Dynamic assignment without money: Optimality of spot mechanisms Online Appendix

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1 Proof of Theorem 4

The proof resembles the one of Theorem 3. In particular, we will use a matrix $\mathbf{A}$ associated with an order over bundles induced by the prices $\boldsymbol{\alpha}$ of a CEEI to prove the existence of certain linear prices. However, the main difference is that we will not be using all the strict orderings induced by $\boldsymbol{\alpha}$.

Let $\mathbf{x}$ be an incentive compatible and Pareto efficient allocation rule. First, we apply the same connection to Ashlagi and Shi (2016) as in the proof of Theorem 3. Hence, in this new environment we treat bundles as objects. An allocation rule $\mathbf{x}$ is A-S Pareto efficient if there is no other allocation rule $\mathbf{x}'$ such that:

1. For each bundle $b \in B$ we have $\int_U x'_b(\mathbf{u})dF = \int_U x_b(\mathbf{u})dF$.
2. For each $\mathbf{u} \in U$ we have $\mathbf{u} \cdot \mathbf{x}'(\mathbf{u}) \geq \mathbf{u} \cdot \mathbf{x}(\mathbf{u})$ and there is a set $A \subset U$ such that $F(A) > 0$ and the inequality is strict for each $\mathbf{u} \in A$.

Similarly to Lemma 1, it is immediate to see that any Pareto efficient allocation rule must

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also be A-S Pareto efficient.[1] Hence, by Theorem 1 of Ashlagi and Shi (2016), we know that, with a continuous distribution F with full relative support, the mechanism $\mathbf{x}$ is a CEEI for some prices $(\alpha_b)_{b\in B} \in (0,\infty]^{|B|}$. Note that these prices, following the definition of Ashlagi and Shi (2016), are strictly positive and some of them can be infinite. We let $\alpha_{\max} := \max_{b\in B} \alpha_b$, and $\alpha_{\min} := \min_{b\in B} \alpha_b$. We start with the following simple observation.

Lemma 1. *If $\mathbf{x}$ is CEEI for prices $\boldsymbol{\alpha}$, then $\alpha_{\min} \leq 1$.*

Proof. If $\alpha_b > 1$ for each bundle b, then for each $\mathbf{q} \in \Delta$, we have $\boldsymbol{\alpha} \cdot \mathbf{q} > 1$. Hence, $\mathbf{x}(\mathbf{u})$ is not affordable for each $\mathbf{u}$, a contradiction. ∎

Now, fix new prices $\hat{\boldsymbol{\alpha}} \in [0,\infty]^{|B|}$. Note that we now allow these prices to be null. We call an allocation rule $\mathbf{x}$ a r-CEEI with prices $\hat{\boldsymbol{\alpha}}$ if, intuitively, it is a CEEI with budget r (instead of 1), i.e., $\arg\max_{\mathbf{q}\in\Delta}\{\mathbf{u}\cdot\mathbf{q} : \hat{\boldsymbol{\alpha}} \cdot \mathbf{q} \leq r\}$. As before, we can similarly define $\hat{\alpha}_{\min}$.

Lemma 2. *If $\mathbf{x}$ is a CEEI with prices $\boldsymbol{\alpha} \in (0,\infty]^{|B|}$, then $\mathbf{x}$ is a r-CEEI with prices $\hat{\boldsymbol{\alpha}} \in [0,\infty]^{|B|}$, budget $r = 1 - \alpha_{\min}$ and $\hat{\alpha}_{\min} = 0$.*

Proof. By Lemma 1, we have $\alpha_{\min} \leq 1$. Then reducing the budget and all the prices by $\alpha_{\min}$ does not change the budget set, and hence $\mathbf{x}$ is still a CEEI under the reduced budget and prices, i.e., a r-CEEI and prices $\hat{\boldsymbol{\alpha}} \in [0,\infty]^{|B|}$ with $\hat{\alpha}_{\min} = 0$ as required. ∎

Fix prices $\boldsymbol{\alpha} \in [0,\infty]^{|B|}$. Let $B_\infty = \{b : \alpha_b = \infty\}$. Note that B_∞ is nonempty when $\alpha_{\max} = \infty$. We denote the highest finite price by $\alpha_{\max^*} := \max_b\{\alpha_b : \alpha_b < \infty\}$, and let $b^{\max^*}$ be a bundle such that $\alpha_b = \alpha_{\max^*}$. The following lemma is useful to construct improving bilateral transfers.

