Supplemental appendices for 'Slow persuasion'

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H Solving for the value function

In this appendix, we explain how to solve for the sender's value function using Proposition 1. We detail in particular how the value is computed in the two-action example, allowing us to draw Figure 1.

Partition $C \setminus D$ into maximal intervals $(R_k)_{k=1}^K$. (In the two-action example, the maximal intervals are $[0, 2/3)$ and $(2/3, 1]$.) Fix a continuity interval R_k. The homogeneous part (without the u) of the differential equation (∂) has general solution $AH_1 - BH_2$ for constants $A, B \in \mathbf{R}$, where

$$H_1(p) := p^\xi (1-p)^{1-\xi}, \xi := 1/2 + \sqrt{1/4 + 2r\sigma^2/\lambda} \quad \text{and} \quad H_2(p) := H_1(1-p).$$

A particular solution may be obtained from formula (6.2) in Coddington (1961, ch. 3). Things are easier when the sender has expected-utility preferences, so that $f(a, \cdot)$ is affine, as u itself is then a particular solution. This is the case in the two-action example, and in the three-action example below. In the expected-utility case, the value function is given on each maximal interval R_k of $C \setminus D$ as

$$v(p) = u(p) + A_{R_k} H_1(p) - B_{R_k} H_2(p) \quad \text{for all } p \in R_k,$$

where the constants $(A_{R_k}, B_{R_k})_{k=1}^K$ are the unique ones that ensure that the properties in Proposition 1 are satisfied: the boundary condition $v = u$ on $\{0, 1\}$, the continuity of v on D, and smooth pasting on $C \cup D$.

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H.1 The two-action example (§4.1)

Here $D = \{2/3\}$, and C contains $[0, 2/3)$ and may or may not contain $[2/3, 1]$. In either case,

$$v(p) = \begin{cases} A_{[0,2/3)}H_1(p) - B_{[0,2/3)}H_2(p) & \text{for } p \in [0, 1/2) \\ \alpha p - \beta + A_{(2/3,1]}H_1(p) - B_{(2/3,1]}H_2(p) & \text{for } p \in (2/3, 1], \end{cases}$$

where $\alpha = 3/2$ and $\beta = 1/2$.[1] The boundary conditions require that $B_{[0,2/3)} = 0$ and $A_{(2/3,1]} = 0$. Continuity of v at $2/3$ requires that

$$A_{[0,2/3)}H_1(2/3) = \alpha(2/3) - \beta - B_{(2/3,1]}H_2(2/3).$$

If $\bar{\lambda}$ is sufficiently high, then $2/3 \in C$, in which case smooth pasting must hold at $2/3$:

$$A_{[0,2/3)}H_1'(2/3) = \alpha - B_{(2/3,1]}H_2'(2/3).$$

Thus the constants are uniquely pinned down.

If $\bar{\lambda}$ is low, then $2/3 \notin C$, in which case $v = u$ on $[2/3, 1]$. Thus $B_{(2/3,1]} = 0$, whence $A_{[0,2/3)}$ is pinned down by the continuity condition.

To determine which case applies for a given value of $\bar{\lambda}$, calculate $A_{[0,2/3)}$ assuming that the first case applies. If

$$A_{[0,2/3)}H_1(2/3) \geq u(1/2) = 1/2,$$

then the first case does indeed apply; if not, then not.

H.2 A three-action example

Consider the flow payoff u depicted in Figure 2. The underlying model has actions $\mathcal{A} = \{0, 1, 3\}$, flow payoff $f_S(a, p) = a$ for the sender, and payoffs $f_D(0, p) = 0$, $f_D(1, p) = 2p - 1$ and $f_D(3, p) = \frac{14}{3}p - 3$ for the decision-maker. Figure 2 depicts the concave envelope, as well as the sender's value function for high and low values of $\bar{\lambda}$.

