, \text{ and} \\
&U_z(m, n1_{(z',c)}) = g(r(z))f(r(z), nq(z')) - cn \text{ for each } n1_{(z',c)} \in X_m.
\end{aligned}
$$

We will establish existence and obtain a characterization of stable matchings for the setting of this section under the following simplifying assumptions. We let $Z = [\underline{z}, \bar{z}]$ with $0 \leq \underline{z} < \bar{z} < \infty$ and assume that $q(\underline{z}) > 0$, $r(\underline{z}) > 0$ and $g(r) > 0$ for each $r > 0$; thus, $g(r(\underline{z})) > 0$. A market satisfying these assumptions as well as the additional specifications described above is a *Rosen market* and denoted by E_{rosen}.

Concerning the existence of stable matchings, note that a Rosen market fails to satisfy two assumptions of our existence result, namely the contract correspondence fails to be compact-valued and the market fails to be bounded. Nevertheless, by considering a sequence of truncated Rosen markets that satisfy our assumptions, we show that stable matchings exist.

Corollary 3 *Every Rosen market has a stable matching.*

We next provide a characterization of stable matchings in Rosen markets that is analogous to the formulation in Rosen (1982). The following concepts are needed. Let $r \in r(Z)$, $q \in q(Z)$ and $w > 0$. If n solves $\max_{n' \in \mathbb{R}_+} [g(r)f(r, n'q) - wn'q]$, then

$$w = g(r)\frac{\partial f(r, nq)}{\partial y} = g(r)\frac{\partial f\left(\frac{r}{nq}, 1\right)}{\partial y}$$

since $\partial f/\partial y$ is homogeneous of degree zero. Thus, there is a continuous function $\phi : r(Z) \times \mathbb{R}_{++} \to \mathbb{R}_{++}$ such that $nq = \phi(r, w)$. The manager's rent is then

$$g(r)\frac{\partial f\left(\frac{r}{nq}, 1\right)}{\partial x} r = g(r)\frac{\partial f\left(\frac{r}{\phi(r,w)}, 1\right)}{\partial x} r.$$

The above functions and formulas are used to define, for each manager of type z, the optimal number of workers of type z' he wants to hire at wage $wq(z')$ and the corresponding rent. Define $n : Z^2 \times \mathbb{R}_{++} \to \mathbb{R}_{++}$ by setting, for each $(z, z', w) \in Z^2 \times \mathbb{R}_{++}$,

$$n(z, z', w) = \frac{\phi(r(z), w)}{q(z')}.$$

Moreover, define $R : Z \times \mathbb{R}_{++} \to \mathbb{R}_{+}$ by setting, for each $(z, w) \in Z \times \mathbb{R}_{++}$,

$$R(z, w) = g(r(z))\frac{\partial f\left(\frac{r(z)}{\phi(r(z),w)}, 1\right)}{\partial x} r(z).$$

Theorem 3 *A matching μ of a Rosen market is stable if and only if there exists $\lambda \in \mathcal{M}(Z^2)$ and $w > 0$ such that*

$$\lambda(B \times Z) + \int_{Z \times B} n(z, z', w)\mathrm{d}\lambda(z, z') = \nu(B) \text{ for each measurable } B \subseteq Z, \quad (1)$$

$$\mathrm{supp}(\lambda) \subseteq \{z \in Z : R(z, w) \geq wq(z)\} \times \{z \in Z : wq(z) \geq R(z, w)\}, \text{ and} \quad (2)$$

$$\mu = \lambda \circ h^{-1}, \quad (3)$$

where $h : Z^2 \to Z \times X$ is defined by setting, for each $(z, z') \in Z^2$,

$$h(z, z') = (z, n(z, z', w)1_{(z', wq(z'))}).$$

As Theorem 3 illustrates, our framework is tractable and our stability notion admits a simple characterization in applied settings; they can therefore be used to clarify important economic questions and highlight what forces might explain them. We give one such example when $q(z) = r(z) = z$ and the technology takes the form $g(z)z^{\alpha}(nz')^{1-\alpha}$ with $\alpha = 1/2$. If $g \equiv 1$, then each individual is indifferent between being a manager or a worker and each individual of type z has an income (wage or rent) equal to $z/2$.[28] In contrast, if $g(z) = z$, then individual income is no longer necessarily linear in the type. For example, when $Z = \{z_1, \ldots, z_4\}$, it is possible to construct a stable matching where individuals of type z_1 and z_2 are workers, individuals of type z_3 and z_4 are managers, each person strictly prefers his occupation to the alternative one and, for some $w > 0$, workers' income is wz while managers' income is $z^3/4w$.[29] In this latter example, any change that leads to a decrease in w causes an increase in the income of those in the top and a decrease in the income of those in the bottom of the income distribution.[30] In addition, as a result of decrease in w, there is less inequality at the bottom (since the function $z \mapsto wz$ describing the income of those in the bottom of the distribution becomes flatter) and more at the top of the income distribution (since the function $z \mapsto z^3/4w$ describing the income of those in the top of the distribution becomes steeper).

6.3 Further applications

In the working paper version we consider additional applications of our framework, which we summarize here, to illustrate its flexibility.

Specifically, we show how our framework can capture the settings of Garicano & Rossi-Hansberg (2004) and Garicano & Rossi-Hansberg (2006), and how it can be extended to accommodate Lucas's (1978) model. Both Garicano & Rossi-Hansberg (2004) and Garicano & Rossi-Hansberg (2006) require feasible matches for managers that depend on the types of the workers hired. This dependence arises because the measure of workers that a manager can hire is determined by the time constraint of

[28] Indeed, if $\alpha = 1/2$ and $g \equiv 1$, then $R(z, w) \geq wq(z)$ if and only if $1/2 \geq w$. It then must be that $w = 1/2$ in any stable matching since otherwise there would be no worker or no managers; thus, $R(z, w) = wq(z) = z/2$ for each $z \in Z$.

[29] If $\alpha = 1/2$, $Z = \{z_1, \ldots, z_4\}$ and g is the identity, then pick $w \in (2z_2, 2z_3)$, which implies that $R(z, w) > wq(z)$ for each $z \in \{z_3, z_4\}$ and $R(z, w) < wq(z)$ for each $z \in \{z_1, z_2\}$. Let ν be such that $\nu(z_3) = \nu(z_4) = 1$, $\nu(z_2) = n(z_4, z_2, w)$ and $\nu(z_3) = n(z_3, z_1, w)$. Then λ such that $\lambda(z_3, z_1) = \lambda(z_4, z_2) = 1$ yields a stable matching. Payoffs are wz for each $z \in \{z_1, z_2\}$ and $R(z, w) = z^3/4w$ for each $z \in \{z_3, z_4\}$.

[30] Such a decrease in w would occur, for example, if $\nu(z_1)$ and $\nu(z_2)$ increase by a small amount.

the manager and is increasing in the quality of the workers. In Garicano & Rossi-Hansberg (2004) all workers have the same quality but in Garicano & Rossi-Hansberg (2006) a manager can hire workers of finitely many different qualities.

In Lucas (1978) there is a capital market in addition to a labor market with occupational choice. The easiest approach to represent this setting is to consider, for each rental price of capital, the resulting market with occupational choice with the amount of capital hired by a firm being included in the contract between the manager and workers. An equilibrium is then a rental price of capital and a matching such that the matching is stable given the rental price and the capital market clears.

7 Concluding remarks

In this paper we provided a formalization of large many-to-one matching markets with occupational choice and a notion of a stable matching for them. This was done with the goal of being able to include the settings of Lucas (1978), Rosen (1982), Garicano & Rossi-Hansberg (2004) and Garicano & Rossi-Hansberg (2006) in our framework, while at the same time extending the two-sided, one-to-one matching setting of Greinecker & Kah (2021).

The large matching markets we consider are, as in Greinecker & Kah (2021), formalized using a distributional approach. Thus, the set of individuals is not explicitly included, rather only the distribution of individuals' types is present in the description of the market. This approach is tractable and this has been illustrated in Section 6.2 in the context of Rosen's (1982) setting where stable matchings are fully characterized.

The above tractability makes our setting potentially useful to address the implications of stability in large labor markets, in particular, for income inequality. We aim to do so in future work.

The representation of Lucas's (1978) setting in our framework required the introduction of capital, which proved to be a relatively easy extension. This suggests that other important elements can be added to our framework.

A Appendix

A.1 Preliminary lemmas

This section presents some lemmas on the support of a measure and on the existence of convergent subsequences for which we could not find a reference. Lemma 1 shows that the support of the image $\mu \circ h^{-1}$ of a measure μ under a homeomorphism h is the image of the support of μ.

Lemma 1 *Let Y and Y' be separable metric spaces, $\mu \in \mathcal{M}(Y)$, $h : Y \to Y'$ be a homeomorphism and $\nu = \mu \circ h^{-1}$. Then $\operatorname{supp}(\nu) = h(\operatorname{supp}(\mu))$ and $\operatorname{supp}(\mu) = h^{-1}(\operatorname{supp}(\nu))$.*

Proof. Note first that $\nu(\operatorname{supp}(\nu)) = \nu(Y') = \mu(h^{-1}(Y')) = \mu(Y) = \mu(\operatorname{supp}(\mu))$ and, since $\operatorname{supp}(\mu) = h^{-1}(h(\operatorname{supp}(\mu)))$,

$$\begin{aligned}\nu(\operatorname{supp}(\nu)) &\geq \nu(h(\operatorname{supp}(\mu))) = \mu(h^{-1}(h(\operatorname{supp}(\mu))))\\ &= \mu(\operatorname{supp}(\mu)) \geq \mu(h^{-1}(\operatorname{supp}(\nu))) = \nu(\operatorname{supp}(\nu)).\end{aligned}$$

Thus,

$$\mu(\operatorname{supp}(\mu)) = \mu(h^{-1}(\operatorname{supp}(\nu))) = \nu(h(\operatorname{supp}(\mu))) = \nu(\operatorname{supp}(\nu)).$$

Since $h^{-1}(\operatorname{supp}(\nu))$ is closed, $\operatorname{supp}(\mu) \subseteq h^{-1}(\operatorname{supp}(\nu))$ and, hence, $h(\operatorname{supp}(\mu)) \subseteq h(h^{-1}(\operatorname{supp}(\nu))) = \operatorname{supp}(\nu)$. Letting f denote the inverse of h, we have that $h(F) = f^{-1}(F)$ is closed for each closed subset F of Y. Thus, it follows that $\operatorname{supp}(\nu) \subseteq h(\operatorname{supp}(\mu))$.

It follows from $\operatorname{supp}(\nu) = h(\operatorname{supp}(\mu))$ that $h^{-1}(\operatorname{supp}(\nu)) = h^{-1}(h(\operatorname{supp}(\mu))) = \operatorname{supp}(\mu)$. ■

Lemma 2 shows that the support correspondence is lower hemicontinuous.

Lemma 2 *If Y is a separable metric space, then the correspondence $\mu \mapsto \operatorname{supp}(\mu)$, from $\mathcal{M}(Y)$ to Y, is lower hemicontinuous.*

Proof. We have that $\mathcal{M}(Y)$ is a separable metrizable space by Varadarajan (1958, Theorem 3.1). The conclusion then follows from (the proof of) Aliprantis & Border (2006, Theorem 17.14, p. 563). ■

Lemma 3 provides conditions for the existence of a convergent subsequence.

Lemma 3 *If Y a separable metrizable space and $\{\mu_k\}_{k=1}^{\infty}$ is a tight sequence in $\mathcal{M}(Y)$ such that, for some $R > 0$, $\mu_k(Y) \leq R$ for all $k \in \mathbb{N}$, then $\{\mu_k\}_{k=1}^{\infty}$ has a convergent subsequence.*

Proof. The proof reduces to the case of probability measures as follows: Suppose first that there is a subsequence $\{\mu_{k_j}\}_{j=1}^{\infty}$ such that $\mu_{k_j}(Y) \to 0$. Then this subsequence converges to the zero measure. Thus, we may assume that there is $\varepsilon > 0$ such that $\mu_k(Y) \geq \varepsilon$ for all but finitely many k. The sequence $\{\mu_k(Y)\}_k$ is bounded, thus we may assume that it converges; let $\theta = \lim_k \mu_k(Y)$. Consider $\{\nu_k\}_{k=1}^{\infty}$ with $\nu_k(B) = \mu_k(B)/\mu_k(Y)$ for each Borel B. This is a tight family of probability measures, so it has a convergent subsequence $\{\nu_{k_j}\}_{j=1}^{\infty}$; let $\nu = \lim_j \nu_{k_j}$, $\mu = \theta\nu$ and B has μ-null boundary, which happens if and only if it has ν-null boundary since $\theta \geq \varepsilon$. Then $\mu_{k_j}(B) = \mu_{k_j}(Y)\mu_{k_j}(B)/\mu_{k_j}(Y) \to \theta\nu(B)$ and, hence, $\mu_{k_j} \to \mu$. ■

A.2 Proof of Theorem 1

In a stable matching of Gale & Shapley's (1962) marriage market, (i) each woman cannot find a man (including the empty man) that she prefers to her husband and who would prefer her to his wife, i.e. each woman cannot find a man in her targets that she prefers to her husband, and (ii) each man cannot find a woman in his targets that he prefers to his wife. It turns out that (ii) implies (i) and Theorem 1 is the analog of this in our setting.

We now turn to the proof of Theorem 1. Note first that $\mathrm{supp}(\mu) \subseteq S(\mu)$ implies that $\mathrm{supp}(\mu) \subseteq S_M(\mu) \cap IR(\mu)$ since $S(\mu) \subseteq S_M(\mu) \cap IR(\mu)$.

Conversely, suppose that $\mathrm{supp}(\mu) \subseteq S_M(\mu) \cap IR(\mu)$. Let $(z, \delta) \in \mathrm{supp}(\mu)$ and assume, in order to reach a contradiction, that $(z, \delta) \notin S(\mu)$. Since $(z, \delta) \in \mathrm{supp}(\mu) \subseteq S_M(\mu) \cap IR(\mu)$, it follows that there is $(z^*, c) \in Z \times C$ and $\bar{z} \in Z$ such that $(z^*, c) \in$

$T^w_{\bar{z}}(\mu)$, $\bar{z} = z$ or $(\bar{z}, \bar{c}) \in \text{supp}(\delta)$ for some $\bar{c} \in C$, (1) $(w, 1_{(z^*,c)}, \mu) \succ_{\bar{z}} (m, \delta, \mu)$ if $\bar{z} = z$ and $\delta \in X$, (2) $(w, 1_{(z^*,c)}, \mu) \succ_{\bar{z}} (w, 1_{(z,\bar{c})}, \mu)$ if $(\bar{z}, \bar{c}) \in \text{supp}(\delta)$ and (3) $(w, 1_{(z^*,c)}, \mu) \succ_{\bar{z}} (s, \delta, \mu)$ if $\bar{z} = z$ and $\delta \in X_\emptyset \setminus X$. Since $(z^*, c) \in T^w_{\bar{z}}(\mu)$, it follows that $c \in \mathbb{C}(z^*, \bar{z})$.

We now show that $(\bar{z}, c) \in T^m_{z^*}(\mu)$. Indeed, we have that $c \in \mathbb{C}(z^*, \bar{z})$ and $(z, \delta) \in \text{supp}(\mu)$. Thus, in case (1), the conclusion follows by condition (c) in the definition of $T^m_{z^*}(\mu)$ since $\bar{z} = z$ and $(w, 1_{(z^*,c)}, \mu) \succ_z (m, \delta, \mu)$; in case (2), the conclusion follows by condition (a) in the definition of $T^m_{z^*}(\mu)$ since $(\bar{z}, \bar{c}) \in \text{supp}(\delta)$ and $(w, 1_{(z^*,c)}, \mu) \succ_{\bar{z}} (w, 1_{(z,\bar{c})}, \mu)$; and, in case (3), the conclusion follows by condition (b) in the definition of $T^m_{z^*}(\mu)$ since $\bar{z} = z$ and $(w, 1_{(z^*,c)}, \mu) \succ_z (s, \delta, \mu)$.

Since $(z^*, c) \in T^w_{\bar{z}}(\mu)$, there is $\tilde{\delta} \in X$ such that $(\bar{z}, c) \in \text{supp}(\tilde{\delta})$ and (a) or (b) or (c) in the definition of $T^w_{\bar{z}}(\mu)$ holds. In either case, we will show that $\text{supp}(\mu) \subseteq S_M(\mu)$ fails, which is a contradiction to $\text{supp}(\mu) \subseteq S_M(\mu) \cap IR(\mu)$.

Suppose that condition (a) in the definition of $T^w_{\bar{z}}(\mu)$ holds. Then, in addition, $\text{supp}(\tilde{\delta}) \setminus \{(\bar{z}, c)\} \subseteq T^m_{z^*}(\mu)$, and there is $(z', c', \delta') \in Z \times C \times X$ such that $(z', \delta') \in \text{supp}(\mu)$, $(z^*, c') \in \text{supp}(\delta')$ and $(m, \tilde{\delta}, \mu) \succ_{z^*} (w, 1_{(z',c')}, \mu)$. Since $(\bar{z}, c) \in T^m_{z^*}(\mu)$, it follows that $(z', \delta') \in \text{supp}(\mu) \setminus S_M(\mu)$ since (ii) of the definition of $S_M(\mu)$ fails. Indeed, $(z', \delta') \in \text{supp}(\mu)$, $(z^*, c') \in \text{supp}(\delta')$, $\text{supp}(\tilde{\delta}) \subseteq T^m_{z^*}(\mu)$ and $(m, \tilde{\delta}, \mu) \succ_{z^*} (w, 1_{(z',c')}, \mu)$.

