Online Appendix to "Collusion Enforcement in Repeated First-Price Auctions"

In this online appendix, we provide the proofs for Claims 2-4, which are needed to complete the proof of Theorem 3. We first present two properties of the revenue function S in Claims 5 and 6 below.

Claim 5. *On the interval* $\left[1-\sqrt{3}d, 1-d\right]$*, the revenue function*

$$S(r) = \frac{1}{3}(1-r)^3 - \frac{3d^2(1-r) - d^2\sqrt{12d^2 - 3(1-r)^2}}{2} + r(1-r^2)$$

has the following derivatives

$$\begin{aligned}
S'(r) &= 2r(1-2r) + \frac{3d^2}{2} + \frac{3d^2(1-r)}{2\sqrt{12d^2-3(1-r)^2}} && (1)\\
S''(r) &= 2 - 8r - \frac{18d^4}{\left(12d^2 - 3(1-r)^2\right)^{3/2}} && (2)\\
S'''(r) &= -8 + \frac{162(1-r)d^4}{\left(12d^2 - 3(1-r)^2\right)^{5/2}} && (3)\\
S^{(4)}(r) &= -\frac{1944d^4(d^2 + (1-r)^2)}{\left(12d^2 - 3(1-r)^2\right)^{7/2}}. && (4)
\end{aligned}$$

Claim 6. *On the interval* $\left[1-\sqrt{3}d, 1-d\right]$*, the first-order derivative* S' *is strictly decreasing.*

We prove Claims 2-6 in Sections A-E, respectively.

A Proof of Claim 2

We proceed by showing that S is increasing if $r < \frac{1}{2}$ and decreasing if $r \geq \frac{1}{2}$. Hence, S is maximized at $r = \frac{1}{2}$.

Differentiating S, we have

$$S'(r) = \begin{cases} 2r(1-2r), & \text{if } r \geq 1-d \\ 2r(1-2r) + \frac{3d^2}{2} + \frac{3d^2(1-r)}{2\sqrt{12d^2-3(1-r)^2}}, & \text{if } 1-\sqrt{3}d < r < 1-d \\ 1-3r^2, & \text{if } r \leq 1-\sqrt{3}d \end{cases}. \tag{5}$$

Case 1. Suppose that $r < \frac{1}{2}$. We use (5) to verify that $S'(r) > 0$. Since $2r(1-2r) > 0$, we have $S'(r) > 0$ for the first case of (5). Similarly, for the second case, we have

$$\begin{aligned} S'(r) =& 2r(1-2r) + \frac{3d^2}{2} + \frac{3d^2(1-r)}{2\sqrt{12d^2-3(1-r)^2}} \\ \geq& 2r(1-2r) \\ >& 0. \end{aligned}$$

For the last case of S' in (5), we have

$$\begin{aligned} S'(r) =& 1-3r^2 \\ \geq& 1-3\left(\frac{1}{2}\right)^2 \\ =& \frac{1}{4}. \end{aligned}$$

This shows that $S'(r) > 0$. Hence, S is increasing if $r < \frac{1}{2}$.

Case 2. Suppose that $r \geq \frac{1}{2}$. By assumption, $d \geq \frac{1}{2}$. This implies that $1-d \leq \frac{1}{2}$, and, therefore, $r \geq 1-d$. Hence, by (5), we have $S'(r) = 2r(1-2r)$. Since $r \geq \frac{1}{2}$, we have $S'(r) \leq 0$. This shows that S is decreasing if $r \geq \frac{1}{2}$.

B Proof of Claim 3

B.1 Case (i): $r \leq 1 - \sqrt{3}d$.

By (5), $S'(r) = 1 - 3r^2$. Hence, S is strictly increasing if $r \leq \frac{\sqrt{3}}{3}$ and strictly decreasing if $r \geq \frac{\sqrt{3}}{3}$.

If $1 - \sqrt{3}d \geq \frac{\sqrt{3}}{3}$, or $d \leq \frac{\sqrt{3}-1}{3}$, then S is strictly increasing on $\left[0, \frac{\sqrt{3}}{3}\right]$ and strictly decreasing on $\left(\frac{\sqrt{3}}{3}, 1 - \sqrt{3}d\right]$. Hence, S is uniquely maximized at $r = \frac{\sqrt{3}}{3}$.

If $1 - \sqrt{3}d \leq \frac{\sqrt{3}}{3}$, or $d \geq \frac{\sqrt{3}-1}{3}$, then S is strictly increasing on $\left[0, 1 - \sqrt{3}d\right]$. Hence, S is uniquely maximized at $r = 1 - \sqrt{3}d$.