Lemma 3. *If $\mathbf{x}$ is a r-CEEI with prices $\boldsymbol{\alpha} \in [0,\infty]^{|B|}$ and $\alpha_{\min} = 0$. Let the set $P \subset B^2$ be defined as follows:*

- *Case 1: if $\alpha_{\max^*} \leq r$, let $P := B\backslash B_\infty \times B_\infty$ with $B_\infty := \{b : \alpha_b = \infty\}$.*
- *Case 2: if $\alpha_{\max^*} > r$ and $r = 0$, let $B_0 := \{b : \alpha_b = 0\}$ and $P := B_0 \times B\backslash B_0$.*
- *Case 3: $\alpha_{\max^*} > r$ and $r > 0$: let $P := \{(b,b') : \alpha_b < \alpha_{b'}\}$.*

Then, for any pair $(b,b') \in P$:

[1] Indeed, the allocation $\mathbf{x}'$ in the definition of A-S Pareto efficiency would also be valid if one uses the definition of Pareto efficiency in Section 5.

1. $\alpha_b < \alpha_{b'}$.

2. There is an open set $f(b,b') \subset U$ *s.t. i)* $u_b < u_{b'}$ *and ii) for some* $m > 0$, $x_b(\mathbf{u}) \geq m$ *for all* $\mathbf{u} \in f(b,b')$.

PROOF. As discussed above, if $\mathbf{x}$ is an incentive compatible and Pareto efficient allocation rule, then, using Lemma 2, it is a r-CEEI with prices $\boldsymbol{\alpha} \in [0,\infty]^{|B|}$ such that $\alpha_{\min} = 0$. We follow each case of the lemma.

Case 1: $\alpha_{\max^*} \leq r$. In this case all bundles with a finite price are affordable. By definition, for each $(b,b') \in P$, we have $\alpha_b < \alpha_{b'}$ so the first condition of the lemma holds.

For each $(b,b') \in P$, let $f(b,b') \subset U$ be the set of utility vectors such that:

- $u_{b'} = 2M + \varepsilon_{b'}$ with $\varepsilon_{b'} \in (0,\bar{\varepsilon})$,
- $u_b = M + \varepsilon_b$ with $\varepsilon_b \in (0,\bar{\varepsilon})$,
- $u_{b''} = \varepsilon_{b''}$ with $\varepsilon_{b''} \in (0,\bar{\varepsilon})$ for each $b'' \neq b, b'$,

where M and $\bar{\varepsilon}$ are some constants. Clearly, the set $f(b,b')$ is open in U as a product of open intervals. For $M > \bar{\varepsilon}$, bundle b' gives the highest utility followed by b, followed by all other bundles. Note that since $(b,b') \in B \backslash B_\infty \times B_\infty$ and $\alpha_{\max^*} \leq r$, b is always affordable under the CEEI while b' is not. Since b gives the highest utility among affordable bundles for $\mathbf{u} \in f(b,b')$, we have $x_b(\mathbf{u}) = 1$ as required.

Case 2: $\alpha_{\max^*} > r$ *and* $r = 0$. In this case, only free bundles are affordable. Since $\alpha_{\min} = 0$, the set B_0 is non-empty. By construction, we have $\alpha_b = 0 < \alpha_{b'}$ for any $(b,b') \in P$ so that the first requirement of the lemma holds.