Clearly C contains $[0, 1/2)$ and $(1/2, 3/4)$, and does not contain $[3/4, 1]$. Thus the value function off D is

$$v(p) = \begin{cases} A_{[0,1/2)}H_1(p) - B_{[0,1/2)}H_2(p) & \text{for } p \in [0, 1/2) \\ \ell + A_{(1/2,3/4)}H_1(p) - B_{(1/2,3/4)}H_2(p) & \text{for } p \in (1/2, 3/4) \\ h, \end{cases}$$

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[1]If C contains $[2/3, 1]$ then the expression for $p \in (2/3, 1]$ holds since (HJB) must be satisfied in the classical sense by Proposition 1. If not, then Proposition 1 requires that $v = u$, which amounts to setting $A_{(2/3,1]} = B_{(2/3,1]} = 0$.
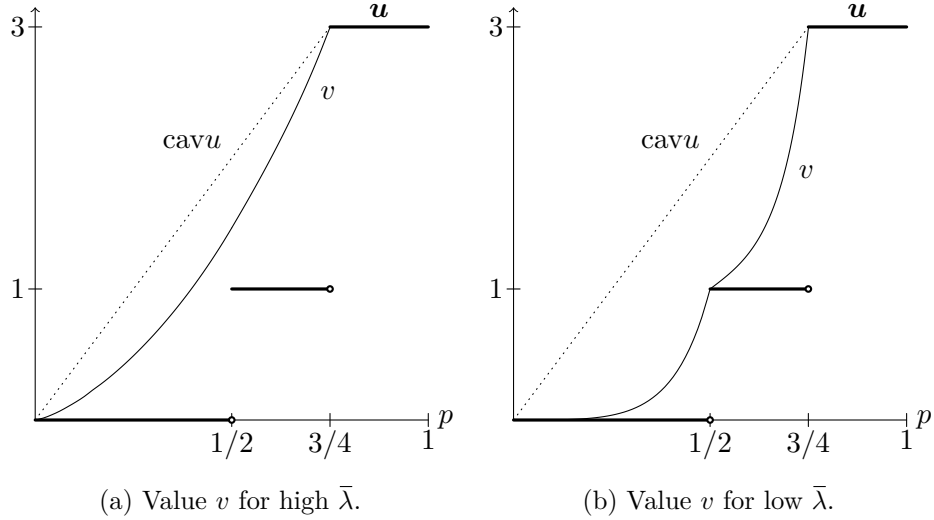
(a) Value v for high $\bar{\lambda}$.   (b) Value v for low $\bar{\lambda}$.

Figure 2: Three-action example.

where $\ell = 1$ and $h = 3$. The boundary condition at $p = 0$ again requires that $B_{[0,1/2)} = 0$. Continuity of v at $1/2$ and at $3/4$ requires that

$$A_{[0,1/2)}H_1(1/2) = \ell + A_{(1/2,3/4)}H_1(1/2) - B_{(1/2,3/4)}H_2(1/2)$$
$$\text{and} \quad \ell + A_{(1/2,3/4)}H_1(3/4) - B_{(1/2,3/4)}H_2(3/4) = h.$$

These are two equations in three unknowns.

If $\bar{\lambda}$ is sufficiently high that $1/2 \in C$, then smooth pasting must hold at $1/2$, giving us the third equation

$$A_{[0,1/2)}H_1'(1/2) = A_{(1/2,3/4)}H_1'(1/2) - B_{(1/2,3/4)}H_2'(1/2).$$

If $\bar{\lambda}$ is sufficiently low that $1/2 \notin C$, then $v(1/2) = u(1/2) = \ell$. We thus obtain a third equation from the requirement that v be continuous at $1/2$:

$$A_{[0,1/2)}H_1(1/2) = \ell.$$

To discern which case applies, compute $A_{[0,1/2)}$ assuming that the first (information arrives fast) case applies. If $A_{[0,1/2)}H_1(1/2) \geq \ell$, then the fast-arrival case does indeed apply; if not, then not.