Suppose next that condition (b) in the definition of $T^w_{\bar{z}}(\mu)$ holds. Then, in addition, $\text{supp}(\tilde{\delta}) \setminus \{(\bar{z}, c)\} \subseteq T^m_{z^*}(\mu)$, and there is $\delta' \in X_\emptyset \setminus X$ such that $(z^*, \delta') \in \text{supp}(\mu)$ and $(m, \tilde{\delta}, \mu) \succ_{z^*} (s, \delta', \mu)$. Since $(\bar{z}, c) \in T^m_{z^*}(\mu)$, it follows that $(z^*, \delta') \in \text{supp}(\mu) \setminus S_M(\mu)$ since (iii) of the definition of $S_M(\mu)$ fails. Indeed, $(z^*, \delta') \in \text{supp}(\mu)$, $\text{supp}(\tilde{\delta}) \subseteq T^m_{z^*}(\mu)$ and $(m, \tilde{\delta}, \mu) \succ_{z^*} (s, \delta', \mu)$.

Finally, suppose that condition (c) in the definition of $T^w_{\bar{z}}(\mu)$ holds. Then, in addition, there is $\delta' \in X_\emptyset \setminus X$ such that $(z^*, \delta') \in \text{supp}(\mu)$, $\text{supp}(\tilde{\delta}) \setminus \{(\bar{z}, c)\} \subseteq T^m_{z^*}(\mu) \cup \text{supp}(\delta')$ and $(m, \tilde{\delta}, \mu) \succ_{z^*} (m, \delta', \mu)$. Since $(\bar{z}, c) \in T^m_{z^*}(\mu)$, it follows that $(z^*, \delta') \in \text{supp}(\mu) \setminus S_M(\mu)$ since (i) of the definition of $S_M(\mu)$ fails. Indeed, $(z^*, \delta') \in \text{supp}(\mu)$, $\text{supp}(\tilde{\delta}) \subseteq T^m_{z^*}(\mu) \cup \text{supp}(\delta')$ and $(m, \tilde{\delta}, \mu) \succ_{z^*} (m, \delta', \mu)$.

A.3 Proof of Theorem 2

The first step in the proof of our existence result consists in the following lemma, which considers the special case where Z, X and C are finite. Our approach in this special case builds on ideas from Section S.10 in Che, Kim & Kojima (2019) but requires many changes since there are externalities in preferences, workers' preferences are not strict and there is occupational choice. There are three main changes which we now briefly describe.[31]

Our approach in the special case where Z, X and C are finite is similar to the one in Che, Kim & Kojima (2019) to the extent that we use a fixed point argument. In their paper, stable matchings are fixed points of a correspondence whose domain consists of pairs of matchings and measures of available workers. In our case, (i) we consider a sequence of correspondences, each of which has a fixed point, but only limit points of the sequence of fixed points will yield a stable matching, (ii) the domain of each correspondence consists of pairs of allocations of types to occupations and matches and measures of available workers and contracts and (iii) the measure of available workers and contracts depends on the allocations of types to occupations and matches in a discontinuous way and, thus, needs to be suitably approximated.

Lemma 4 *If E is a rational and continuous market such that Z, X and C are finite, then E has a stable matching.*

Proof. Note first that $Z_\emptyset$, $X_\emptyset$ and Δ are also finite. Define $\bar{\tau} \in \mathbb{R}^{Z\times\Delta}$ by setting, for each $(z, a, \delta) \in Z \times \Delta$,

$$\bar{\tau}(z, a, \delta) = \begin{cases} 0 & \text{if } \{z\} \times \operatorname{supp}(\delta) \not\subseteq \operatorname{graph}(\mathbb{C}) \text{ and } a \neq w, \\ 0 & \text{if } \operatorname{supp}(\delta_Z) \times \{z\} \times \operatorname{supp}(\delta_C) \not\subseteq \operatorname{graph}(\mathbb{C}) \text{ and } a = w, \\ \nu(z) & \text{otherwise.} \end{cases}$$

Let $\bar{\kappa} \in \mathbb{R}^{Z\times Z\times C}$ be such that $\bar{\kappa}(z, z', c) = \nu(z')$ if $(z, z', c) \in \operatorname{graph}(\mathbb{C})$, and $\bar{\kappa}(z, z', c) = 0$ otherwise.

[31] See the working paper version for a more detailed outline of the proof of Theorem 2.

Define

$$\mathcal{T} = \{\tau \in \mathbb{R}_+^{Z\times\Delta} : \tau(z,a,\delta) \leq \bar{\tau}(z,a,\delta) \text{ and } \sum_{(a,\delta)\in\Delta} \tau(z,a,\delta) \leq \nu(z)$$
$$\text{for each } (z,a,\delta) \in Z\times\Delta\} \text{ and}$$
$$\mathcal{K} = \{\kappa \in \mathbb{R}_+^{Z\times Z\times C} : \kappa(z,z',c) \leq \bar{\kappa}(z,z',c) \text{ for each } (z,z',c) \in Z\times Z\times C\}.$$

Note that $\mathcal{T}$ and $\mathcal{K}$ are nonempty, convex, and compact subsets of Euclidean spaces. Each $\tau \in \mathcal{T}$ is an allocation of types to occupations and matches and for each $\kappa \in \mathcal{K}$, we interpret $\kappa(z,z',c)$ as the measure of workers of type z' and contract c that are available to z. Below, we will consider allocations τ that maximize preferences subject to the constraints that each manager type does not hire more workers than available to him (given by κ) and that the measure of each worker type allocated to a manager does not exceed the manager's demand (given by some reference μ).

Let $u : Z\times\Delta\times\mathcal{M}(Z\times X_\emptyset) \to \mathbb{R}$ be a continuous utility function that represent preferences. We normalize so that $u \geq 1$. For each $n \in \mathbb{N}$, let $u_n = u^n$. Since $x \mapsto x^n$ is strictly increasing on $[1,\infty)$, u_n and u represent the same preferences.

Define $d : \mathcal{T} \to \mathbb{R}_+^{Z\times X_\emptyset}$ by setting, for each $\tau \in \mathcal{T}$ and $(z,\delta) \in Z\times X_\emptyset$,

$$d(\tau)(z,\delta) = \begin{cases} \tau(z,m,\delta) & \text{if } \delta \in X, \\ \tau(z,s,\delta) & \text{if } \delta \in X_\emptyset \setminus X. \end{cases}$$

The function d is continuous. We abuse notation and, for each $(z,a,\delta,\tau) \in Z\times\Delta\times\mathcal{T}$, write $u(z,a,\delta,\tau)$ for $u(z,a,\delta,d(\tau))$ and analogously for u_n. We also write $(a,\delta,\tau) \succ_z (a',\delta',\tau)$ for $(a,\delta,d(\tau)) \succ_z (a',\delta',d(\tau))$, where $(a',\delta') \in \Delta$.

For each $n \in \mathbb{N}$, let $D_n : \mathcal{T}\times\mathcal{K} \rightrightarrows \mathcal{T}$ be defined by setting, for each $(\mu,\kappa) \in \mathcal{T}\times\mathcal{K}$,

$$D_n(\mu,\kappa) = \{\tau \in \mathcal{T} : \tau \in \arg\max_{\tau'\in\mathcal{T}} \sum_{z\in Z,(a,\delta)\in\Delta} u_n(z,a,\delta,\mu)\tau'(z,a,\delta)$$
$$\text{subject to } \sum_{(a,\delta)\in\Delta} \tau'(z,a,\delta) = \nu(z) \text{ for all } z \in Z,$$
$$\sum_{\delta\in X} \tau'(z,m,\delta)\delta(z',c) \leq \kappa(z,z',c) \text{ for all } (z,z',c) \in Z\times Z\times C, \text{ and}$$
$$\tau'(z,w,1_{(z',c)}) \leq \sum_{\delta\in X} \mu(z',m,\delta)\delta(z,c) \text{ for all } (z,z',c) \in Z\times Z\times C\}.$$

Claim 1 *D_n is upper hemicontinuous with nonempty, compact and convex values.*

Proof. It follows by the linearity of the objective function together with the convexity of the constraint set that D_n has convex values. It follows from Berge's maximum theorem that D_n is upper hemicontinuous with nonempty and compact values. To see this, first note that the objective function is continuous and that the constraint set, denoted by $\Phi(\mu, \kappa)$, is contained in the compact set $\mathcal{T}$. It is clear that Φ is upper hemicontinuous with compact values. To see that Φ has nonempty values, define $\bar{\tau} \in \mathcal{T}$ as follows. For each $z \in Z$, let $c_z \in \mathbb{C}(z, \emptyset)$, $\bar{\tau}(z, s, 1_{(\emptyset, c_z)}) = \nu(z)$ and $\bar{\tau}(z, a, \delta) = 0$ for each $(a, \delta) \in \Delta \backslash \{(s, 1_{(\emptyset, c_z)})\}$. We then have that $\bar{\tau} \in \Phi(\mu, \kappa)$ for each $(\mu, \kappa) \in \mathcal{T} \times \mathcal{K}$. Finally, to see that Φ is lower hemicontinuous, let $(\mu, \kappa) \in \mathcal{T} \times \mathcal{K}$, $O \subseteq \mathcal{T}$ be an open set such that $\Phi(\mu, \kappa) \cap O \neq \emptyset$, and $\tau \in \Phi(\mu, \kappa) \cap O$. Let $\hat{\tau} = \lambda\tau + (1-\lambda)\bar{\tau} \in O$ for some $\lambda \in (0, 1)$. Note that for each $z \in Z$, $\sum_{(a,\delta)\in\Delta} \hat{\tau}(z, a, \delta) = \nu(z)$, $\sum_{\delta\in X} \hat{\tau}(z, m, \delta)\delta(z', c) < \kappa(z, z', c)$ for each $(z, z', c) \in Z \times Z \times C$ such that $\kappa(z, z', c) > 0$ and $\hat{\tau}(z, w, 1_{(z',c)}) < \sum_{\delta\in X} \mu(z', m, \delta)\delta(z, c)$ for each $(z, z', c) \in Z \times Z \times C$ such that $\sum_{\delta\in X} \mu(z', m, \delta)\delta(z, c) > 0$. Thus, there is an open neighborhood V of (μ, κ) such that $\hat{\tau} \in \Phi(\mu', \kappa') \cap O$ for each $(\mu', \kappa') \in V$. ■

Claim 2 *If $(\mu, \kappa) \in \mathcal{T} \times \mathcal{K}$, $\tau \in D_n(\mu, \kappa)$ and $(z, a, \delta') \in Z \times \Delta$ is such that $\tau(z, a, \delta') > 0$, then $\tau(z, w, 1_{(\hat{z},\hat{c})}) = \sum_{\delta\in X} \mu(\hat{z}, m, \delta)\delta(z, \hat{c})$ for each $(\hat{z}, \hat{c}) \in Z \times C$ such that $(w, 1_{(\hat{z},\hat{c})}, \mu) \succ_z (a, \delta', \mu)$.*

Proof. If not, then $\tau(z, w, 1_{(\hat{z},\hat{c})}) < \sum_{\delta\in X} \mu(\hat{z}, m, \delta)\delta(z, \hat{c})$ for some $(\hat{z}, \hat{c}) \in Z \times C$ such that $(w, 1_{(\hat{z},\hat{c})}, \mu) \succ_z (w, 1_{(z',c')}, \mu)$. Then $\sum_{\delta\in X} \mu(\hat{z}, m, \delta)\delta(z, \hat{c}) > 0$ and, hence, $(\hat{z}, z, \hat{c}) \in \text{graph}(\mathbb{C})$. Thus, increase $\tau(z, w, 1_{(\hat{z},\hat{c})})$ and decrease $\tau(z, a, \delta')$ by the same amount $\varepsilon \in (0, \tau(z, a, \delta'))$ such that $\tau(z, w, 1_{(\hat{z},\hat{c})}) + \varepsilon < \sum_{\delta\in X} \mu(\hat{z}, m, \delta)\delta(z, \hat{c})$. This increases the objective function in $D_n(\mu, \kappa)$ while satisfying the constraints, thus contradicting $\tau \in D_n(\mu, \kappa)$. ■

For each $\mu \in \mathcal{T}$ and $(z, z', c) \in Z \times Z \times C$, let

$$W(z, z', c, \mu) = \{(a, \delta) \in \Delta : u(z', w, 1_{(z,c)}, \mu) > u(z', a, \delta, \mu)\}.$$

Let $g : \mathcal{T} \to \mathcal{K}$ be defined by setting, for each $\mu \in \mathcal{T}$ and $(z, z', c) \in Z \times Z \times C$,

$$g(\mu)(z,z',c) = \begin{cases} \sum_{(a,\delta)\in W(z,z',c,\mu)} \mu(z',a,\delta) & \text{if } (z,z',c) \in \text{graph}(\mathbb{C}), \\ 0 & \text{otherwise.} \end{cases}$$

To see that $g(\mu) \in \mathcal{K}$, first note that if $(z, z', c) \notin \text{graph}(\mathbb{C})$, $g(\mu)(z, z', c) = 0$. If $(z, z', c) \in \text{graph}(\mathbb{C})$, then, since $\mu \in \mathcal{T}$, $0 \le g(\mu)(z, z', c) \le \nu(z') = \bar{\kappa}(z, z', c)$.

The function g may fail to be continuous and, thus, we will consider a continuous approximation to it. For each $n \in \mathbb{N}$ and $(z, z', c) \in Z \times Z \times C$, let $\alpha_{n,(z,z',c)} : \Delta \times \mathcal{T} \to [0, 1]$ be defined by setting, for each $(a, \delta, \mu) \in \Delta \times \mathcal{T}$,

$$\alpha_{n,(z,z',c)}(a,\delta,\mu) = n \max\left\{0, \min\left\{u(z',w,1_{(z,c)},\mu) - u(z',a,\delta,\mu), \frac{1}{n}\right\}\right\}.$$

Let $g_n : \mathcal{T} \to \mathcal{K}$ be defined by setting, for each $\mu \in \mathcal{T}$ and $(z, z', c) \in Z \times Z \times C$,

$$g_n(\mu)(z,z',c) = \begin{cases} \sum_{(a,\delta)\in \Delta} \alpha_{n,(z,z',c)}(a,\delta,\mu)\mu(z',a,\delta) & \text{if } (z,z',c) \in \text{graph}(\mathbb{C}), \\ 0 & \text{otherwise.} \end{cases}$$

We have that g_n is continuous since $\alpha_{n,(z,z',c)}$ is continuous for each $(z, z', c) \in Z \times Z \times C$. Note that

$$\alpha_{n,(z,z',c)}(a,\delta,\mu) \in \begin{cases} \{0\} & \text{if } u(z',a,\delta,\mu) \ge u(z',w,1_{(z,c)},\mu), \\ (0,1) & \text{if } u(z',w,1_{(z,c)},\mu) - \frac{1}{n} < u(z',a,\delta,\mu) < u(z',w,1_{(z,c)},\mu), \\ \{1\} & \text{if } u(z',a,\delta,\mu) \le u(z',w,1_{(z,c)},\mu) - \frac{1}{n}. \end{cases}$$

Hence, it follows that

$$g_n(\mu)(z,z',c) \le g(\mu)(z,z',c) \tag{4}$$

for each $\mu \in \mathcal{T}$ and $(z, z', c) \in Z \times Z \times C$ since

$$g(\mu)(z,z',c) = \begin{cases} \sum_{(a,\delta)\in \Delta} \alpha_{(z,z',c)}(a,\delta,\mu)\mu(z',a,\delta) & \text{if } (m,w,c) \in \text{graph}(\mathbb{C}), \\ 0 & \text{otherwise} \end{cases}$$

with

$$\alpha_{(z,z',c)}(a,\delta,\mu) = \begin{cases} 1 & \text{if } u(z',a,\delta,\mu) < u(z',w,1_{(z,c)},\mu), \\ 0 & \text{otherwise.} \end{cases}$$

To see that $g_n(\mu) \in \mathcal{K}$, note that $0 \leq g_n(\mu) \leq g(\mu) \leq \bar{\kappa}$.

Let $f_n : \mathcal{T} \to \mathcal{K}$ be defined by setting, for each $\mu \in \mathcal{T}$ and $(z, z', c) \in Z \times Z \times C$,

$$f_n(\mu)(z, z', c) = \mu(z', w, 1_{(z,c)}) + \frac{1}{n} g_n(\mu)(z, z', c).$$

To see that $f_n(\mu) \in \mathcal{K}$, note that if $(z, z', c) \notin \text{graph}(\mathbb{C})$,

$$\mu(z', w, 1_{(z,c)}) = g_n(\mu)(z, z', c) = 0,$$

and hence $f_n(\mu)(z, z', c) = 0$. If $(z, z', c) \in \text{graph}(\mathbb{C})$, then

$$\begin{aligned} 0 &\leq \mu(z', w, 1_{(z,c)}) + \frac{1}{n} g_n(\mu)(z, z', c) \\ &\leq \mu(z', w, 1_{(z,c)}) + g(\mu)(z, z', c) \leq \nu(z') = \bar{\kappa}(z, z', c). \end{aligned}$$

We have that f_n is continuous since so is g_n.

Let $\Psi_n : \mathcal{T} \times \mathcal{K} \rightrightarrows \mathcal{T} \times \mathcal{K}$ be defined by setting, for each $(\mu, \kappa) \in \mathcal{T} \times \mathcal{K}$,

$$\Psi_n(\mu, \kappa) = D_n(\mu, \kappa) \times \{f_n(\mu)\}.$$

It follows from the continuity of f_n and from Claim 1 that Ψ_n is upper hemicontinuous with nonempty, compact and convex values. Hence, by Kakutani fixed point theorem, let (μ_n, κ_n) be a fixed point of Ψ_n. Thus, $\mu_n \in D_n(\mu_n, \kappa_n)$ and $\kappa_n = f_n(\mu_n)$.