For each $(b,b') \in P$ we now define the set $f(b,b') \subset U$ in the same way as in Case 1. Note that since $(b,b') \in B_0 \times B \backslash B_0$ and $r = 0$, bundle b is always affordable under the CEEI while b' is not. Since b gives the highest utility among affordable bundles for $\mathbf{u} \in f(b,b')$, we have $x_b(\mathbf{u}) = 1$ as required.

Case 3: $\alpha_{\max^*} > r$ *and* $r > 0$. In this case, note that we can normalize the budget to one by dividing all the prices by r and obtain the same CEEI. So, in what follows, we assume that $r = 1$. Remember that we have at least one free bundle since $\alpha_{\min} = 0$. By definition of P, the first requirement of the lemma holds.

Fix constants $M, \bar{\varepsilon}, \bar{\delta}$. Remember that $b^{\max^*}$ is a bundle such that $\alpha_{b^{\max^*}} = \alpha_{\max^*}$. For each $(b,b') \in P$ let $f(b,b')$ be the set of utility vectors such that:

- $u_{b''} = \alpha_{b''} + \varepsilon_{b''}$ with $\varepsilon_{b''} \in (0, \bar{\varepsilon})$ for $b'' \neq b, b^{\max^*}$ such that $\alpha_{b''} < \infty$,
- $u_{b''} = M + \varepsilon_{b''}$ with $\varepsilon \in (0, \bar{\varepsilon})$ for $b'' \in B_\infty$,
- $u_b = \alpha_b + \delta_b + \varepsilon_b$ with $\varepsilon_b \in (0, \bar{\varepsilon})$,
- $u_{b^{\max^*}} = \alpha_{b^{\max^*}} + \delta_{b^{\max^*}} + \varepsilon_{b^{\max^*}}$ with $\varepsilon_{b^{\max^*}} \in (0, \bar{\varepsilon})$.

In words, utility vectors in $f(b, b')$ assign to each bundle b'' an utility equal to the bundle's price $\alpha_{b''}$ (or a large constant if this price is infinite) perturbed by some positive constant. For each bundle $b'' \neq b^{\max^*}$ with price $\alpha_{b''} < \infty$, let $s(\alpha_{b''})$ be the next strictly highest price, possibly infinite, i.e., $s(\alpha_{b''}) := \min_{b'}\{\alpha_{b'} : \alpha_{b'} > \alpha_{b''}\}$. We can choose positive constants M, δ_b, $\delta_{b^{max}}$, and $\bar{\varepsilon}$, so that they satisfy the following constraints:

(i) For each b'' such that $\alpha_{b''} \neq \alpha_{b^{\max^*}}$ we have:

$$\alpha_{b''} + \delta_b + \bar{\varepsilon} < s(\alpha_{b''}). \tag{1.1}$$

And $M > \alpha_{b^{\max^*}} + \delta_{b^{\max^*}} + \bar{\varepsilon}$.

(ii)

$$\delta_b > \bar{\varepsilon}. \tag{1.2}$$

(iii) If $\alpha_b > 0$ and $b \neq b^{max^*}$, then for each $b'' \neq b, b^{\max^*}$ such that $\alpha_{b''} > 0$ we have:

$$\frac{\delta_b}{\alpha_b} > \frac{\delta_{b^{\max^*}} + \bar{\varepsilon}}{\alpha_{b^{\max^*}}} + \left(\frac{1}{\alpha_b} - \frac{1}{\alpha_{b^{\max^*}}}\right)\bar{\varepsilon} > \frac{\delta_{b^{\max^*}}}{\alpha_{b^{\max^*}}} > \frac{\bar{\varepsilon}}{\alpha_{b''}}. \tag{1.3}$$

The constraint 1.1 makes sure that the ranking induced by the perturbed utilities is consistent with the strict ranking induced by prices $\boldsymbol{\alpha}$. Constraint 1.2 implies that bundle b is the most attractive bundle among all bundles with the same price. Constraint 1.3 implies that bundles b and $b^{\max^*}$ deliver the highest utility per unit of artificial currency among all non-free bundles with finite price, and, roughly speaking, b is sufficiently more attractive than $b^{\max^*}$.[2] Clearly, the set $f(b, b')$ is open in U as a product of open intervals in $\mathbb{R}$.