I   Generic uniqueness of long-run beliefs

We claimed in §5.1 that provided $v(p_0) > u(p_0)$, generically, all best replies of the sender induce the same long-run beliefs (viz. the beliefs $\{p^-, p^+\}$ defined

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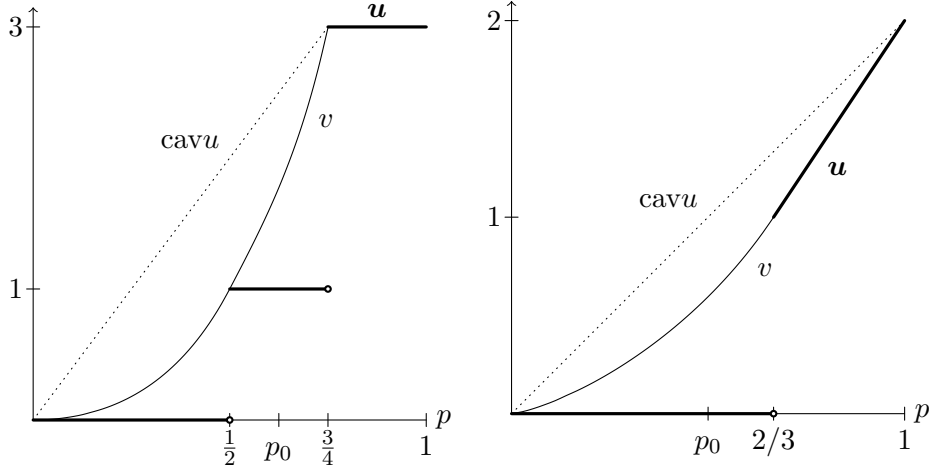
(a) Three-action example from Figure 2.　(b) Two-action example from §4.1.

Figure 3: Knife-edge cases in which long-run beliefs are not unique.

in Corollary 3).

To see how uniqueness can fail, consider the three-action example in supplemental appendix H.2. Figure 3a depicts the knife-edge case in which $\bar{\lambda}$ is such that the fast-information value function with the convex-flat shape in Figure 2a touches u at $1/2$.[2] In this case, the sender is indifferent between providing and not providing information at $1/2$, and strictly prefers to do so on $(0, 1/2)$ and $(0, 3/4)$. The best reply $\Lambda^\star$ from Corollary 2 stops at $1/2$, inducing the long-run beliefs $\{p^-, p^+\} = \{1/2, 3/4\}$ from Corollary 3. But since the sender is indifferent at $1/2$, she also has a best reply that provides information at $1/2$, which induces long-run beliefs $\{0, 3/4\}$.

This scenario is non-generic in the sense that slightly increasing $\bar{\lambda}$ puts us back in Figure 2a, where the sender strictly prefers to provide information at full tilt at $1/2$, whereas slightly decreasing $\bar{\lambda}$ puts us in Figure 2b, where she strictly prefers to stop at $1/2$.

Similarly, Figure 3b depicts the case in §4.1 in which $\bar{\lambda}$ has exactly the value needed for the fast-information value function with the convex shape in Figure 1a to just touch u at $2/3$. In this example, there is more multiplicity: the sender is indifferent on $[1/2, 1]$, so has best replies that induce any mean-p_0 distribution of long-run beliefs supported on $\{0\} \cup [2/3, 1]$. (The best reply $\Lambda^\star$ induces the beliefs $\{0, 2/3\}$.) Again, perturbing $\bar{\lambda}$ makes the sender's

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[2]We thank Jeff Ely for pointing out this scenario.

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preference strict at $2/3$, so that long-run induced beliefs are unique (either $\{0, 2/3\}$ or $\{0, 1\}$).

The non-genericity of multiplicity in these examples is a general phenomenon. Multiplicity occurs for some prior p_0 with $v(p_0) > u(p_0)$ precisely if the sender is indifferent between stopping and continuing at some $p \in (0,1)$ and weakly prefers to continue on a neighbourhood of p. In such cases, her preference becomes strict when $\overline{\lambda}$ is perturbed slightly.

