Strategic Exits in Stochastic Partnerships: The Curse of Profitability[*]

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Abstract

We study dynamic partnerships where the output evolves stochastically, each player can exit at any time, and players who have exited continue to accrue some benefits if the remaining players keep contributing to the partnership. Players can strategically exit to free-ride on their partners' contributions, knowing that it may trigger subsequent exits of their partners. We characterize the unique Pareto-optimal equilibrium. When players have sufficiently large free-riding incentives and a medium level of mutual reliance, this equilibrium exhibits a *curse of profitability*: An increase in the partnership's output may strictly harm all the players. Another main finding is that Pareto-improvement can be achieved if any player commits not to exit first.

Keywords: partnerships, strategic exits, curse of profitability, dynamic coordination, stochastic stopping games.

JEL Codes: C73, D70, L22.

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1 Introduction

In some partnerships, partners who have exited continue to accrue some benefits as long as the remaining partners keep contributing to the partnership. In a startup firm, for instance, co-founders who have ceased investment can still benefit from the startup's later success, including monetary returns (if they still retain some shares of the startup) and reputation gains. In a cartel, firms that have departed can still benefit from the low quantities or high prices maintained by those that remain in the cartel. In a financial institution facing liquidity strains, investors who cease injecting liquidity will suffer less loss if the other investors fulfill the liquidity demands. In an environmental agreement, nations that have withdrawn still benefit from the reduction of greenhouse gases by the remaining participants.

These partnerships face the common problem of *strategic exiting* — partners may exit to save their private contribution costs while relying on the continued contributions of others. Notice that a partner's exit makes it more difficult for the remaining partners to operate the partnership and thus may trigger them to exit as well. Such a *ripple effect*, in turn, determines whether a partner would like to strategically exit in the first place.

This paper builds a framework to investigate the dynamics of cooperation in partnerships where exited partners continue to benefit from the partnership's output. In particular, we focus on stochastic partnerships — the partnership's flow output, which we refer to as its level of *profitability*, changes stochastically over time. This captures the fact that a partnership's output is usually affected by some evolving external factors.[1]

The main finding of this paper is that the partnership may be subject to a *curse of profitability* — under some conditions that we specify later, a more profitable partnership leaves *all* the partners strictly worse off. Intuitively, higher profitability is a double-edged sword. On the one hand, it means the partnership generates more output. On the other hand, if some partners exit, higher profitability makes the remaining partners more willing to keep operating the partnership, which stimulates strategic exiting in the first place. Moreover, partners have incentives to pre-empt each other since they prefer to be the free-riders (those who exit while others remain), and because of

[1] For instance, a startup faces evolving market competition and financing environment; a cartel faces fluctuating market demand and is subject to technology shocks; a financial institution faces changing market sentiment and regulatory environment; an environmental agreement faces changing public political attitudes.

that, the free-riders exit "too early" in equilibrium. As a consequence, all the players — including the free-riders — may suffer from high profitability.

Our baseline model features two players running a joint project whose profitability level evolves according to a Brownian motion. Each player can exit at any time to save his contribution cost.[2] We refer to the player who exits first as the *first mover* and his partner as the *second mover*.[3] The ripple effect is that the second mover, finding it more difficult to run the project after the first mover exits, may choose to exit as well and thus terminate the project.

Section 3 analyzes the pure-strategy Markov perfect equilibria (MPE), where each player decides when to exit based on the project's current profitability level and whether his partner has already exited. Theorem 1 shows that in the *unique* Pareto-optimal equilibrium, increasing the partnership's profitability level may strictly decrease both players' continuation value. This finding formalizes this paper's core insight regarding the curse of profitability in partnerships. Moreover, we find that the curse of profitability occurs if and only if the players have sufficiently large free-riding incentives and a medium level of mutual reliance (Corollary 1).

Section 4 studies whether and how the issue of strategic exiting can be mitigated when one player commits not to exit first. Departing from the baseline model, we analyze an alternative setting where one player, referred to as Susan, commits not to exit first and becomes the designated second mover, while the other player, Frank, is designated as the first mover. This section presents two main findings. First, Theorem 2 shows that Frank may strategically exit only when the partnership's profitability exceeds a certain level, which explains some exit patterns observed in practice. For example, serial entrepreneurs often play a pivotal role during a startup's early stages but strategically exit as the startup gains traction, shifting their focus to the next startup in their pipelines.[4] Second, Theorem 3 shows that Susan's no-first-exit commitment can lead to a Pareto-improvement over the baseline model, as her benefit from avoiding pre-emption outweighs her cost of forgoing the option to exit first.

Section 5 examines the robustness of the paper's main result. The curse of profitability is shown

[2]In the baseline model, exits are assumed *irreversible* (i.e., exited partners cannot re-enter the partnership), capturing the idea that re-entry is either impossible or costly in many real-world partnerships. Online Appendix B.3 of Xu (2025) shows that this paper's main result remains true under some conditions when re-entry is possible but costly.

[3]The identities of the first and second movers are endogenously determined by the players' strategies.

[4]One prominent example is Peter Thiel, who left PayPal after its successful acquisition by eBay and soon transitioned his investment into other ventures, including Facebook.

to persist across various generalizations, including settings with more than two players, asymmetric payoff structure among players, the possibility for exited players to re-enter the partnership, and the relaxation of several innocuous modeling assumptions.

Related Literature

Broadly speaking, this paper contributes to the study of dynamic incentives in cooperation. The most distinctive feature of this paper is that players can irreversibly exit and continue to free-ride on others' contributions. Hence, this paper is related to the following strands of literature.

First, it is related to the literature on dynamic games where players have exit options. Many papers in this literature feature rippling exits in equilibrium, driven by either payoff externalities (i.e., a player's exit alters others' payoffs) or information externalities (i.e., a player's exit conveys information to others).[5] Our paper builds on payoff externalities but introduces a key novelty: we consider *two-way externalities* — players who exit early harm the remaining players but are also harmed if the remaining players later exit. Hence, players in our paper are concerned about the ripple effect triggered by their own exits, while such a concern is absent in the existing literature. Because of that, our paper gives rise to new economic forces like the curse of profitability.

Second, this paper is related to the literature on dynamic contribution games, where players exert effort over time to build a common stock of public goods (Admati & Perry, 1991; Fershtman & Nitzan, 1991; Marx & Matthews, 2000; Georgiadis, 2015). In that literature, a player's contribution can encourage others to contribute more in the future; in our paper, similarly, a player's decision to stay in the partnership can encourage others to stay.[6] Some papers in that literature also highlight economic forces under which stronger fundamentals of a partnership paradoxically lead to worse outcomes. For example, Curello (2023) studies a dynamic contribution game where a player's effort stochastically increases the stock of public goods, while the opportunity cost of effort rises with the stock. Due to the opportunity cost, a high stock of public goods may result in low effort and, consequently, low continuation values for the players. Similarly, Ramos and

[5] For models with payoff externalities, see Jovanovic and MacDonald (1994) in industry shakeouts, Cetemen, Urgun, and Yariv (2023) in collective search, etc. For models with information externalities, see Chamley and Gale (1994), Rosenberg, Solan, and Vieille (2007), Moscarini and Squintani (2010), Murto and Välimäki (2011), Guo and Roesler (2018), Margaria (2020), Awaya and Krishna (2021), Kirpalani and Madsen (2023), etc.

[6] Model-wise, papers in that literature can be viewed as dynamic games with an endogenous state variable — the stock of public goods. In our paper, the endogenous state variable is the number of remaining players.

Sadzik (2023) study a dynamic contribution game where players accumulate relational capital. In that paper, a high level of relational capital may weaken players' relational incentives because of a cap on how much relational capital they can accumulate. Our paper contributes to this line of inquiry by introducing a novel economic force under which strong fundamentals of a partnership can have adverse effects — high profitability of a partnership may incentivize strategic exits, as players anticipate that the remaining players will be motivated to continue operating the project.

Third, this paper is related to voluntary partnership games, where players repeatedly face the prisoner's dilemma and have the option to opt out (Ghosh & Ray, 1996; Fujiwara-Greve & Okuno-Fujiwara, 2009; McAdams, 2011). Despite the similarity, the purpose of exiting is opposite — players in our paper strategically exit to free-ride on others' efforts, while in those papers, the intention of an exit is to punish a free-rider.

Finally, this paper adds to the applications of continuous-time stopping games (also referred to as real options games), especially those concerning pre-emption in different contexts (Fudenberg & Tirole, 1985; Dutta & Rustichini, 1993; Grenadier, 1996; Weeds, 2002; Bobtcheff, Bolte, & Mariotti, 2017; Riedel & Steg, 2017; Thomas, 2021).

2 Baseline Model

2.1 Payoff

	Stay	Exit
Stay	$X_t - c\ ,\ X_t - c$	$\beta X_t - \kappa c\ ,\ \alpha X_t$
Exit	$\alpha X_t\ ,\ \beta X_t - \kappa c$	$0\ ,\ 0$

Table 1: Flow payoff at time t in the baseline model

Time is continuous with an infinite horizon, indexed by $t \in [0, \infty)$. Two players ($i = 1, 2$) form a partnership to run a joint project. Player i's realized lifetime utility is $\Pi_i = \int_0^\infty e^{-rt}\pi_{it}dt$, where $r > 0$ is the common discount rate of the players and π_{it} is his flow payoff at time t. Players' flow payoffs are given in Table 1. If both players stay in the partnership, they each pay a flow contribution cost of $c > 0$ and receive a flow revenue of $X_t > 0$. We interpret $X_t \in \mathcal{X} = \mathbb{R}^+$ as the project's level of *profitability*. It changes over time, following a geometric Brownian motion,

$dX_t/X_t = \mu dt + \sigma dZ_t$, where $\mu < r$, $\sigma > 0$, and Z_t is a standard Wiener process.[7] If Player i ("he") exits while Player j ("she") operates the project alone, two changes in payoff happen. On the one hand, Player i saves his contribution cost while continuing to enjoy a flow revenue of αX_t with $\alpha > 0$. We refer to α as the *free-riding parameter* as it measures a player's benefit from free-riding. On the other hand, Player j's flow cost increases to κc with $\kappa \geq 1$, as she now has to take on the additional responsibilities that Player i would have carried out. Her flow revenue is also changed to βX_t with $\beta > 0$.[8] The change in her revenue can be attributed to two potential factors: the loss of synergy resulting from the other player's exit, which tends to reduce β, or the increase in her control over the project, which typically raises β. To streamline later analysis, we introduce a parameter $\lambda := \kappa/\beta$, allowing Player j's flow payoff to be equivalently expressed as $\beta(X_t - \lambda c)$. We refer to λ as the *reliance parameter*, as a higher value of λ indicates greater difficulty for a single player to operate the project, or equivalently, a higher degree of mutual reliance between the players in operating the project. Finally, if both players exit, their payoffs are normalized to zero. We place two assumptions on the parameters.

Assumption 1. $\alpha + \beta \leq 2$ *and* $\kappa \geq 2$.

Assumption 2. $\alpha < \beta$.

Assumption 1 says that the partnership with a solo contributor generates (weakly) less revenue and incurs (weakly) more cost than that with two contributors, capturing the idea that the players create synergy when contributing to the partnership together. Because of this assumption, strategic exiting is socially inefficient since players' total flow payoff with two contributors, $2(X_t - c)$, is always higher than that with only one contributor, $(\alpha + \beta)X_t - \kappa c$. Assumption 2 says that a free-rider receives less revenue than a contributor, which is realistic for many real-world partnerships. We relegate the discussion of the less realistic situation where $\alpha \geq \beta$ to Online Appendix B.5 of Xu (2025), where we show that the main insights of this paper remain intact except that some additional discussion of the parameters is needed.

It is also worth noticing that these two assumptions, when put together, determine the domains for the parameters (α, β, λ). The domain of α is $(0, 1)$. Given the value of α, the domain of β is

[7] Online Appendix B.4 of Xu (2025) shows that this paper's main result continues to hold if X_t follows a more general diffusion process.

[8] This encompasses the special case $\beta = 1$, where Player j's flow revenue remains unchanged after Player i exited.

$(\alpha, 2-\alpha]$, and the domain of λ is $[\underline{\lambda}, \infty)$ where $\underline{\lambda} := 2/(2-\alpha) > 1$.

2.2 Timeline

Players choose when to exit the partnership, and their past actions are perfectly observed. To allow players to instantaneously react to their partners' actions, we formulate the model as a two-stage dynamic game *à la* Murto and Välimäki (2013).[9]

Stage 1. In Stage 1, each player chooses when to exit, given that neither has exited yet. Player i's strategy in this stage is an $\mathcal{H}_t$-adapted stopping time τ^i, where $\mathcal{H}_t$ contains all the information about the public history, including the history of the state variable during $[0, t]$ and the history of players' actions during $[0, t)$. Stage 1 ends at $\tau := \min\{\tau^1, \tau^2\}$. It is possible, however, that both players attempt to exit at the same time (i.e., $\tau^1 = \tau^2$) in Stage 1. In case that happens, we make the following tie-breaking assumption: Only one player (selected at random by a coin flip or other fair randomization device) can successfully exit.[10] Given this assumption, whether or not tie-breaking is necessary, there is only one player exiting in Stage 1. We call this player the first mover ("he").

Stage 2. After the first mover exits, the game immediately proceeds to Stage 2, where the remaining player, whom we refer to as the second mover ("she"), chooses when to exit. Her strategy in Stage 2 is an $\mathcal{H}_t$-adapted stopping time $\tau^s \geq \tau$. The second mover may choose to exit *immediately* after the first mover, i.e., $\tau^s = \tau$. If that happens, we refer to it as a *de facto joint exit*. Hence, the loser of the coin flip (if any) in Stage 1 is effectively given an opportunity to take back her initial decision to exit. If she still opts to exit, her exit is formally treated as happening in Stage 2 to maintain consistency.

3 Equilibrium

This section contains the paper's main results and is organized as follows. Section 3.1 specifies the equilibrium concept, pure-strategy Markov perfect equilibrium (MPE). Unless otherwise specified, an "equilibrium" in this paper refers to a pure-strategy MPE. Sections 3.2 and 3.3 use backward

[9]It is a common practice in the literature to transform a continuous-time game with irreversible actions into a game with discrete stages. See also Bulow and Klemperer (1994), Akcigit and Liu (2016), etc.

[10]This tie-breaking assumption is common in stopping games (Dutta & Rustichini, 1993; Grenadier, 1996; Abreu & Gul, 2000; Weeds, 2002; Murto, 2004). See Online Appendix B.6 of Xu (2025) for more discussion.

induction to characterize the *unique* Pareto-optimal equilibrium. Following the characterization, Section 3.4 studies the properties of this equilibrium, especially the curse of profitability. Section 3.5 discusses non-Pareto-optimal equilibria. Finally, Section 3.6 establishes that the *unique* Pareto-optimal pure-strategy MPE is also the *unique* Pareto-optimal subgame-perfect Nash equilibrium (SPNE), suggesting that the main results of this paper can be applied more broadly to the equilibrium concept of SPNE.[11]

3.1 Equilibrium Concept

We focus on pure-strategy Markov perfect equilibrium (MPE), where a player's exit decision is based on (i) the current state X_t and (ii) whether the other player has already exited. Since we formulate the model as a two-stage dynamic game, the players' strategy profile in a pure-strategy MPE can be represented by a tuple $(\mathcal{X}^1, \mathcal{X}^2, \mathcal{X}^s)$. In Stage 1, Player i chooses an *exit region* $\mathcal{X}^i \subseteq \mathcal{X}$, meaning that he intends to exit at time t if and only if $X_t \in \mathcal{X}^i$. In Stage 2, the second mover, whether it is Player 1 or Player 2, faces the same single-player decision problem. As we will show later, the solution to this decision problem is unique, and therefore, there is no need to distinguish the two players' strategies in Stage 2. We describe the second mover's strategy in Stage 2 as an *exit region* $\mathcal{X}^s \subseteq \mathcal{X}$, meaning that she exits at time t if and only if $X_t \in \mathcal{X}^s$.