[2]Note that there are positive constants M, δ_b, $\delta_{b^{\max^*}}$, and $\bar{\varepsilon}$ satisfying (i), (ii) and (iii). Indeed, one can set δ_b, $\bar{\varepsilon}$ small enough and M high enough so that (i) holds. With an even smaller $\bar{\varepsilon}$, (ii) holds. Finally, with $\bar{\varepsilon}$ small again and $\delta_{b^{\max^*}}$ small, (iii) holds true. Also note that since we have assumed that $\alpha_{\max^*} < \infty$, then $\alpha_{b^{\max^*}} < \infty$ so that constraint 1.3 is indeed true.

We now show that if $\mathbf{u} \in f(b, b')$, then $x_b(\mathbf{u}) = m$ for some $m > 0$. We begin by showing that, in the CEEI, there does not exist $b'' \neq b, b^{\max^*}$ and $\mathbf{u} \in f(b, b')$ such that $\alpha_{b''} > 0$ and $x_{b''}(\mathbf{u}) > 0$. For the sake of contradiction, suppose such b'' and $\mathbf{u}$ exist. Consider reducing expenditures of such agents on b'' by $\eta > 0$, and increasing their expenditures on $b^{\max^*}$ by η. So their probability share of b'' decreases by $\eta/\alpha_{b''}$, and their probability share of $b^{\max^*}$ increases by $\eta/\alpha_{b^{\max^*}} \leq \eta/\alpha_{b''}$. To keep the sum of probability shares equal to 1, increase the share of any free bundle by $\eta/\alpha_{b''} - (\eta/b^{\max^*})$. For a sufficiently small $\eta > 0$, such transfer of mass is feasible and increases the utility of agents with $\mathbf{u} \in f(b, b')$ by constraint 1.1 above, a contradiction to the allocation being a CEEI.

First, suppose $\alpha_b = 0$. Then, given the above result, an agent with $\mathbf{u} \in f(b, b')$ must spend her entire budget on $b^{\max^*}$ in purchasing a $1/\alpha_{b^{\max^*}} < 1$ probability share of $b^{\max^*}$, and complete the allocation with the free bundle b in purchasing a $1 - (1/\alpha_{b^{\max^*}}) > 0$ probability share of b because $\delta_b > \bar{\varepsilon}_{b''}$ for each $b'' \neq b$ such that $\alpha_{b''} = 0$.

Second, suppose $\alpha_b > 0$. Notice that, because of constraint 1.3, an agent with $\mathbf{u} \in f(b, b')$ must allocate the entire budget between bundles b and $b^{\max^*}$, and potentially complete the allocation with a share of a free bundle which delivers the highest utility, denoted by b_0. Specifically, she solves the following optimization problem:

$$\max_{0 \leq z \leq 1} (\alpha_{b^{\max^*}} + \delta_{b^{\max^*}} + \varepsilon_{b^{\max^*}}) \frac{z}{\alpha_{b^{\max^*}}} + (\alpha_b + \delta_b + \varepsilon_b) \frac{1-z}{\alpha_b} + \varepsilon_{b_0} \left(1 - \frac{z}{\alpha_{b^{\max^*}}} - \frac{1-z}{\alpha_b}\right),$$

subject to

$$1 - \frac{z}{\alpha_{b^{\max^*}}} - \frac{1-z}{\alpha_b} \geq 0.$$

Given the constraint 1.3, the objective is linearly decreasing in z. If $\alpha_b \geq 1$, then the constraint does not bind and optimally $x_b(\mathbf{u}) = 1/\alpha_b$, i.e., the entire budget is spent on b. If $\alpha_b < 1$, then the constraint binds, which implies that $x_{b_0}(\mathbf{u}) = 0$, and the budget is split between $b^{\max^*}$ and b such that

$$x_b(\mathbf{u}) = \frac{\alpha_{b^{\max^*}}(1 - \alpha_b)}{\alpha_{b^{\max^*}} - \alpha_b}.$$