B.2 Case (ii): $1 - \sqrt{3}d \leq r \leq 1 - d$.

We first evaluate S' at the two end points of the interval $[1 - \sqrt{3}d, 1 - d]$. By (1),

$$\begin{aligned}
S'\left(1 - \sqrt{3}d\right) =& 2\left(1 - \sqrt{3}d\right)\left(1 - 2\left(1 - \sqrt{3}d\right)\right) \\
&+ \frac{3d^2}{2} + \frac{3d^2\left(1 - \left(1 - \sqrt{3}d\right)\right)}{2\sqrt{12d^2 - 3\left(1 - \left(1 - \sqrt{3}d\right)\right)^2}} \\
=& 2\left(1 - \sqrt{3}d\right)\left(-1 + 2\sqrt{3}d\right) + \frac{3d^2}{2} + \frac{3d^2 \cdot \sqrt{3}d}{2\sqrt{12d^2 - 3 \cdot 3d^2}} \\
=& -2 + 6\sqrt{3}d - 9d^2.
\end{aligned}$$

Hence,

$$S'\left(1 - \sqrt{3}d\right) > 0 \iff d > \frac{\sqrt{3} - 1}{3} \approx 0.24, \tag{6}$$

where we use the assumption that $d < \frac{1}{2}$. Similarly,

$$\begin{aligned}
S'(1 - d) =& 2(1 - d)(1 - 2(1 - d)) + \frac{3d^2}{2} + \frac{3d^2(1 - (1 - d))}{2\sqrt{12d^2 - 3(1 - (1 - d))^2}} \\
=& 2(1 - d)(-1 + 2d) + \frac{3d^2}{2} + \frac{3d^2 \cdot d}{2\sqrt{12d^2 - 3 \cdot d^2}} \\
=& -2 + 6d - 2d^2.
\end{aligned}$$

Hence,

$$S'\left(1-d\right)>0 \iff d>\frac{3-\sqrt{5}}{2}\approx 0.38. \tag{7}$$

By Claim 6, S' is strictly decreasing on $[1-\sqrt{3}d, 1-d]$. We now use this fact and (6) and (7) to prove Case (ii) of Claim 3.

If $\frac{3-\sqrt{5}}{2} \leq d < \frac{1}{2}$, then by (7) we have $S'\left(1-d\right) > 0$. Since S' is decreasing, we have $S'(r) > 0$ for all $r \in \left[1-\sqrt{3}d, 1-d\right]$. Thus, S is strictly increasing and is uniquely maximized at $r = 1-d$.

If $\frac{\sqrt{3}-1}{3} < d < \frac{3-\sqrt{5}}{2}$, then by (6) and (7) we have $S'\left(1-\sqrt{3}d\right) > 0$ and $S'\left(1-d\right) < 0$, respectively. Since $S'' < 0$, there exists a unique solution $r_0 \in \left(1-\sqrt{3}d, 1-d\right)$ to $S'(r) = 0$. Moreover, $S'(r) > 0$ for all $r < r_0$ and $S'(r) < 0$ for all $r > r_0$. This implies that S is strictly increasing on $\left[1-\sqrt{3}d, r_0\right]$ and strictly decreasing on $[r_0, 1-d]$. Hence, S is uniquely maximized at r_0.

If $d \leq \frac{\sqrt{3}-1}{3}$, then by (6) we have $S'\left(1-\sqrt{3}d\right) < 0$. Since S' is decreasing, we have $S'(r) < 0$ for all $r \in \left[1-\sqrt{3}d, 1-d\right]$. That is, S is strictly decreasing and is uniquely maximized at $r = 1-\sqrt{3}d$.

This completes the proof of Case (ii).

B.3 Case (iii): $r \geq 1-d$.

By (5), $S'(r) = 2r(1-2r)$. Since $1-d \geq \frac{1}{2}$, we have $S'(r) = 2r(1-2r) < 0$ for all $r \geq 1-d$. Hence, S is strictly decreasing and is uniquely maximized at $r = 1-d$.