3.2 Stage 2

In Stage 2, the second mover faces the following optimal stopping problem: She gets a flow payoff of $\beta(X_t - \lambda c)$ until she exits, at which point she collects a zero lump-sum payoff. As is standard for a time-homogeneous stopping problem of this sort, the second mover's optimal strategy is a (stationary) Markovian decision rule, which can be represented by an *exit region* $\mathcal{X}^s \subseteq \mathcal{X}$, as introduced in Section 3.1. This optimal decision rule induces her a value function that we denote by $S(x)$, reflecting her continuation value at time t if $X_t = x$. The value function must satisfy the following Hamilton-Jacobi-Bellman equation,

$$S(x) = \max\left\{0, \beta(x - \lambda c) + (1 - r)S(x) + S'(x)\mu x + \frac{\sigma^2}{2}S''(x)x^2\right\}, \tag{1}$$

[11] Although the results can be applied more broadly, we still focus on pure-strategy MPE in Section 3, as it drastically simplifies the analysis.

where 0 is the continuation value of exiting and $\beta(x-\lambda c)+(1-r)S(x)+S'(x)\mu x+[\sigma^2 S''(x)x^2]/2$ is the continuation value of staying. The following claim describes the solution to this problem.

Claim 1. *The second mover's optimal exit region is $\mathcal{X}^s = (0, x^*]$ with the exit threshold being $x^* := [(r-\mu)\gamma]/[r(\gamma-1)] \cdot \lambda c$, and her value function is*

$$S(x) = \begin{cases} \frac{\beta}{r-\mu} \cdot x - \frac{\beta\lambda c}{r} + \frac{\beta\lambda c}{r(1-\gamma)(x^*)^\gamma} \cdot x^\gamma & \text{if } x > x^*, \\ 0 & \text{if } x \le x^*, \end{cases}$$

where $\gamma := \left(\sigma^2 - 2\mu - \sqrt{(\sigma^2-2\mu)^2 + 8r\sigma^2}\right)/(2\sigma^2) < 0$.

Proof. See Appendix A.1. □

Claim 1 suggests that the second mover's optimal exit region takes a threshold form — she exits when the partnership's profitability level X_t falls below x^*. Notably, this threshold is proportional to λ, indicating that the second mover is more inclined to exit under a higher level of mutual reliance. Moreover, when $x > x^*$, the second mover's value function $S(x)$ can be decomposed into two parts: the first two terms represent her expected future payoff if she never exits, whereas the third term reflects her option value of exiting. In addition, the value of γ is determined to make $S(x)$ satisfy the ODE, $S(x) = \beta(x-\lambda c) + (1-r)S(x) + S'(x)\mu x + [\sigma^2 S''(x)x^2]/2$ for any $x > x^*$.[12]

Knowing the second mover's response in Stage 2, we can derive the first mover's continuation value upon exit. After exiting, he continues to receive a flow payoff of αX_t until the second mover terminates the project, i.e., the next moment that X_t falls below x^*. Let $F(x)$ denote the first mover's continuation value upon exit at time t if $X_t = x$. When $x \le x^*$, we have $F(x) = 0$ because the first mover's exit will immediately trigger the second mover to exit and terminate the project. When $x > x^*$, the value function $F(x)$ must satisfy the following Feynman-Kac formula,

$$F(x) = \alpha x + (1-r)F(x) + F'(x)\mu x + \frac{\sigma^2}{2}F''(x)x^2.$$

The following claim provides the closed-form solution of $F(x)$.

[12]As the proof will show, the general solution to this ODE is $S(x) = \beta\,[x/(r-\mu) - \lambda c/r] + k_1 x^\gamma + k_2 x^\eta$ where $\gamma < 0$ and $\eta > 0$ are the two roots of $\Gamma(y) := \mu y + [\sigma^2 y(y-1)]/2 - r$. Determining the values of k_1 and k_2 yields the closed-form solution of $S(x)$ presented in Claim 1.

Claim 2. *The first mover's continuation value upon exit is*

$$F(x) = \begin{cases} \frac{\alpha}{r-\mu} \cdot x - \frac{\alpha}{(r-\mu)(x^*)^{\gamma-1}} \cdot x^\gamma & \text{if } x > x^*, \\ 0 & \text{if } x \leq x^*. \end{cases} \quad (2)$$

Proof. See Appendix A.2. □

Like what we did, we can interpret $F(x)$ by decomposing it into two parts when $x > x^*$. The first term represents the first mover's expected future payoff if the project is never terminated, whereas the second term reflects the loss from possible termination of the project.

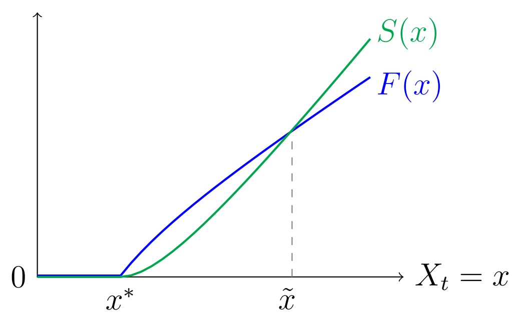


Figure 1: Illustration of $F(x)$ and $S(x)$. The value $\tilde{x}$ is the only intersection of $F(x)$ and $S(x)$ in the interval (x^*, ∞).

Figure 1 depicts the two value functions in the same place to highlight their properties and comparison. The function $F(x)$ has a "kink" at $x = x^*$, where it has a zero left derivative and a strictly positive right derivative. This kink does not violate the principle of optimal stopping because the threshold x^* is not chosen by the first mover, and therefore, smooth pasting does not apply. Crucially, this kink suggests the occurrence of first-mover advantage for an interval of x, as formally shown in Lemma 1.

Lemma 1. *There exists a unique* $\tilde{x} \in (x^*, \infty)$ *such that*

$$\begin{aligned} F(x) &= S(x) && \text{for } x \in (0, x^*], \\ F(x) &> S(x) && \text{for } x \in (x^*, \tilde{x}), \\ F(x) &= S(x) && \text{for } x = \tilde{x}, \\ F(x) &< S(x) && \text{for } x \in (\tilde{x}, \infty). \end{aligned}$$

Proof. See Appendix A.3. □

Lemma 1 indicates that a *first-mover advantage* arises in the interval $(x^*, \tilde{x})$, as also illustrated by Figure 1. Intuitively, when $x > x^*$, the first mover's payoff differs from the second mover's in two aspects — he saves the contribution cost, but also receives less revenue than the second mover due to Assumption 2. The partnership's profitability level does not affect the first aspect but is proportional to the second aspect. Hence, the first-mover advantage arises when $X_t \in (x^*, \tilde{x})$ as the first aspect dominates, but not when $X_t > \tilde{x}$ as the second aspect becomes dominant.

3.3 Stage 1

Since the second mover's optimal strategy in Stage 2 is unique (up to the indeterminacy at the threshold x^*), we can induce backward to Stage 1, where the players face the following stopping game. As long as no one has exited, each player receives a flow payoff of $X_t - c$. If one player chooses to exit at time t, he collects a continuation value of $F(X_t)$ as the first mover, while the remaining player receives a continuation value of $S(X_t)$ as the second mover.

In principle, an equilibrium should specify players' strategies in both Stage 1 (i.e., $\mathcal{X}^1$ and $\mathcal{X}^2$) and Stage 2 (i.e., $\mathcal{X}^s$). However, to save notation, we omit $\mathcal{X}^s$ when specifying an equilibrium in Section 3.3. This is because $\mathcal{X}^s$ is identical for every equilibrium and does not play an important role in analyzing the players' interaction in Stage 1.

Lemma 2. *In Stage 1 of any pure-strategy MPE, both players either always exit or always contribute for all the values of x in the interval $(x^*, \tilde{x})$. That is, the entire interval $(x^*, \tilde{x})$ is either included in or excluded from both players' exit regions in Stage 1.*

Proof. See Appendix A.4. □

Lemma 2 is due to the effect of pre-emption. Notice that $(x^*, \tilde{x})$ is a connected set of values of x that features first-mover advantage. In the presence of first-mover advantage, once a player intends to exit, his partner will react by choosing to exit slightly earlier than he does; unraveling thus occurs as the pre-emption exercise diffuses to the entire connected set where first-mover advantage exists. With this lemma, any equilibrium must belong to one of the following two types.

Definition 1. *Two types of pure-strategy MPE:*

(a) A ***cooperative equilibrium*** *is a pure-strategy MPE where* $(x^*, \tilde{x}) \cap \mathcal{X}^i = \emptyset$ *for* $i = 1, 2$*;*

(b) A ***pre-emptive equilibrium*** *is a pure-strategy MPE where* $(x^*, \tilde{x}) \subseteq \mathcal{X}^i$ *for* $i = 1, 2$.[13]

3.3.1 Cooperative Equilibria

To begin with, we characterize the *socially optimal outcome*, which will play an important role in analyzing cooperative equilibria. Think about a social planner who wants to maximize the players' total welfare by choosing when each player irreversibly exits. Because strategic exiting is socially inefficient due to Assumption 1, the socially optimal outcome is one where both players jointly terminate the project when the state X_t falls below some threshold. By solving the optimal stopping problem with the flow payoff being $X_t - c$ and the lump-sum exit payoff being zero, the optimal exit threshold is $x^{**} := [(r-\mu)\gamma]/[r(\gamma-1)] \cdot c$. This induces each player a value function $V_c(x)$, representing his continuation value at time t if $X_t = x$.

$$V_c(x) = \begin{cases} \frac{1}{r-\mu} \cdot x - \frac{c}{r} + \frac{c}{r(1-\gamma)(x^{**})^\gamma} \cdot x^\gamma & \text{when } x > x^{**}, \\ 0 & \text{when } x \leq x^{**}. \end{cases} \tag{3}$$

The derivation of x^{**} and $V_c(x)$ is almost identical to that for Claim 1 and is thus omitted. Like before, when $x > x^{**}$, the first two terms in $V_c(x)$ correspond to each player's expected future payoff if the project is never terminated, whereas the third term reflects the option value from terminating the project. Notably, the threshold x^{**} differs from x^* derived in Claim 1 because x^{**} is the optimal exit threshold with two contributors, whereas x^* is the optimal threshold with only one contributor. Indeed, these two thresholds satisfy $x^* = \lambda x^{**}$ — as players become more reliant on each other, the gap between the two thresholds gets larger.

Having specified the socially optimal outcome, the next lemma establishes its connection with cooperative equilibria.

Lemma 3. *If a cooperative equilibrium exists, there must be a cooperative equilibrium characterized by* $\mathcal{X}^1 = \mathcal{X}^2 = (0, x^{**}]$. *This equilibrium implements the socially optimal outcome.*

[13] In a pre-emptive equilibrium, both players intend to exit in Stage 1 when $X_t \in (x^*, \tilde{x})$. However, under the tie-breaking assumption, the one who (fails the coin-flip and) proceeds to Stage 2 will continue to contribute until the next time that the process X_t reaches x^*.

Proof. See Appendix A.5. □

The intuition of Lemma 3 is as follows. Since the strategy profile $\mathcal{X}^1 = \mathcal{X}^2 = (0, x^{**}]$ implements the socially optimal outcome, the value function that it generates to each player must be point-wise higher than any other strategy profile that satisfies the necessary condition of a cooperative equilibrium as in Definition 1. Hence, among all the strategy profiles that are potential cooperative equilibria, the one with $\mathcal{X}^1 = \mathcal{X}^2 = (0, x^{**}]$ is *least* vulnerable to strategic exiting. In other words, players who are deterred from strategic exiting in any cooperative equilibrium must also be deterred from doing so under $\mathcal{X}^1 = \mathcal{X}^2 = (0, x^{**}]$. Therefore, Lemma 3 suggests that the existence of a cooperative equilibrium boils down to whether $\mathcal{X}^1 = \mathcal{X}^2 = (0, x^{**}]$ is an equilibrium. That is, we only need to check whether $V_c(x)$, the value function generated by this strategy profile to each player, satisfies $V_c(x) \geq F(x)$ for all $x \in (x^{**}, \infty)$ so that strategic exiting is never a profitable deviation for each player. This paves the way for the next lemma, which establishes the key properties of cooperative equilibria.

Lemma 4. *(1) There exists a cooperative equilibrium if and only if* $\lambda \geq \lambda^* := [(1-(1-\alpha)^\gamma)/(\alpha\gamma)]^{\frac{1}{1-\gamma}}$.
(2) If $\lambda = \lambda^*$*, then* $\mathcal{X}^1 = \mathcal{X}^2 = (0, x^{**}]$ *is the unique cooperative equilibrium (up to outcome equivalence and a zero-measured set).*
(3) If $\lambda > \lambda^*$*, then there are multiple cooperative equilibria. Among all cooperative equilibria, the unique one that Pareto-dominates any other cooperative equilibrium (up to outcome equivalence and a zero-measured set) is* $\mathcal{X}^1 = \mathcal{X}^2 = (0, x^{**}]$.[14]

Proof. See Appendix A.6. □

Henceforth, we refer to $\mathcal{X}^1 = \mathcal{X}^2 = (0, x^{**}]$ as the *Pareto-optimal cooperative equilibrium* (if existing). Lemma 4 shows that a cooperative equilibrium (in particular, the Pareto-optimal cooperative equilibrium) exists when the reliance parameter λ is sufficiently large. Intuitively, when it is more difficult for the second mover to run the project alone, players will be deterred from strategic exiting in the first place. This finding is also illustrated by Figure 2. Notice that as the reliance parameter λ increases, the threshold x^* gets farther away from x^{**}, and therefore,

[14] The uniqueness in this lemma is up to outcome equivalence because any asymmetric strategy profile satisfying $\mathcal{X}^1 \cup \mathcal{X}^2 = (0, x^{**}]$ generates the same outcome as $\mathcal{X}^1 = \mathcal{X}^2 = (0, x^{**}]$, as the players de facto jointly exit in the interval $(0, x^{**}]$. The uniqueness is also up to a zero-measured set because the players are indifferent between whether or not to exit at the threshold x^{**}.

the continuation value from strategic exiting, $F(x)$, becomes point-wise smaller. When λ is large, as in Panels (a) and (b), $F(x)$ is point-wise (weakly) smaller than $V_c(x)$, indicating that strategic exiting is never a profitable deviation. When λ is small, as in Panel (c), $F(x)$ intersects with $V_c(x)$, so there exist some values of x where players strictly benefit from strategic exiting.

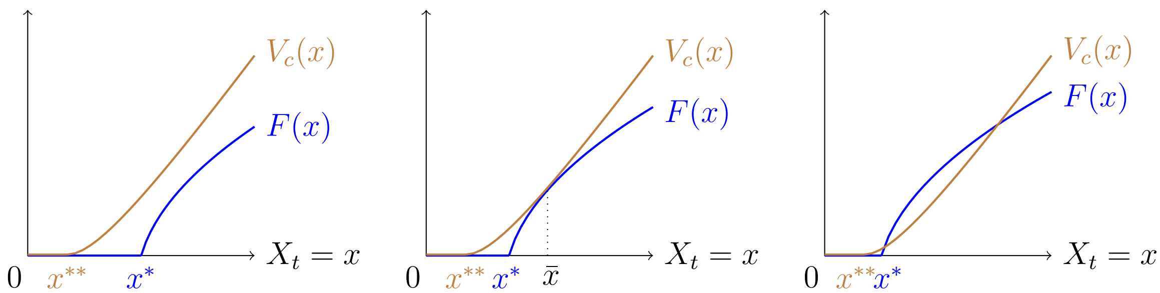


Figure 2: Illustration of how the value of λ affects the existence of a cooperative equilibrium. A cooperative equilibrium exists if $V_c(x) \geq F(x), \forall x$. In Panel (b), $\bar{x}$ is the tangent point of $V_c(x)$ and $F(x)$.