Since $\mathcal{T} \times \mathcal{K}$ is compact, taking a subsequence if necessary, we may assume that $\{(\mu_n, \kappa_n)\}_{n=1}^{\infty}$ converges; let $(\mu, \kappa) = \lim_{n\to\infty}(\mu_n, \kappa_n)$. For each n, we have $\kappa_n = f_n(\mu_n)$, and so

$$\kappa(z, z', c) = \lim_{n\to\infty} f_n(\mu_n)(z, z', c) = \mu(z', w, 1_{(z,c)}) \tag{5}$$

for each $(z, z', c) \in Z \times Z \times C$. Let

$$\mu^* = d(\mu)$$

and $\mu_n^* = d(\mu_n)$ for each $n \in \mathbb{N}$.

For each $z \in Z$ and $n \in \mathbb{N}$, it follows from $\mu_n \in D_n(\mu_n, \kappa_n)$ that

$$\sum_{(z',c)\in Z\times C} \mu_n(z, w, 1_{(z',c)}) \leq \sum_{(z',c)\in Z\times C} \sum_{\delta\in X} \mu_n(z', m, \delta)\delta(z, c) \leq \sum_{(z',c)\in Z\times C} \kappa_n(z', z, c).$$

By (5), $\lim_n \sum_{(z',c)\in Z\times C} \kappa_n(z',z,c) = \sum_{(z',c)\in Z\times C} \mu(z,w,1_{(z',c)})$ and, hence,

$$\sum_{(z',c)\in Z\times C} \mu(z,w,1_{(z',c)}) = \sum_{(z',c)\in Z\times C} \sum_{\delta\in X} \mu(z',m,\delta)\delta(z,c) \text{ for each } z\in Z.$$

Thus, for each $z\in Z$,

$$\sum_{(z',\delta)\in Z\times X} \mu^*(z',\delta)\delta_Z(z) = \sum_{(z',c)\in Z\times C} \sum_{\delta\in X} \mu(z',m,\delta)\delta(z,c) = \sum_{(z',c)\in Z\times C} \mu(z,w,1_{(z',c)}). \tag{6}$$

Claim 3 *μ^* is a matching.*

Proof. Condition 1 follows because if $(z,\delta)\in Z\times X_\emptyset$ is such that $\mu^*(z,\delta)>0$, then $\mu(z,a,\delta)>0$ for some $a\neq w$ and $\{z\}\times \mathrm{supp}(\delta)\subseteq \mathrm{graph}(\mathbb{C})$ since $\mu\in\mathcal{T}$.

Condition 2 holds since, for each $z\in Z$, $\mu=\lim_n \mu_n$ and $\mu_n\in D_n(\mu_n,\kappa_n)$ for each $n\in\mathbb{N}$ imply that

$$\begin{aligned}\nu(z) &= \sum_{\delta\in X}\mu(z,m,\delta) + \sum_{\delta\in X_\emptyset\setminus X}\mu(z,s,\delta) + \sum_{(z',c)\in Z\times C}\mu(z,w,1_{(z',c)})\\ &= \sum_{\delta\in X}\mu^*(z,\delta) + \sum_{\delta\in X_\emptyset\setminus X}\mu^*(z,\delta) + \sum_{(z',\delta)\in Z\times X}\mu^*(z',\delta)\delta_Z(z),\end{aligned}$$

the last equality following by (6). ■

Claim 4 *If $(z,z',c,\delta)\in Z\times Z\times C\times X$ is such that $(z,\delta)\in \mathrm{supp}(\mu^*)$ and $(z',c)\in \mathrm{supp}(\delta)$, then $\mu(z',w,1_{(z,c)})>0$.*

Proof. We have that $\sum_{\delta'\in X}\mu(z,m,\delta')\delta'(z',c)\geq \mu(z,m,\delta)\delta(z',c)>0$ and, thus, $\kappa(z,z',c)>0$ since $\mu_n\in D_n(\mu_n,\kappa_n)$ for each $n\in\mathbb{N}$. Hence, (5) implies that $\mu(z',w,1_{(z,c)})>0$. ■

Claim 5 *$\mathrm{supp}(\mu^*)\subseteq IR(\mu^*)$.*

Proof. Suppose not; then there exists $(z^*,\delta^*)\in \mathrm{supp}(\mu^*)\setminus IR(\mu^*)$. We claim that there exists $z\in Z$, $(a,\delta)\in\Delta$ and $c'\in\mathbb{C}(z,\emptyset)$ such that

1. $\mu(z,a,\delta)>0$ and

2. $(s, 1_{(\emptyset,c')}, \mu^*) \succ_z (a, \delta, \mu^*)$.

Indeed, in cases (i) and (iii) of the definition of $IR(\mu^*)$, let $(z, \delta) = (z^*, \delta^*)$ in both cases and $a = m$ in case (i) and $a = s$ in case (iii). In case (ii) of the definition of $IR(\mu^*)$, we have that $\delta^* \in X$ and there exist $(z', c) \in \text{supp}(\delta^*)$ and $c' \in \mathbb{C}(z', \emptyset)$ such that $(s, 1_{(\emptyset,c')}, \mu^*) \succ_{z'} (w, 1_{(z^*,c)}, \mu^*)$. Claim 4 implies that $\mu(z', w, 1_{(z^*,c)}) > 0$; hence, let $z = z'$, $a = w$ and $\delta = 1_{(z^*,c)}$.

We then have that $\mu_n(z, a, \delta) > 0$ and $(s, 1_{(\emptyset,c')}, \mu_n^*) \succ_z (a, \delta, \mu_n^*)$ for n sufficiently large. Then decrease $\mu_n(z, a, \delta)$ and increase $\mu_n(z, s, 1_{(\emptyset,c')})$ by the same amount $\varepsilon \in (0, \mu_n(z, a, \delta))$ to increase the objective function in $D_n(\mu_n, \kappa_n)$ while satisfying the constraints. But this is a contradiction to $\mu_n \in D_n(\mu_n, \kappa_n)$. ∎

Claim 6 *If $(z, z', c) \in Z \times Z \times C$ is such that $(z', c) \in T_z^m(\mu^*)$, then there is $N_{z,z',c} \in \mathbb{N}$ such that $\sum_{\delta \in X} \mu_n(z, m, \delta)\delta(z', c) < \kappa_n(z, z', c)$ for each $n \geq N_{z,z',c}$.*

Proof. Let $(z', c) \in T_z^m(\mu^*)$. Then $c \in \mathbb{C}(z, z')$. In case (a) of the definition of $T_z^m(\mu^*)$, there exists $(\tilde{z}, \tilde{c}, \tilde{\delta}) \in Z \times C \times X$ such that $(\tilde{z}, \tilde{\delta}) \in \text{supp}(\mu^*)$, $(z', \tilde{c}) \in \text{supp}(\tilde{\delta})$ and $(w, 1_{(z,c)}, \mu^*) \succ_{z'} (w, 1_{(\tilde{z},\tilde{c})}, \mu^*)$. Hence, $\mu(z', w, 1_{(\tilde{z},\tilde{c})}) > 0$ by Claim 4.

In cases (b) and (c) of the definition of $T_z^m(\mu^*)$, there exists $(a, \delta') \in \Delta$ such that $a \neq w$, $(z', \delta') \in \text{supp}(\mu^*)$ and $(w, 1_{(z,c)}, \mu^*) \succ_{z'} (a, \delta', \mu^*)$. Thus, letting $a = w$ and $\delta' = 1_{(\tilde{z},\tilde{c})}$ in case (a), it follows that, in all cases, there exists $(a, \delta') \in \Delta$ such that $(z', a, \delta') \in \text{supp}(\mu)$ and $(w, 1_{(z,c)}, \mu^*) \succ_{z'} (a, \delta', \mu^*)$.

Let $N_{z,z',c} \in \mathbb{N}$ be such that $\mu_n(z', a, \delta') > 0$ and $(w, 1_{(z,c)}, \mu_n^*) \succ_{z'} (a, \delta', \mu_n^*)$ for each $n \geq N_{z,z',c}$. Thus, for each $n \geq N_{z,z',c}$,

$$\mu_n(z', w, 1_{(z,c)}) = \sum_{\delta \in X} \mu_n(z, m, \delta)\delta(z', c)$$

by Claim 2 and, since $\alpha_{n,(z,z',c)}(a, \delta', \mu_n) > 0$,

$$\begin{aligned}
\kappa_n(z, z', c) &= \mu_n(z', w, 1_{(z,c)}) + \frac{1}{n} g_n(\mu_n)(z, z', c) \\
&\geq \mu_n(z', w, 1_{(z,c)}) + \frac{1}{n} \alpha_{n,(z,z',c)}(a, \delta', \mu_n)\mu_n(z', a, \delta') \\
&> \mu_n(z', w, 1_{(z,c)}) = \sum_{\delta \in X} \mu_n(z, m, \delta)\delta(z', c).
\end{aligned}$$

∎

Claim 7 $\text{supp}(\mu^*) \subseteq S_M(\mu^*)$.

Proof. Suppose not; then there exists $(z^*, \delta^*) \in \text{supp}(\mu^*) \setminus S_M(\mu^*)$. We claim that there exists $z \in Z$, $(a, \delta) \in \Delta$ and $\delta' \in X$ such that

1. $\mu(z, a, \delta) > 0$,
2. $\text{supp}(\delta') \subseteq T_z^m(\mu^*) \cup \text{supp}(\delta)$ if $a = m$ and $\text{supp}(\delta') \subseteq T_z^m(\mu^*)$ if $a \neq m$, and
3. $(m, \delta', \mu^*) \succ_z (a, \delta, \mu^*)$.

Indeed, in cases (i) and (iii) of the definition of $S_M(\mu^*)$, let $(z, \delta) = (z^*, \delta^*)$ in both cases and $a = m$ in case (i) and $a = s$ in case (iii). In case (ii) of the definition of $S_M(\mu^*)$, we have that $\delta^* \in X$ and there exist $(z', c) \in \text{supp}(\delta^*)$ and $\delta' \in X$ such that $\text{supp}(\delta') \subseteq T_{z'}^m(\mu^*)$ and $(m, \delta', \mu^*) \succ_{z'} (w, 1_{(z^*,c)}, \mu^*)$. Claim 4 implies that $\mu(z', w, 1_{(z^*,c)}) > 0$; hence, let $z = z'$, $a = w$ and $\delta = 1_{(z^*,c)}$.

Note that $\{z\} \times \text{supp}(\delta') \subseteq \text{graph}(\mathbb{C})$ since $\{z\} \times T_z^m(\mu^*) \subseteq \text{graph}(\mathbb{C})$ and, when $a = m$, $(z, \delta) \in \text{supp}(\mu^*)$ and thus $\{z\} \times \text{supp}(\delta) \subseteq \text{graph}(\mathbb{C})$.

Let $\theta = 1$ if $\text{supp}(\delta) \cap \text{supp}(\delta') = \emptyset$; otherwise, let $(\bar{z}, \bar{c}) \in \text{supp}(\delta) \cap \text{supp}(\delta')$ be such that $\delta(\bar{z}, \bar{c})/\delta'(\bar{z}, \bar{c}) \leq \delta(z, c)/\delta'(z, c)$ for each $(z, c) \in \text{supp}(\delta) \cap \text{supp}(\delta')$ and define

$$\theta = \min \left\{ 1, \frac{\delta(\bar{z}, \bar{c})}{\delta'(\bar{z}, \bar{c})} \right\}.$$

Let $k \in \mathbb{N}$ be such that $k\theta > 1$.

There is $N \in \mathbb{N}$ such that, for each $n \geq N$,

(i) $\text{supp}(\mu) \subseteq \text{supp}(\mu_n)$,

(ii) $\sum_{\hat{\delta} \in X} \mu_n(z, m, \hat{\delta})\hat{\delta}(z', c) < \kappa_n(z, z', c)$ for each $(z', c) \in T_z^m(\mu^*)$, and

(iii) $u_n(z, m, \delta', \mu_n) \geq k u_n(z, a, \delta, \mu_n)$.

Indeed, (i) is clear since Z and Δ are finite. As for (ii), take $N \geq \max_{(z',c) \in Z \times C} N_{z,z',c}$ where, for each $(z', c) \in Z \times C$, $N_{z,z',c}$ is given by Claim 6. Finally, for (iii), we have that $u(z,m,\delta',\mu)/u(z,a,\delta,\mu) > 1$ and $u(z,m,\delta',\mu_n)/u(z,a,\delta,\mu_n) \geq \beta$ for some $\beta > 1$ for all n sufficiently large. Hence,

$$\frac{u_n(z,m,\delta',\mu_n)}{u_n(z,a,\delta,\mu_n)} = \left(\frac{u(z,m,\delta',\mu_n)}{u(z,a,\delta,\mu_n)}\right)^n \geq \beta^n > k$$

for all n sufficiently large.

Fix $n \geq N$ and let $c^* \in \mathbb{C}(z,\emptyset)$. For each $\varepsilon > 0$, define π_ε by setting, for each $(\hat{z},\hat{a},\hat{\delta}) \in Z \times \Delta$,

$$\pi_\varepsilon(\hat{z},\hat{a},\hat{\delta}) = \begin{cases} \mu_n(z,a,\delta) - \varepsilon & \text{if } \hat{z} = z, \hat{a} = a \text{ and } \hat{\delta} = \delta, \\ \mu_n(z,m,\delta') + \theta\varepsilon & \text{if } \hat{z} = z, \hat{a} = m \text{ and } \hat{\delta} = \delta', \\ \mu_n(z,s,1_{(\emptyset,c^*)}) + (1-\theta)\varepsilon & \text{if } \hat{z} = z, \hat{a} = s \text{ and } \hat{\delta} = 1_{(\emptyset,c^*)}, \\ \mu_n(\hat{z},\hat{a},\hat{\delta}) & \text{otherwise.} \end{cases}$$

By (i), $\mu_n(z,a,\delta) > 0$. We have that, for each $\varepsilon \in (0,\mu_n(z,a,\delta))$, $\pi_\varepsilon(\hat{z},\hat{a},\hat{\delta}) \geq 0$ for each $(\hat{z},\hat{a},\hat{\delta}) \in Z \times \Delta$, $\pi_\varepsilon(\hat{z},w,1_{(z',c)}) \leq \mu_n(\hat{z},w,1_{(z',c)}) \leq \sum_{\hat{\delta}\in X} \mu_n(z',m,\hat{\delta})\hat{\delta}(\hat{z},c)$ for each $(\hat{z},z',c) \in Z \times Z \times C$ and

$$\sum_{(\hat{a},\hat{\delta})\in\Delta} \pi_\varepsilon(\hat{z},\hat{a},\hat{\delta}) = \sum_{(\hat{a},\hat{\delta})\in\Delta} \mu_n(\hat{z},\hat{a},\hat{\delta}) = \nu(\hat{z})$$

for each $\hat{z} \in Z$. In particular, $\pi_\varepsilon \in \mathcal{T}$.

We also have that, for some $\varepsilon \in (0,\mu_n(z,a,\delta))$,

$$\sum_{\hat{\delta}\in X} \pi_\varepsilon(\hat{z},m,\hat{\delta})\hat{\delta}(z',c) \leq \kappa_n(\hat{z},z',c) \text{ for all } (\hat{z},z',c) \in Z \times Z \times C. \tag{7}$$

First, note that it is enough to consider $\hat{z} = z$ and that, for each $(z',c) \in Z \times C$,

$$\sum_{\hat{\delta}\in X} \pi_\varepsilon(z,m,\hat{\delta})\hat{\delta}(z',c) = \sum_{\hat{\delta}\in X} \mu_n(z,m,\hat{\delta})\hat{\delta}(z',c) + \varepsilon\left(-\delta(z',c)1_m(a) + \theta\delta'(z',c)\right).$$

Thus, (7) holds if $(z',c) \notin \operatorname{supp}(\delta')$.

If $a = m$ and $(z',c) \in \operatorname{supp}(\delta') \cap \operatorname{supp}(\delta)$, the definition of $(\bar{z},\bar{c})$ implies that:

$$\sum_{\hat{\delta}\in X} \pi_\varepsilon(z,m,\hat{\delta})\hat{\delta}(z',c) \leq \sum_{\hat{\delta}\in X} \mu_n(z,m,\hat{\delta})\hat{\delta}(z',c) \leq \kappa_n(z,z',c).$$

If $a \neq m$ and $(z', c) \in \text{supp}(\delta')$ or if $a = m$ and $(z', c) \in \text{supp}(\delta') \setminus \text{supp}(\delta)$, then $(z', c) \in T_z^m(\mu^*)$ and, thus, $\sum_{\hat{\delta} \in X} \mu_n(z, m, \hat{\delta})\hat{\delta}(z', c) < \kappa_n(z, z', c)$ by (ii). Hence, there is $\varepsilon(z', c) > 0$ such that

$$\sum_{\hat{\delta} \in X} \pi_\varepsilon(z, m, \hat{\delta})\hat{\delta}(z', c) = \sum_{\hat{\delta} \in X} \mu_n(z, m, \hat{\delta})\hat{\delta}(z', c) + \varepsilon\theta\delta'(z', c) < \kappa_n(z, z', c)$$

for each $0 < \varepsilon < \varepsilon(z', c)$. Thus, letting $B = \text{supp}(\delta')$ if $a \neq m$, $B = \text{supp}(\delta') \setminus \text{supp}(\delta)$ if $a = m$ and $0 < \varepsilon < \min_{(z',c) \in B} \varepsilon(z', c)$, we have that $\sum_{\hat{\delta} \in X} \pi_\varepsilon(\hat{z}, m, \hat{\delta})\hat{\delta}(z', c) \leq \kappa_n(\hat{z}, z', c)$ for each $(\hat{z}, z', c) \in Z \times Z \times C$.