J   Piecewise continuity is merely tie-breaking

We asserted in §2.3 that provided the decision-maker's flow payoff f_D is non-degenerate in a mild sense, it is without loss of optimality for her to restrict attention to piecewise continuous Markov strategies $A : [0,1] \to \Delta(\mathcal{A})$.

To justify this claim, begin by recalling from §3 that the decision-maker best-replies to a Markov strategy of the sender by myopically maximising $f_D(a, p)$ at each p. Fix two actions $a, a' \in \mathcal{A}$, and write

$$\psi(p) := f_D(a, p) - f_D(a', p)$$

for their payoff difference. Say that ψ *strictly up-crosses* at $p \in (0,1)$ iff $\psi(p) = 0$ and for any $\varepsilon > 0$, there are $p' \in (p - \varepsilon, p)$ and $p'' \in (p, p + \varepsilon)$ such that $\psi(p') < 0 < \psi(p'')$, *strictly down-crosses* if the reverse inequalities hold, and simply *strictly crosses* if either is the case. Write $K \subseteq (0,1)$ for the set on which ψ strictly crosses. We claim that given some weak non-degeneracy condition on f_D, the crossing set K is discrete, so that the decision-maker strictly prefers to switch actions only on a discrete set. (It suffices to consider only two arbitrary actions $a, a' \in \mathcal{A}$ because $\mathcal{A}$ is finite.)

To see what can go wrong, suppose that $f_D(a, p) = 0$ and that $p \mapsto f_D(a', p)$ is a typical path of a standard Brownian motion. Then ψ is continuous, but the strict crossing set K is non-empty with no isolated points (see e.g. Theorem 9.6 in Karatzas and Shreve (1991, ch. 2)). This preference dithers maniacally, wishing to switch actions back and forth extremely frequently.

As a first pass, observe that if ψ is monotone, or more generally if ψ or $-\psi$ has the single-crossing property ($\psi(p) \geq (>) 0$ implies $\psi(p') \geq (>) 0$ for $p < p'$), then K is empty or a singleton, so certainly discrete. These assumptions are satisfied by expected-utility preferences.

A weak non-degeneracy condition that suffices is *local single-crossing:* for each $p \in K$, we have either $\psi \geq 0$ or $\psi \leq 0$ on a left-neighbourhood of p, and similarly on a right-neighbourhood. Then each $p \in K$ is manifestly the unique strict crossing of ψ on a neighbourhood, hence isolated. A sufficient condition

for this is *local monotonicity:* for each $p \in K$, we have $\psi(p-\varepsilon) \leq 0 \leq \psi(p+\varepsilon)$ for all sufficiently small $\varepsilon > 0$, or the reverse inequality.

K A very brief introduction to viscosity solutions

Crandall (1997), Katzourakis (2015) and Crandall, Ishii and Lions (1992) provide overviews of the theory of viscosity solutions of second-order differential equations. Moll (2017), Evans (2010, ch. 10), Calder (2018) and Bressan (2011) give easier treatments that deal mostly with first-order equations.

The general idea of viscosity solutions is as follows. If w is a viscosity solution of (HJB), then it must satisfy (HJB) in the classical sense on any neighbourhood on which w'' exists and is continuous. If w'' does not exist at $p \in [0, 1]$, we require instead that (HJB) hold with the appropriate inequality when $w''(p)$ is replaced by $\phi''(p)$ for some twice continuously differentiable local approximation ϕ to w at p. (The formal definition was given in §4.2.)

K.1 Illustration of the definition

Consider the three-action example from supplemental appendix H.2 (Figure 2). Write $\mathcal{C}^2$ for the set of twice continuously differentiable functions $(0, 1) \to \mathbf{R}$. Begin by observing that v is continuous, hence upper and lower semi-continuous.