As a side comment, one may wonder how to derive the closed-form solution of the threshold λ^*. As depicted in Panel (b), λ^* makes the corresponding $F(x)$ tangentially intersect with $V_c(x)$ at some $\bar{x} > x^*$, owing to the strict concavity of $F(x)$ and the strict convexity of $V_c(x)$ when $x > x^*$. In the proof, we exploit this geometric property to derive the closed-form solution of λ^* (and also $\bar{x}$) from solving two simultaneous equations, $F(\bar{x}; \lambda^*) = V_c(\bar{x})$ and $F'(\bar{x}; \lambda^*) = V_c'(\bar{x})$.

3.3.2 Pre-emptive Equilibria

Next, we turn to pre-emptive equilibria, in which both players intend to exit in Stage 1 when $X_t \in (x^*, \tilde{x})$. Unlike cooperative equilibria, whose existence depends on the values of parameters, a pre-emptive equilibrium always exists — in particular, $\mathcal{X}^1 = \mathcal{X}^2 = (0, \tilde{x})$ is always an equilibrium. To see why this is true, notice that if $X_t \in (0, x^*]$, given that the other player always exits in Stage 1, a de facto joint exit is unavoidable no matter a player exits or not in Stage 1; if $X_t \in (x^*, \tilde{x})$, a player finds it optimal to exit in Stage 1 because of the first-mover advantage; if $X_t \in [\tilde{x}, \infty)$, staying in the partnership is each player's dominant strategy in Stage 1. However, this equilibrium may be Pareto-dominated by another pre-emptive equilibrium, as the next lemma suggests.

Lemma 5. *(1) A pre-emptive equilibrium always exists.*

(2) If $\lambda \le \lambda^{**} := [r(\gamma-1)]/[(r-\mu)\gamma]$*, then* $\mathcal{X}^1 = \mathcal{X}^2 = (0, \tilde{x})$ *is the unique pre-emptive equilibrium (up to outcome equivalence and a zero-measured set).*

(3) If $\lambda > \lambda^{**}$*, then there are multiple pre-emptive equilibria. Among all pre-emptive equilibria, the unique one that Pareto-dominates any other pre-emptive equilibrium (up to outcome equivalence and a zero-measured set) takes the form of* $\mathcal{X}^1 = \mathcal{X}^2 = (0, x^0] \cup (x^*, \tilde{x})$ *with* $x^0 \in (0, x^*)$*.*[15]

Proof. See Appendix A.7. □

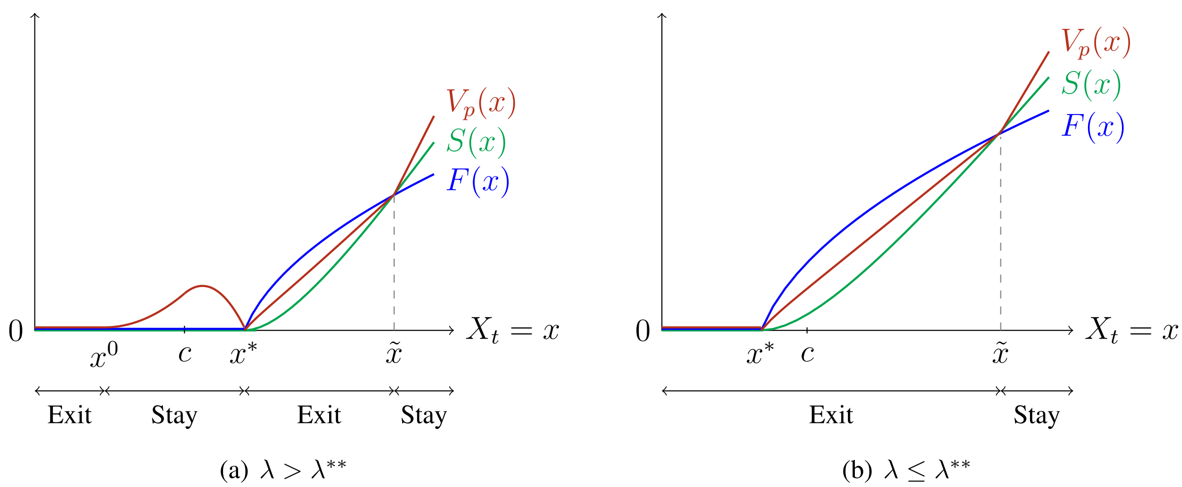


Figure 3: Illustration of the Pareto-optimal pre-emptive equilibrium. In each panel, the players' Stage-1 exit regions in this equilibrium are labeled below the horizontal axis — in Panel (a), for instance, both players intend to exit in Stage 1 if and only if $X_t \in (0, x^0] \cup (x^*, \tilde{x})$. Also depicted are $V_p(x)$, each player's expected continuation value in Stage 1 under this equilibrium, and $F(x)$ and $S(x)$, their respective continuation values once entering Stage 2.

Lemma 5 characterizes the unique *Pareto-optimal pre-emptive equilibrium*, which we depict in Figure 3. The next two paragraphs are dedicated to explaining why it is uniquely Pareto-optimal among all pre-emptive equilibria.

As will be shown in the proof, it is dominant for both players to stay in the partnership when $X_t \ge \tilde{x}$ in Stage 1. Meanwhile, both players exit in Stage 1 when $X_t \in (x^*, \tilde{x})$ by the definition of a pre-emptive equilibrium. Hence, what remains undetermined in a pre-emptive equilibrium is the players' Stage-1 strategies when $X_t \in (0, x^*]$. Notice that when $X_t \in (0, x^*]$, any player's exit always triggers a de facto joint exit. As a consequence, finding the Pareto-optimal pre-emptive

[15]The uniqueness is up to outcome equivalence and a zero-measured set for the same reason as in Footnote 14.

equilibrium boils down to solving the following single-player stopping problem within the interval $(0, x^*]$: the flow payoff is $X_t - c$, the exit payoff is zero, and there is an additional constraint that the continuation value is fixed at zero when $X_t = x^*$. This constraint stems from the fact that in any pre-emptive equilibrium, both players intend to exit when X_t is epsilon-above x^*, yielding each of them an expected continuation value arbitrarily close to $[F(x^*) + S(x^*)]/2 = 0$.

The solution to this single-player stopping problem depends on the value of λ. If $\lambda > \lambda^{**}$, as in Panel (a), it is optimal to run the project when X_t belongs to the interval $(x^0, x^*]$. This is because the condition $\lambda > \lambda^{**}$ is equivalent to $x^* > c$, which indicates that when $c < X_t < x^*$, the project still generates a positive flow payoff. It is valuable to exploit such a payoff until X_t falls below the threshold x^0, which is, again, determined by the value matching and smooth pasting conditions, as in standard single-player stopping problems. By contrast, if $\lambda \leq \lambda^{**}$, as in Panel (b), it is optimal to terminate the project when $X_t \leq x^*$ because the flow payoff is non-positive.

To intuitively understand this equilibrium in the case of $\lambda > \lambda^{**}$, we divide $\mathcal{X}$ into four sections according to players' Stage-1 strategies. When X_t is *very low* ($X_t \in (0, x^0]$), the players jointly exit, as it is no longer worthwhile to run the project. When X_t is *moderately low* ($X_t \in (x^0, x^*]$), both players stay in the partnership to exploit the project's payoff, knowing that anyone's exit will immediately trigger the other player's exit and thus the project's termination. When X_t is *moderately high* ($X_t \in (x^*, \tilde{x})$), both players intend to exit, but only the one who wins the coin flip succeeds in exiting while the other one will stay in the partnership until x^* is reached again — in other words, this is the region of pre-emptive strategic exiting. Finally, when X_t is *very high* ($X_t \in [\tilde{x}, \infty)$), both players find it dominant to stay in the partnership. This equilibrium pattern is notable in that players' exit regions do not admit a threshold form, indicating that their incentives to exit are non-monotonic in the partnership's profitability. In particular, they both intend to exit when X_t is moderately high but prefer to stay when X_t is moderately low. As a result, their equilibrium continuation value, denoted by $V_p(x)$, is non-monotonic in x when $\lambda > \lambda^{**}$. We will revisit this finding in Section 3.4 when discussing the curse of profitability.

3.3.3 Pareto-Optimal Equilibrium

Lemma 2 suggests that an equilibrium must be either a cooperative equilibrium or a pre-emptive equilibrium. Lemmas 4 and 5 identify the unique *Pareto-optimal cooperative equilibrium* (if existing) and the unique *Pareto-optimal pre-emptive equilibrium* (always existing), respectively. Moreover, it is not difficult to see that the Pareto-optimal cooperative equilibrium, if it exists, Pareto-dominates the Pareto-optimal pre-emptive equilibrium because it implements the socially optimal outcome. The above arguments combined point to a unique equilibrium that Pareto-dominates any other equilibrium (up to outcome equivalence and a zero-measured set). Depending on the value of λ, this Pareto-optimal equilibrium falls into one of the following three scenarios.

(1) If $\lambda \geq \lambda^*$, it is the Pareto-optimal cooperative equilibrium. Players' exit regions in Stage 1 are $\mathcal{X}^1 = \mathcal{X}^2 = (0, x^{**}]$.

(2) If $\lambda^{**} < \lambda < \lambda^*$, it is the Pareto-optimal pre-emptive equilibrium. Players' exit regions in Stage 1 are in the form of $\mathcal{X}^1 = \mathcal{X}^2 = (0, x^0] \cup (x^*, \tilde{x})$.

(3) If $\lambda < \lambda^*$ and $\lambda \leq \lambda^{**}$, it is the Pareto-optimal pre-emptive equilibrium. Players' exit regions in Stage 1 are $\mathcal{X}^1 = \mathcal{X}^2 = (0, \tilde{x})$.

Notice that Scenario (2) occurs if and only if $\lambda^{**} < \lambda^*$. However, neither λ^* nor λ^{**} are primitive parameters. Recall that α, which measures the players' free-riding incentives, is a primitive parameter that determines the value of λ^*. We use the value of α to determine which of the aforementioned scenarios arise in the equilibrium characterization. As shown in Appendix A.8, there exists a unique $\underline{\alpha} := \{\alpha | \lambda^* = \lambda^{**}\}$ such that $\lambda^{**} < \lambda^*$ holds if and only if $\alpha > \underline{\alpha}$. This allows us to characterize the unique Pareto-optimal equilibrium as described in the following theorem.

Theorem 1. *A pure-strategy MPE always exists. Moreover, there uniquely exists a Pareto-optimal pure-strategy MPE (up to outcome equivalence and a zero-measured set), which is characterized as follows.*[16]

(1) In Stage 1, the players' exit regions $(\mathcal{X}^1, \mathcal{X}^2)$ *are*

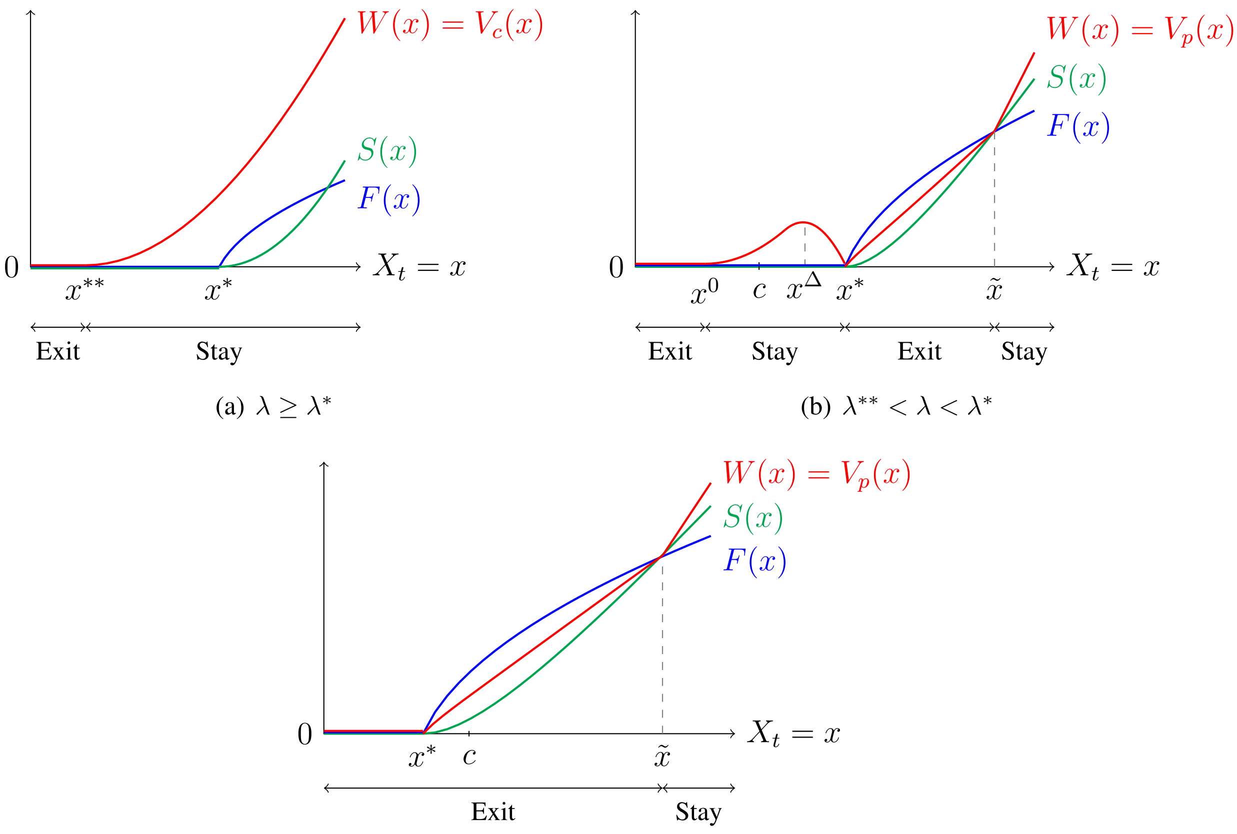
$$\mathcal{X}^1 = \mathcal{X}^2 = \begin{cases} (0, x^{**}] & \text{if } \lambda \geq \lambda^*, \\ (0, x^0] \cup (x^*, \tilde{x}) & \text{if } \alpha > \underline{\alpha} \text{ and } \lambda \in (\lambda^{**}, \lambda^*), \\ (0, \tilde{x}) & \text{if (i) } \alpha > \underline{\alpha} \text{ and } \lambda \leq \lambda^{**} \text{ or (ii) } \alpha \leq \underline{\alpha} \text{ and } \lambda < \lambda^*. \end{cases}$$

(2) In Stage 2, whoever becomes the second mover adopts the exit region $\mathcal{X}^s = (0, x^*]$.

Proof. See Appendix A.8. □

[16] The uniqueness is up to outcome equivalence and a zero-measured set for the same reason as in Footnote 14.

Based on this equilibrium, let $W(x)$ denote each player's Stage-1 continuation value at time t when $X_t = x$. It equals $V_c(x)$ if the Pareto-optimal cooperative equilibrium exists and $V_p(x)$ if not. Figure 4 depicts this equilibrium when $\alpha > \underline{\alpha}$.



(a) $\lambda \geq \lambda^*$

(b) $\lambda^{**} < \lambda < \lambda^*$

(c) $\lambda \leq \lambda^{**}$

Figure 4: Illustration of the Pareto-optimal equilibrium when $\alpha > \underline{\alpha}$. In each panel, the players' equilibrium exit regions in Stage 1 are labeled below the horizontal axis. Also depicted are $W(x)$, each player's expected continuation value in Stage 1 under this equilibrium, and $F(x)$ and $S(x)$, their respective continuation values once entering Stage 2. In Panel (b), x^Δ corresponds to the maximum of $W(x)$ in the interval $[x^0, x^*]$. Notably, when $\alpha \leq \underline{\alpha}$, this figure can still illustrate the Pareto-optimal equilibrium, except that Panel (b) will disappear.