Finally, note that

$$\sum_{\hat{z} \in Z, (\hat{a}, \hat{\delta}) \in \Delta} u_n(\hat{z}, \hat{a}, \hat{\delta}, \mu_n)\pi_\varepsilon(\hat{z}, \hat{a}, \hat{\delta}) > \sum_{\hat{z} \in Z, (\hat{a}, \hat{\delta}) \in \Delta} u_n(\hat{z}, \hat{a}, \hat{\delta}, \mu_n)\mu_n(\hat{z}, \hat{a}, \hat{\delta})$$

since $u_n(z, m, \delta', \mu_n) \geq ku_n(z, a, \delta, \mu_n)$ by (iii) since $n \geq N$, $u_n(z, s, 1_{(\emptyset, c^*)}, \mu_n) \geq 1$ and, hence,

$$\sum_{\hat{z} \in Z, (\hat{\delta}, \hat{a}) \in \Delta} u_n(\hat{z}, \hat{a}, \hat{\delta}, \mu_n) \left(\pi_\varepsilon(\hat{z}, \hat{a}, \hat{\delta}) - \mu_n(\hat{z}, \hat{a}, \hat{\delta})\right) \geq u_n(z, a, \delta, \mu_n)\varepsilon\left(-1 + k\theta\right) > 0.$$

In conclusion $\mu_n \notin D_n(\mu_n, \kappa_n)$, a contradiction. ■

It follows from Claims 3, 5 and 7 that μ^* is a stable matching. ■

In the remainder of the proof, we extend Lemma 4 using three limit arguments. The first one considers the case where X is $\mathcal{M}_R(Z \times C)$ for some $R > 0$ to dispense with the finiteness of C. The second one replaces $X = \mathcal{M}_R(Z \times C)$ with a general X satisfying our assumptions in the case where Z is finite.

The finiteness of Z is important in the second limit result to represent each preference relation $\succ_z$ with a continuous and bounded (e.g. by 1 below and 2 above) utility function $u : Z \times \Delta \times \mathcal{M}(Z \times X_\emptyset) \to [1, 2]$. Such function can then be extended by replacing X with $\mathcal{M}_R(Z \times C)$ in the definition of its domain and is then modified by adding a utility penalty for managers who choose a workforce δ at a distance greater than $1/k$ from X. This then defines a market to which the conclusion of the previous limit argument applies.

The final limit argument then dispenses with the finiteness of Z. Our first and third limit results have analogs in Greinecker & Kah (2021) but are more involved due to the presence of many-to-one matching and occupational choice. There is no analog to our second limit result in their work.

The following lemma unifies the common elements of the above three limit results.

Lemma 5 *Let* $\{(E_k, \mu_k)\}_{k=1}^{\infty}$ *be such that* $E_k = (Z_k, \nu_k, C_k, \mathbb{C}_k, X_k, (\succ_{z,k})_{z\in Z_k})$ *is a market and* μ_k *is a stable matching for* E_k *for each* $k \in \mathbb{N}$. *Let* $E = (Z, \nu, C, \mathbb{C}, X, (\succ_z)_{z\in Z})$ *be a rational, continuous, bounded and rich market such that* $\nu_k \to \nu$ *and, for each* $k \in \mathbb{N}$, $Z_k \subseteq Z$, $\mathrm{supp}(\nu_k) \subseteq \mathrm{supp}(\nu)$, $C_k \subseteq C$, $\mathbb{C}_k(z, z') \subseteq \mathbb{C}(z, z')$ *for each* $(z, z') \in Z_k \times Z_{\emptyset,k}$ *and* $X_k \subseteq X$. *Then:*

1. $\{\mu_k\}_{k=1}^{\infty}$ *has a convergent subsequence in* $\mathcal{M}(Z \times X_\emptyset)$.

Suppose further that $\{\mu_k\}_{k=1}^{\infty}$ *converges and let* $\mu = \lim_k \mu_k$. *Then:*

2. μ *is a matching for* E.

Suppose further that $\succ_{z,k}$ *is the restriction of* $\succ_z$ *to* $\Delta_k \times \mathcal{M}(Z_k \times X_{\emptyset,k})$ *for each* $z \in Z_k$. *Then:*

3. $\mathrm{supp}(\mu) \subseteq IR(\mu)$.

4. $\mathrm{supp}(\mu) \subseteq S_M(\mu)$ *if*

   (a) *for each* $(z, \delta, \mu) \in Z \times X \times \mathcal{M}(Z \times X_\emptyset)$, $\delta' \in \Lambda(z, \delta, \mu)$, *open neighborhood* $V_{\delta'}$ *of* δ', *subsequence* $\{\mu_{k_j}\}_{j=1}^{\infty}$ *of* $\{\mu_k\}_{k=1}^{\infty}$ *and sequence* $\{(z_{k_j}, \delta_{k_j})\}_{j=1}^{\infty}$ *such that* $(z_{k_j}, \delta_{k_j}) \to (z, \delta)$ *and* $(z_{k_j}, \delta_{k_j}) \in Z_{k_j} \times X_{k_j}$ *for each* $j \in \mathbb{N}$, *there exists* $J \in \mathbb{N}$ *such that* $\{\gamma \in X_{k_j} : \{z_{k_j}\} \times \mathrm{supp}(\gamma) \subseteq \mathrm{graph}(\mathbb{C}_{k_j})\} \cap \Lambda(z_{k_j}, \delta_{k_j}, \mu_{k_j}) \cap V_{\delta'} \neq \emptyset$ *for each* $j \geq J$, *and*

   (b) *for each* $(z, \mu) \in Z \times \mathcal{M}(Z \times X_\emptyset)$, $\delta' \in \Lambda_0(z, \mu)$, *open neighborhood* $V_{\delta'}$ *of* δ', *subsequence* $\{\mu_{k_j}\}_{j=1}^{\infty}$ *of* $\{\mu_k\}_{k=1}^{\infty}$ *and sequence* $\{z_{k_j}\}_{j=1}^{\infty}$ *such that* $z_{k_j} \to z$ *and* $z_{k_j} \in Z_{k_j}$ *for each* $j \in \mathbb{N}$, *there exists* $J \in \mathbb{N}$ *such that* $\{\gamma \in X_{k_j} : \{z_{k_j}\} \times \mathrm{supp}(\gamma) \subseteq \mathrm{graph}(\mathbb{C}_{k_j})\} \cap \Lambda_0(z_{k_j}, \mu_{k_j}) \cap V_{\delta'} \neq \emptyset$ *for each* $j \geq J$.

Proof. We divide the proof into several parts corresponding to the ones in the statement of the lemma.

Part 1: Since $\mathcal{M}(Z \times X_\emptyset)$ is a separable metrizable space, it suffices to show that $\{\mu_k\}_{k=1}^\infty$ is tight; this follows by Lemma 3 together with $\mu_k(Z \times X_\emptyset) \leq \nu_k(Z)$ for each $k \in \mathbb{N}$ and the fact that $\{\nu_k(Z)\}_{k=1}^\infty$ converges (to $\nu(Z)$) and, hence, is bounded.

Let $\varepsilon > 0$. Since $\{\nu_k\}_{k=1}^\infty$ is tight, there exists a compact subset K_Z of Z such that $\nu_k(Z \setminus K_Z) \leq \varepsilon$ for all k.

For each $n \in \mathbb{N}$, let K_n be a compact subset of Z such that $\hat{\nu}(Z \setminus K_n) < \varepsilon/(n2^n)$ for each $\hat{\nu} \in \{\nu_k\}_{k=1}^\infty$. Let $K_n^\emptyset = K_n \cup \{\emptyset\}$ and let $D_n = \bigcup_{(z,z') \in K_Z \times K_n^\emptyset} \mathbb{C}(z, z')$. Note that D_n is compact since $\mathbb{C}$ is continuous and compact-valued, and K_Z and $K_n^\emptyset$ are compact.

Define

$$K_X = \left\{ \delta \in X : \delta(Z \times C \setminus K_n \times D_n) \leq \frac{1}{n} \text{ for each } n \in \mathbb{N} \right\}.$$

Then K_X is closed since if $\delta_j \to \delta$ and $\delta_j \in K_X$ for each $j \in \mathbb{N}$, then $\delta \in X$ since X is closed and, for each $n \in \mathbb{N}$, $\delta(Z \times C \setminus K_n \times D_n) \leq \liminf_j \delta_j(Z \times C \setminus K_n \times D_n) \leq 1/n$ since $Z \times C \setminus K_n \times D_n$ is open. Hence, $\delta \in K_X$. In addition, K_X is tight since, for each $\eta > 0$, there is $n \in \mathbb{N}$ such that $1/n < \eta$ and, thus, $\delta(Z \times C \setminus K_n \times D_n) \leq 1/n < \eta$ for each $\delta \in K_X$. Let $R > 0$ be such that $X \subseteq \mathcal{M}_R(Z \times C)$. Since K_X is a closed and tight subset of $\mathcal{M}_R(Z \times C)$, it follows that K_X is compact.

For each $n \in \mathbb{N}$, let

$$K_{X,n} = \left\{ \delta \in X : \delta(Z \times C \setminus K_n \times D_n) > \frac{1}{n} \text{ and } \delta(Z \times C \setminus K_j \times D_j) \leq \frac{1}{j} \right.$$
$$\left. \text{for each } j = 1, \ldots, n-1 \right\}.$$

Then $X \setminus K_X = \cup_{n=1}^\infty K_{X,n}$ and the family $\{K_{X,n}\}_{n=1}^\infty$ is pairwise disjoint. Fix $k \in \mathbb{N}$. For each $n \in \mathbb{N}$, we have that

$$\begin{aligned} \frac{\varepsilon}{n2^n} > \nu_k(Z \setminus K_n) &\geq \int_{Z \times X} \delta((Z \setminus K_n) \times C) \mathrm{d}\mu_k(z, \delta) \\ &\geq \int_{K_Z \times K_{X,n}} \delta((Z \setminus K_n) \times C) \mathrm{d}\mu_k(z, \delta) \\ &= \int_{K_Z \times K_{X,n}} \delta(Z \times C \setminus K_n \times D_n) \mathrm{d}\mu_k(z, \delta) > \frac{1}{n} \mu_k(K_Z \times K_{X,n}), \end{aligned}$$

where the equality follows because $\delta(Z\times C\setminus K_n\times D_n)=\delta((Z\setminus K_n)\times C)+\delta(K_n\times(C\setminus D_n))$ and condition 1 of a matching implies that, for each $(z,\delta)\in\text{supp}(\mu_k)\cap(K_Z\times X)$, $\text{supp}(\delta)\cap(K_n\times C)\subseteq D_n$ and, thus, $\delta(K_n\times(C\setminus D_n))=0$. Hence,

$$\varepsilon=\sum_{n=1}^{\infty}\frac{\varepsilon}{2^n}>\sum_{n=1}^{\infty}\mu_k(K_Z\times K_{X,n})=\mu_k(K_Z\times(X\setminus K_X)).$$

Note that $\cup_{z\in K_Z}\mathbb{C}(z,\emptyset)\subseteq D_1$ and let $K_X^{\emptyset}=K_X\cup\{1_{(\emptyset,c)}:c\in D_1\}$. Then $K_X^{\emptyset}$ is compact since both K_X and D_1 are compact. Moreover, $\mu_k(K_Z\times(X_\emptyset\setminus(K_X^{\emptyset}\cup X)))=\mu_k(K_Z\times\{1_{(\emptyset,c)}:c\notin D_1\})=0$ where the first equality follows since $(X_\emptyset\setminus K_X^{\emptyset})\cap(X_\emptyset\setminus X)=(X_\emptyset\setminus K_X^{\emptyset})\cap\{1_{(\emptyset,c)}:c\in C\}=\{1_{(\emptyset,c)}:c\notin D_1\}$ and the second by condition 1 of a matching since $\cup_{z\in K_Z}\mathbb{C}(z,\emptyset)\subseteq D_1$. Then, for each $k\in\mathbb{N}$,

$$\begin{aligned}\mu_k(Z\times X_\emptyset\setminus K_Z\times K_X^{\emptyset})&=\mu_k((Z\setminus K_Z)\times X_\emptyset)+\mu_k(K_Z\times(X_\emptyset\setminus(K_X^{\emptyset}\cup X)))\\&+\mu_k(K_Z\times((X_\emptyset\setminus K_X^{\emptyset})\cap X))\le\nu_k(Z\setminus K_Z)+0+\mu_k(K_Z\times(X\setminus K_X))<2\varepsilon,\end{aligned}$$

where $\mu_k((Z\setminus K_Z)\times X_\emptyset)\le\nu_k(Z\setminus K_Z)$ because of condition 2 of a matching.

Part 2: We first consider condition 2 of the definition of a matching. Let $\pi:Z\times X_\emptyset\to Z$ be the projection of $Z\times X_\emptyset$ onto Z and note that, for each Borel subset B of Z, $\nu_M(B)+\nu_S(B)=\mu(B\times X_\emptyset)=\mu\circ\pi^{-1}(B)$ and $\nu_{M,k}(B)+\nu_{S,k}(B)=\mu_k(B\times X_\emptyset)=\mu_k\circ\pi^{-1}(B)$ for each $k\in\mathbb{N}$. Since π is continuous, $\mu_k\circ\pi^{-1}\to\mu\circ\pi^{-1}$. Indeed, for each bounded and continuous $f:Z\to\mathbb{R}$, $\int_Z f\mathrm{d}\mu_k\circ\pi^{-1}=\int_{Z\times X_\emptyset}f\circ\pi\mathrm{d}\mu_k\to\int_{Z\times X_\emptyset}f\circ\pi\mathrm{d}\mu=\int_Z f\mathrm{d}\mu_k\circ\pi^{-1}$ since $f\circ\pi:Z\times X_\emptyset\to\mathbb{R}$ is bounded and continuous. Hence, since $\mathcal{M}(Z\times X_\emptyset)$ is metrizable, $\nu_M+\nu_S=\mu\circ\pi^{-1}=\lim_k\mu_k\circ\pi^{-1}=\lim_k\nu_{M,k}+\nu_{S,k}$.

Also, for each Borel subset B of Z, $\nu_W(B)=\int_{Z\times X}\delta(B\times C)\mathrm{d}\mu(z,\delta)$ and $\nu_{W,k}(B)=\int_{Z\times X}\delta(B\times C)\mathrm{d}\mu_k(z,\delta)$ for each $k\in\mathbb{N}$. We show that $\nu_{W,k}\to\nu_W$. Let $B\subseteq Z$ be closed and $f:X\to\mathbb{R}$ be defined by setting, for each $\delta\in X$, $f(\delta)=\delta(B\times C)$. Then f is bounded and upper semi-continuous. Hence, by (a suitable adaptation of) Aliprantis & Border (2006, Theorem 15.5), $\limsup_k\nu_{W,k}(B)=\limsup_k\int_{Z\times X}f\mathrm{d}\mu_k\le\int_{Z\times X}f\mathrm{d}\mu=\nu_W(B)$ and it follows that $\nu_W=\lim_k\nu_{W,k}$ as claimed. Thus,

$$\nu_M+\nu_W+\nu_S=\lim_k\nu_{M,k}+\lim_k\nu_{W,k}+\lim_k\nu_{S,k}=\lim_k(\nu_{M,k}+\nu_{W,k}+\nu_{S,k})=\nu.$$

Condition 1 holds because, by Carmona & Podczeck (2009, Lemma 12), for each $(z,\delta)\in\text{supp}(\mu)$ and $(z',c)\in\text{supp}(\delta)$, there exists a subsequence $\{\mu_{k_j}\}_{j=1}^{\infty}$ of $\{\mu_k\}_{k=1}^{\infty}$

and corresponding $\{(z_{k_j}, \delta_{k_j}, z'_{k_j}, c_{k_j})\}_{j=1}^{\infty}$ such that $(z_{k_j}, \delta_{k_j}, z'_{k_j}, c_{k_j}) \to (z, \delta, z', c)$ and, for each $j \in \mathbb{N}$, $(z_{k_j}, \delta_{k_j}) \in \text{supp}(\mu_{k_j})$ and $(z'_{k_j}, c_{k_j}) \in \text{supp}(\delta_{k_j})$. Hence, $c_{k_j} \in \mathbb{C}_{k_j}(z_{k_j}, z'_{k_j}) \subseteq \mathbb{C}(z_{k_j}, z'_{k_j})$ and, since $\mathbb{C}$ is continuous, $c \in \mathbb{C}(z, z')$.

Part 3: Let $(z, \delta) \in \text{supp}(\mu)$ and suppose that $(z, \delta) \notin IR(\mu)$. Then either (i) there exists $c \in \mathbb{C}(z, \emptyset)$ such that $(s, 1_{(\emptyset, c)}, \mu) \succ_z (a(\delta), \delta, \mu)$ where $a(\delta) = m$ if $\delta \in X$ and $a(\delta) = s$ if $\delta \in X_\emptyset \setminus X$, or (ii) there exists $(z', c) \in \text{supp}(\delta)$ and $c' \in \mathbb{C}(z', \emptyset)$ such that $(s, 1_{(\emptyset, c')}, \mu) \succ_{z'} (w, 1_{(z,c)}, \mu)$.

