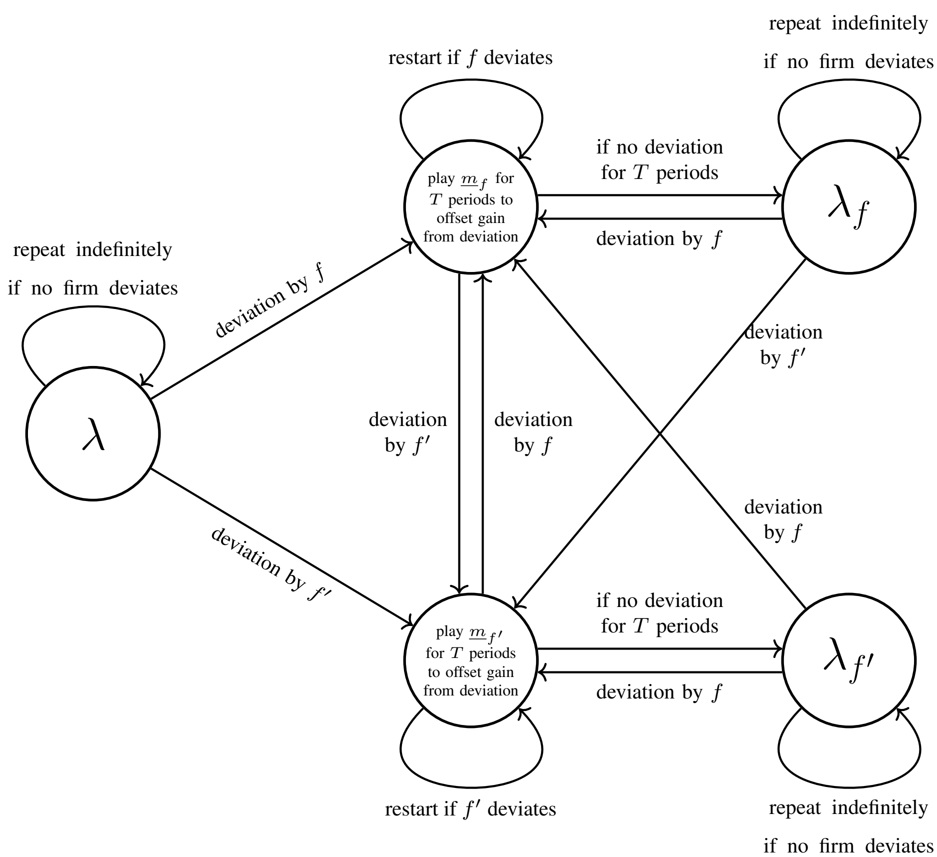
A Supplementary Appendix

A.1 Proof of Theorem 2

The proof of Theorem 2 follows a similar construction as Fudenberg and Maskin (1986) and Abreu, Dutta and Smith (1994). Due to the notational complexity of the proof, I first illustrate the self-enforcing matching process using a phase diagram, before proceeding with the full proof of Theorem 2.

A.1.1 A Phase Diagram Illustration

Consider a case when there are only two firms $\mathcal{F} = \{f, f'\}$. Below is a phase diagram illustrating the self-enforcing matching process.



In this phase diagram, $\lambda \in \Lambda^*$ is the random matching we want to sustain on the path of play. The matching $\underline{m}_f \in \arg\min_{m \in M^\circ_{\mathcal{R}}} \max_{W \subseteq D_f(m), |W| \leq q_f} u_f(W)$ is the minmax matching for firm f, and $\underline{m}_f$ is defined similarly.

The random matchings λ_f and $\lambda_{f'}$ are "firm-specific punishments" that are played after the minmax phase. In particular, they are random matchings that guarantee the following properties:

$$u_f(\lambda_f) < u_f(\lambda) \text{ and } u_f(\lambda_f) < u_{f'}(\lambda_f)$$

In other words, each firm prefers the on-path randomization λ over their own firm-specific punishments, and each firm prefers the other firm being punished over being punished itself.

The existence of λ_f and $\lambda_{f'}$ can be shown by resorting to the non-equivalent utilities (NEU) condition in Abreu, Dutta and Smith (1994): Observe that for each firm, when it is unmatch, it is indifferent towards how the other firm f' matches with workers, so their utilities cannot be positive affine transformation of another. Lemma 1 and Lemma 2 in Abreu, Dutta and Smith (1994) then ensure the existence of λ_f and $\lambda_{f'}$ that satisfy the properties above.