3.4 Properties of the Unique Pareto-Optimal Equilibrium

3.4.1 Curse of Profitability

The most noteworthy property of the unique Pareto-optimal equilibrium is the possibility of a *curse of profitability* — increasing the project's level of profitability may render both players strictly worse off. As depicted in Figure 4(b), each player's Stage-1 continuation value $W(x)$ strictly decreases in x when $x \in [x^\Delta, x^*]$, where $x^\Delta := \arg\max_{x\in[x^0,x^*]} W(x)$.[17] This property of $W(x)$ is driven by the fact that for the equilibrium depicted in Figure 4(b), players intend to exit when X_t is moderately high (i.e., $X_t \in (x^*, \tilde{x})$) but prefer to stay when X_t is moderately low (i.e., $X_t \in (x^0, x^*]$), as explained after Lemma 5.

What is the intuition behind the curse of profitability? A larger X_t is a double-edged sword. While it means the project generates higher revenue, it also makes it less challenging for the second mover to run the project alone, which stimulates strategic exiting in the first place. Furthermore, the harm of strategic exiting is amplified by players' pre-emptive incentives. As a consequence, players can still cooperate when $X_t \in (x^0, x^*]$, but as X_t increases to approach x^*, they suddenly become enemies and pre-empt each other, letting go of the benefits from cooperation.

Corollary 1. *The curse of profitability occurs in the unique Pareto-optimal equilibrium if and only if players have sufficiently large free-riding incentives (i.e., $\alpha > \underline{\alpha}$) and a medium level of mutual reliance (i.e., $\lambda^{**} < \lambda < \lambda^*$).*

Corollary 1 indicates two necessary and sufficient conditions for the curse of profitability to occur. First, players' free-riding incentives should be sufficiently large. Otherwise (i.e., $\alpha \le \underline{\alpha}$), strategic exiting is likely not a concern of the players, who will find it easy to cooperate as in Figure 4(a); when it does become a concern, their mutual reliance must be so low such that there is already no scope for two players to cooperate given that one player cannot run the project alone, as in Figure 4(c). Second, players should have a medium level of mutual reliance. Under high mutual reliance ($\lambda \ge \lambda^*$), as in Figure 4(a), strategic exiting can be avoided because running the project alone is too difficult. Under low mutual reliance ($\lambda \le \lambda^{**}$), as in Figure 4(c), running the project alone is relatively easy, so if a single player does not want to do it, there must also be no value of cooperation for two players — specifically, when $X_t < x^*$, it must follow that $X_t < c$.

3.4.2 Comparative Statics on Existence of Cooperative Equilibrium

Corollary 2. *Taking other parameters as given, there exists α^* (or β^*, κ^*, r^*, μ^*, σ^*) such that a cooperative equilibrium exists if and only if $\alpha \le \alpha^*$ (or $\beta \le \beta^*$, $\kappa \ge \kappa^*$, $r \le r^*$, $\mu \le \mu^*$, $\sigma \ge \sigma^*$).*

[17] Indeed, even the ex-post first mover suffers from the curse of profitability. The ex-post first mover's realized continuation value is $W(x)\mathbb{1}(x \notin (x^*, \tilde{x})) + F(x)\mathbb{1}(x \in (x^*, \tilde{x}))$, which also decreases in x when $x \in [x^\Delta, x^*]$.

Proof. See Appendix A.9. □

We can interpret Corollary 2 as follows: The partnership's ability to sustain cooperation benefits from smaller α (free-riding incentive), smaller β (second mover's revenue), larger κ (second mover's cost), smaller r (discount rate), smaller μ (drift in the project's profitability), and larger σ (volatility in the project's profitability). As a side note, it does not depend on c (contribution cost).

The comparative statics related to α, β, and κ are intuitive. A cooperative equilibrium is easier to sustain if the second mover is less motivated to run the project alone (i.e., smaller β, larger κ), which, in turn, may deter strategic exiting from happening in the first place. Also, lower free-riding incentives (i.e., smaller α) make cooperation easier. The comparative statics related to r, μ, and σ involve the following trade-off. Increasing μ, increasing σ, and decreasing r all have two opposite effects. On the one hand, they increase a player's continuation value upon cooperation;[18] on the other hand, they also increase a player's temptation to deviate as they make the second mover more motivated to run the project alone. It turns out that the second effect dominates for μ, whereas the first effect dominates for σ and r. Therefore, a larger μ makes cooperation harder, while a larger σ and a smaller r facilitate cooperation.

3.4.3 Comparative Statics on Players' Welfare

To measure the players' welfare, we use $W(x)$, their Stage-1 expected continuation value in the unique Pareto-optimal equilibrium. One feature of this equilibrium is its discontinuity in the value of λ — the equilibrium switches from cooperative to pre-emptive when λ crosses the threshold λ^*. Indeed, as suggested by Corollary 2, this feature also applies to the other parameters. Because of such discontinuity, the change of a parameter has two effects on players' welfare. The first-order effect is to change the type of equilibrium (i.e., cooperative or pre-emptive), which works in a discontinuous manner when the parameter crosses a certain threshold, as suggested by Corollary 2. The second-order effect is that each parameter plays a role in the players' value function within a specific type of equilibrium.

As reported in Figure 5, we use numerical examples to elaborate on the above point. Take α as an example. The first-order effect is that players' welfare faces a discontinuous drop at

[18] For σ, in particular, a decision maker in a stopping problem benefits from the high variance of the stochastic state if his flow payoff is weakly convex and non-decreasing in the stochastic state (Villeneuve, 2007).

$\alpha = \alpha^* \approx 0.697$, as the Pareto-optimal equilibrium switches from cooperative to pre-emptive. The second-order effect is that, when $\alpha > \alpha^*$, players benefit from a larger α; this does not happen when $\alpha \leq \alpha^*$, as the parameter α does not enter the value function of a cooperative equilibrium. Combining these two effects, we can infer that, under some parametric values, players' welfare is non-monotonic in the value of α. We also perform similar analyses for the other five parameters, as shown in the other panels of Figure 5. Under some parametric values, players' welfare is non-monotonic in β, κ, and μ, strictly increasing in σ, and strictly decreasing in r. As a side note, the parameter c (flow cost) does not have a first-order effect, and its second-order impact is always negative. Hence, increasing c always decreases $W(x)$ in a continuous manner.

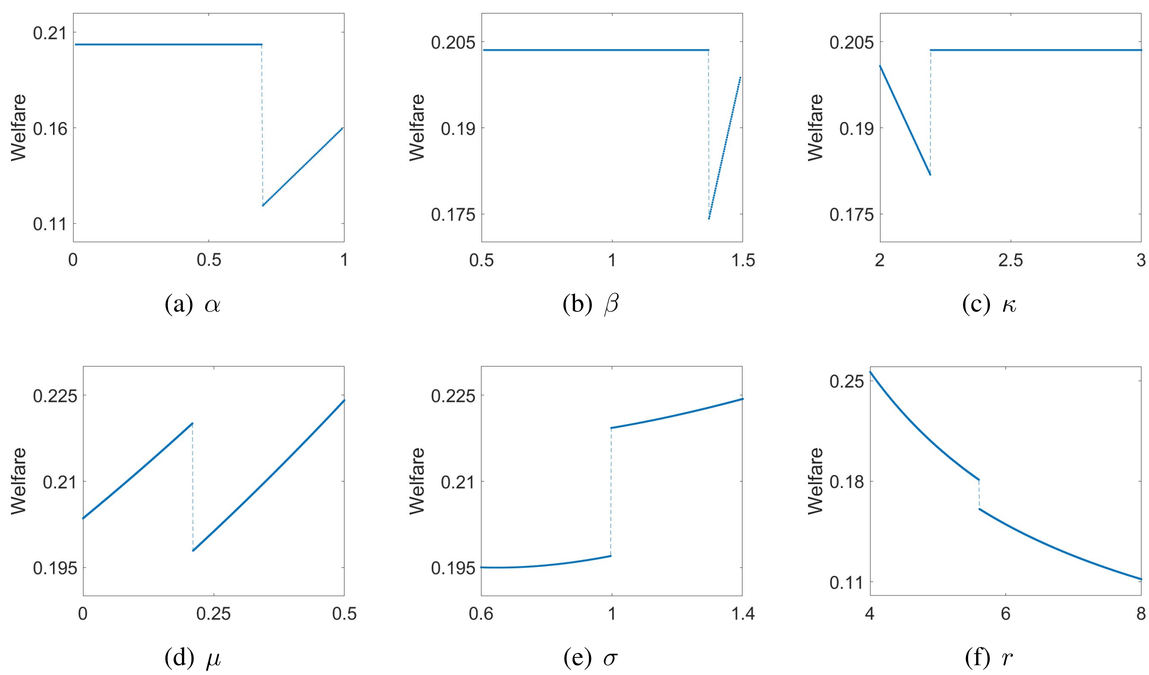


Figure 5: Numerical examples of players' welfare as a function of six parameters. Welfare is measured by $W(x)$. The parametric values are as follows. Panel (a): $\beta = 1, \kappa = 2, \mu = 0, \sigma = 1, r = 5, c = 1$, and $x = 2$. Panel (b): $\alpha = 0.5, \kappa = 2, \mu = 0, \sigma = 1, r = 5, c = 1$, and $x = 2$. Panel (c): $\alpha = 0.5, \beta = 1.5, \mu = 0, \sigma = 1, r = 5, c = 1$, and $x = 2$. Panel (d): $\alpha = 0.5, \beta = 1.5, \kappa = 2.2, \sigma = 1, r = 5, c = 1$, and $x = 2$. Panel (e): $\alpha = 0.5, \beta = 1.5, \kappa = 2.2, \mu = 0.2, r = 5, c = 1$, and $x = 2$. Panel (f): $\alpha = 0.5, \beta = 1.5, \kappa = 2.2, \mu = 0, \sigma = 1, c = 1$, and $x = 2$.

3.5 Non-Pareto-Optimal Equilibria

Our previous analysis focused on the Pareto-optimal equilibrium. This subsection serves dual purposes. First, we provide a more complete characterization of equilibria, including those that are *not* Pareto-optimal. Second, we use the characterization to demonstrate that the curse of profitability does *not* depend on the equilibrium selection of Pareto-optimality.

For ease of exposition, we disregard two sources of equilibrium multiplicity in this subsection.[19] First, we regard two equilibria as identical if they are *outcome-equivalent* in the sense of generating the same exit process on the equilibrium path. This simplification aims to avoid discussing asymmetric equilibria because every asymmetric equilibrium is outcome-equivalent to a symmetric equilibrium.[20] Second, if the Stage-1 exit regions of two symmetric equilibria differ only in *a zero-measured set*, we also regard them as being essentially identical.

Non-Pareto-optimal cooperative equilibria. Lemma 4 suggests multiple cooperative equilibria when $\lambda > \lambda^*$. Notice that the Pareto-optimal cooperative equilibrium $\mathcal{X}^1 = \mathcal{X}^2 = (0, x^{**}]$ takes a threshold form, i.e., both players intend to exit in Stage 1 if and only if X_t is below a certain threshold. The following proposition characterizes *all* cooperative equilibria (including the non-Pareto-optimal ones) that take a threshold form.[21]

Proposition 1. *When $\lambda > \lambda^*$, the set of cooperative equilibria that take a threshold form is $\{\mathcal{X}^1 = \mathcal{X}^2 = (0, x^c] \mid x^c \in [x^{**}, k(\lambda) \cdot x^{**}]\}$, where $k(\lambda)$ is determined by $\gamma \cdot (k(\lambda))^{1-\gamma} + (1 - \gamma) \cdot (k(\lambda))^{-\gamma} = (1-\alpha)^\gamma + \alpha\gamma\lambda^{1-\gamma}$. Moreover, $k(\lambda)$ strictly increases in λ, and $k(\lambda) < \lambda$.*

Proof. See Appendix A.10. □

Proposition 1 identifies a continuum of cooperative equilibria in the threshold form, with the threshold x^c ranging from x^{**} to $k(\lambda) \cdot x^{**}$. When the threshold $x^c > x^{**}$, the corresponding

[19] Indeed, the uniqueness of Pareto-optimal equilibrium in Theorem 1 also hinges on disregarding these two sources of equilibrium multiplicity, as we detailed in Footnote 14.

[20] To see this point, given any asymmetric cooperative equilibrium, the players only exit in the interval $(0, x^*]$, i.e., $\mathcal{X}^1 \neq \mathcal{X}^2$ and $\mathcal{X}^1 \cup \mathcal{X}^2 \subseteq (0, x^*]$. Since $(0, x^*]$ is the interval where a de facto joint exit is triggered, this asymmetric equilibrium must be outcome-equivalent to a symmetric equilibrium where both players' exit regions in Stage 1 are $\mathcal{X}^1 \cup \mathcal{X}^2$. The same logic applies to any asymmetric pre-emptive equilibrium $(\mathcal{X}^1, \mathcal{X}^2)$, which must be outcome-equivalent to a symmetric equilibrium where both players' Stage-1 exit regions are $\mathcal{X}^1 \cup \mathcal{X}^2$.

[21] There are cooperative equilibria that do not take the threshold form. One such example is $\mathcal{X}^1 = \mathcal{X}^2 = (0, \underline{x}^c] \cup [\bar{x}^c, x^c]$ with $x^c > \bar{x}^c > \underline{x}^c > c$. The reason why players cooperate in Stage 1 in the interval $(\underline{x}^c, \bar{x}^c)$ is the same as the Pareto-optimal pre-emptive equilibrium. Although such construction is possible, it does not provide new insight to this paper. Therefore, we do not attempt to fully characterize the cooperative equilibria with a non-threshold form.

equilibrium is Pareto-dominated because the players exit "too early" due to miscoordination. As the threshold x^c increases from x^{**} to $k(\lambda) \cdot x^{**}$, the corresponding Stage-1 value function for each player becomes point-wise lower — intuitively, the players become worse off if their threshold to terminate the project gets farther away from the optimal threshold x^{**}. When $x^c = k(\lambda) \cdot x^{**}$, the value function generated by $\mathcal{X}^1 = \mathcal{X}^2 = (0, k(\lambda) \cdot x^{**}]$ tangentially intersects with $F(x)$, which explains why the threshold x^c cannot go beyond $k(\lambda) \cdot x^{**}$.

Two features of the set of cooperative equilibria are worth mentioning. First, $k(\lambda) < \lambda$ indicates that the threshold x^c is always strictly smaller than x^* (which equals λx^{**}) — hence, both players always exit at de facto the same time, even for the Pareto-dominated cooperative equilibria. Second, the fact that $k(\lambda)$ is strictly increasing suggests the following: As λ increases, the set of cooperative equilibria switches from being empty to non-empty at λ^* and then keeps expanding.

Non-Pareto-optimal pre-emptive equilibria. Lemma 5 suggests multiple pre-emptive equilibria when $\lambda > \lambda^{**}$. The following proposition characterizes *all* pre-emptive equilibria that take a threshold form in the interval $(0, x^*]$, i.e., given that $X_t \in (0, x^*]$, both players intend to exit in Stage 1 if and only if X_t is below a threshold.[22]

Proposition 2. *When $\lambda > \lambda^{**}$, the set of pre-emptive equilibria that take a threshold form in the interval $(0, x^*]$ is $\{\mathcal{X}^1 = \mathcal{X}^2 = (0, x^p] \cup (x^*, \tilde{x}) \mid x^p \in [x^0, x^*]\}$.*

Proof. See Appendix A.11. □

Proposition 2 identifies a continuum of pre-emptive equilibria in the threshold form in $(0, x^*]$, with the threshold x^p ranging from x^0 to x^*. When the threshold $x^p > x^0$, the equilibrium is Pareto-dominated because the players, again, terminate the project too early due to miscoordination.