Consider case (i) first. The continuity of $(\succ_z)_{z \in Z}$ and $\mathbb{C}$ implies that there are open neighborhoods V_c, V_z, V_δ and V_μ of c, z, δ and μ, respectively, such that $(s, 1_{(\emptyset, \hat{c})}, \hat{\mu}) \succ_{\hat{z}} (a(\delta), \hat{\delta}, \hat{\mu})$ and $\mathbb{C}(\hat{z}, \emptyset) \cap V_c \neq \emptyset$ for each $\hat{c} \in V_c$, $\hat{z} \in V_z$, $\hat{\delta} \in V_\delta$ and $\hat{\mu} \in V_\mu$. Since $(z, \delta) \in \text{supp}(\mu)$, it follows that $0 < \mu(V_z \times V_\delta) \leq \liminf_k \mu_k(V_z \times V_\delta)$; hence, for each k sufficiently large, $\mu_k(V_z \times V_\delta) > 0$ and $\mu_k \in V_\mu$. This means that, for any such k, there exist $(\hat{z}, \hat{\delta}) \in \text{supp}(\mu_k) \cap (V_z \times V_\delta)$ and $\hat{c} \in \mathbb{C}(\hat{z}, \emptyset) \cap V_c$. But then $(s, 1_{(\emptyset, \hat{c})}, \mu_k) \succ_{\hat{z}} (a(\delta), \hat{\delta}, \mu_k)$ and, hence, $(s, 1_{(\emptyset, \hat{c})}, \mu_k) \succ_{\hat{z}, k} (a(\delta), \hat{\delta}, \mu_k)$, contradicting the individual rationality of μ_k.

Consider next case (ii). The continuity of $(\succ_z)_{z \in Z}$ and $\mathbb{C}$ implies that there are open neighborhoods $V_{c'}$, V_c, V_z, $V_{z'}$ and V_μ of c', c, z, z' and μ, respectively, such that $(s, 1_{(\emptyset, \tilde{c})}, \hat{\mu}) \succ_{\tilde{z}} (w, 1_{(\hat{z}, \hat{c})}, \hat{\mu})$ and $\mathbb{C}(\emptyset, \tilde{z}) \cap V_{c'} \neq \emptyset$ for each $\tilde{c} \in V_{c'}$, $\hat{c} \in V_c$, $\hat{z} \in V_z$, $\tilde{z} \in V_{z'}$ and $\hat{\mu} \in V_\mu$. Since $(z', c) \in \text{supp}(\delta)$, there is an open neighborhood V_δ of δ such that $\text{supp}(\hat{\delta}) \cap (V_{z'} \times V_c) \neq \emptyset$ for each $\hat{\delta} \in V_\delta$ by Lemma 2. Since $\mu_k \to \mu$ and $(z, \delta) \in \text{supp}(\mu)$, it follows that $0 < \mu(V_z \times V_\delta) \leq \liminf_k \mu_k(V_z \times V_\delta)$; hence, for all k sufficiently large, $\mu_k(V_z \times V_\delta) > 0$ and $\mu_k \in V_\mu$. This means that, for any such k, there exists $(\hat{z}, \hat{\delta}) \in \text{supp}(\mu_k) \cap (V_z \times V_\delta)$, $(\tilde{z}, \hat{c}) \in \text{supp}(\hat{\delta}) \cap (V_{z'} \times V_c)$ and $\tilde{c} \in \mathbb{C}(\emptyset, \tilde{z}) \cap V_{c'}$. But then $(s, 1_{(\emptyset, \tilde{c})}, \mu_k) \succ_{\tilde{z}} (w, 1_{(\hat{z}, \hat{c})}, \mu_k)$ and, hence, $(s, 1_{(\emptyset, \tilde{c})}, \mu_k) \succ_{\tilde{z}, k} (w, 1_{(\hat{z}, \hat{c})}, \mu_k)$, contradicting the individual rationality of μ_k.

Part 4: In this proof, to avoid confusion, we write $T_z^m(\mu; E')$ for $T_z^m(\mu)$ in a market E'.

Let $(z, \delta) \in \text{supp}(\mu)$ and suppose that $(z, \delta) \notin S_M(\mu)$. Then there exists $\delta' \in X$ such that either (i) $\text{supp}(\delta') \subseteq T_z^m(\mu) \cup \text{supp}(\delta)$ and $(m, \delta', \mu) \succ_z (a(\delta), \delta, \mu)$, where

$a(\delta) = m$ if $\delta \in X$ and $a(\delta) = s$ if $\delta \in X_\emptyset \setminus X$,[32] or (ii) there exists $(z', c) \in \text{supp}(\delta)$ such that $\text{supp}(\delta') \subseteq T^m_{z'}(\mu)$ and $(m, \delta', \mu) \succ_{z'} (w, 1_{(z,c)}, \mu)$.

Consider case (i) first. Let V_z, $V_{\delta'}$, V_δ and V_μ be open neighborhoods of z, δ', δ and μ, respectively, such that $(m, \gamma', \bar{\mu}) \succ_{\bar{z}} (a(\delta), \gamma, \bar{\mu})$ for each $\bar{z} \in V_z$, $\gamma' \in V_{\delta'}$, $\gamma \in V_\delta$ and $\bar{\mu} \in V_\mu$. Let, by the richness of E, $\tilde{V}_z$, $\tilde{V}_\delta$ and $\tilde{V}_\mu$ be open neighborhoods of z, δ and μ, respectively, such that $\Lambda(\bar{z}, \gamma, \bar{\mu}) \cap V_{\delta'} \neq \emptyset$ for each $(\bar{z}, \gamma, \bar{\mu}) \in \tilde{V}_z \times \tilde{V}_\delta \times \tilde{V}_\mu$.

By Carmona & Podczeck (2009, Lemma 12), there is a subsequence $\{\mu_{k_j}\}_{j=1}^\infty$ of $\{\mu_k\}_{k=1}^\infty$ and corresponding sequence $\{(z_{k_j}, \delta_{k_j})\}_{j=1}^\infty$ such that $(z_{k_j}, \delta_{k_j}) \to (z, \delta)$ and $(z_{k_j}, \delta_{k_j}) \in \text{supp}(\mu_{k_j})$ for each $j \in \mathbb{N}$.

Let $J \in \mathbb{N}$ be such that $\mu_{k_j} \in V_\mu \cap \tilde{V}_\mu$, $z_{k_j} \in V_z \cap \tilde{V}_z$, $\delta_{k_j} \in V_\delta \cap \tilde{V}_\delta$ and $\{\gamma \in X_{k_j} : \{z_{k_j}\} \times \text{supp}(\gamma) \subseteq \text{graph}(\mathbb{C}_{k_j})\} \cap \Lambda(z_{k_j}, \delta_{k_j}, \mu_{k_j}) \cap V_{\delta'} \neq \emptyset$ for all $j \geq J$. Let $j \geq J$ and let $\delta'_{k_j} \in \{\gamma \in X_{k_j} : \{z_{k_j}\} \times \text{supp}(\gamma) \subseteq \text{graph}(\mathbb{C}_{k_j})\} \cap \Lambda(z_{k_j}, \delta_{k_j}, \mu_{k_j}) \cap V_{\delta'}$. Then $\text{supp}(\delta'_{k_j}) \subseteq T^m_{z_{k_j}}(\mu_{k_j}; E) \cup \text{supp}(\delta_{k_j})$ and $(m, \delta'_{k_j}, \mu_{k_j}) \succ_{z_{k_j}} (a(\delta), \delta_{k_j}, \mu_{k_j})$. It then follows that $\text{supp}(\delta'_{k_j}) \subseteq T^m_{z_{k_j}}(\mu_{k_j}; E_{k_j}) \cup \text{supp}(\delta_{k_j})$ and $(m, \delta'_{k_j}, \mu_{k_j}) \succ_{z_{k_j}, k_j} (a(\delta), \delta_{k_j}, \mu_{k_j})$. But this contradicts the stability of μ_{k_j}.

Consider next case (ii). Let V_z, $V_{z'}$, $V_{\delta'}$, V_c and V_μ be open neighborhoods of z, z', δ', c and μ respectively, such that $(m, \hat{\delta}', \hat{\mu}) \succ_{\hat{z}'} (w, 1_{(\hat{z},\hat{c})}, \hat{\mu})$ for each $\hat{z} \in V_z$, $\hat{z}' \in V_{z'}$, $\hat{\delta}' \in V_{\delta'}$, $\hat{c} \in V_c$ and $\hat{\mu} \in V_\mu$. Let, by the richness of E, $\tilde{V}_{z'}$ and $\tilde{V}_\mu$ be open neighborhoods of z' and μ, respectively, such that $\Lambda_0(\hat{z}', \hat{\mu}) \cap V_{\delta'} \neq \emptyset$ for each $(\hat{z}', \hat{\mu}) \in \tilde{V}_{z'} \times \tilde{V}_\mu$.

By Carmona & Podczeck (2009, Lemma 12), there is a subsequence $\{\mu_{k_j}\}_{j=1}^\infty$ of $\{\mu_k\}_{k=1}^\infty$ and corresponding sequence $\{(z_{k_j}, \delta_{k_j}, z'_{k_j}, c_{k_j})\}_{j=1}^\infty$ such that $(z_{k_j}, \delta_{k_j}) \in \text{supp}(\mu_{k_j})$ and $(z'_{k_j}, c_{k_j}) \in \text{supp}(\delta_{k_j})$ for each $j \in \mathbb{N}$ and $(z_{k_j}, \delta_{k_j}, z'_{k_j}, c_{k_j}) \to (z, \delta, z', c)$.

Let $J \in \mathbb{N}$ be such that $\delta_{k_j} \in V_\delta$, $z_{k_j} \in V_z$, $z'_{k_j} \in V_{z'} \cap \tilde{V}_{z'}$, $c_{k_j} \in V_c$, $\mu_{k_j} \in V_\mu \cap \tilde{V}_\mu$ and $\{\gamma \in X_{k_j} : \{z'_{k_j}\} \times \text{supp}(\gamma) \subseteq \text{graph}(\mathbb{C}_{k_j})\} \cap \Lambda_0(z'_{k_j}, \mu_{k_j}) \cap V_{\delta'} \neq \emptyset$ for all $j \geq J$. Let $j \geq J$ and let $\delta'_{k_j} \in \{\gamma \in X_{k_j} : \{z'_{k_j}\} \times \text{supp}(\gamma) \subseteq \text{graph}(\mathbb{C}_{k_j})\} \cap \Lambda_0(z'_{k_j}, \mu_{k_j}) \cap V_{\delta'}$. Then $\text{supp}(\delta'_{k_j}) \subseteq T^m_{z'_{k_j}}(\mu_{k_j}; E)$ and $(m, \delta'_{k_j}, \mu_{k_j}) \succ_{z'_{k_j}} (w, 1_{(z_{k_j}, c_{k_j})}, \mu_{k_j})$. It then follows that $\text{supp}(\delta'_{k_j}) \subseteq T^m_{z'_{k_j}}(\mu_{k_j}; E_{k_j})$ and $(m, \delta'_{k_j}, \mu_{k_j}) \succ_{z'_{k_j}, k_j} (w, 1_{(z_{k_j}, c_{k_j})}, \mu_{k_j})$. But this

[32] Note that when $\delta \in X_\emptyset \setminus X$ and $\delta' \in X$, $\text{supp}(\delta') \subseteq T^m_z(\mu) \cup \text{supp}(\delta)$ if and only if $\text{supp}(\delta') \subseteq T^m_z(\mu)$.

contradicts the stability of μ_{k_j}. ∎

The second step in the proof of our existence result consists in the following lemma, which considers the special case where Z is finite and $X = \mathcal{M}_R(Z \times C)$ for some $R > 0$.

Lemma 6 *If E is a rational and continuous market such that Z is finite and $X = \mathcal{M}_R(Z \times C)$ for some $R > 0$, then E has a stable matching.*

Proof. For each $(z, z') \in Z \times Z_\emptyset$, let $\{c^n_{z,z'}\}_{n=1}^\infty$ be a dense subset of $\mathbb{C}(z, z')$. For each $k \in \mathbb{N}$, define $\mathbb{C}_k(z, z') = \{c^n_{z,z'} : n \leq k\}$ and $C_k = \cup_{(z,z') \in Z \times Z_\emptyset} \mathbb{C}_k(z, z')$.

In addition, enumerate $\mathbb{Q} = \{q_1, q_2, \ldots\}$ and, for each $k \in \mathbb{N}$, let X_k be the set of $\delta \in \mathcal{M}_R(Z \times C)$ such that $\text{supp}(\delta)$ is a subset of $Z \times C_k$ and, for each $(z, c) \in Z \times C_k$, $\delta(z, c) \in \{q_n : n \leq k\}$. Let $X_{\emptyset,k} = X_k \cup \{1_{(\emptyset,c)} : c \in C_k\}$, $X_{m,k} = X_k$, $X_{s,k} = \{1_{(\emptyset,c)} : c \in C_k\}$, $X_{w,k} = \{1_{(z,c)} : (z, c) \in Z \times C_k\}$ and $\Delta_k = \{(a, \delta) : \delta \in X_{a,k}\}$.

For each $k \in \mathbb{N}$, let $E_k = (Z, \nu, C_k, \mathbb{C}_k, X_k, (\succ_z)_{z \in Z})$ be a market where $\succ_z$ is restricted to $\Delta_k \times \mathcal{M}(Z \times X_{\emptyset,k})$ for each $z \in Z$. Let $\mu_k \in \mathcal{M}(Z \times X_{\emptyset,k})$ be a stable matching in E_k, which exists by Lemma 4 since Z, C_k and X_k are finite.

It follows by part 1 of Lemma 5 that we may assume that $\{\mu_k\}_{k=1}^\infty$ converges; let $\mu = \lim_k \mu_k$. It then follows by parts 2 and 3 of Lemma 5 that μ is a matching and that $\text{supp}(\mu) \subseteq IR(\mu)$.

The following claim will be used to show that condition (a) of part 4 of Lemma 5 holds.

Claim 8 *Let $(\tilde{z}, \tilde{c}) \in T^m_z(\mu)$ and $V_{\tilde{c}}$ be an open neighborhood of $\tilde{c}$. Then, for all k sufficiently large, there exists $c_k \in \mathbb{C}_k(z, \tilde{z})$ such that $(\tilde{z}, c_k) \in T^m_z(\mu_k) \cap (\{\tilde{z}\} \times V_{\tilde{c}})$.*

Proof. Let $(\tilde{z}, \tilde{c}) \in T^m_z(\mu)$ and $V_{\tilde{c}}$ be an open neighborhood of $\tilde{c}$. Then $\tilde{c} \in \mathbb{C}(z, \tilde{z})$ and either (i) there exists $(\hat{z}, \hat{\delta}, \hat{c})$ such that $(\hat{z}, \hat{\delta}) \in \text{supp}(\mu)$, $(\tilde{z}, \hat{c}) \in \text{supp}(\hat{\delta})$ and $(w, 1_{(z,\tilde{c})}, \mu) \succ_{\tilde{z}} (w, 1_{(\hat{z},\hat{c})}, \mu)$, or (ii) there exists $\tilde{\delta} \in X_\emptyset$ such that $(\tilde{z}, \tilde{\delta}) \in \text{supp}(\mu)$ and $(w, 1_{(z,\tilde{c})}, \mu) \succ_{\tilde{z}} (a(\tilde{\delta}), \tilde{\delta}, \mu)$, where $a(\tilde{\delta}) = s$ if $\tilde{\delta} \in X_\emptyset \setminus X$ and $a(\tilde{\delta}) = m$ if $\tilde{\delta} \in X$.

Consider case (i) first. Let $O_{\tilde{c}}$, $O_{\hat{c}}$, $O_{\hat{\delta}}$ and O_μ be open neighborhoods of $\tilde{c}$, $\hat{c}$, $\hat{\delta}$ and μ, respectively, such that $(w, 1_{(z,\tilde{c}')}, \mu') \succ_{\tilde{z}} (w, 1_{(\hat{z},\hat{c}')}, \mu')$ and $\text{supp}(\hat{\delta}') \cap (\{\tilde{z}\} \times O_{\hat{c}}) \neq$

$\emptyset$ for each $\tilde{c}' \in O_{\tilde{c}}$, $\hat{c}' \in O_{\hat{c}}$, $\hat{\delta}' \in O_{\hat{\delta}}$ and $\mu' \in O_\mu$. Since $0 < \mu(\{\hat{z}\} \times O_{\hat{\delta}}) \leq \liminf_k \mu(\{\hat{z}\} \times O_{\hat{\delta}})$, it follows that, for each k sufficiently large, there is $\hat{\delta}_k \in O_{\hat{\delta}}$ such that $(\hat{z}, \hat{\delta}_k) \in \operatorname{supp}(\mu_k)$ and, for some $\hat{c}_k \in O_{\hat{c}}$, $(\tilde{z}, \hat{c}_k) \in \operatorname{supp}(\hat{\delta}_k)$. In addition, $\mu_k \in O_\mu$ and there exists $c_k \in \mathbb{C}_k(z, \tilde{z}) \cap O_{\tilde{c}} \cap V_{\tilde{c}}$ since, respectively, $\mu_k \to \mu$ and $\mathbb{C}_k(z, \tilde{z})$ increases to a dense subset of $\mathbb{C}(z, \tilde{z})$. Then $(w, 1_{(z,c_k)}, \mu_k) \succ_{\tilde{z}} (w, 1_{(\hat{z},\hat{c}_k)}, \mu_k)$ and, hence, $(\tilde{z}, c_k) \in T^m_z(\mu_k) \cap (\{\tilde{z}\} \times V_{\tilde{c}})$ for all k sufficiently large.