C Proof of Claim 5

On the interval $\left[1-\sqrt{3}d, 1-d\right]$, we have

$$S(r) = \frac{1}{3}(1-r)^3 - \frac{3d^2(1-r) - d^2\sqrt{12d^2 - 3(1-r)^2}}{2} + r(1-r^2).$$

Differentiating S, we have

$$
\begin{aligned}
S'(r) =& -(1-r)^2 - \frac{-3d^2 - d^2 \cdot \frac{6(1-r)}{2\sqrt{12d^2-3(1-r)^2}}}{2} + 1 - 3r^2 \\
=& 2r(1-2r) + \frac{3d^2 + \frac{3d^2(1-r)}{\sqrt{12d^2-3(1-r)^2}}}{2} \\
=& 2r(1-2r) + \frac{3d^2}{2} + \frac{3d^2(1-r)}{2\sqrt{12d^2-3(1-r)^2}}.
\end{aligned}
$$

Further differentiating S', we have

$$
\begin{aligned}
S''(r) =& 2 - 8r + \frac{3d^2}{2} \frac{-\sqrt{12d^2-3(1-r)^2} - (1-r)\frac{6(1-r)}{2\sqrt{12d^2-3(1-r)^2}}}{12d^2-3(1-r)^2} \\
=& 2 - 8r + \frac{3d^2}{2} \frac{-(12d^2-3(1-r)^2) - 3(1-r)^2}{(12d^2-3(1-r)^2)^{3/2}} \\
=& 2 - 8r - \frac{18d^4}{(12d^2-3(1-r)^2)^{3/2}},
\end{aligned}
$$

$$
\begin{aligned}
S'''(r) =& -8 + 18d^4 \cdot \frac{3}{2} \cdot \frac{6(1-r)}{(12d^2-3(1-r)^2)^{5/2}} \\
=& -8 + \frac{162(1-r)d^4}{(12d^2-3(1-r)^2)^{5/2}},
\end{aligned}
$$

and

$$
\begin{aligned}
S^{(4)}(r) =& 162d^4 \cdot \frac{-(12d^2-3(1-r)^2)^{5/2} - (1-r) \cdot \frac{5}{2} \cdot (12d^2-3(1-r)^2)^{3/2} \cdot 6(1-r)}{(12d^2-3(1-r)^2)^5} \\
=& 162d^4 \cdot \frac{-(12d^2-3(1-r)^2)^{5/2} - 15 \cdot (1-r)^2 \cdot (12d^2-3(1-r)^2)^{3/2}}{(12d^2-3(1-r)^2)^5} \\
=& 162d^4 \cdot \frac{-(12d^2-3(1-r)^2) - 15(1-r)^2}{(12d^2-3(1-r)^2)^{7/2}} \\
=& 162d^4 \cdot \frac{-12\left(d^2+(1-r)^2\right)}{(12d^2-3(1-r)^2)^{7/2}} \\
=& -\frac{1944d^4(d^2+(1-r)^2)}{(12d^2-3(1-r)^2)^{7/2}}.
\end{aligned}
$$

D Proof of Claim 6

We show that $S'' < 0$ on $[1-\sqrt{3}d, 1-d]$. We proceed in two steps. We break the interval $[1-\sqrt{3}d, 1-d]$ into two subintervals,

$$[1-\sqrt{3}d, 1-\sqrt{5/2}d] \qquad \text{and} \qquad [1-\sqrt{5/2}d, 1-d].$$

In Step 1, we show that S'' is negative on the first subinterval $[1-\sqrt{3}d, 1-\sqrt{5/2}d]$. In Step 2, we show that S'' is decreasing on the second subinterval $[1-\sqrt{5/2}d, 1-d]$. This implies that S'' is negative on $[1-\sqrt{3}d, 1-d]$.

D.1 Step 1

Using the expression for S'' in Claim 5, we show that an upper bound for S'' is strictly negative. By (2), for all r in the first subinterval $[1-\sqrt{3}d, 1-\sqrt{5/2}d]$, we have

$$\begin{aligned}
S''(r) =& 2 - 8r - \frac{18d^4}{\left(12d^2 - 3(1-r)^2\right)^{3/2}} \\
\leq& 2 - 8\left(1-\sqrt{3}d\right) - \frac{18d^4}{\left(12d^2 - 3\left(1-\left(1-\sqrt{5/2}d\right)\right)^2\right)^{3/2}} \\
=& 2 - 8\left(1-\sqrt{3}d\right) - \frac{18d^4}{\left(12d^2 - 3\cdot\frac{5}{2}d^2\right)^{3/2}} \\
=& 2 - 8\left(1-\sqrt{3}d\right) - \frac{18d^4}{\frac{27}{2\sqrt{2}}d^3} \\
=& -6 + 8\sqrt{3}d - \frac{4\sqrt{2}}{3}d \\
<& -6 + 4\sqrt{3} - \frac{2\sqrt{2}}{3} \approx -0.015 \\
<& 0,
\end{aligned} \tag{8}$$

where, since the second term on the right-hand side of the first line, $-8r$, is decreasing in r and the last term

$$-\frac{18d^4}{\left(12d^2 - 3(1-r)^2\right)^{3/2}}$$

is increasing in r, the second line follows from applying $r \geq 1-\sqrt{3}d$ and $r \leq 1-\sqrt{5/2}d$ to the second and the last terms, respectively, the next three lines follow from simplifying terms, and the seventh from $d < \frac{1}{2}$. This shows that $S''(r) < 0$ for all $r \in [1-\sqrt{3}d, 1-\sqrt{5/2}d]$.