We highlight two features of these equilibria. First, all these equilibria exhibit two disjoint intervals in both players' exit regions in Stage 1 — when X_t is in the interval $(0, x^p]$, players jointly abandon the project as they exit at de facto the same time; when X_t is in the interval $(x^*, \tilde{x})$, players intend to exit out of pre-emptive motive, so the coin flip loser, after entering Stage 2, will still run the project until X_t falls below x^*. Second, given that $\lambda > \lambda^{**}$, increasing λ also

[22] There are pre-emptive equilibria that do not take the threshold form in the interval $(0, x^*]$. One such example is $\mathcal{X}^1 = \mathcal{X}^2 = (0, x^p] \cup (\bar{x}^p, \tilde{x})$ with $c \le x^p < \bar{x}^p < x^*$. For the same reason as stated in Footnote 21, we do not attempt to fully characterize all pre-emptive equilibria without the aforementioned threshold form.

enlarges the set of pre-emptive equilibria with the aforementioned threshold form. As λ increases, x^* increases since it equals λx^{**}, and therefore, x^0 also decreases because of the larger option value for the players to wait when $X_t < c$. This further indicates that the possible range of the threshold x^p (i.e., $[x^0, x^*]$) is enlarged as λ increases.

Curse of profitability for non-Pareto-optimal equilibria. All pre-emptive equilibria characterized in Proposition 2, except for the degenerate one with $x^p = x^*$ (i.e., the one characterized by $\mathcal{X}_1 = \mathcal{X}_2 = (0, \tilde{x})$), are subject to the curse of profitability — in any of these equilibria, each player's Stage-1 continuation value is zero at $x = x^*$ but strictly positive in (x^p, x^*). Moreover, if $\alpha > \underline{\alpha}$ and $\lambda^{**} < \lambda < \lambda^*$, no cooperative equilibrium exists, so it suffices to only consider these pre-emptive equilibria. These arguments give rise to the following corollary, suggesting that the equilibrium selection of Pareto-optimality is not essential to the paper's main insight.

Corollary 3. *(1) If $\lambda > \lambda^{**}$, the curse of profitability occurs in a continuum of equilibria.*
*(2) If $\alpha > \underline{\alpha}$ and $\lambda^{**} < \lambda < \lambda^*$, the curse of profitability occurs in **all** the equilibria characterized in this subsection, except for the degenerate pre-emptive equilibrium $\mathcal{X}_1 = \mathcal{X}_2 = (0, \tilde{x})$.*

Comments on Pareto-optimality criterion. Having seen the equilibrium multiplicity, one may wonder about the rationale for the Pareto-optimality criterion, besides its advantage in selecting a unique equilibrium. Here is another justification. In our setting, Pareto-optimality is equivalent to *(strong) renegotiation-proofness à la* Farrell and Maskin (1989), where the players can renegotiate the continuation play at any moment in Stage 1. This argument is backed up by Safronov and Strulovici (2018), which shows that when there exists a unique Pareto-optimal continuation value profile, players can always renegotiate their continuation play to achieve it. Intuitively, regardless of the currently prescribed continuation play, one player can propose a switch to the continuation play that achieves the Pareto-optimal continuation value profile, and the other player will approve this proposal. Therefore, Pareto-optimality is a reasonable selection criterion when players can communicate and renegotiate, which is very common for real-world partnerships.

3.6 Comments on the Equilibrium Concept

The following proposition shows that the equilibrium characterized in Theorem 1 is also the unique Pareto-optimal SPNE (up to outcome equivalence and a zero-measured set). Hence, the main results of this paper can be applied more broadly if we use SPNE as the equilibrium concept.

Proposition 3. *The equilibrium characterized in Theorem 1 Pareto-dominates any other SPNE.*

Proof. See Appendix A.12 □

As the proof will show, Proposition 3 is built on two arguments. First, there does not exist a non-Markovian SPNE that makes players better off compared to the equilibrium characterized in Theorem 1. Why? In Stage 2, the second mover faces a single-player time-homogeneous stopping problem, whose optimal decision rule must be Markovian. In Stage 1, no one has ever exited, and because of that, there is no variance in players' past actions that we can condition on to use non-Markovian strategies. This stands in contrast to canonical repeated games, where it is valuable to punish a player for his past defections. Second, the proof also shows that introducing mixed strategies brings new MPEs but cannot improve players' continuation values in equilibrium.

4 Fixed Exit Order

In some partnerships, certain partners refrain from exiting the partnership unless others have already left. Such a *no-first-exit commitment* could arise from these partners' reputation concerns, the partnership's rules about exit priority, or some other factors. Making such a commitment imposes a direct (negative) effect on this partner, as she forgoes the option to exit first. However, it may also generate an indirect (positive) effect — the commitment may mitigate the pre-emptive tension within the partnership, making other partners exit less aggressively.

This section studies such a commitment in two steps. First, Theorem 2 delineates the indirect effect by analyzing how others react to a partner's no-first-exit commitment. Second, Theorem 3 identifies the situations where the indirect effect outweighs the direct effect, thereby demonstrating the possibility of a Pareto-improvement resulting from such a commitment.

4.1 Setup

Consider an alternative setting where players' exit order is fixed — one player is designated as the second mover and never exits first, while the other is designated as the first mover. To distinguish from the baseline model, in Section 4, we refer to the players as Frank, the designated first mover, and Susan, the designated second mover. The game proceeds in a Stackelberg manner. In Stage 1, Frank chooses an $\tilde{\mathcal{H}}_t$-adapted stopping time $\tilde{\tau}^f$, where $\tilde{\mathcal{H}}_t$ contains information about the public history up to time t. After Frank exits, Stage 2 immediately starts, and Susan chooses an $\tilde{\mathcal{H}}_t$-adapted stopping time $\tilde{\tau}^s \geq \tilde{\tau}^f$.[23] The equilibrium concept is SPNE.

4.2 Equilibrium

Susan's decision problem is identical to the second mover's in the baseline model. She exits if and only if $X_t \in \mathcal{X}^s = (0, x^*]$. We then induce backward to Frank's stopping problem in Stage 1: He receives a flow payoff of $X_t - c$ until he exits, upon which he receives a lump-sum payoff of $F(X_t)$. Since this problem is time-homogeneous, it is optimal for Frank to adopt a (stationary) Markovian strategy, which can be represented by an exit region $\mathcal{X}^f \subseteq \mathcal{X}$. Let $U_f(x)$ and $U_s(x)$ denote Frank's and Susan's continuation values in Stage 1, respectively, from Frank's optimal exit region $\mathcal{X}^f$. The following Hamilton-Jacobi-Bellman equation must hold.

$$U_f(x) = max\{F(x), x - c + (1-r)U_f(x) + U_f'(x)\mu x + \frac{\sigma^2}{2}U_f''(x)x^2\},$$

where $F(x)$ is his continuation value of exiting and $x - c + (1-r)U_f(x) + U_f'(x)\mu x + [\sigma^2 U_f''(x)x^2]/2$ is his continuation value of staying. The solution to Frank's stopping problem is embedded in the following theorem.

Theorem 2. *There is a unique SPNE (up to a zero-measured set), as characterized below.*

(1) In Stage 1, Frank's exit region is

$$\mathcal{X}^f = \begin{cases} (0, x^{**}] & \text{if } \lambda \geq \lambda^*, \\ (0, x'] \cup [x'', x'''] & \text{if } \lambda < \lambda^*, \end{cases}$$

[23] Notably, this setting can be cast as the baseline model with the addition that one player commits to $\mathcal{X}^i = \emptyset$.

where the three thresholds $x' < x'' < x'''$ are well defined.

(2) In Stage 2, Susan's exit region is $\mathcal{X}^s = (0, x^*]$.

Proof. See Appendix A.13. □

Figure 6 illustrates Frank's optimal exit strategy. If $\lambda \geq \lambda^*$, he finds it optimal to implement the socially optimal outcome. On the equilibrium path, both players exit at de facto the same time when X_t falls below x^{**}. Strategic exiting is never profitable for Frank in this case because $V_c(x) \geq F(x)$ for any x. If $\lambda < \lambda^*$, Frank benefits from strategic exiting because $F(x) > V_c(x)$ for some x. Like other stopping problems, his optimal exit thresholds (x', x'', and x''') are determined by value matching and smooth pasting conditions of $U_f(x)$ and $F(x)$ at the three thresholds, as explained in more detail in Appendix A.13.

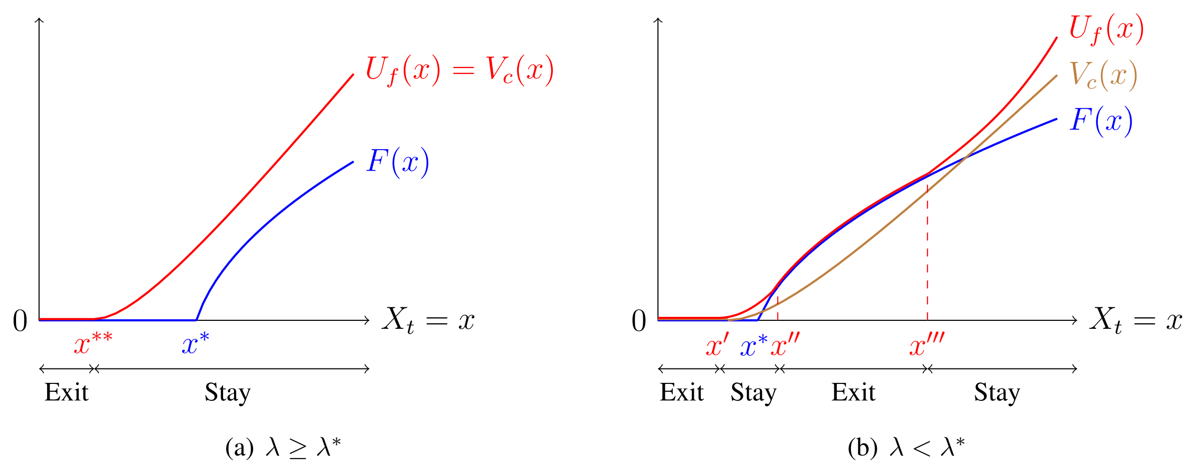


Figure 6: Illustration of Frank's exit strategy in the unique SPNE. In each panel, Frank's equilibrium exit region is labeled below the horizontal axis.

To intuitively understand Frank's optimal strategy in the case of $\lambda < \lambda^*$, we divide $\mathcal{X}$ into four sections.[24] When X_t is *very low* ($X_t \in (0, x']$), Frank initiates a de facto joint exit, as it is no longer worthwhile to run the project. When X_t is *moderately low* ($X_t \in (x', x'')$), Frank finds it worthwhile to let the project operate, and he is also deterred from strategic exiting because Susan would have found it challenging to run the project alone. When X_t is *moderately high* ($X_t \in [x'', x''']$), Frank finds it optimal to strategically exit, knowing that Susan will be motivated to

[24] Recall that we also divided $\mathcal{X}$ into four sections in the baseline model when analyzing the Pareto-optimal preemptive equilibrium in the case of $\lambda \geq \lambda^{**}$ (see Section 3.3.2). Despite the similarity, the two division exercises differ in both the interpretations of the four sections and the thresholds to determine the four sections.

run the project alone. Finally, when X_t is *very high* ($X_t \in (x''', \infty)$), Frank stays in the partnership so as to avoid the discount in revenue due to Assumption 2. Notably, Frank's optimal exit strategy does not admit a threshold form, which is uncommon in the literature on optimal stopping. It occurs in our setting because of the non-standard exit payoff $F(x)$, which is kinked at x^* (see Figure 1).

The key takeaway from the above finding is that strategic exiting occurs *only* when the partnership's profitability exceeds the threshold x''.[25] Specifically, Frank strategically exits when X_t is moderately high, but not when it is moderately low. This finding explains some exit patterns observed in practice, particularly among serial entrepreneurs. Many startups have "partners" consisting of founders and early-stage investors — typically, founders do not leave a startup before investors. Some investors, known as serial entrepreneurs, may strategically exit a startup and switch gears to the next startup in their pipelines. In a startup's early development stages, serial entrepreneurs often play a pivotal role in guiding the startup towards viability. However, as the startup gains traction and becomes more profitable, serial entrepreneurs may choose to strategically exit to save resources for new ventures. One prominent example is Peter Thiel, who left PayPal after its acquisition by eBay and soon transitioned his investment into other ventures, including Facebook.

Having established Frank's response to Susan's no-first-exit commitment, we next examine whether such a commitment can lead to a Pareto-improvement. Compared to the baseline model, Frank is better off by being the designated first mover. How about Susan? Intuitively, Susan's no-first-exit commitment has two opposite effects on her welfare. The direct (negative) effect is a consequence of her forgoing the option to exit first. The indirect (positive) effect is that her commitment prevents pre-emption and makes Frank exit less aggressively than the baseline model, as formalized by the following lemma.

Lemma 6. *If $\lambda < \lambda^*$, Frank's strategic exit region, $[x'', x''']$, is a strict subset of $(x^*, \tilde{x})$, a player's Stage-1 pre-emptive exit region in the baseline model. In other words, $x'' > x^*$ and $x''' < \tilde{x}$.*

Proof. See Appendix A.14. □

[25] Exceeding the threshold x'' is necessary but not sufficient for strategic exiting, as it does not occur if the partnership's profitability exceeds not only x'' but also an even higher threshold x'''.

The next theorem identifies the situations where the positive effect outweighs the negative one, implying that Susan's no-first-exit commitment can lead to a Pareto-improvement.[26]

Theorem 3. *If $\lambda < \lambda^*$ and $\beta \leq 1$, there exist x^s such that $U_s(x) > W(x)$, $\forall x \in (x^s, \infty)$. In other words, Susan strictly benefits from the no-first-exit-commitment when $X_t \in (x^s, \infty)$.*

Proof. See Appendix A.15. □

As suggested by Theorem 3, Susan benefits from the no-first-exit commitment when $\lambda < \lambda^*$, which makes pre-emption occur in the baseline model, $\beta \leq 1$, which ensures a substantial benefit from avoiding pre-emption (because the social welfare loss from strategic exiting is large), and $X_t > x^s$, which makes the negative effect mild since the partnership's current profitability is high.

5 Robustness of Main Result

To succinctly illustrate the insight regarding the curse of profitability, we develop a concise baseline model. However, it is worth emphasizing that the core insight of this paper remains valid across more general settings and alternative specifications. We compile the discussion in Online Appendix B of Xu (2025), with the key messages summarized below.

More than two players. Many real-world partnerships involve more than two partners. In such cases, the ripple effect of an exit becomes more complex: an initial exit may trigger a second exit, which may further trigger a third, and so on. Hence, when considering a strategic exit, a player must correctly anticipate the potential ripple effect, which in turn depends on the strategic exit decisions of others. In the appendix, we generalize the model to more than two players. Although the equilibrium characterization becomes less tractable in this generalized setting, we introduce an algorithm to find *stage-wise Pareto-optimal equilibria* — a modified solution concept tailored to this generalization. We also establish sufficient conditions under which the curse of profitability arises in *any* stage-wise Pareto-optimal equilibrium.

Asymmetric payoffs. In reality, it is common for partners to derive different payoffs from a partnership. This asymmetry can lead to players' divergent incentives in strategic exits. In the

[26] This finding is similar to earlier studies on how *sequentiality* of moves, compared with *simultaneity*, promotes cooperation in games with strategic complementarities (Zhou & Chen, 2015).

appendix, we generalize the model to an asymmetric payoff structure and show that the unique Pareto-optimal equilibrium remains subject to the curse of profitability.

Re-entry. In some situations, partners who have exited a partnership may have the option to return, typically by incurring an additional cost. The appendix investigates a generalized setting where re-entry is allowed upon payment of a lump-sum cost. In equilibrium, re-entry occurs when the partnership's profitability reaches a high level. We demonstrate that the curse of profitability persists when the re-entry cost is sufficiently large.

Relaxation of innocuous assumptions. Our baseline model made several innocuous assumptions to facilitate the analysis. In the appendix, the curse of profitability is shown to persist under a range of extensions: when the partnership's profitability follows a more general diffusion process than the Brownian motion used in the baseline model, when a free-rider earns higher revenue than a contributor, and when alternative tie-breaking rules are adopted in Stage 1 of the game.