Consider a p in whose vicinity v is twice continuously differentiable, e.g. $p = 2/5$. We may easily find $\phi_1, \phi_2 \in \mathcal{C}^2$ such that $\phi_1 - v$ and $v - \phi_2$ are locally minimised at p, as in Figure 4a. But in particular, we may choose $\phi \in \mathcal{C}^2$ to coincide with v on a neighbourhood of p. Then $\phi - v$ and $v - \phi$ are *both* locally minimised at p, and $\phi''(p) = v''(p)$. Since v is a viscosity sub-solution (super-solution) by Theorem 1, and $u(p) = u^\star(p) = u_\star(p)$, it follows that

$$v(p) \leq (\geq) \, u(p) + \bar{\lambda}\frac{p^2(1-p)^2}{2r\sigma^2} \max\{0, v''(p)\}.$$

So (HJB) must be satisfied in the classical sense at p.

Next consider a point at which v'' is undefined, e.g. $p = 1/2$. There are many $\phi \in \mathcal{C}^2$ such that $\phi - v$ has a local minimum at p; an example is depicted in Figure 4b. Since v is a viscosity sub-solution of (HJB) and $u^\star(p) = u(p)$, we must have

$$v(p) \leq u(p) + \bar{\lambda}\frac{p^2(1-p)^2}{2r\sigma^2} \max\{0, \phi''(p)\}$$

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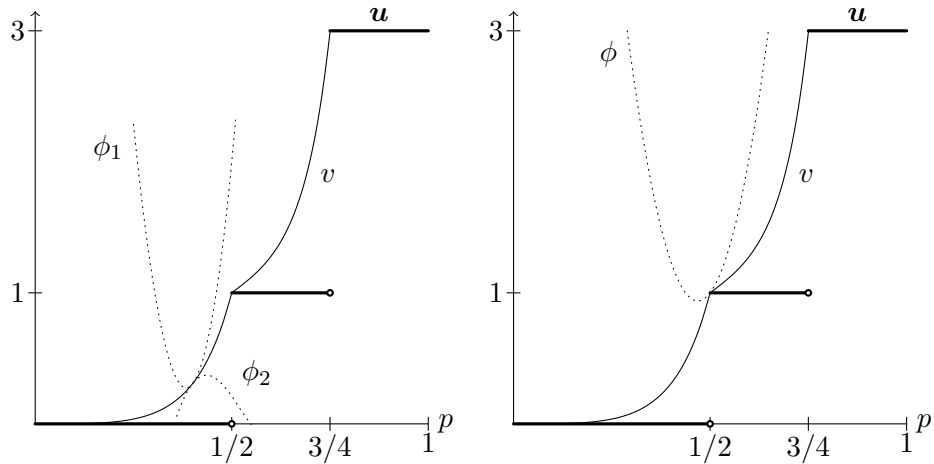
(a) $\phi_1, \phi_2 \in \mathcal{C}^2$ such that $\phi_1 - v$ and $v - \phi_2$ have local minima at $2/5$.

(b) $\phi \in \mathcal{C}^2$ such that $\phi - v$ has a local minimum at $1/2$.
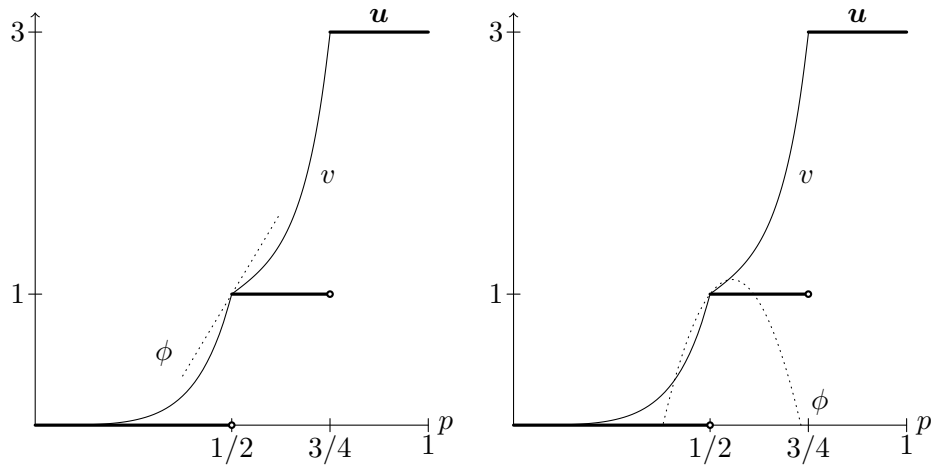
(c) $\phi \in \mathcal{C}^2$ such that $\phi - v$ has a local minimum at $1/2$ and $\phi''(1/2) = 0$.