Summarizing, for each $\mathbf{u} \in f(b, b')$ we have

$$x_b(\mathbf{u}) \geq \min \left\{1 - \frac{1}{\alpha_{b^{\max^*}}}, \frac{1}{\alpha_b}, \frac{\alpha_{b^{\max^*}}(1 - \alpha_b)}{\alpha_{b^{\max^*}} - \alpha_b}\right\} := m > 0.$$

as required. ∎

Last, we show that from the sets $f(b, b')$ as in Lemma 3, we can create an open cone of preferences with positive mass having the same property. The reader interested in the proof of Theorem 4 can skip the proof since it mostly relies on topology arguments.

Lemma 4. *Fix a r-CEEI $\mathbf{x}$ with prices $\boldsymbol{\alpha}$. If the distribution F has full relative support, then for each pair $(b, b') \in P$ and its associated open set $f(b, b')$ from Lemma 3, there exists an open cone $C(b, b') \in \mathcal{C}$ such that $F(C(b, b')) > 0$ and for each $\mathbf{u} \in C(b, b')$ and some $m > 0$ we have $x_b(u) \geq m$.*

Proof. Fix a pair $(b, b') \in P$ and the associated open set $f(b, b') \subset U$ from Lemma 3.[3] In the sequel, we recall that Proj_D stands for the projection from U into D, i.e.,

$$\mathrm{Proj}_D(\mathbf{u}) := (u_b - \frac{\sum_b u_b}{|B|})_b.$$

Note that, from Lemma 3, under a r-CEEI, for any $\mathbf{u} \in f(b, b')$, then $x_b(\mathbf{u}') \geq m > 0$. For any $\mathbf{u}' = \lambda \mathbf{u} - \xi \mathbf{1}$ with $\lambda > 0$ and $\xi \in \mathbb{R}$ since the choices are invariant to linear transformations of $\mathbf{u}$, we also have $x_b(\mathbf{u}') \geq m$. In words, rescaling and translating the cardinal utilities will not impact the optimal choice of the agent in a CEEI. Given $\lambda > 0$, we denote $X_\lambda := \{\mathbf{u}' \in U : \mathbf{u}' = \lambda \mathbf{u} \text{ for some } \mathbf{u} \in f(b, b')\}$. Note that for any $\lambda > 0$, X_λ is open in U (since the function $\mathbf{u} \mapsto \lambda \mathbf{u}$ is an homeomorphism). Now, let us consider $\mathcal{Z} := \cup_{\lambda > 0} X_\lambda$. Note that, as a union of open sets, $\mathcal{Z}$ is open in U. Let $C := \mathrm{Proj}_D(\mathcal{Z})$. Here again, for any $\mathbf{u} \in C$, we must have $x_b(\mathbf{u}) \geq m$ since such $\mathbf{u}$ are simple linear transformations of utility vectors in $f(b, b')$.

We first claim that C is a cone. Take any $\mathbf{u}' \in C$ and any $\lambda > 0$. We must show that $\lambda \mathbf{u}' \in C$. Indeed, since $\mathbf{u}' \in C$, we must have that for some $\mathbf{u} \in \mathcal{Z}$, $\mathrm{Proj}_D(\mathbf{u}) = \mathbf{u}'$. Hence, $\mathrm{Proj}_D(\lambda \mathbf{u}) = \lambda \mathrm{Proj}_D(\mathbf{u}) = \lambda \mathbf{u}'$ where the first equality uses the linearity of Proj_D. Since, by definition of set $\mathcal{Z}$, it must be that $\lambda \mathbf{u}$ belongs to $\mathcal{Z}$, $\mathrm{Proj}_D(\lambda \mathbf{u}) = \lambda \mathbf{u}'$ implies that $\lambda \mathbf{u}' \in \mathrm{Proj}_D(\mathcal{Z}) = C$, as claimed.