6 Conclusion

In this paper, we study dynamic partnerships where partners can strategically exit to free-ride on others' efforts. We highlight a curse of profitability — when players have sufficiently large free-riding incentives and a medium level of mutual reliance, an increase in the partnership's profitability may leave all partners strictly worse off. Additionally, we show that if any player commits not to exit first, it can lead to a Pareto-improvement.

Our framework is tractable and can offer insights into other questions concerning partnerships of this nature. For instance, a companion paper, Xu (2023), studies a deterministic partnership where partners can choose their effort levels over time. We investigate the optimal way for the partners to monitor each other's efforts and show that imperfect monitoring can, counterintuitively, facilitate cooperation.

A Proofs

A.1 Proof of Claim 1

Since (1) the second mover's flow payoff, $\beta X_t - \kappa c$, is weakly convex and non-decreasing in X_t, and (2) her lump-sum exit payoff is a constant, her optimal exit strategy must take a threshold form according to Villeneuve (2007). Let x^* denote her exit threshold. The value function $S(x)$ must take the following form,

$$S(x) = \begin{cases} \beta\left(\frac{x}{r-\mu} - \frac{\lambda c}{r}\right) + k_1 x^\gamma + k_2 x^\eta & \text{if } x > x^*, \\ 0 & \text{if } x \leq x^*, \end{cases} \tag{A1}$$

where $S(x) = \beta\left[x/(r-\mu) - \lambda c/r\right] + k_1 x^\gamma + k_2 x^\eta$ is the general solution to the ODE, $S(x) = \beta(x - \lambda c) + (1-r)S(x) + S'(x)\mu x + [\sigma^2 S''(x)x^2]/2$. In this general solution, $\eta = (\sigma^2 - 2\mu + \sqrt{(\sigma^2-2\mu)^2 + 8r\sigma^2})/(2\sigma^2) > 0$ and $\gamma = (\sigma^2 - 2\mu - \sqrt{(\sigma^2-2\mu)^2 + 8r\sigma^2})/(2\sigma^2) < 0$ are the two roots of $\Gamma(y) = \mu y + [\sigma^2 y(y-1)]/2 - r$. We know that $\Gamma(y)$ has two roots of different signs because it is a convex parabola with $\Gamma(0) = -r < 0$.

We need to determine three parameters in (A1), x^*, k_1, and k_2. First, the boundary condition, $\lim_{x\to\infty} \{S(x) - \beta\left[x/(r-\mu) - \lambda c/r\right]\} = 0$, pins down $k_2 = 0$. This is because when $X_t \to \infty$, the option value of exit approaches zero, and thus $S(X_t)$ should be arbitrarily close to $\beta\left[X_t/(r-\mu) - \lambda c/r\right]$, the second mover's continuation value if she never exercises the exit option. Second, the values of x^* and k_1 are jointly pinned down by the value matching condition, $S(x^*) = 0$, and the smooth pasting condition, $S'(x^*) = 0$, as expanded below.

$$\beta\left(\frac{x^*}{r-\mu} - \frac{\lambda c}{r}\right) + k_1 (x^*)^\gamma = 0 \tag{A2}$$

$$\beta\frac{1}{r-\mu} + \gamma k_1 (x^*)^{\gamma-1} = 0. \tag{A3}$$

From (A2)$\times\gamma-$ (A3)$\times x^*$, we get $x^* = [(r-\mu)\gamma]/[r(\gamma-1)] \cdot \lambda c$. Substituting this expression into (A2) pins down the value of k_1 and thus the closed-form solution of $S(x)$ in Claim 1.

A.2 Proof of Claim 2

When $x \leq x^*$, the first mover's exit immediately triggers the second mover to terminate the project, and thus $F(x) = 0$. When $x > x^*$, the general solution to the Feynman-Kac formula is $F(x) = \alpha x/(r-\mu) + k_3 x^\gamma + k_4 x^\eta$. The boundary condition $\lim_{x\to\infty} [F(x) - \alpha x/(r-\mu)] = 0$ pins down $k_4 = 0$ for the same reason as Appendix A.1. The value of k_3 is pinned down by $F(x^*) = 0$.

A.3 Proof of Lemma 1

Claim 3. ***(a)*** *The function $F(x)$ is kinked at $x = x^*$ and strictly concave when $x \in (x^*, \infty)$;* ***(b)*** *The function $S(x)$ is differentiable at $x = x^*$ and strictly convex when $x \in (x^*, \infty)$.*

Proof. The left derivative of $F(x)$ at x^* is 0, while the right derivative is

$$F'_+(x^*) = \frac{\alpha}{r-\mu} - \frac{\alpha\gamma}{(r-\mu)(x^*)^{\gamma-1}}(x^*)^{\gamma-1} = \frac{\alpha(1-\gamma)}{r-\mu} > 0,$$

so $F(x)$ has a kink at x^*. Also, when $x > x^*$, we have

$$F''(x) = -\frac{\alpha\gamma(\gamma-1)}{(r-\mu)(x^*)^{\gamma-1}} x^{\gamma-2} < 0,$$

justifying the strict concavity argument. Differentiability of $S(x)$ at x^* comes directly from the smooth pasting condition, $S'(x^*) = 0$. Strict convexity of $S(x)$ when $x > x^*$ comes from

$$S''(x) = -\frac{\beta\lambda c\gamma}{r(x^*)^\gamma} x^{\gamma-2} > 0.$$

□

Denote $\Delta(x) = F(x) - S(x)$. It is bounded by the two asymptotic lines:

$$\Delta(x) < \frac{\alpha}{r-\mu}x - \frac{\beta}{r-\mu}x + \frac{\beta\lambda c}{r} = \frac{\alpha-\beta}{r-\mu}x + \frac{\beta\lambda c}{r}. \tag{A4}$$

Therefore, when x is sufficiently large (i.e., $x > [\beta\lambda c(r-\mu)]/[r(\beta-\alpha)]$), the RHS of (A4) is negative and thus $\Delta(x) < 0$. Meanwhile, Claim 3 indicates that the right derivative of $\Delta(x)$ is positive at $x = x^*$, which implies that $\Delta(x^* + \epsilon) > 0$ with $\epsilon > 0$ arbitrarily small. By the continuity of $\Delta(x)$, the function $\Delta(x)$ must admit at least one root in the interval (x^*, ∞). Indeed, $\Delta(x)$ has only one root in the interval (x^*, ∞). This is because $\Delta(x)$ is strictly concave due to the convexity of $S(x)$ and the concavity of $F(x)$ as shown in Claim 3. A strictly concave function can admit at most two roots, which are x^* and $\tilde{x}$ in our case. Strict concavity of $\Delta(x)$ also indicates that $\Delta(x) > 0$ for $x \in (x^*, \tilde{x})$ and $\Delta(x) < 0$ for $x \in (\tilde{x}, \infty)$.

A.4 Proof of Lemma 2

First of all, players' exit regions must be identical in the interval with the first-mover advantage. Suppose, by contradiction, $x \in (x^*, \tilde{x})$ while x falls in $\mathcal{X}^i$ but not $\mathcal{X}^j$. Then Player j will be better off by deviating to exit when $X_t = x$, as $[F(x) + S(x)]/2 > S(x)$. Hence, if Lemma 2 does not hold, for any $\epsilon > 0$, there must be x^+ and x^- in the interval $(x^*, \tilde{x})$ such that: (1) both players exit when $X_t = x^-$ and stay when $X_t = x^+$; (2) x^+ and x^- are very close so that $|F(x^+) - F(x^-)| < \epsilon$, $|S(x^+) - S(x^-)| < \epsilon$, and $|V(x^+) - V(x^-)| < \epsilon$ where $V(\cdot)$ is each player's value function in the equilibrium. Notice that $V(\cdot)$ is continuous since the stochastic state variable X_t has a continuous path and can evolve in both directions. Since $V(x^-) = [F(x^-) + S(x^-)]/2$, we infer that $V(x^+) < V(x^-) + \epsilon = [F(x^-) + S(x^-)]/2 + \epsilon < [F(x^+) + S(x^+)]/2 + 2\epsilon$. Together with the fact that $F(x^+)$ is strictly larger than $S(x^+)$ due to first-mover advantage, the above inequality indicates that $V(x^+) < F(x^+)$ when ϵ is sufficiently small. This contradicts the presumption that both players choose to stay when $X_t = x^+$.

A.5 Proof of Lemma 3

It suffices to show that any cooperative equilibrium (if existing) must be weakly Pareto-dominated by $\mathcal{X}^1 = \mathcal{X}^2 = (0, x^{**}]$, which implements the socially optimal outcome. Since the socially optimal outcome maximizes the total welfare of the two players, if a cooperative equilibrium violates the above statement, it must satisfy the following condition: there exists some $x^\#$ such that $F(x^\#) > V_c(x^\#)$. Suppose by contradiction, such an $x^\#$ exists. It cannot be true that $x^\# \leq x^*$ because such an $x^\#$ gives $F(x^\#) = 0$. By definition of a cooperative equilibrium, such an $x^\#$ also cannot fall in the interval of $(x^*, \tilde{x})$. Finally, $x^\# \geq \tilde{x}$ cannot hold as well. If it holds, we have $S(x^\#) \geq F(x^\#) > V_c(x^\#)$, where the first inequality comes from Lemma 1. An immediate consequence is that $S(x^{\#}) + F(x^{\#}) > 2V_c(x^{\#})$, which cannot be true because $V_c(x^{\#})$ is the highest possible social welfare.

A.6 Proof of Lemma 4

Step 1. As is stated in the paragraph before Lemma 4, the existence of a cooperative equilibrium boils down to whether $\mathcal{X}^1 = \mathcal{X}^2 = (0, x^{**}]$ is an equilibrium. Hence, we want to show that $V_c(x) \geq F(x)$ holds for all $x \in (x^{**}, \infty)$ if and only if $\lambda \geq \lambda^*$.

Equation (2) indicates that $F(x)$ is point-wise (weakly) decreasing as λ increases. Meanwhile, $F(x)$ is strictly concave and $V_c(x)$ is strictly convex for $x > x^*$ according to Claim 3. These properties imply the existence of a threshold λ^* under which $V_c(x)$ tangentially intersects with $F(x)$ at some $\bar{x} > x^*$. Hence, $F(\bar{x}; \lambda^*) = V_c(\bar{x})$ and $F'(\bar{x}; \lambda^*) = V_c'(\bar{x})$, which are equivalent to

$$-\frac{c}{r} + \frac{1-\alpha}{r-\mu} \cdot \bar{x} + K \cdot (\bar{x})^{\gamma} \;=\; 0 \tag{A5}$$

$$\frac{1-\alpha}{r-\mu} \cdot \bar{x} + \gamma K \cdot (\bar{x})^{\gamma} \;=\; 0, \tag{A6}$$

where

$$K := \frac{c}{r(1-\gamma)(x^{**})^{\gamma}} + \frac{\alpha}{(r-\mu)(x^*)^{\gamma-1}} = \frac{\left(1 - \alpha\gamma(\lambda^*)^{1-\gamma}\right) c}{r(1-\gamma)(x^{**})^{\gamma}}.$$

We subtract (A6) from (A5)$*\gamma$ and get

$$\bar{x} = \frac{1}{1-\alpha}\frac{r-\mu}{r}\frac{\gamma}{\gamma-1}c = \frac{1}{1-\alpha}x^{**}. \tag{A7}$$

Plugging (A7) into (A6), we get $1-\alpha\gamma(\lambda^*)^{1-\gamma} = (1-\alpha)^{\gamma}$, which yields $\lambda^* = \{[1-(1-\alpha)^{\gamma}]/(\alpha\gamma)\}^{\frac{1}{1-\gamma}}$.

Step 2. To complete the analysis, we need to verify that $\lambda^* > \underline{\lambda}$, where $\underline{\lambda}$ is the lower bound of λ's domain as defined in Section 2. Indeed, showing this inequality is not trivial. We start with the following claim.

Claim 4. *λ^* is strictly decreasing in γ.*

Proof. We first replace $1/(1-\alpha)$ by z and let $\lambda^* = f(\gamma) := \{(z^{1-\gamma} - z)/[-(z-1)\gamma]\}^{\frac{1}{1-\gamma}}$. We would like to show that $f'(\gamma) < 0$. Let $g(\gamma) = [f(\gamma)]^{1-\gamma}$ and $h(\gamma) = ln(g(\gamma))$. Since

$$\begin{aligned} f'(\gamma) &= \frac{1}{1-\gamma} g(\gamma)^{\frac{1}{1-\gamma}-1} g'(\gamma) + g(\gamma)^{\frac{1}{1-\gamma}} ln(g(\gamma)) \frac{1}{(1-\gamma)^2} \\ &= \frac{g(\gamma)^{\frac{1}{1-\gamma}}}{1-\gamma} \left[\frac{g'(\gamma)}{g(\gamma)} + \frac{ln(g(\gamma))}{1-\gamma} \right] \\ &= \frac{f(\gamma)}{1-\gamma} \left[h'(\gamma) - \frac{h(1)-h(\gamma)}{1-\gamma} \right], \end{aligned}$$

it suffices to show that

$$h'(\gamma) - \frac{h(1)-h(\gamma)}{1-\gamma} < 0. \tag{A8}$$

Notice that $[h(1) - h(\gamma)]/(1-\gamma)$ is the slope of the secant line between γ and 1 on the curve of $h(\cdot)$, one sufficient condition for (A8) to hold is that $h(\gamma)$ is convex; i.e., $g(\gamma)$ is log-convex. To prove the log-convexity of $g(\gamma)$, we only need to show that $z^{\gamma y} g(\gamma)$ is convex for any $y \in \mathbb{R}$.[27] We adopt Taylor Expansion w.r.t. γ on $z^{\gamma y} g(\gamma)$ as below

$$\begin{aligned} z^{\gamma y} g(\gamma) &= \frac{z}{(z-1)\gamma} \left(z^{\gamma y} - z^{\gamma y - \gamma} \right) \\ &= \frac{z}{(z-1)\gamma} \left[\sum_{n=0}^{\infty} (y ln(z))^n \gamma^n - \sum_{n=0}^{\infty} ((y-1) ln(z))^n \gamma^n \right] \\ &= \frac{z}{(z-1)} \sum_{n=1}^{\infty} (ln(z))^n \left[y^n - (y-1)^n \right] \gamma^{n-1}. \end{aligned}$$

Since $z > 1$, and $y^n - (y-1)^n > 0$ for all $n \geq 1$ and $y \in \mathbb{R}$, we conclude that $z^{\gamma y} g(\gamma)$ is convex. $\square$

With the above claim, it suffices to show $\underline{\lambda} \leq \lim_{\gamma \to 0} \lambda^*$. Applying L'Hospital Rule gives $\lim_{\gamma \to 0} \lambda^* = [-ln(1-\alpha)]/\alpha$. Hence, it remains to show that for any $\alpha \in (0, 1)$, $2/(2-\alpha) \leq [-ln(1-\alpha)]/\alpha$, or equivalently, $2\alpha/(2-\alpha) \leq -ln(1-\alpha)$. Denote LHS$(\alpha) := 2\alpha/(2-\alpha)$ and RHS$(\alpha) := -ln(1-\alpha)$. We can show that: (1) LHS(0) = RHS(0) = 0; (2) LHS$'(0)$ = RHS$'(0) = 1$; and (3) LHS$(\cdot)$ is strictly concave while RHS$(\cdot)$ is strictly convex. These three conditions combined conclude the proof of $\lambda^* > \underline{\lambda}$.

[27] See Page 70 of Niculescu and Persson (2006) for reference.

The above two steps prove Statement (1) of the lemma. The remaining two statements immediately follow because $\mathcal{X}^1 = \mathcal{X}^2 = (0, x^{**}]$ implements the socially optimal outcome (Lemma 3), and thus generates a value function to each player that is point-wise higher than any other cooperative equilibrium. Besides, the multiplicity in Statement (3) is explicitly explained in Section 3.5.