(d) $\phi \in \mathcal{C}^2$ for which $v - \phi$ does not have a local minimum at $1/2$.

Figure 4: Functions $\phi \in \mathcal{C}^2$ that approximate v locally.

for any such ϕ. In fact, ϕ can be chosen so that $\phi''(p) \leq 0$: the ϕ depicted in Figure 4c is affine, so has $\phi''(p) = 0$. The sub-solution condition therefore requires precisely that

$$v(p) \leq \inf_{\substack{\phi \in \mathcal{C}^2: \\ \phi - v \text{ loc. min. at } p}} \left\{ u(p) + \bar{\lambda} \frac{p^2(1-p)^2}{2r\sigma^2} \max\{0, \phi''(p)\} \right\} = u(p),$$

which holds (with equality, in fact).

By contrast, there are no $\phi \in \mathcal{C}^2$ such that $v - \phi$ has a local minimum at p; a (failed) attempt to find such a ϕ is drawn in Figure 4d. The fact that v is a viscosity super-solution of (HJB) therefore has no bite at $p = 1/2$.

K.2   Some properties of viscosity solutions

There are other non-classical notions of 'solution' of a differential equation, most importantly distributional solutions (e.g. Evans (2010, chs. 5–9)). But for many differential equations, including HJB equations, viscosity solutions are the appropriate notion. The chief reasons are twofold: viscosity solutions exist, and they satisfy a comparison principle.

Begin with existence. Many HJB equations, including ours, fail to have a classical solution. Many also fail to have non-classical solutions of e.g. the distributional variety. By contrast, HJB equations always have a viscosity solution.

The other virtue of viscosity solutions is that they satisfy a comparison principle (also called a 'maximum principle') of the following kind: if $\underline{w}$ is a sub-solution on (a, b), $\overline{w}$ is a super-solution on (a, b), and $\underline{w} \leq \overline{w}$ on $\{a, b\}$, then $\underline{w} \leq \overline{w}$ on (a, b). (See Crandall, Ishii and Lions (1992, Theorem 3.3).) Classical sub- and super-solutions also satisfy a comparison principle, but other non-classical notions of 'solution' do not.

The comparison principle may be used to obtain uniqueness results; a standard one is that the HJB equation has at most one viscosity solution with the right boundary conditions satisfying a linear-growth condition. It follows that the value function is the unique solution with the right boundary conditions and linear growth. (See Fleming and Soner (2006, ch. V).) We use the comparison principle in this manner in the proofs of Lemmata 1 and 3 (appendices B and D).

The comparison principle may also be used to establish the continuity of solutions, and thus of the value function. In particular, suppose that we have shown that the upper (lower) semi-continuous envelope $v^\star$ ($v_\star$) of the value v is a sub-solution (super-solution) of the HJB equation, and that $v_\star = v^\star$

on $\{0, 1\}$. (We do precisely this in the proof of Theorem 1 in appendix A.) A comparison principle then yields $v^\star \le v_\star$, which since $v_\star \le v \le v^\star$ implies that v is itself a viscosity solution, hence continuous.

In our proof of Theorem 1 (appendix A), we eschew this approach in favour of a direct proof that v is continuous. We do this because we are not aware of a comparison principle that applies assuming only piecewise continuity of u. The closest result that we know of is Theorem 3.3 in Soravia (2006), which would be applicable under the additional hypotheses that u has only *finitely* many discontinuities and satisfies $u(p) \in [u(p-) \wedge u(p+), u(p-) \vee u(p+)]$ at every $p \in (0, 1)$.

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