Now, we show that C is open in D in order to eventually show that C is open in $\mathcal{C}$. This comes from the feature that Proj_D is an open map together with the fact that $\mathcal{Z}$ is open in U.[4] Finally, we want to show that our cone C is open in $\mathcal{C}$, i.e., $C \cap \tilde{D}$ is open in $\tilde{D}$. This is true since, as we just claimed, C is open in D and so $C \cap \tilde{D}$ is open in $\tilde{D}$ by definition of the

[3] Remember that the sets P and $f(b, b')$ change depending on the values of the cutoffs $\boldsymbol{\alpha}$ as shown in the proof of Lemma 3.

[4] Proj_D is a continuous mapping under our topologies and it is surjective and linear. By the open mapping theorem, Proj_D is an open mapping, i.e., for any open set $\mathcal{O}$ in U, $\mathrm{Proj}_D(\mathcal{O})$ is open in D.

relative topology. Thus, we can set $C(b, b') := C$. The open cone $C(b, b')$ satisfies $x_b(\mathbf{u}) \geq m$ for any $\mathbf{u} \in C(b, b')$. Since F has full relative support and $C(b, b')$ is open in $\tilde{D}$, we have $F(C(b, b')) > 0$. ∎

We are now equipped with all the lemmas to prove Theorem 4. Similar to the proof of Theorem 3, the proof relies on constructing feasible bilateral transfers whenever there is no solution to a well constructed set of linear inequalities. The proof is divided into several cases depending on the value of $\alpha_{\max}$ and $\alpha_{\min}$. For each of them, we will show that we can construct a spot MRB which induces the same allocation as the initial CEEI. As in the proof of Proposition 5, we will construct a matrix $\mathbf{A}$ associated to a strict ordering over bundles in B and consider the system of linear inequalities $\mathbf{Ap} < 0$. We will show that, if such system has no solution, there exists $\mathbf{y}$ such that $\mathbf{y} \geq \mathbf{0}$, $\mathbf{y} \neq \mathbf{0}$ and $\mathbf{A}^T\mathbf{y} = 0$, and that such $\mathbf{y}$ can be used to construct feasible improving bilateral transfers for a positive mass of agents (using Lemma 4) so that $\mathbf{x}$ is not Pareto efficient.

Proof. Suppose $\mathbf{x}$ is an incentive compatible and Pareto efficient allocation rule. By Lemma 2, $\mathbf{x}$ is a r-CEEI with prices $\boldsymbol{\alpha} \in [0, \infty]^{|B|}$ such that $\alpha_{\min} = 0$. Fix the set P of pairs of bundles as defined in Lemma 3. First, we show that there exist linear prices $\bar{\boldsymbol{\alpha}}$ such that $\bar{\alpha}_b < \bar{\alpha}_{b'}$ for each $(b, b') \in P$.

Construct the matrix $\mathbf{A}$ as in the proof of Proposition 5 so that each row of $\mathbf{A}$ corresponds to a pair of bundles $(b, b') \in P$ and each column corresponds to a generalized object in O. For the sake of contradiction suppose such linear prices $\bar{\boldsymbol{\alpha}}$ do not exist. Then, as in Proposition 5, there exists $\mathbf{y}$ such that $\mathbf{y} \geq \mathbf{0}$, $\mathbf{y} \neq \mathbf{0}$ and $\mathbf{A}^T\mathbf{y} = 0$, and, in what follows, we use $\mathbf{y}$ to construct improving bilateral transfers for a positive mass of agents.