A.7 Proof of Lemma 5

Statement (1) holds because $\mathcal{X}^1 = \mathcal{X}^2 = (0, \tilde{x})$ is always a pre-emptive equilibrium, as we explain in the paragraph before the lemma. The proof of Statements (2) and (3) is mainly completed in the explanatory paragraphs following the lemma, except for the two missing pieces below.

First, for the same reason as Appendix A.5, it is dominant for each player to stay in the partnership when $X_t \geq \tilde{x}$. Intuitively, any player cannot benefit from exiting the partnership when $X_t = x^{\#} \geq \tilde{x}$. By contradiction, if that were true, the second mover would benefit even more since $S(x^{\#}) \geq F(x^{\#})$ — such a situation where both players benefit from one's exit is impossible because strategic exiting is socially inefficient due to Assumption 1.

Second, if $\lambda > \lambda^{**}$, Section 3.5 explicitly explains the multiplicity of pre-emptive equilibria.

A.8 Proof of Theorem 1

The paragraphs preceding Theorem 1 already prove the theorem, except for the following discussion on how the value of α affects λ^*, λ^{**}, and their relative magnitude. Notice that λ can be any number in $[\underline{\lambda}, \infty)$. In Step 3 of Appendix A.6, we already show that $\underline{\lambda} < \lambda^*$ always holds. Besides, we have the following claim.

Claim 5. *(1) The value of λ^* strictly increases in α; also, $\lim\limits_{\alpha \to 0} \lambda^* = 1$ and $\lim\limits_{\alpha \to 1} \lambda^* = +\infty$.*
(2) The value of $\underline{\lambda}$ strictly increases in α; also, $\lim\limits_{\alpha \to 0} \underline{\lambda} = 1$ and $\lim\limits_{\alpha \to 1} \underline{\lambda} = 2$.
(3) The value of λ^{**} does not depend on α; also, we have $\lambda^{**} > 1$.

Proof. (1) From the expression of λ^*, we want to show that $[1 - (1 - \alpha)^\gamma]/(\alpha\gamma)$ strictly increases in α. This is true because $\frac{\partial \left[\frac{1-(1-\alpha)^\gamma}{\alpha\gamma}\right]}{\partial \alpha} = \{(1 - \alpha)^{\gamma-1}[\alpha(1 + \gamma) - 1] - 1\}/(\gamma\alpha^2) > 0$. By applying L'Hospital rule, we also get $\lim\limits_{\alpha \to 0} \lambda^* = 1$ and $\lim\limits_{\alpha \to 1} \lambda^* = +\infty$.
(2) This is straightforward from the expression $\underline{\lambda} = 2/(2 - \alpha)$.

(3) The value of λ^{**} does not depend on α because α does not show up in its expression. The fact that $\lambda^{**} > 1$ comes from a classic result in optimal stopping problems that a decision maker will stop when the myopic return is negative. Specifically, from Appendix A.1, x^* must satisfy the homogenous ODE, $\beta(x^* - \lambda c) = rS(x^*) - S'(x^*)\mu x^* - [\sigma^2 S''(x^*)(x^*)^2]/2$. Because of value matching and smooth pasting, we know that $S(x^*) = S'(x^*) = 0$; meanwhile, $S''(x^*) > 0$ because $S(\cdot)$ is strictly convex. Plugging these terms into the homogenous ODE, we have $x^* - \lambda c < 0$, which is equivalent to $[(r-\mu)\gamma]/[r(\gamma-1)] < 1$ according to the closed-form of x^*. Since λ^{**} is the inverse of $[(r-\mu)\gamma]/[r(\gamma-1)]$, we conclude that $\lambda^{**} > 1$. □

From Claim 5, we can infer that there exists a unique $\underline{\alpha} \in (0,1)$ such that the values of λ^* and λ^{**} are identical. Therefore, we discuss the equilibrium characterization as follows.

Case 1: $\alpha \in (0, \underline{\alpha}]$. In this case, it follows that $\lambda^{**} \geq \lambda^*$. Therefore, the value of λ should fall in either Scenario (1) (i.e., $\lambda \geq \lambda^*$) or Scenario (3) (i.e., $\lambda < \lambda^*$).

Case 2: $\alpha \in (\underline{\alpha}, 1)$. In this case, it follows that $\lambda^{**} < \lambda^*$. The characterization now depends on the relative magnitude of $\underline{\lambda}$ and λ^{**}. We further consider two sub-cases.

Case 2.1: $\lambda^{**} \geq 2$. In this sub-case, it follows that $\underline{\lambda} < \lambda^{**}$ always holds. Hence, the value of λ covers all three different scenarios.

Case 2.2: $\lambda^{**} < 2$. In this sub-case, there exists $\bar{\alpha}$ such that $\underline{\lambda} = \lambda^{**}$. When $\alpha \in (\underline{\alpha}, \bar{\alpha})$, it follows that $\underline{\lambda} < \lambda^{**} < \lambda^*$, so the value of λ still covers all three different scenarios as Case 2.1. When $\alpha \in [\bar{\alpha}, 1)$, it follows that $\lambda^{**} \leq \underline{\lambda} < \lambda^*$, so Scenario (3) disappears — the value of λ should fall in either Scenario (1) (i.e., $\lambda \geq \lambda^*$) or Scenario (2) (i.e., $\lambda < \lambda^*$).

The characterization in Theorem 1 encompasses all the above cases.

A.9 Proof of Corollary 2

A cooperative equilibrium can be sustained if and only if $\lambda \geq \lambda^*$. Notice that λ is determined by β and κ; λ^* is determined by α and γ, while γ is determined by r, μ, and σ.

For α: Claim 5 in Appendix A.8 already proves that λ^* strictly increases in α.

For β and κ: It is straightforward from the expression $\lambda = \kappa/\beta$.

For r, μ, and σ: Since Claim 4 in Appendix A.6 already proves that λ^* strictly decreases in γ, it suffices to show that γ strictly decreases in r and μ and strictly increases in σ. Notice that γ is the negative root of $\Gamma(y;\mu,\sigma)=\mu y+[\sigma^2 y(y-1)]/2-r$. The above results can be shown by the implicit function theorem since $\frac{\partial\Gamma}{\partial y}|_{y=\gamma}<0$, $\frac{\partial\Gamma}{\partial r}|_{y=\gamma}<0$, $\frac{\partial\Gamma}{\partial\mu}|_{y=\gamma}<0$, and $\frac{\partial\Gamma}{\partial\sigma}|_{y=\gamma}>0$.

For c: Neither λ nor λ^* depends on c.

A.10 Proof of Proposition 1

First, the threshold x^c cannot be smaller than x^{**}. Otherwise, any player will be better off by exiting when $X_t \in (x^c, x^{**})$.

Second, we want to derive the function $k(\lambda)$ such that the threshold x^c cannot be larger than $k(\lambda)\cdot x^{**}$. Notice that each cooperative equilibrium $\mathcal{X}^1=\mathcal{X}^2=(0,x^c]$ generates a continuation value

$$\hat{V}_c(x;x^c)=-\frac{c}{r}\left[1-\left(\frac{x}{x^c}\right)^{\gamma}\right]+\frac{x}{r-\mu}\left[1-\left(\frac{x}{x^c}\right)^{\gamma-1}\right].$$

It suffices to jointly solve $\bar{x}$ and x^c from the simultaneous equations $F(\bar{x})=\hat{V}_c(\bar{x};x^c)$ and $F'(\bar{x})=\hat{V}_c'(\bar{x};x^c)$. Using similar derivation as in Lemma 4, we can get $\bar{x}=x^{**}/(1-\alpha)$. Plug this into $F'(\bar{x})=\hat{V}_c'(\bar{x};x^c)$, which can be written as

$$\frac{\gamma c}{r}\cdot\left(\frac{\bar{x}}{x^c}\right)^{\gamma}+\frac{(1-\alpha)\bar{x}}{r-\mu}=\frac{\gamma\bar{x}}{r-\mu}\cdot\left(\frac{\bar{x}}{x^c}\right)^{\gamma-1}-\frac{\alpha\gamma\bar{x}}{r-\mu}\cdot\left(\frac{\bar{x}}{x^*}\right)^{\gamma-1},$$

we have

$$(\gamma-1)(1-\alpha)\cdot\left(\frac{\bar{x}}{x^c}\right)^{\gamma}+(1-\alpha)=\gamma\cdot\left(\frac{\bar{x}}{x^c}\right)^{\gamma-1}-\alpha\gamma[\lambda(1-\alpha)]^{1-\gamma}.$$

This equation is satisfied if we let $x^c=k(\lambda)x^{**}$ while the function $k(\lambda)$ satisfies

$$\gamma\cdot(k(\lambda))^{1-\gamma}+(1-\gamma)\cdot(k(\lambda))^{-\gamma}=(1-\alpha)^{\gamma}+\alpha\gamma\lambda^{1-\gamma}. \tag{A9}$$

Since the LHS of (A9) strictly decreases in $k(\lambda)$ and the RHS strictly decreases in λ, we further infer that (i) the function $k(\lambda)$ is well-defined, and (ii) by the implicit function theorem, $k(\lambda)$ strictly increases in λ. Also, to see why $k(\lambda)<\lambda$, notice that the LHS is strictly smaller than the RHS if $k(\lambda)=\lambda$. Finally, when $\lambda=\lambda^*$, we have $k(\lambda^*)=1$ by the definition of λ^* in Lemma 4.

A.11 Proof of Proposition 2

The threshold x^p cannot be smaller than x^0 because otherwise, any player prefers to exit when $X_t \in (x^p, x^0)$. Also, by construction, x^p cannot exceed x^*. Besides these two requirements, there are no restrictions on the value of x^p, because as long as $x^p \in [x^0, x^*]$, each player's value function in the interval $[x^p, x^*]$ is non-negative. When $x^p \geq c$, this argument holds trivially. When $x^p < c$, this argument still holds for the following reason. Since each player's flow payoff in the interval $[x^0, x^p)$ is always negative, we can infer that the value function $\tilde{V}_p(x; x^p)$, which is generated by $\mathcal{X}^1 = \mathcal{X}^2 = (0, x^p] \cup (x^*, \tilde{x})$ will have a kink at x^p with strictly positive right derivative, which implies that $\tilde{V}_p(x; x^p) \geq 0$ for $x \in [x^p, x^*]$.

A.12 Proof of Proposition 3

The proof includes two parts corresponding to two gaps between pure-strategy MPEs and SPNEs.

Part 1: Non-Markovian SPNE. The proof is mostly contained in the paragraph following Proposition 3. For Stage 1, in particular, we provide a more formal proof. Since the only relevant history in Stage 1 is the trajectory of the process, a non-Markovian SPNE indicates that players' continuation play at some time t must depend on $(X_s^{'})_{s \in [0,t)}$. However, conditional on the value of X_t, the process $(X_s^{'})_{s \in [0,t)}$ is payoff-irrelevant for the continuation game. Hence, conditional on X_t, the set of achievable continuation value profiles should not depend on $(X_s^{'})_{s \in [0,t)}$. Therefore, introducing non-Markovian SPNEs cannot enlarge the set of achievable continuation value profiles at $t = 0$.

Part 2: Mixed-Strategy MPE. By contradiction, suppose there exists a mixed-strategy MPE that generates a continuation value profile $(\tilde{W}^1(x), \tilde{W}^2(x))$ such that $\tilde{W}^1(x) > W(x)$ for some x.

Case 1: $\lambda \geq \lambda^*$. Notice that $W(x) \geq F(x)$ for any x, while $S(x) > W(x)$ is possible for some high value of x. It must be the case that Player 2 exits for some large $x^\#$ where $S(x^\#) > W(x^\#)$ to make Player 1's continuation value larger than $W(x^\#)$. However, this violates Player 2's rationality because it is dominant for Player 2 to stay under such a high $x^\#$.

Case 2: $\lambda < \lambda^*$. First, both players find it dominant to stay when $X_t \in [\tilde{x}, \infty)$. Second, we want to show that both players exit with probability one when $X_t \in (x^*, \tilde{x})$, even when they are allowed to use mixed strategies. Notice that there must exist an interval (x^1, x^2) such that $F(x) > V_c(x)$ when $x \in (x^1, x^2)$. These two thresholds are the intersections of $F(x)$ and $V_c(x)$ as shown in

Figure 2(b). Suppose, by contradiction, Player 2 uses a mixed strategy for some $X_t \in (x^1, x^2)$. It must follow that $\tilde{W}^2(X_t) \geq F(X_t)$. However, this further indicates that $\tilde{W}^1(X_t) < F(X_t)$ because $\tilde{W}^1(X_t) + \tilde{W}^2(X_t) \leq 2V_c(X_t) < 2F(X_t)$, where the last inequality comes from $X_t \in (x^1, x^2)$. Therefore, Player 1 must exit with probability one instead, which, in turn, disproves Player 2's using mixed strategy for X_t. Moreover, players' incentives to pre-empt each other still exist even when mixed strategies are allowed. Hence, the logic of Lemma 2 continues to work — triggered by the fact that both players exit with probability one in the interval (x^1, x^2), they will do the same for the entire interval $(x^*, \tilde{x})$ in any (possibly mixed-strategy) MPE. Finally, when $X_t \in (0, x^*]$, players always exit at de facto the same time, attaining identical continuation value. For that reason, mixed strategies may introduce new MPEs, but cannot improve players' continuation value beyond the Pareto-optimal equilibrium.

A.13 Proof of Theorem 2

We want to show that $U_f(x)$ generated by $\mathcal{X}^f$ satisfies the HJB equation. According to Strulovici and Szydlowski (2015), it suffices to check three conditions: (1) $U_f(x) \geq F(x)$, $\forall x$; (2) $U_f(x)$ is everywhere continuous and first-order differentiable; (3) $U_f(x) \geq (1-r)U_f(x) + x - c + U_f'(x)\mu x + [\sigma^2 U_f''(x)x^2]/2$ whenever $U_f(x) = F(x)$. When $\lambda \geq \lambda^*$, we have $U_f(x) = V_c(x)$, and it is not difficult to check that all three conditions are satisfied. When $\lambda < \lambda^*$, we want to construct three thresholds (x', x'', x''') that satisfy the corresponding smooth pasting and value matching conditions. For x''', its closed-form is exactly pinned down by Equations (A5) and (A6). Hence, $x''' = x^{**}/(1-\alpha)$ as in (A7). For x' and x'', the construction takes the following two steps.

Step 1: Existence. Let the general solution of $U_f(x)$ for $x \in [x', x'']$ be $U_f(x) = -c/r + x/(r-\mu) + k_5 x^\gamma + k_6 x^\eta$. Notice that k_6 is not necessarily zero as the boundary condition when $x \to \infty$ no longer holds. The value matching and smooth pasting conditions for these two thresholds are

$$
\begin{aligned}
-\frac{c}{r}+\frac{x'}{r-\mu}+k_5\cdot(x')^{\gamma}+k_6\cdot(x')^{\eta} &= 0 && \text{(A10)}\\
\frac{x'}{r-\mu}+\gamma k_5\cdot(x')^{\gamma}+\eta k_6\cdot(x')^{\eta} &= 0 && \text{(A11)}\\
-\frac{c}{r}+\frac{(1-\alpha)x''}{r-\mu}+(k_5-k_3)\cdot(x'')^{\gamma}+k_6\cdot(x'')^{\eta} &= 0 && \text{(A12)}\\
\frac{(1-\alpha)x''}{r-\mu}+\gamma(k_5-k_3)\cdot(x'')^{\gamma}+\eta k_6\cdot(x'')^{\eta} &= 0. && \text{(A13)}
\end{aligned}
$$

Claim 6. $k_6 > 0$ *and* $k_5 > k_3$.