A.1.2 Complete Proof of Theorem 2

Fix $\lambda^0 \in \Lambda^*$. Define $u^0 = u(\lambda^0)$ and $U^* \equiv \{u(\lambda) : \lambda \in \Lambda^*\}$. Observe that for firms in $\mathcal{F} \cap \mathcal{R}$, the set $M^\circ_{\mathcal{R}}$ satisfies the non-equivalent utilities (NEU) condition in Abreu, Dutta and Smith (1994): holding $f \in \mathcal{F} \cap \mathcal{R}$ unmatched, f is indifferent towards how another firm $f' \in \mathcal{F} \cap \mathcal{R}$ matches with workers, so their utilities cannot be positive affine transformation of another. Lemma 1 and Lemma 2 in Abreu, Dutta and Smith (1994) then ensure the existence of vectors $\{u^f : f \in \mathcal{F} \cap \mathcal{R}\} \subseteq U^*$, such that

$$u^f_f < u^0_f \text{ and } u^f_f < u^{f'}_f$$

for all $f, f' \in \mathcal{F} \cap \mathcal{R}$ and $f \neq f'$. Let $\lambda^f \in \Lambda^*$ be the distribution over $M_{\mathcal{R}}$ that give rise to the payoff vector u^f for each f. In addition, for each $f \in \mathcal{F} \cap \mathcal{R}$, let

$$\underline{m}_f \in \arg\min_{m \in M^\circ_{\mathcal{R}}} \max_{W \subseteq D_f(m), |W| \leq q_f} u_f(W)$$

be the stage-game recommendation to minmax firm f.

Consider the matching process represented by the automaton $(\Theta, \gamma^0, f, \gamma)$, where

1. $\Theta = \big\{\theta(e, m) : e \in \mathcal{F} \cap \mathcal{R} \cup \{0\}, m \in M_{\mathcal{R}}\big\} \cup \big\{\underline{\theta}(f, t) : f \in \mathcal{F} \cap \mathcal{R},\ 0 \leq t < L\big\}$ is the set of all states;
2. γ^0 is the initial distribution over states, which satisfies $\gamma^0(\theta(0, m)) = \lambda^0(m)$ for all

$m \in M^{\circ}_{\mathcal{R}}$;

3. $O : \Theta \to M$ is the output function, where $O(\theta(e, m)) = m$ and $O(\underline{\theta}(f, t)) = \underline{m}_f$;
4. $\kappa : \Theta \times M \to \Delta(\Theta)$ is the transition function. For states $\{\underline{\theta}(f, t) | 0 \le t < L - 1\}$, κ is defined as

$$\kappa\big(\underline{\theta}(f,t), m'\big) = \begin{cases} \underline{\theta}(f', 0) & \text{if } m' \neq \underline{m}_f;\ m' = [\underline{m}_{f'}, (f', W)] \text{ for some } f' \in \mathcal{F} \cap \mathcal{R} \text{ and } W \subseteq \mathcal{W} \\ \underline{\theta}(f, t+1) & \text{otherwise} \end{cases}$$

For states $\underline{\theta}(f, L - 1)$, the transition is defined as

$$\kappa\big(\underline{\theta}(f, L{-}1), m'\big) = \begin{cases} \underline{\theta}(f', 0) & \text{if } m' \neq \underline{m}_f;\ m' = [\underline{m}_{f'}, (f', W)] \text{ for some } f' \in \mathcal{F} \cap \mathcal{R} \text{ and } W \subseteq \mathcal{W} \\ \gamma^f & \text{otherwise} \end{cases}$$

where for each $f \in \mathcal{F} \cap \mathcal{R}$, p^f is the distribution over states that satisfies $\gamma^f(\theta(f, m)) = \lambda^f(m)$ for all $m \in M$.

For states $\theta(e, m)$, the transition is

$$\kappa\big(\theta(e,m), m'\big) = \begin{cases} \underline{\theta}(f', 0) & \text{if } m' \neq \underline{m}_f;\ m' = [\underline{m}_{f'}, (f', W)] \text{ for some } f' \in \mathcal{F} \cap \mathcal{R} \text{ and } W \subseteq \mathcal{W} \\ \gamma^e & \text{otherwise} \end{cases}$$

where the distributions γ^e are defined as above.

Note that owing to the identifiability of deviating firm (Lemma 2), for any $\theta \in \Theta$ and matching $m' \neq O(\theta)$, we can uniquely identify the firm responsible for this deviation, so the transition above is well-defined.

Note that no firms in $\mathcal{T}$ wish to deviate, since they are always matched with their top coalition workers; no workers want to deviate since all recommended matchings are in $M^{\circ}_{\mathcal{R}}$. It remains to verify no firm $f \in \mathcal{F} \cap \mathcal{R}$ has incentives to deviate. Choose a number $Z > \sup_{\{m \in M, f \in \mathcal{F} \cap \mathcal{R}\}} u_f(m)$

For states of the form $\{\theta(e, m)\}$: Consider a one-shot deviation (f, W). There are two cases to consider.