Consider next case (ii). Let $O_{\tilde{c}}$, $O_{\tilde{\delta}}$ and O_μ be open neighborhoods of $\tilde{c}$, $\tilde{\delta}$ and μ, respectively, such that $(w, 1_{(z,\tilde{c}')}, \mu') \succ_{\tilde{z}} (a(\tilde{\delta}), \tilde{\delta}', \mu')$ for each $\tilde{c}' \in O_{\tilde{c}}$, $\tilde{\delta}' \in O_{\tilde{\delta}}$ and $\mu' \in O_\mu$. Since $0 < \mu(\{\tilde{z}\} \times O_{\tilde{\delta}}) \leq \liminf_k \mu_k(\{\tilde{z}\} \times O_{\tilde{\delta}})$, it follows that, for each k sufficiently large there is $\tilde{\delta}_k \in O_{\tilde{\delta}}$ such that $(\tilde{z}, \tilde{\delta}_k) \in \operatorname{supp}(\mu_k)$. In addition, $\mu_k \in O_\mu$ and there exists $c_k \in \mathbb{C}_k(z, \tilde{z}) \cap O_{\tilde{c}} \cap V_{\tilde{c}}$ since, respectively, $\mu_k \to \mu$ and $\mathbb{C}_k(z, \tilde{z})$ increases to a dense subset of $\mathbb{C}(z, \tilde{z})$. Then $(w, 1_{(z,c_k)}, \mu_k) \succ_{\tilde{z}} (a(\tilde{\delta}), \tilde{\delta}_k, \mu_k)$ and, hence, $(\tilde{z}, c_k) \in T^m_z(\mu_k) \cap (\{\tilde{z}\} \times V_{\tilde{c}})$ for all k sufficiently large. ■

We now show that condition (a) of part 4 of Lemma 5 holds. Let $(z, \delta, \mu) \in Z \times X \times \mathcal{M}(Z \times X_\emptyset)$, $\delta' \in \Lambda(z, \delta, \mu)$, $V_{\delta'}$ be an open neighborhood of δ' and $\{(z_{k_j}, \delta_{k_j}, \mu_{k_j})\}_{j=1}^\infty$ be a sequence such that $(z_{k_j}, \delta_{k_j}, \mu_{k_j}) \to (z, \delta, \mu)$ and $(z_{k_j}, \delta_{k_j}, \mu_{k_j}) \in Z_{k_j} \times X_{k_j} \times \mathcal{M}(Z_{k_j} \times X_{\emptyset,k_j})$ for each $j \in \mathbb{N}$.

In particular, $\operatorname{supp}(\delta') \subseteq T^m_z(\mu) \cup \operatorname{supp}(\delta)$ and we may assume that $\operatorname{supp}(\delta')$ is finite, i.e. $\delta' = \sum_{(\tilde{z},\tilde{c}) \in \operatorname{supp}(\delta')} a(\tilde{z}, \tilde{c}) 1_{(\tilde{z},\tilde{c})}$ for some $a = (a(\tilde{z}, \tilde{c}))_{(\tilde{z},\tilde{c}) \in \operatorname{supp}(\delta')}$. Let V_a be an open neighborhood of a and, for each $(\tilde{z}, \tilde{c}) \in \operatorname{supp}(\delta')$, $V_{(\tilde{z},\tilde{c})}$ be an open neighborhood of $(\tilde{z}, \tilde{c})$ be such that

$$\sum_{(\tilde{z},\tilde{c}) \in \operatorname{supp}(\delta')} \hat{a}(\tilde{z}, \tilde{c}) 1_{(z(\tilde{z},\tilde{c}), c(\tilde{z},\tilde{c}))} \in V_{\delta'}$$

whenever $(z(\tilde{z}, \tilde{c}), c(\tilde{z}, \tilde{c})) \in V_{(\tilde{z},\tilde{c})}$ for each $(\tilde{z}, \tilde{c}) \in \operatorname{supp}(\delta')$ and $\hat{a} \in V_a$. Let $\hat{a} = (\hat{a}(\tilde{z}, \tilde{c}))_{(\tilde{z},\tilde{c}) \in \operatorname{supp}(\delta')} \in \mathbb{Q}_+^{|\operatorname{supp}(\delta')|} \cap V_a$ and $V_{\tilde{c}}$ be an open neighborhood of $\tilde{c}$ such that $\{\tilde{z}\} \times V_{\tilde{c}} \subseteq V_{(\tilde{z},\tilde{c})}$.

For each $(\tilde{z}, \tilde{c}) \in \operatorname{supp}(\delta') \cap T^m_z(\mu)$, and for each k sufficiently large, let $c_k(\tilde{z}, \tilde{c}) \in \mathbb{C}_k(z, \tilde{z})$ be such that $(\tilde{z}, c_k(\tilde{z}, \tilde{c})) \in T^m_z(\mu_k) \cap (\{\tilde{z}\} \times V_{\tilde{c}})$, which exists by Claim 8.

If $(\tilde{z}, \tilde{c}) \in \operatorname{supp}(\delta') \setminus T^m_z(\mu)$, then $\delta \in X$, $(\tilde{z}, \tilde{c}) \in \operatorname{supp}(\delta)$ and $0 < \delta(\{\tilde{z}\} \times V_{\tilde{c}}) \leq \liminf_j \delta_{k_j}(\{\tilde{z}\} \times V_{\tilde{c}})$. Hence, for each j sufficiently large, let $c_{k_j}(\tilde{z}, \tilde{c}) \in V_{\tilde{c}}$ be such that $(\tilde{z}, c_{k_j}(\tilde{z}, \tilde{c})) \in \mathrm{supp}(\delta_{k_j})$.

Let $J' \in \mathbb{N}$ be such that, for each $j \geq J'$, $(\tilde{z}, c_{k_j}(\tilde{z}, \tilde{c})) \in T_z^m(\mu_{k_j}) \cap (\{\tilde{z}\} \times V_{\tilde{c}})$ if $(\tilde{z}, \tilde{c}) \in \mathrm{supp}(\delta') \cap T_z^m(\mu)$ and $(\tilde{z}, c_{k_j}(\tilde{z}, \tilde{c})) \in \mathrm{supp}(\delta_{k_j}) \cap (\{\tilde{z}\} \times V_{\tilde{c}})$ if $(\tilde{z}, \tilde{c}) \in \mathrm{supp}(\delta') \setminus T_z^m(\mu)$. Thus, letting $\delta'_{k_j} = \sum_{(\tilde{z}, \tilde{c}) \in \mathrm{supp}(\delta')} \hat{a}(\tilde{z}, \tilde{c}) 1_{(\tilde{z}, c_{k_j}(\tilde{z}, \tilde{c}))}$ for each $j \geq J'$, we have that $\delta'_{k_j} \in V_{\delta'}$ and $\mathrm{supp}(\delta'_{k_j}) \subseteq T_z^m(\mu_{k_j}) \cup \mathrm{supp}(\delta_{k_j})$. Since $\{\hat{a}(\tilde{z}, \tilde{c}) : (\tilde{z}, \tilde{c}) \in \mathrm{supp}(\delta')\}$ is finite, it follows that there is $J > J'$ such that $\delta'_{k_j} \in X_{k_j}$ for each $j \geq J$.

An analogous argument shows that condition (b) of part 4 of Lemma 5 also holds. Hence, it follows that $\mathrm{supp}(\mu) \subseteq S_M(\mu)$. This together with the fact that μ is a matching and $\mathrm{supp}(\mu) \subseteq IR(\mu)$ shoes that μ is stable. ■

The next step of the proof of Theorem 2 extends Lemma 6 by requiring only that E be rich.

Lemma 7 *If E is a rational, continuous, bounded and rich market such that Z is finite, then E has a stable matching.*

Proof. It follows by Debreu (1964, Proposition 3) and by the finiteness of Z that there exists a continuous function $u : Z \times \Delta \times \mathcal{M}(Z \times X_\emptyset) \to [1, 2]$ such that $(a, \delta, \mu) \mapsto u(z, a, \delta, \mu)$ represents $\succeq_z$ for each $z \in Z$, using the fact that $[1, 2]$ and the extended reals are homeomorphic.

Let $R > 0$ be such that $X \subseteq \mathcal{M}_R(Z \times C)$, $\Delta^* = (\{m\} \times \mathcal{M}_R(Z \times C)) \cup (\{w\} \times X_w) \cup (\{s\} \times X_s)$ and $X^* = \mathcal{M}_R(Z \times C) \cup \{1_{(\emptyset, c)} : c \in C\}$. By Tietze Extension Theorem, let $U : Z \times \Delta^* \times \mathcal{M}(Z \times X^*) \to [1, 2]$ be a continuous extension of u.

Let ρ be a metric on $\mathcal{M}_R(Z \times C)$. For each $k \in \mathbb{N}$, let

$$\Delta_k = \{m\} \times \{\delta \in \mathcal{M}_R(Z \times C) : \rho(\delta, X) \geq k^{-1}\}.$$

Let, by Urysohn's Lemma, $g_k : \Delta^* \to [0, 1]$ be a continuous function such that $g_k^{-1}(1) = \Delta$ and $g_k^{-1}(0) = \Delta_k$. Then define $U_k : Z \times \Delta^* \times \mathcal{M}(Z \times X^*) \to \mathbb{R}$ by setting, for each $(z, a, \delta, \mu) \in Z \times \Delta^* \times \mathcal{M}(Z \times X^*)$, $U_k(z, a, \delta, \mu) = g_k(a, \delta) U(z, a, \delta, \mu)$.

Consider the market $E_k = (Z, \nu, C, \mathbb{C}, \mathcal{M}_R(Z \times C), U_k)$, i.e. E_k is equal to E except that X is replaced with $\mathcal{M}_R(Z \times C)$ and u with U_k. Since E_k is rational and continuous with Z finite and $X = \mathcal{M}_R(Z \times C)$, then E_k has a stable matching μ_k by Lemma 6.

Let $E^* = (Z, \nu, C, \mathbb{C}, \mathcal{M}_R(Z \times C), U)$. To avoid confusion, we write $IR(\mu; E')$ for $IR(\mu)$ and $S_M(\mu; E')$ for $S_M(\mu)$ whenever μ is a matching of a market E'. It follows by part 1 of Lemma 5 that we may assume that $\{\mu_k\}_{k=1}^{\infty}$ converges; let $\mu = \lim_k \mu_k$. It then follows by part 2 of Lemma 5 that μ is a matching of E^*.

The proof of part 3 of Lemma 5 implies that $\text{supp}(\mu) \subseteq IR(\mu; E^*)$ since the requirement that $\succ_{z,k}$ is the restriction of $\succ_z$ to $\Delta_k \times \mathcal{M}(Z_k \times X_{\emptyset,k})$ for each $z \in Z_k$ can be replaced with the following condition: $(s, \delta, \hat{\mu}) \succ_{z,k} (a, \delta', \hat{\mu})$ for each $k \in \mathbb{N}$, $z \in Z_k$, $\delta \in X_{s,k}$, $(a, \delta') \in \Delta_k$ and $\hat{\mu} \in \mathcal{M}(Z_k \times X_{\emptyset,k})$ such that $(s, \delta, \hat{\mu}) \succ_z (a, \delta', \hat{\mu})$. This condition holds because $U_k(z, a, \delta', \hat{\mu}) \leq U(z, a, \delta', \hat{\mu})$ and $U_k(z, s, 1_{(\emptyset,\hat{c})}, \hat{\mu}) = U(z, s, 1_{(\emptyset,\hat{c})}, \hat{\mu})$ for each $k \in \mathbb{N}$, $z \in Z$, $(a, \delta') \in \Delta^*$, $\hat{c} \in C$ and $\hat{\mu} \in \mathcal{M}(Z \times X^*)$ since $(s, 1_{(\emptyset,\hat{c})}) \in \Delta$ and, hence, $g_k(s, 1_{(\emptyset,\hat{c})}) = 1$.

We have that μ belongs to $\mathcal{M}(Z \times X_\emptyset)$. Indeed, let $k \in \mathbb{N}$ and $(z, \delta) \in \text{supp}(\mu_k) \cap \mathcal{M}(Z \times C)$. If $\delta \in X$ and $\rho(\delta, X) \geq k^{-1}$, then let $c \in \mathbb{C}(z, \emptyset)$ and $\delta' = 1_{(\emptyset,c)}$ to obtain that $\text{supp}(\delta') \subseteq T_z^s(\mu_k)$ and $U_k(z, s, \delta', \mu) = U(z, s, \delta', \mu) > 0 = U_k(z, m, \delta, \mu)$, the latter since $(s, \delta') \in \Delta$ and, thus, $g_k(s, \delta') = 1$, $U(z, s, \delta', \mu) \in (0, 1)$ and $g_k(m, \delta) = 0$. But this contradicts the stability of μ_k. Hence, it follows that $\rho(\delta, X) < k^{-1}$.

Thus, for each $k \in \mathbb{N}$,

$$\text{supp}(\mu_k) \subseteq (Z \times \{\delta \in \mathcal{M}_R(Z \times C) : \rho(\delta, X) \leq k^{-1}\}) \cup (Z \times \{1_{(\emptyset,c)} : c \in C\}).$$

Hence, $\text{supp}(\mu) \subseteq Z \times X_\emptyset$ as claimed.

It then follows that μ is a matching of E and that $\text{supp}(\mu) \subseteq IR(\mu; E)$ since $IR(\mu; E^*) \cap (Z \times X_\emptyset) \subseteq IR(\mu; E)$. Claim 9, which is analogous to part 4 of Lemma 5, shows that $\text{supp}(\mu) \subseteq S_M(\mu; E)$.

Claim 9 $\text{supp}(\mu) \subseteq S_M(\mu; E)$.

Proof. Let $(z, \delta) \in \text{supp}(\mu)$ and suppose that $(z, \delta) \notin S_M(\mu; E)$. Then there exists $\delta' \in X$ such that either (i) $\text{supp}(\delta') \subseteq T_z^m(\mu) \cup \text{supp}(\delta)$ and $U(z, m, \delta', \mu) > U(z, a(\delta), \delta, \mu)$, where $a(\delta) = m$ if $\delta \in X$ and $a(\delta) = s$ if $\delta \in X_\emptyset \setminus X$ (see Footnote 32), or (ii) there exists $(z', c) \in \text{supp}(\delta)$ such that $\text{supp}(\delta') \subseteq T_{z'}^m(\mu)$ and $U(z', m, \delta', \mu) > U(z', w, 1_{(z,c)}, \mu)$.

Consider case (i) first. Let $V_{\delta'}$, V_δ and V_μ be open neighborhoods of δ', δ and μ, respectively, such that $U(z, m, \gamma', \bar{\mu}) > U(z, a(\delta), \gamma, \bar{\mu})$ for each $\gamma' \in V_{\delta'}$, $\gamma \in V_\delta$ and $\bar{\mu} \in V_\mu$. Let, by the richness of E, $\tilde{V}_\delta$ and $\tilde{V}_\mu$ be open neighborhoods of δ and μ, respectively, such that $\Lambda(z, \gamma, \bar{\mu}) \cap V_{\delta'} \neq \emptyset$ for each $(\gamma, \bar{\mu}) \in \tilde{V}_\delta \times \tilde{V}_\mu$.

By Carmona & Podczeck (2009, Lemma 12), there is a subsequence $\{\mu_{k_j}\}_{j=1}^\infty$ of $\{\mu_k\}_{k=1}^\infty$ and corresponding sequence $\{\delta_{k_j}\}_{j=1}^\infty$ such that $\delta_{k_j} \to \delta$ and $(z, \delta_{k_j}) \in \operatorname{supp}(\mu_{k_j})$ for each $j \in \mathbb{N}$.

Let $J \in \mathbb{N}$ be such that $\mu_{k_j} \in V_\mu \cap \tilde{V}_\mu$ and $\delta_{k_j} \in V_\delta \cap \tilde{V}_\delta$ for all $j \geq J$ and, for each $j \geq J$, let $\delta'_{k_j} \in \Lambda(z, \delta_{k_j}, \mu_{k_j}) \cap V_{\delta'}$. Then, for each $j \geq J$,

$$U_{k_j}(z, m, \delta'_{k_j}, \mu_{k_j}) = U(z, m, \delta'_{k_j}, \mu_{k_j}) > U(z, a(\delta), \delta_{k_j}, \mu_{k_j}) \geq U_{k_j}(z, a(\delta), \delta_{k_j}, \mu_{k_j})$$

since $\delta'_{k_j} \in X$ by the definition of Λ, and $\operatorname{supp}(\delta'_{k_j}) \subseteq T_z^m(\mu_{k_j}) \cup \operatorname{supp}(\delta_{k_j})$. But this contradicts the stability of μ_{k_j}.

Now assume there exists $(z', c) \in \operatorname{supp}(\delta)$ and $\delta' \in X$ such that $\operatorname{supp}(\delta') \subseteq T_{z'}^m(\mu)$ and $U(z', m, \delta', \mu) > U(z', w, 1_{(z,c)}, \mu)$. Let $V_{\delta'}$, V_c and V_μ be open neighborhoods of δ', c and μ respectively, such that $U(z', m, \hat{\delta}', \hat{\mu}) > U(z', w, 1_{(z,\hat{c})}, \hat{\mu})$ for each $\hat{\delta}' \in V_{\delta'}$, $\hat{c} \in V_c$ and $\hat{\mu} \in V_\mu$. Let, by the richness of E, $\tilde{V}_\mu$ be an open neighborhood of μ such that $\Lambda_0(z', \hat{\mu}) \cap V_{\delta'} \neq \emptyset$ for each $\hat{\mu} \in \tilde{V}_\mu$.