Proof. Let (A13) $-$ (A12)$*\gamma$ and (A13) $-$ (A12)$*\eta$, we have

$$
\frac{c\gamma}{r}+(1-\gamma)\frac{(1-\alpha)x''}{r-\mu}+k_6\cdot(\eta-\gamma)(x'')^{\eta}=0 \qquad \text{(A14)}
$$

$$
\frac{c\eta}{r}+(1-\eta)\frac{(1-\alpha)x''}{r-\mu}+(k_5-k_3)\cdot(\gamma-\eta)(x'')^{\gamma}=0 \qquad \text{(A15)}
$$

By construction, we require that $x'' < x''' = x^{**}/(1-\alpha)$. Plugging it into (A14) yields $k_6 > 0$. Plugging it into (A15), we get

$$
\frac{c\eta}{r}+(1-\eta)\frac{(1-\alpha)x''}{r-\mu} > \frac{c(\eta-\gamma)}{r(1-\gamma)} > 0,
$$

which indicates that $k_5 > k_3$. □

From (A11) $-$ (A10)$*\gamma$ and (A11) $-$ (A10)$*\eta$, we have

$$
\begin{aligned}
\frac{c\gamma}{r}+(1-\gamma)\frac{x'}{r-\mu}+k_6\cdot(\eta-\gamma)(x')^{\eta} &= 0\\
\frac{c\eta}{r}+(1-\eta)\frac{x'}{r-\mu}+k_5\cdot(\gamma-\eta)(x')^{\gamma} &= 0.
\end{aligned}
$$

We can thus express k_5 and k_6 as functions of x':

$$
\begin{aligned}
k_6(x') &= \frac{c\gamma}{r(\gamma-\eta)}(x')^{-\eta}+\frac{1-\gamma}{(r-\mu)(\gamma-\eta)}(x')^{1-\eta} && \text{(A16)}\\
k_5(x') &= \frac{c\eta}{r(\eta-\gamma)}(x')^{-\gamma}+\frac{1-\eta}{(r-\mu)(\eta-\gamma)}(x')^{1-\gamma}. && \text{(A17)}
\end{aligned}
$$

We then construct the following function that takes z as a parameter,

$$\tilde{U}(x; z) = -\frac{c}{r} + \frac{x}{r-\mu} + k_5(z)x^\gamma + k_6(z)x^\eta. \tag{A18}$$

It suffices to find a value of z such that $\tilde{U}(x; z)$ tangentially intersects with $F(x)$ in the interval (x^*, ∞) — after doing so, Equations (A10) to (A13) are satisfied by letting $x' = z$ and x'' be the tangent point. Denote $\tilde{\Delta}(x; z) = \tilde{U}(x; z) - F(x)$.

Claim 7. *For any $z > 0$, $\tilde{\Delta}(x; z)$ is strictly convex in x.*

Proof. $\tilde{\Delta}''(x; z) = \gamma(\gamma-1)(k_5-k_3)x^{\gamma-2}+\eta(\eta-1)k_6x^{\eta-2} > 0$, as we already know from Claim 6 that $k_6 > 0$ and $k_5 > k_3$, together with $\gamma < 0$ and $\eta > 1$. □

On one hand, $\tilde{\Delta}(x; x^{**}) = V_c(x)$. Since $\lambda < \lambda^*$, we infer that $\inf_{x\in(x^*,\infty)} \tilde{\Delta}(x; x^{**}) < 0$ as $V_c(x)$ (non-tangentially) intersects with $F(x)$. On the other hand, for $\epsilon > 0$ sufficiently small, it is not difficult to see that $\inf_{x\in[x^*,\infty)} \tilde{\Delta}(x; \epsilon) > 0$. By continuity of $\tilde{\Delta}(x; z)$ w.r.t. z, there must exist $x' \in (0, x^{**})$ such that $\inf_{x\in[x^*,\infty)} \tilde{\Delta}(x; x') = 0$. According to the strict convexity of $\tilde{\Delta}(\cdot; x')$ (Claim 7), $\tilde{\Delta}(x^*; x') > 0$, and $\tilde{\Delta}(\infty; x') = \infty$, we know the infimum is uniquely attainable. Let this point of infimum be x''. We can verify that $\tilde{\Delta}(x''; x') = 0$, $\tilde{\Delta}'(x''; x') = 0$, and $\tilde{\Delta}(x; x') > 0$ for $x \in [x^*, \infty)/\{x''\}$. In other words, $\tilde{U}(x; x')$ smoothly pastes with $F(x)$ at x' and x'', while satisfying $\tilde{U}(x; x') > F(x)$ for $x \in (x', x'')$.

To conclude on the existence, we finally verify that the constructed x'' is consistent with the presumption that $x'' < x'''$. Combining (A14) and (A16), we get

$$\frac{c\gamma}{r}\left[(x'')^\eta - (x')^\eta\right] + \frac{(1-\gamma)}{r-\mu}\left[x'(x'')^\eta - (1-\alpha)x''(x')^\eta\right] = 0,$$

which gives us $x'' < x'/(1-\alpha) < x^{**}/(1-\alpha) = x'''$.

Step 2: Uniqueness. To prove the uniqueness of z satisfying $\inf_{x\in[x^*,\infty)} \tilde{\Delta}(x; z) = 0$, it suffices to show that $\inf_{x\in[x^*,\infty)} \tilde{\Delta}(x; z)$ is single-crossing w.r.t. z (i.e., crosses the horizontal axis only once) when $\inf_{x\in[x^*,\infty)} \tilde{\Delta}(x; z) = 0$. By the Envelope Theorem, we only need to show that $\frac{\partial\tilde{\Delta}}{\partial z}(x'', x')$ is either always positive or always negative.

$$
\begin{aligned}
\frac{\partial \tilde{\Delta}}{\partial z}(x'', x') \cdot x' &= k_5'(x')x'(x'')^\gamma + k_6'(x')x'(x'')^\eta \\
&= (1-\gamma)k_5 \cdot (x'')^\gamma + (1-\eta)k_6 \cdot (x'')^\eta + \frac{c}{r(\eta-\gamma)}\left[\gamma\left(\frac{x''}{x'}\right)^\eta - \eta\left(\frac{x''}{x'}\right)^\gamma\right] \\
&= \frac{c}{r} + k_3 \cdot (1-\gamma)(x'')^\gamma + \frac{c}{r(\eta-\gamma)}\left[\gamma\left(\frac{x''}{x'}\right)^\eta - \eta\left(\frac{x''}{x'}\right)^\gamma\right] \\
&= k_3 \cdot (1-\gamma)(x'')^\gamma + \frac{c}{r(\eta-\gamma)}\left[\eta - \gamma + \gamma\left(\frac{x''}{x'}\right)^\eta - \eta\left(\frac{x''}{x'}\right)^\gamma\right] \\
&< k_3 \cdot (1-\gamma)(x'')^\gamma + \frac{c}{r(\eta-\gamma)}[\eta - \gamma + \gamma - \eta] \\
&= k_3 \cdot (1-\gamma)(x'')^\gamma < 0.
\end{aligned}
$$

The first equality results from (A18). The second equality is obtained by plugging in the derivatives of $k_5(x)$ and $k_6(x)$ according to (A16) and (A17). The third equality makes use of (A12) and (A13). The fourth equality combines like terms. The first inequality holds because the function $\Phi(y) = \gamma y^\eta - \eta y^\gamma$, when $y \geq 1$, strictly decreases in y, while $x'' > x'$. The last inequality comes from $k_3 < 0$. We eventually conclude that the single-crossing condition holds, so there exists a unique pair of (x', x'') satisfying Equations (A10) to (A13).

A.14 Proof of Lemma 6

For $x''' < \tilde{x}$. By the construction of x''', we have $F(x''') > S(x''')$ — otherwise Frank will not benefit from strategic exiting at x'''. This directly implies $x''' < \tilde{x}$ because $F(x) \geq S(x), \forall x \geq \tilde{x}$.

For $x'' > x^*$. By the construction of x'', we have $F(x'') > 0 = F(x^*)$. The argument $x'' > x^*$ immediately follows.

A.15 Proof of Theorem 3

Step 1: $x''' < \tilde{x}$. This is because x''' is smaller than the largest intersection of $V_c(x)$ and $F(x)$ according to Theorem 2, while $\tilde{x}$ must be larger than that intersection since $V_c(\tilde{x}) > S(\tilde{x}) = F(\tilde{x})$ due to the fact that $V_c(x)$ maximizes social welfare. The intuition is that Frank exits less aggressively than in the baseline model due to pre-emption being avoided.

Step 2: $U_s(\tilde{x}) > W(\tilde{x})$. When $\lambda < \lambda^*$, the Pareto-optimal equilibrium in the baseline model is pre-emptive so $W(\tilde{x}) = V_p(\tilde{x}) = F(\tilde{x}) = S(\tilde{x})$. Hence, it suffices to show that $U_s(\tilde{x}) > S(\tilde{x})$. To compare $U_s(\tilde{x})$ and $S(\tilde{x})$, notice that: (1) $S(\tilde{x})$ is equivalent to the continuation value of a player (when $X_t = \tilde{x}$) who keeps receiving a flow payoff of $\beta X_t - \kappa c$ until exogenously exiting at x''' with a lump-sum payoff of $S(x''')$; (2) $U_s(\tilde{x})$ is equivalent to the continuation value of a player (when $X_t = \tilde{x}$) who keeps receiving a flow payoff of $X_t - c$ until exogenously exiting at x''' with a lump-sum payoff of $S(x''')$. These two scenarios have the same lump-sum payoff when exogenous exiting happens at x''', but the flow payoff in the first scenario is lower than the second one, as we assume $\beta \leq 1$. Hence, we conclude that $U_s(\tilde{x}) > S(\tilde{x})$.

Step 3: $U_s(x) > W(x)$ for $x > \tilde{x}$. For $x > \tilde{x}$, $W(x)$ equals the continuation value of a player (when $X_t = x$) who keeps receiving a flow payoff of $X_t - c$ until exogenously exiting at $\tilde{x}$ with a lump-sum payoff of $W(\tilde{x})$. Meanwhile, $U_s(x)$ equals the continuation value of a player (when $X_t = x$) who keeps receiving a flow payoff of $X_t - c$ until exogenously exiting at $\tilde{x}$ with a lump-sum payoff of $U_s(\tilde{x})$. These two arguments, together with $U_s(\tilde{x}) > W(\tilde{x})$ that we show in Step 2, indicate that $U_s(x) > W(x)$ for $x > \tilde{x}$.

Step 4: Existence of x^s. Steps 2 and 3 show that $U_s(x) > W(x)$ when $x \geq \tilde{x}$. According to the continuity of $U_s(x)$ and $W(x)$, there must exist $x^s < \tilde{x}$ such that $U_s(x) \geq W(x)$ when $x \geq x^s$.

References

Abreu, D., & Gul, F. (2000). Bargaining and reputation. *Econometrica*, *68*(1), 85–117.

Admati, A. R., & Perry, M. (1991). Joint projects without commitment. *Review of Economic Studies*, *58*(2), 259–276.

Akcigit, U., & Liu, Q. (2016). The role of information in innovation and competition. *Journal of the European Economic Association*, *14*(4), 828–870.

Awaya, Y., & Krishna, V. (2021). Startups and upstarts: Disadvantageous information in r&d. *Journal of Political Economy*, *129*(2), 000–000.

Bobtcheff, C., Bolte, J., & Mariotti, T. (2017). Researcher's dilemma. *The Review of Economic Studies*, *84*(3), 969–1014.

Bulow, J., & Klemperer, P. (1994). Rational frenzies and crashes. *Journal of political Economy*, *102*(1), 1–23.

Cetemen, D., Urgun, C., & Yariv, L. (2023). Collective progress: Dynamics of exit waves. *Journal of Political Economy*, *131*(9), 2402–2450.

Chamley, C., & Gale, D. (1994). Information revelation and strategic delay in a model of investment. *Econometrica*, 1065–1085.

Curello, G. (2023). Incentives for collective innovation. *arXiv:2109.01885*.

Dutta, P. K., & Rustichini, A. (1993). A theory of stopping time games with applications to product innovations and asset sales. *Economic Theory*, *3*(4), 743–763.

Farrell, J., & Maskin, E. (1989). Renegotiation in repeated games. *Games and Economic Behavior*, *1*(4), 327–360.

Fershtman, C., & Nitzan, S. (1991). Dynamic voluntary provision of public goods. *European Economic Review*, *35*(5), 1057–1067.

Fudenberg, D., & Tirole, J. (1985). Preemption and rent equalization in the adoption of new technology. *Review of Economic Studies*, *52*(3), 383–401.

Fujiwara-Greve, T., & Okuno-Fujiwara, M. (2009). Voluntarily separable repeated prisoner's dilemma. *Review of Economic Studies*, *76*(3), 993–1021.

Georgiadis, G. (2015). Projects and team dynamics. *Review of Economic Studies*, *82*(1), 187–218.

Ghosh, P., & Ray, D. (1996). Cooperation in community interaction without information flows. *Review of Economic Studies*, *63*(3), 491–519.

Grenadier, S. R. (1996). The strategic exercise of options: Development cascades and overbuilding in real estate markets. *Journal of Finance*, *51*(5), 1653–1679.

Guo, Y., & Roesler, A.-K. (2018). Private learning and exit decisions in collaboration. *Mimeo*.

Jovanovic, B., & MacDonald, G. M. (1994). The life cycle of a competitive industry. *Journal of Political Economy*, *102*(2), 322–347.

Kirpalani, R., & Madsen, E. (2023). Strategic investment evaluation. *Theoretical Economics*, *18*(3), 1141–1180.

Margaria, C. (2020). Learning and payoff externalities in an investment game. *Games and Economic Behavior*, *119*, 234–250.

Marx, L. M., & Matthews, S. A. (2000). Dynamic voluntary contribution to a public project. *Review of Economic Studies*, *67*(2), 327–358.

McAdams, D. (2011). Performance and turnover in a stochastic partnership. *American Economic Journal: Microeconomics*, *3*(4), 107–42.

Moscarini, G., & Squintani, F. (2010). Competitive experimentation with private information: The survivor's curse. *Journal of Economic Theory*, *145*(2), 639–660.

Murto, P. (2004). Exit in duopoly under uncertainty. *RAND Journal of Economics*, 111–127.

Murto, P., & Välimäki, J. (2011). Learning and information aggregation in an exit game. *Review of Economic Studies*, *78*(4), 1426–1461.

Murto, P., & Välimäki, J. (2013). Delay and information aggregation in stopping games with private information. *Journal of Economic Theory*, *148*(6), 2404–2435.

Niculescu, C., & Persson, L.-E. (2006). *Convex functions and their applications*. Springer.

Ramos, J., & Sadzik, T. (2023). Partnership with persistence. *Available at SSRN 3308865*.

Riedel, F., & Steg, J.-H. (2017). Subgame-perfect equilibria in stochastic timing games. *Journal of Mathematical Economics*, *72*, 36–50.

Rosenberg, D., Solan, E., & Vieille, N. (2007). Social learning in one-arm bandit problems. *Econometrica*, *75*(6), 1591–1611.

Safronov, M., & Strulovici, B. (2018). Contestable norms. *Mimeo*.

Strulovici, B., & Szydlowski, M. (2015). On smoothness of value functions and existence of optimal strategies in diffusion models. *Journal of Economic Theory*, *159*, 1016–1055.

Thomas, C. (2021). Strategic experimentation with congestion. *American Economic Journal: Microeconomics*, *13*(1), 1–82.

Villeneuve, S. (2007). On threshold strategies and the smooth-fit principle for optimal stopping problems. *Journal of applied probability*, *44*(1), 181–198.

Weeds, H. (2002). Strategic delay in a real options model of r&d competition. *Review of Economic Studies*, *69*(3), 729–747.

Xu, B. (2023). Partnership stability and the benefit of imperfect monitoring. *Available at SSRN 4208031*.

Xu, B. (2025). Strategic exits in stochastic partnerships: The curse of profitability. *Available at SSRN 3847105*.

Zhou, J., & Chen, Y.-J. (2015). Key leaders in social networks. *Journal of Economic Theory*, *157*, 212–235.