D.2 Step 2

By (4), $S^{(4)}(r) < 0$ for all $r \in [1-\sqrt{3}d, 1-d]$. Hence, S''' is decreasing. By (3),

$$\begin{aligned} S'''(1-\sqrt{5/2}d) &= -8 + \frac{162(1-(1-\sqrt{5/2}d))d^4}{\left(12d^2 - 3(1-(1-\sqrt{5/2}d))^2\right)^{5/2}} \\ &= -8 + \frac{162 \cdot \sqrt{\frac{5}{2}}d \cdot d^4}{\left(12d^2 - 3 \cdot \frac{5}{2} \cdot d^2\right)^{5/2}} \\ &= -\frac{8}{3}(3-\sqrt{5}) < 0. \end{aligned}$$

Hence, $S'''(r) < 0$ for all $r \in [1-\sqrt{5/2}d, 1-d]$. This implies that S'' is decreasing on the interval $[1-\sqrt{5/2}d, 1-d]$.

E Proof of Claim 4

Total differentiating (29) in the main text, the equation that defines r^*, we have

$$\begin{aligned} 0 = &\left(2 - 8r - \frac{18\left(\frac{1-\delta}{\delta}\right)^2}{\left(12\frac{1-\delta}{\delta} - 3(1-r)^2\right)^{3/2}}\right) \mathrm{d}r \\ &+ \frac{3}{2}\left(1 + (1-r)\frac{6\frac{1-\delta}{\delta} - 3(1-r)^2}{(12\frac{1-\delta}{\delta} - 3(1-r)^2)^{3/2}}\right)\left(-\frac{1}{\delta^2}\right) \mathrm{d}\delta, \end{aligned}$$

where in the first line we use (2) for $S''(r)$. Rearranging terms yields

$$\frac{\mathrm{d}r}{\mathrm{d}\delta} = \frac{\frac{3}{2\delta^2}\left(1 + (1-r)\frac{6\frac{1-\delta}{\delta} - 3(1-r)^2}{(12\frac{1-\delta}{\delta} - 3(1-r)^2)^{3/2}}\right)}{2 - 8r - \frac{18\left(\frac{1-\delta}{\delta}\right)^2}{\left(12\frac{1-\delta}{\delta} - 3(1-r)^2\right)^{3/2}}}. \tag{9}$$

By Claim 6, the denominator on the right-hand side of (9) is negative. Hence, it suffices to show that the numerator is positive. Since $r > 1 - \sqrt{\frac{3(1-\delta)}{\delta}}$, by rearranging terms, we have

$$3\frac{1-\delta}{\delta} > (1-r)^2. \tag{10}$$

Hence, the bracketed term of the numerator becomes

$$\begin{aligned}
&1 + (1-r)\frac{6\frac{1-\delta}{\delta} - 3(1-r)^2}{(12\frac{1-\delta}{\delta} - 3(1-r)^2)^{3/2}} \\
=&1 + (1-r)\frac{-(1-r)^2}{(12\frac{1-\delta}{\delta} - 3(1-r)^2)^{3/2}} + (1-r)\frac{6\frac{1-\delta}{\delta} - 2(1-r)^2}{(12\frac{1-\delta}{\delta} - 3(1-r)^2)^{3/2}} \\
>&1 - \frac{(1-r)^3}{(12\frac{1-\delta}{\delta} - 3(1-r)^2)^{3/2}} \\
\geq&1 - \frac{(1-r)^3}{(4(1-r)^2 - 3(1-r)^2)^{3/2}} \\
\geq&0,
\end{aligned}$$

where the second line follows from rearranging terms, the third from applying (10) to conclude that the second term on the second line is greater than 0, the fourth from applying (10) to substitute $3(1-r)^2$ for $12\frac{1-\delta}{\delta}$ in the denominator on the third line, and the last from simplifying terms. This shows that the numerator of (9) is positive. Hence, $\frac{\mathrm{d}r}{\mathrm{d}\delta} < 0$.