By Carmona & Podczeck (2009, Lemma 12), there is a subsequence $\{\mu_{k_j}\}_{j=1}^\infty$ of $\{\mu_k\}_{k=1}^\infty$ and corresponding sequence $\{(\delta_{k_j}, c_{k_j})\}_{j=1}^\infty$ such that $(\delta_{k_j}, c_{k_j}) \to (\delta, c)$, $(z, \delta_{k_j}) \in \operatorname{supp}(\mu_{k_j})$ and $(z', c_{k_j}) \in \operatorname{supp}(\delta_{k_j})$ for each $j \in \mathbb{N}$.

Let $J \in \mathbb{N}$ be such that $\delta_{k_j} \in V_\delta$, $c_{k_j} \in V_c$ and $\mu_{k_j} \in V_\mu \cap \tilde{V}_\mu$ for all $j \geq J$ and, for each $j \geq J$, let $\delta'_{k_j} \in \Lambda_0(z', \mu_{k_j}) \cap V_{\delta'}$. Then, for each $j \geq J$, $U_{k_j}(z', m, \delta'_{k_j}, \mu_{k_j}) = U(z', m, \delta'_{k_j}, \mu_{k_j}) > U(z', w, 1_{(z, c_{k_j})}, \mu_{k_j}) \geq U_{k_j}(z', w, 1_{(z, c_{k_j})}, \mu_{k_j})$ since $\delta'_{k_j} \in X$ by the definition of Λ_0, and $\operatorname{supp}(\delta'_{k_j}) \subseteq T_{z'}^m(\mu_{k_j})$. But this contradicts the stability of μ_{k_j}. ■

It follows by $\operatorname{supp}(\mu) \subseteq IR(\mu; E)$ and by Claim 9 that $\operatorname{supp}(\mu) \subseteq S_M(\mu; E) \cap IR(\mu; E)$. Thus, μ is stable. ■

We now complete the proof of our existence result.

Proof of Theorem 2. Let $\{\nu_k\}_{k=1}^\infty$ be such that $\nu_k \to \nu$ and $\operatorname{supp}(\nu_k)$ is a finite subset of Z for each $k \in \mathbb{N}$. Define $Z_k = \text{supp}(\nu_k)$, $Z_{\emptyset,k} = Z_k \cup \{\emptyset\}$, $X_k = X \cap \mathcal{M}(Z_k \times C)$, $X_{\emptyset,k} = X_k \cup \{1_{(\emptyset,c)} : c \in C\}$, $X_{m,k} = X_k$, $X_{s,k} = \{1_{(\emptyset,c)} : c \in C\}$, $X_{w,k} = \{1_{(z,c)} : (z,c) \in Z_k \times C\}$ and $\Delta_k = \{(a,\delta) : \delta \in X_{a,k}\}$ for each $k \in \mathbb{N}$. Note that X_k is closed for each $k \in \mathbb{N}$.

For each $k \in \mathbb{N}$, let $\tilde{E}_k = (Z_k, \nu_k, C, \mathbb{C}, X_k, (\succ_z)_{z\in Z})$ be a market where $\succ_z$ is restricted to $\Delta_k \times \mathcal{M}(Z_k \times X_{\emptyset,k})$ for each $z \in Z$. Furthermore, let E_k be exactly as $\tilde{E}_k$, except with X in place of X_k and Z in place of Z_k; more precisely, $E_k = (Z, \nu_k, C, \mathbb{C}, X, (\succ_z)_{z\in Z})$.

Claim 10 *For each $k \in \mathbb{N}$, if μ is a stable matching of $\tilde{E}_k$, then μ is a stable matching of E_k.*

Proof. In this proof, to avoid confusion, we write $IR(\mu; E)$ for $IR(\mu)$ and $S_M(\mu; E)$ for $S_M(\mu)$ whenever μ is a matching of a market E.

Let $k \in \mathbb{N}$ and μ be a stable matching of $\tilde{E}_k$. Clearly, μ is a matching of E_k and $\text{supp}(\mu) \subseteq IR(\mu; E_k)$. We show that $\text{supp}(\mu) \subseteq S_M(\mu; E_k)$. Suppose not; then let $(z, \delta) \in \text{supp}(\mu) \setminus S_M(\mu; E_k)$.

First suppose that there exists $\delta' \in X$ such that $\text{supp}(\delta') \subseteq T^m_z(\mu) \cup \text{supp}(\delta)$ and $(m, \delta', \mu) \succ_z (a(\delta), \delta, \mu)$, where $a(\delta) = m$ if $\delta \in X$ and $a(\delta) = s$ if $\delta \in X_\emptyset \setminus X$. We claim that $\delta' \in X_k$, i.e. that $\text{supp}(\delta') \subseteq Z_k \times C$, from which we obtain a contradiction to the stability of μ in $\tilde{E}_k$.

Note that $\text{supp}(\bar{\delta}) \subseteq Z_k \times C$ whenever $\bar{\delta} \in X$ and $(\bar{z}, \bar{\delta}) \in \text{supp}(\mu)$ for some $\bar{z} \in Z_k$ since μ is stable in $\tilde{E}_k$. Thus, it follows that $\text{supp}(\delta') \cap \text{supp}(\delta) \subseteq Z_k \times C$ since if $\text{supp}(\delta') \cap \text{supp}(\delta) \neq \emptyset$, then $\delta \in X$. We also have that $\text{supp}(\delta') \cap T^m_z(\mu) \subseteq Z_k \times C$. Indeed, if $(z', c) \in T^m_z(\mu)$, then $(z', \bar{c}) \in \text{supp}(\bar{\delta})$ and $(\bar{z}, \bar{\delta}) \in \text{supp}(\mu)$ for some $\bar{c} \in C$, $\bar{z} \in Z_k$ and $\bar{\delta} \in X$ whenever $\text{supp}(\delta') \cap T^m_z(\mu) \neq \emptyset$; hence, $z' \in Z_k$. Thus, $\text{supp}(\delta') = (\text{supp}(\delta') \cap \text{supp}(\delta)) \cup (\text{supp}(\delta') \cap T^m_z(\mu)) \subseteq Z_k \times C$ as desired.

Now suppose that there exists $\delta' \in X$ and $(z', c) \in \text{supp}(\delta)$ such that $\text{supp}(\delta') \subseteq T^m_{z'}(\mu)$ and $(m, \delta', \mu) \succ_{z'} (w, 1_{(z,c)}, \mu)$. As above, we obtain a contradiction to the stability of μ in $\tilde{E}_k$ by showing that $\delta' \in X_k$. To establish this claim, it suffices to show that $T^m_{z'}(\mu) \subseteq Z_k \times C$. If $(\tilde{z}, \tilde{c}) \in T^m_{z'}(\mu)$, then $(\tilde{z}, \bar{c}) \in \text{supp}(\bar{\delta})$ and $(\bar{z}, \bar{\delta}) \in \text{supp}(\mu)$

for some $\bar{c} \in C$, $\bar{z} \in Z_k$ and $\bar{\delta} \in X$; hence, $(\tilde{z}, \tilde{c}) \in Z_k \times C$ as required. ■

For each $k \in \mathbb{N}$, let $\mu_k \in \mathcal{M}(Z \times X_{\emptyset,k})$ be a stable matching in E_k, which exists by Lemma 7 (since Z_k is finite) and Claim 10.

It follows by part 1 of Lemma 5 that we may assume that $\{\mu_k\}_{k=1}^{\infty}$ converges; let $\mu = \lim_k \mu_k$. It then follows by parts 2 – 4 of Lemma 5 that μ is a matching and that $\mathrm{supp}(\mu) \subseteq S_M(\mu) \cap IR(\mu)$. Hence, μ is stable. ■

A.4 Proof of Corollary 3

Let E be a Rosen market. For each $k \in \mathbb{N}$, let $\mathbb{C}_k \equiv [0, k]$, $X_k = \{n1_{(z,c)} : (z, c) \in Z \times C \text{ and } n \in [0, k]\}$ and E_k be equal to E except for these changes to $\mathbb{C}_k$ and X_k. It follows by Theorem 2 that there exists a stable matching μ_k of E_k.

Claim 11 $\mathrm{supp}(\mu_k) \subseteq Z \times X$ *for each* $k \in \mathbb{N}$.

Proof. Suppose not; then let $(z, \delta) \in \mathrm{supp}(\mu_k) \cap (Z \times (X_\emptyset \setminus X))$. Let $\varepsilon > 0$ be such that $g(r(z))q(z)\theta\left(r(z)/q(z)\right) - \varepsilon > 0$. Then $(z, \varepsilon) \in T_z^m(\mu_k)$ since $(z, \delta) \in \mathrm{supp}(\mu)$ and $U_z(w, 1_{(z,\varepsilon)}) = \varepsilon > 0 = U_z(s, \delta)$. Thus, letting $\delta' = 1_{(z,\varepsilon)}$, it follows that $\mathrm{supp}(\delta') \subseteq T_z^m(\mu_k)$ and $U_z(m, \delta') = g(r(z))q(z)\theta\left(r(z)/q(z)\right) - \varepsilon > 0 = U_z(s, \delta)$. Hence, $(z, \delta) \notin S(\mu_k)$, a contradiction to the stability of μ_k. ■

Claim 12 *There exist* $K, M \in \mathbb{N}$ *such that, for each* $k \geq K$ *and* $(z, \delta) \in \mathrm{supp}(\mu_k)$, $\delta(Z \times C) \leq M$ *and* $\delta(Z \times ([0, 1/M) \cup (M, \infty))) = 0$.

Proof. Suppose not; then, for each $j \in \mathbb{N}$, there exists $k_j \geq j$ and $(z_{k_j}, \delta_{k_j}) \in \mathrm{supp}(\mu_{k_j}) \subseteq Z \times X$ such that $\delta_{k_j}(Z \times C) > j$ or $\delta_{k_j}(Z \times ([0, 1/j) \cup (j, \infty))) > 0$.

Suppose first that $\delta_{k_j}(Z \times C) > j$ holds for infinitely many js. Taking a subsequence if needed, we may assume that $\delta_{k_j}(Z \times C) > j$ holds for each j. Thus, for some $(z'_{k_j}, c_{k_j}, n_{k_j}) \in Z \times C \times [0, k_j]$, $\delta_{k_j} = n_{k_j} 1_{(z'_{k_j}, c_{k_j})}$ with $n_{k_j} > j$. We have that

$$\begin{aligned} U_{z_{k_j}}(m, n_{k_j} 1_{(z'_{k_j}, c_{k_j})}) &\leq g(r(\bar{z}))f(r(\bar{z}), n_{k_j} q(\bar{z})) - c_{k_j} n_{k_j} \\ &= \left[g(r(\bar{z}))q(\bar{z})\theta\left(\frac{r(\bar{z})}{n_{k_j} q(\bar{z})}\right) - c_{k_j} \right] n_{k_j}. \end{aligned}$$

Since μ_{k_j} is stable, it follows that $U_{z_{k_j}}(m, n_{k_j}1_{(z'_{k_j},c_{k_j})}) \geq 0$ for each j; hence,

$$0 \leq c_{k_j} \leq g(r(\bar{z}))q(\bar{z})\theta\left(\frac{r(\bar{z})}{n_{k_j}q(\bar{z})}\right).$$

Since $n_{k_j} \to \infty$, it follows that $g(r(\bar{z}))q(\bar{z})\theta\left(r(\bar{z})/n_{k_j}q(\bar{z})\right) \to 0$ and, hence, $c_{k_j} \to 0$. Since $g(r(\underline{z}))q(\underline{z})\theta\left(r(\underline{z})/q(\underline{z})\right) > 0$, let $\varepsilon > 0$ be such that

$$g(r(\underline{z}))q(\underline{z})\theta\left(\frac{r(\underline{z})}{q(\underline{z})}\right) - \varepsilon > 0.$$

We have that $(z'_{k_j}, c_{k_j} + \varepsilon) \in T^m_{z'_{k_j}}(\mu_{k_j})$ for each j and that

$$U_{z'_{k_j}}(m, 1_{(z'_{k_j},c_{k_j}+\varepsilon)}) \geq g(r(\underline{z}))q(\underline{z})\theta\left(\frac{r(\underline{z})}{q(\underline{z})}\right) - c_{k_j} - \varepsilon > c_{k_j}$$

for all j sufficiently large. But this contradicts the stability of μ_{k_j}.

It follows from what has been shown above that $\delta_{k_j}(Z \times ([0, 1/j) \cup (j, \infty))) > 0$ holds for each j sufficiently large. Thus, for some $(z'_{k_j}, c_{k_j}, n_{k_j}) \in Z \times C \times [0, k_j]$, $\delta_{k_j} = n_{k_j}1_{(z'_{k_j},c_{k_j})}$ with $c_{k_j} > j$ or $c_{k_j} < 1/j$. First, suppose that $c_{k_j} < 1/j$ holds for infinitely many js. Note that $(z'_{k_j}, 1/j) \in T^m_{z'_{k_j}}(\mu_{k_j})$ and

$$U_{z'_{k_j}}(m, 1_{(z'_{k_j},\frac{1}{j})}) \geq g(r(\underline{z}))q(\underline{z})\theta\left(\frac{r(\underline{z})}{q(\underline{z})}\right) - \frac{1}{j} > \frac{1}{j} > c_{k_j}$$

for j sufficiently large, contradicting the stability of μ_{k_j}.

Now suppose that $c_{k_j} > j$ for all j sufficiently large. Since μ_{k_j} is stable, we then have that

$$0 \leq U_{z_{k_j}}(m, n_{k_j}1_{(z'_{k_j},c_{k_j})}) \leq \left[g(r(\bar{z}))q(\bar{z})\theta\left(\frac{r(\bar{z})}{n_{k_j}q(\bar{z})}\right) - c_{k_j}\right] n_{k_j}.$$

Thus, $n_{k_j} \to 0$ as $c_{k_j} \to \infty$ and, hence,

$$U_{z_{k_j}}(m, n_{k_j}1_{(z'_{k_j},c_{k_j})}) \leq g(r(z_{k_j}))f(r(z_{k_j}), n_{k_j}q(z_{k_j})) \to 0.$$

Let $\varepsilon > 0$ be such that

$$g(r(\underline{z}))q(\underline{z})\theta\left(\frac{r(\underline{z})}{q(\underline{z})}\right) - \varepsilon > 0.$$

We have that $(z_{k_j}, \varepsilon) \in T^m_{z_{k_j}}(\mu_{k_j})$ and that

$$U_{z_{k_j}}(m, 1_{(z_{k_j},\varepsilon)}) \geq g(r(\underline{z}))q(\underline{z})\theta\left(\frac{r(\underline{z})}{q(\underline{z})}\right) - \varepsilon > 0$$

for all j sufficiently large. But this contradicts the stability of μ_{k_j}. ■

Claim 12 implies that, for each $k \geq K$, the payoff of a manager in μ_k is bounded above by $\max_{n\in[0,M]} g(r(\bar{z}))f(r(\bar{z}), nq(\bar{z})) = \max_{n\in[0,M]} g(r(\bar{z}))nq(\bar{z})\theta\left(r(\bar{z})/nq(\bar{z})\right)$. In addition, the payoff of a manager is bounded below by $(1/2)g(r(\underline{z}))q(\underline{z})\theta\left(r(\underline{z})/q(\underline{z})\right)$, since if $(z, \delta) \in \mathrm{supp}(\mu_k)$ and $U_z(m, \delta) < (1/2)g(r(\underline{z}))q(\underline{z})\theta\left(r(\underline{z})/q(\underline{z})\right)$, then, letting $\varepsilon > 0$ be such that $g(r(\underline{z}))q(\underline{z})\theta\left(r(\underline{z})/q(\underline{z})\right) - \varepsilon > 2U_z(m, \delta)$, it follows that $(z, U_z(m, \delta) + \varepsilon) \in T_z^m(\mu_k)$ and

$$\begin{aligned} U_z(m, 1_{(z,U_z(m,\delta)+\varepsilon)}) &= g(r(z))f(r(z), q(z)) - U_z(m, \delta) - \varepsilon \\ &\geq g(r(\underline{z}))q(\underline{z})\theta\left(\frac{r(\underline{z})}{q(\underline{z})}\right) - U_z(m, \delta) - \varepsilon > U_z(m, \delta), \end{aligned}$$

which contradicts the stability of μ_k.

The payoff of a worker in μ_k is bounded below by $1/M$; since by Claim 11 there is no unemployment, it follows that

$$\min\{U_z(m, n1_{(z',c)}), U_{z'}(w, 1_{(z,c)})\} \geq \min\left\{\frac{1}{M}, \frac{1}{2}g(r(\underline{z}))q(\underline{z})\theta\left(\frac{r(\underline{z})}{q(\underline{z})}\right)\right\} \tag{8}$$

for each $(z, n1_{(z',c)}) \in \mathrm{supp}(\mu_k)$ and $k \geq K$.

Let

$$\bar{M} = \max\left\{M, \max_{n\in[0,M]} g(r(\bar{z}))nq(\bar{z})\theta\left(\frac{r(\bar{z})}{nq(\bar{z})}\right), \frac{2}{g(r(\underline{z}))q(\underline{z})\theta\left(\frac{r(\underline{z})}{q(\underline{z})}\right)}\right\},$$

$n(z, z', c)$ be the solution of $\max_{n\in\mathbb{R}_+}\left[g(r(z))nq(z')\theta\left(r(z)/nq(z')\right) - cn\right]$ for each $z, z' \in Z$ and $c \in [1/\bar{M}, \bar{M} + 1]$ and $\bar{n} = \max_{(z,z',c)\in Z^2\times[1/\bar{M},\bar{M}+1]} n(z, z', c)$; the existence of $\bar{n}$ follows by the compactness of $Z^2 \times [1/\bar{M}, \bar{M} + 1]$ and the continuity of $(z, z', c) \mapsto n(z, z', c)$.

