Equilibrium Contracts and Boundedly Rational Expectations: Supplementary Appendix

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A A Brief Introduction to d-separation

We briefly introduce the concept of d-separation, a result from the Bayesian network literature that allows us to check, for any given model \( \mathcal{R} \), whether two variables (or two sets of variables) are independent when conditioning on a third variable (or set of variables). For simple models \( \mathcal{R} \) it can be used as visual inspection tool; for complex models, there exists an algorithm for checking d-separation (Geiger et al. 1990). Define a path \( \tau \) in \( \mathcal{R} \) as a sequence of nodes so that any adjacent nodes are linked in \( \mathcal{R} \); \( \tau \) is a directed path if the links between any two adjacent nodes in \( \tau \) point in the same direction (from the former to the latter or vice versa). A node \( j \) is a descendant of node \( i \) if there exists a directed path from \( i \) to \( j \). For convenience, we use the notation \( i \rightarrow j \) instead of \( iRj \) in this section. The following definitions and result are adopted from Pearl (2009).

**Definition 8.** A path \( \tau \) is blocked in \( \mathcal{R} = (R, N) \) by a set of variables \( M \subseteq N \) if and only if one of the following condition holds:

(a) \( \tau \) contains variables \( i, m, j \) with \( m \in M \) so that \( i \rightarrow m \rightarrow j \) or \( i \leftarrow m \rightarrow j \), or

(b) \( \tau \) contains variables \( i, m, j \) so that \( i \rightarrow m \leftarrow j \), \( m \notin M \), and no descendant of \( m \) is in \( M \).

**Figure A1:** Objective model \( \mathcal{R}^* \) from Figure 1 (left) and objective model \( \mathcal{R}^* \) from Figure 3 (right).

To illustrate, consider the DAG \( \mathcal{R}^* \) from Figure 1 in the paper, reproduced here on the left of Figure A1. The path \( \tau = 0 \rightarrow 2 \leftarrow 1 \rightarrow 3 \rightarrow 4 \) between the nodes 0 and 4 is blocked by node 1 and node 3, but not by node 2. To see this, note that conditions (a) and (b) are both satisfied if we define \( M = \{1\} \), or \( M = \{3\} \); however, none of the conditions is satisfied if we define \( M = \{2\} \).

**Definition 9.** Let \( \mathcal{R} = (R, N) \) be a DAG and \( M', M'' \), \( M \) disjoint subsets of \( N \). \( M' \) and \( M'' \) are d-separated by \( M \) in \( \mathcal{R} \), if \( M \) blocks every path between any node in \( M' \) and any node in \( M'' \).
Consider the DAG $R^*$ from Figure 3 in the paper, reproduced here on the right of Figure A1. We check whether the nodes 0 and 4 are $d$-separated in $R^*$ by $M = \{2\}$. For this, we have to consider three paths, $\tau = 0 \to 2 \to 4$, $\tau' = 0 \to 2 \leftarrow 1 \to 3 \to 4$, and $\tau'' = 0 \to 2 \to 3 \to 4$. By condition (a) in Definition 8, the paths $\tau$ and $\tau''$ are blocked by $M = \{2\}$. In contrast, the path $\tau'$ is not blocked by $M = \{2\}$. Hence, the nodes 0 and 4 are not $d$-separated in $R^*$ by $M = \{2\}$. However, they are $d$-separated in $R^*$ by $M = \{1, 2\}$, $M = \{2, 3\}$, or $M = \{1, 2, 3\}$. Suppose, for example, that $M = \{1, 2\}$. Now not only the paths $\tau$ and $\tau''$ are blocked according to condition (a) in Definition 1, but also path $\tau'$ (we see this from the segment $2 \leftarrow 1 \to 3$). The implication of $d$-separation is given in the following result.

**Proposition 10** (Implications of $d$-separation). If the variables 0 and $n$ are $d$-separated by variable $i$ in $R$, then $p_R(x_n \mid x_0, x_i; q) = p_R(x_n \mid x_i; q)$ for all $q \in \Delta(A)$ and all triples $x_0, x_i, x_n$. If the variables 0 and $n$ are not $d$-separated by variable $i$ in $R$, then $x_0$ and $x_n$ are dependent conditional on $x_i$ for at least one distribution compatible with $R$. 
References