Case 1: $f \neq e$. Without deviation, f has value $(1 - \delta)u_f(m) + \delta u^e_f$. After deviation, f yields less than $(1-\delta)Z + \delta(1-\delta^L)\underline{u}^{\mathcal{R}}_f + \delta^{L+1}u^f_f$. There is no profitable one-shot deviation for f if

$$(1 - \delta)u_f(m) + \delta u^e_f \ge (1 - \delta)Z + \delta(1 - \delta^L)\underline{u}^{\mathcal{R}}_f + \delta^{L+1}u^f_f$$

As $\delta \to 1$, the LHS converges to u_f^e while the RHS converges to u_f^f. By construction, $u_f^e > u_f^f$, so such deviations are not profitable for δ high enough.

Case 2: $f = e$. Without deviation, f has value $(1-\delta)u_f(m) + \delta u_f^f$. After deviation, f yields less than $(1-\delta)Z + \delta(1-\delta^L)\underline{u}_f^{\mathcal{R}} + \delta^{L+1}u_f^f$. There is no profitable one-shot deviation for f if

$$(1-\delta)u_f(m) + \delta u_f^f \geq (1-\delta)Z + \delta(1-\delta^L)\underline{u}_f^{\mathcal{R}} + \delta^{L+1}u_f^f.$$

The inequality is equivalent to

$$Z - u_f(m) \leq \delta(1 + \ldots + \delta^{L-1})[u_f^f - \underline{u}_f^{\mathcal{R}}]$$

By construction, $u_f^f - \underline{u}_f^{\mathcal{R}} > 0$. Choose L large enough so that $L(u_f^f - \underline{u}_f^{\mathcal{R}}) > Z - u_f(m)$. As $\delta \to 1$, the LHS remains unchanged while the RHS converges to $L(u_f^f - \underline{u}_f^{\mathcal{R}})$, so such deviations are not profitable for δ high enough.

For states of the form $\{\underline{\theta}(f,t)\}$: Consider a one-shot deviation (f', W). There are two cases to consider.

Case 1: $f' \neq f$. Without deviation, firm f' has payoff $(1-\delta^{L-t})u_{f'}(\underline{m}_f) + \delta^{L-t}u_{f'}^f$. After deviation, f' has payoff less than $(1-\delta)Z + \delta(1-\delta^L)\underline{u}_{f'}^{\mathcal{R}} + \delta^{L+1}u_{f'}^{f'}$. There is no profitable one-shot deviation for f' if

$$(1-\delta^{L-t})u_{f'}(\underline{m}_f) + \delta^{L-t}u_{f'}^f \geq (1-\delta)Z + \delta(1-\delta^L)\underline{u}_{f'}^{\mathcal{R}} + \delta^{L+1}u_{f'}^{f'}$$

As $\delta \to 1$, the LHS converges to $u_{f'}^f$ for all $0 \leq t \leq L$, while the RHS converges to $u_{f'}^{f'}$. By construction $u_{f'}^f > u_{f'}^{f'}$. So the above inequality holds for sufficiently high δ.

Case 2: $f' = f$. Without deviation, firm f' has payoff $(1-\delta^{L-t})\underline{u}_{f'}^{\mathcal{R}} + \delta^{L-t}u_{f'}^{f'}$. When deviating from $\underline{m}_{f'}$, f' can obtain at most $\underline{u}_f^{\mathcal{R}}$. So its payoff from deviation is at most

$$(1-\delta)\underline{u}_{f'}^{\mathcal{R}} + \delta(1-\delta^L)\underline{u}_{f'}^{\mathcal{R}} + \delta^{L+1}u_{f'}^{f'} = (1-\delta^{L+1})\underline{u}_{f'}^{\mathcal{R}} + \delta^{L+1}u_{f'}^{f'}$$

Firm f' has no profitable deviation if $(1-\delta^{L-t})\underline{u}_{f'}^{\mathcal{R}} + \delta^{L-t}u_{f'}^{f'} \geq (1-\delta^{L+1})\underline{u}_{f'}^{\mathcal{R}} + \delta^{L+1}u_{f'}^{f'}$, or

$$u_{f'}^{f'} \geq \underline{u}_{f'}^{\mathcal{R}}.$$

This is true by construction. So f' has no profitable one-shot deviation.

We have verified that there is no profitable one-shot deviation in any states of the automaton. This completes the proof.

References

Abreu, Dilip, Prajit K. Dutta, and Lones Smith, "The folk theorem for repeated games: a NEU condition," *Econometrica*, 1994, *62* (4), 939–948.

Fudenberg, Drew and Eric Maskin, "The folk theorem in repeated games with discounting or with incomplete information," *Econometrica*, 1986, pp. 533–554.