Let $k > \max\{K, \bar{M} + 1, \bar{n}\}$ and $\mu = \mu_k$.

Claim 13 *μ is a stable matching of E.*

Proof. We will explicitly indicate the market we are considering in the stability set of μ, and thus write $S_M(\mu; E)$ and $S_M(\mu; E_k)$. We use analogous notation for $IR(\mu)$ and $T_z^m(\mu)$ for each $z \in Z$.

We first claim that, for each $(z, z', c) \in Z^2 \times C$, if $(z', c) \in T^m_z(\mu; E)$, then $(z', \bar{M} + 1) \in T^m_z(\mu; E_k)$. Indeed, $(z', c) \in T^m_z(\mu; E)$ implies that $c = U_{z'}(w, 1_{(z,c)}) > U_{z'}(a, \delta)$ for some $(a, \delta) \in \Delta$ such that:

(a) If $a = w$, then $\delta = 1_{(\hat{z},\hat{c})}$ with $(\hat{z}, \hat{n}1_{(z',\hat{c})}) \in \text{supp}(\mu)$ and, thus, $U_{z'}(w, \delta) = \hat{c} \leq M$ by Claim 12.

(b) If $a = s$, then $U_{z'}(s, \delta) = 0$.

(c) If $a = m$, then $(z', \delta) \in \text{supp}(\mu)$ and, thus, $U_{z'}(m, \delta) \leq M$ by Claim 12.

Hence, $U_{z'}(a, \delta) \leq M \leq \bar{M}$ and it follows that $(z', \bar{M}+1) \in T^m_z(\mu; E_k)$ since $k > \bar{M}+1$.

We now establish that μ is a stable matching of E. Let $(z, \delta) \in \text{supp}(\mu)$. Since μ is a stable matching of E_k, $(z, \delta) \in S_M(\mu; E_k) \cap IR(\mu; E_k)$ and $\delta \in X$ by Claim 11. We have that $U_{z'}(s, \delta') = 0$ for each $(z', \delta') \in Z \times X_s$ and, thus, $IR(\mu; E_k) \subseteq IR(\mu; E)$. Hence, $(z, \delta) \in IR(\mu; E)$.

It thus remains to show that $(z, \delta) \in S_M(\mu; E)$. Let $\delta = n1_{(\tilde{z},c)}$ and let (i) $(\hat{z}, \hat{\delta}) = (z, \delta)$ and $a = m$ or (ii) $(\hat{z}, \hat{\delta}) = (\tilde{z}, 1_{(z,c)})$ and $a = w$. Let $\delta' \in X$ be such that $\text{supp}(\delta') \subseteq T^m_{\hat{z}}(\mu; E)$ and let $\delta' = n^* 1_{(z^*,c^*)}$. Note that $(z^*, c^*) \in T^m_{\hat{z}}(\mu; E)$ implies that $c^* \geq 1/\bar{M}$ by (8). If $c^* \leq \bar{M} + 1$, then $(z^*, c^*) \in T^m_{\hat{z}}(\mu; E_k)$ and

$$U_{\hat{z}}(m, \delta') = U_{\hat{z}}(m, n^* 1_{(z^*,c^*)}) \leq U_{\hat{z}}(m, n(\hat{z}, z^*, c^*) 1_{(z^*,c^*)}) \leq U_{\hat{z}}(a, \hat{\delta}),$$

where the last inequality follows from $(z, \delta) \in S_M(\mu; E_k)$. If $c^* > \bar{M} + 1$, then $(z^*, \bar{M} + 1) \in T^m_{\hat{z}}(\mu; E_k)$ and

$$\begin{aligned} &U_{\hat{z}}(m, \delta') = U_{\hat{z}}(m, n^* 1_{(z^*,c^*)}) \leq U_{\hat{z}}(m, n^* 1_{(z^*,\bar{M}+1)}) \leq \\ &U_{\hat{z}}(m, n(\hat{z}, z^*, \bar{M} + 1) 1_{(z^*,\bar{M}+1)}) \leq U_{\hat{z}}(a, \hat{\delta}), \end{aligned}$$

where the last inequality follows from $(z, \delta) \in S_M(\mu; E_k)$.

Finally, let $\delta' \in X$ be such that $\text{supp}(\delta') \subseteq \text{supp}(\delta)$ in case (i). Then $\delta' = n' 1_{(\tilde{z},c)}$ for some $n' \in \mathbb{R}_+$. Since $1/M \leq c \leq M$ by Claim 12, it follows that

$$U_z(m, \delta') = U_z(m, n' 1_{(\tilde{z},c)}) \leq U_z(m, n(z, \tilde{z}, c) 1_{(\tilde{z},c)}) \leq U_z(m, \delta),$$

where the last inequality follows from $(z, \delta) \in S_M(\mu; E_k)$. This concludes the proof that $(z, \delta) \in S_M(\mu; E)$ and establishes the claim. ■

A.5 Proof of Theorem 3

In this section, we show that the conditions in the statement of Theorem 3 are necessary and sufficient for μ to be a stable matching of the Rosen market.[33] Note that the function h is an homeomorphism between Z^2 and $h(Z^2)$.

Sufficiency. Let $\mu = \lambda \circ h^{-1}$ for some w and λ as in the statement of the Theorem. To see that μ is a matching, note that for each measurable B,

$$\begin{aligned}
&\mu(B \times X) + \int_{Z\times X} \delta(B \times C)\mathrm{d}\mu(z,\delta) \\
&= \lambda \circ h^{-1}(B \times X) + \int_{Z\times X} \delta(B \times C)\mathrm{d}\lambda \circ h^{-1}(z,\delta) \\
&= \lambda(B \times Z) + \int_{Z\times B} n(z,z',w)\mathrm{d}\lambda(z,z') = \nu(B).
\end{aligned}$$

We now show that μ is stable by establishing that $\mathrm{supp}(\mu) \subseteq S_M(\mu) \cap IR(\mu)$. Let $(z,\delta) \in \mathrm{supp}(\mu)$; then $\delta = n(z,z',w)1_{(z',wq(z'))}$ for some $z' \in Z$ and $(z,z') \in \mathrm{supp}(\lambda)$ by Lemma 1. To see that $(z,\delta) \in IR(\mu)$, note that $U_z(m, n(z,z',w)1_{(z',wq(z'))}) = R(z,w) > 0$ and $U_{z'}(w, 1_{(z,wq(z'))}) = wq(z') > 0$.

Suppose that $(z, n(z,z',w)1_{(z',wq(z'))}) \notin S_M(\mu)$. Then either there exists $(z^*,c^*) \in T^m_z(\mu) \cup \{(z', wq(z'))\}$ such that $U_z(m, n(z,z^*,c^*/q(z^*))1_{(z^*,c^*)}) > R(z,w)$ or there exists $(z^*,c^*) \in T^m_{z'}(\mu)$ such that $U_{z'}(m, n(z',z^*,c^*/q(z^*))1_{(z^*,c^*)}) > wq(z')$. If $(z^*,c^*) = (z', wq(z'))$, then $U_z(m, n(z,z^*,c^*/q(z^*))1_{(z^*,c^*)}) = R(z,w)$. Thus, $(z^*,c^*) \in T^m_z(\mu) \cup T^m_{z'}(\mu)$ and, hence, $c^* > wq(z^*)$; indeed, condition (b) of $T^m_z(\mu) \cup T^m_{z'}(\mu)$ cannot happen since $\mathrm{supp}(\mu) \subseteq Z \times X$, condition (a) implies $c^* > wq(z^*)$ and condition (c) implies that $z^* \in \mathrm{proj}_1(\mathrm{supp}(\lambda))$ and $c^* > R(z^*,w)$ and, thus, that $c^* > wq(z^*)$ since then $R(z^*,w) \geq wq(z^*)$. If $(z^*,c^*) \in T^m_z(\mu)$, then $U_z(m, n(z,z^*,c^*/q(z^*))1_{(z^*,c^*)}) < R(z,w) = U_z(m, n(z,z',w)1_{(z',wq(z'))})$. If $(z^*,c^*) \in T^m_{z'}(\mu)$, then

$$U_{z'}(m, n(z',z^*,\tfrac{c^*}{q(z^*)})1_{(z^*,c^*)}) < R(z',w) \leq wq(z'),$$

the last inequality holding since $z' \in \mathrm{proj}_2(\mathrm{supp}(\lambda))$. Thus, $(z, n(z,z',w)1_{(z',wq(z'))}) \in S_M(\mu)$, and hence μ is stable.

[33] See the working paper version for an illustration of Theorem 3 and its proof in the Cobb-Douglas case.

Necessity. Let μ be a stable matching of a Rosen market. We first show that $\text{supp}(\mu) \subseteq h(Z^2)$. Let $z, z', \hat{z}, \tilde{z} \in Z$ and $\hat{n}, \tilde{n}, c(z), c(z') \in \mathbb{R}_+$ be such that $(\hat{z}, \hat{n}1_{(z,c(z))})$ and $(\tilde{z}, \tilde{n}1_{(z',c(z'))})$ belong to $\text{supp}(\mu)$. Suppose, in order to reach a contradiction, that $c(z)/c(z') \neq q(z)/q(z')$; for concreteness, $c(z) > (c(z')/q(z'))q(z)$ and let $w = c(z')/q(z')$. It follows that

$$U_{\hat{z}}(m, \hat{n}1_{(z,c(z))}) < \max_n U_{\hat{z}}(m, n1_{(z,wq(z))}) = R(\hat{z}, w) = \max_n U_{\hat{z}}(m, n1_{(z',wq(z'))}).$$

Thus, there is $\varepsilon > 0$ such that $U_{\hat{z}}(m, \hat{n}1_{(z,c(z))}) < R(\hat{z}, w + \varepsilon)$. Since $(w + \varepsilon)q(z') = c(z') + \varepsilon q(z') > c(z')$, it follows that $(z', (w+\varepsilon)q(z')) \in T^m_{\hat{z}}(\mu)$. Thus, $\delta' = n(\hat{z}, z', w+\varepsilon)1_{(z',(w+\varepsilon)q(z'))}$ is such that $\text{supp}(\delta') \subseteq T^m_{\hat{z}}(\mu)$ and

$$U_{\hat{z}}(m, \delta') = R(\hat{z}, w + \varepsilon) > U_{\hat{z}}(m, \hat{n}1_{(z,c(z))}).$$

But this contradicts the stability of μ. It then follows that $c(z)/c(z') = q(z)/q(z')$ and, again letting $w = c(z')/q(z')$, that $c(z) = wq(z)$.

It also follows that $\hat{n} = n(\hat{z}, z, w)$ since otherwise $\delta' = n(\hat{z}, z, w)1_{(z,wq(z))}$ is such that $\text{supp}(\delta') \subseteq \text{supp}(\hat{n}1_{(z,wq(z))})$ and $U_{\hat{z}}(m, \delta') > U_{\hat{z}}(m, \hat{n}1_{(z,wq(z))})$ and, thus, contradicts the stability of μ.

Let $h^{-1} : h(Z^2) \to Z^2$ be the inverse of h and define $\lambda = \mu \circ (h^{-1})^{-1}$. Then (1) and (3) follow.

To see (2), let $(z, z') \in \text{supp}(\lambda)$, which implies that $(z, n(z, z', w)1_{(z',wq(z'))}) \in \text{supp}(\mu)$ by Lemma 1. If $R(z, w) < wq(z)$, then let $\varepsilon > 0$ be such that $wq(z) - \varepsilon > R(z, w)$ and note that $(z, wq(z) - \varepsilon) \in T^m_z(\mu)$ since $(z, n(z, z', w)1_{(z',wq(z'))}) \in \text{supp}(\mu)$ and $U_z(w, 1_{(z,wq(z)-\varepsilon)}) > U_z(m, n(z, z', w)1_{(z',wq(z'))})$. Thus,

$$\begin{aligned}&U_z(m, n(z, z, w)1_{(z,wq(z)-\varepsilon)}) > U_z(m, n(z, z, w)1_{(z,wq(z))})\\ &= R(z, w) = U_z(m, n(z, z', w)1_{(z',wq(z'))}),\end{aligned}$$

contradicting the stability of μ. Hence, $R(z, w) \geq wq(z)$.

Similarly, if $wq(z') < R(z', w)$, then let $\varepsilon > 0$ be such that

$$U_{z'}(m, n(z', z', w)1_{(z',wq(z')+\varepsilon)}) > wq(z').$$

Note that $(z', wq(z') + \varepsilon) \in T^m_{z'}(\mu)$ since $(z, n(z,z',w)1_{(z',wq(z'))}) \in \mathrm{supp}(\mu)$ and $U_{z'}(w, 1_{(z',wq(z')+\varepsilon)}) > U_{z'}(w, 1_{(z,wq(z'))})$. Thus,

$$U_{z'}(m, n(z',z',w)1_{(z',wq(z')+\varepsilon)}) > wq(z') = U_{z'}(w, 1_{(z,wq(z'))}),$$

contradicting the stability of μ. Hence, $wq(z') \geq R(z', w)$.

A.6 Non-existence example

We show that without the boundedness assumptions on X, a stable matching need not exist, even when stability is defined via strong domination.

Consider the following market E, where for simplicity we omit contracts and preferences do not depend on the matching. Let $Z = [0,1]$, let ν be the uniform distribution and let $X = \mathcal{M}(Z)$. Preferences are given by $u_z(m,\delta) = \delta(Z)$, $u_z(w, 1_{z'}) = z'$ and $u_z(s, 1_\emptyset) = 0$ for each $z, z' \in Z$ and $\delta \in X$. Then E is rational, continuous and rich but not bounded and it has no stable matching as we next show.

Suppose that E has a stable matching μ. First note that $\mu(Z \times (X_\emptyset \setminus X)) = 0$. If not, then let $\hat{Z} = \{z \in Z : (z, 1_\emptyset) \in \mathrm{supp}(\mu)\}$ and $z \in \hat{Z}$ be such that $z > 0$. Then $\hat{Z} \subseteq T^m_z(\mu)$ and $\hat{Z}$ is closed. Thus, letting $\nu|_{\hat{Z}}$ be the restriction of ν to $\hat{Z}$ (i.e. $\nu|_{\hat{Z}}(B) = \nu(B \cap \hat{Z})$ for each Borel subset B of Z), it follows that $\mathrm{supp}(\nu|_{\hat{Z}}) \subseteq T^m_z(\mu)$ which, together with $(m, \nu|_{\hat{Z}}) \succ_z (s, 1_\emptyset)$, contradicts the stability of μ.

Next note that if $(z,\delta) \in \mathrm{supp}(\mu) \cap \big((Z \setminus \{1\}) \times X\big)$, then $\delta(Z) = 0$. To see this, suppose that $(z,\delta) \in \mathrm{supp}(\mu)$ with $z < 1$, $\delta \in X$ and $\delta(Z) > 0$. Note that for all $z' \in \mathrm{supp}(\delta)$, $z' \in T^m_{z^*}(\mu)$ for each $z^* > z$ since $(w, z^*) \succ_{z'} (w, z)$; thus, $\mathrm{supp}(\delta) \subseteq T_{z^*}(\mu)$. Since $\mu((z,1] \times X_\emptyset) + \int_{Z \times X} \delta((z,1])\mathrm{d}\mu(z,\delta) = \nu((z,1]) > 0$, it follows that either $\mathrm{supp}(\mu) \cap \big((z,1] \times X_\emptyset\big) \neq \emptyset$ or $\mathrm{supp}(\hat{\delta}) \cap \big((z,1]\big) \neq \emptyset$ for some $(\hat{z}, \hat{\delta}) \in \mathrm{supp}(\mu)$. Let $z^* > z$ be such that either $(z^*, \delta^*) \in \mathrm{supp}(\mu)$ for some $\delta^* \in X_\emptyset$ or $z^* \in \mathrm{supp}(\hat{\delta})$ for some $(\hat{z}, \hat{\delta}) \in \mathrm{supp}(\mu)$. Then consider $\delta' = n\delta$ where n is such that $\delta'(Z) = n\delta(Z) > \max\{\delta^*(Z), 1\}$. We have that $\mathrm{supp}(\delta') = \mathrm{supp}(\delta) \subseteq T_{z^*}(\mu)$ and $(m, \delta') \succ_{z^*} (m, \delta^*)$ if $(z^*, \delta^*) \in \mathrm{supp}(\mu)$ and $\delta^* \in X$, $(m, \delta') \succ_{z^*} (s, \delta^*)$ if $(z^*, \delta^*) \in \mathrm{supp}(\mu)$ and $\delta^* \in X_\emptyset \setminus X$, and $(m, \delta') \succ_{z^*} (w, 1_{\hat{z}})$ if $z^* \in \mathrm{supp}(\hat{\delta})$ and $(\hat{z}, \hat{\delta}) \in \mathrm{supp}(\mu)$. This contradicts the stability of μ.

It follows by the above claims that $\mu(Z \times (X_\emptyset \setminus X)) = 0$ and that

$$\int_{Z\times X} \delta(Z)\mathrm{d}\mu(z,\delta) = \int_{\mathrm{supp}(\mu)\cap((Z\setminus\{1\})\times X)} \delta(Z)\mathrm{d}\mu(z,\delta) = 0.$$

Thus, $\mu(Z\times X) = \nu(Z) = 1$ and, since $\delta = 0$ for each $(z,\delta) \in \mathrm{supp}(\mu)\cap\big((Z\setminus\{1\})\times X\big)$, it follows that $\mathrm{supp}(\mu) = Z \times \{0\}$, where $0 \in \mathcal{M}(Z)$ denotes the zero measure on Z. But then $Z \subseteq T_1^m(\mu)$ and $(m,\nu) \succ_1 (m,0)$, contradicting the stability of μ.

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