For each $(b, b') \in P$, Lemma 4 guarantees that there exists an open cone $C(b, b')$ such that $F(C(b, b')) > 0$, and for each $\mathbf{u} \in C(b, b')$ we have $u_b < u_{b'}$ and $x_b(\mathbf{u}) \geq m > 0$. For each $(b, b') \in P$, consider a transfer of a probability mass $(\epsilon / F(C(b, b')))y_{b,b'}$ from b to b' for agents with $\mathbf{u} \in C(b, b')$ at their random allocation $\mathbf{x}(\mathbf{u})$. By construction of $C(b, b')$, this is an improving bilateral transfer given a sufficiently small $\epsilon > 0$. Moreover, because $\mathbf{A}^T\mathbf{y} = 0$, by Lemma 7 these transfers do not change the allocated mass of each object. Therefore, $\mathbf{x}$ is not Pareto efficient, which is a contradiction. It follows that there exist linear cutoffs $\bar{\boldsymbol{\alpha}}$ such that $\bar{\alpha}_b < \bar{\alpha}_{b'}$ for each $(b, b') \in P$. Without loss of generality assume that $\max_b \bar{\alpha}_b = 1$.

To finish the proof, we construct a spot MRB mechanism that implements r-CEEI $\mathbf{x}$. By Proposition 3, $\mathbf{x}$ is a MRB mechanism $L = (\hat{\boldsymbol{\alpha}}, \mathcal{G})$, where cutoffs $\hat{\boldsymbol{\alpha}}$ are the normalized prices $\boldsymbol{\alpha}$ as in the proof of Proposition 3. Using the linear prices $\bar{\boldsymbol{\alpha}}$, we now construct a collection of distributions $\mathcal{G}'$ such that the spot MRB mechanism with $L' = (\bar{\boldsymbol{\alpha}}, \mathcal{G}')$ implements the allocation rule $\mathbf{x}$. For each distribution $G_{\mathbf{x}(\mathbf{u})} \in \mathcal{G}$, let the corresponding distribution $G'_{\mathbf{x}(\mathbf{u})} \in \mathcal{G}'$ assign probability $x_b(\mathbf{u})$ to $\bar{\alpha}_b$ instead of $\hat{\alpha}_b$. We now consider the same cases as in Lemma 3.

Case 1: $\alpha_{\max^*} \leq r$. In the r-CEEI, each agent is assigned her favorite bundle out of the ones with a finite price. In the spot MRB with $L' = (\bar{\boldsymbol{\alpha}}, \mathcal{G}')$, each agent can also receive the same bundle and cannot receive a positive share of any bundle with an infinite price because those bundles keep having the highest prices under the linear cutoffs $\bar{\boldsymbol{\alpha}}$ and the distributions in $\mathcal{G}'$ put probability 1 on the budget strictly below these prices. Hence, the induced allocation rule must be the same.

Case 2: $\alpha_{\max^*} > r$ *and* $r = 0$. In the r-CEEI, each agent is assigned her favorite free bundle. Similarly to the previous case, each agent can also receive the same bundle in the spot MRB with $L' = (\bar{\boldsymbol{\alpha}}, \mathcal{G}')$. Moreover, she cannot receive a positive share of any other bundle because those bundles have strictly higher prices than the ones of the free bundles under the linear cutoffs $\bar{\boldsymbol{\alpha}}$ and the distributions in $\mathcal{G}'$ put probability 1 on the budget strictly below these prices. Hence, the induced allocation rule must be the same.

Case 3: $\alpha_{\max^*} > r$ *and* $r > 0$. Note that in $L' = (\bar{\boldsymbol{\alpha}}, \mathcal{G}')$, for each realization of a random budget, the set of affordable bundles is the same as in L for each distribution because the linear cutoffs $\bar{\boldsymbol{\alpha}}$ have the same strict order as prices $\boldsymbol{\alpha}$. Then, we have that, for each distribution and for each set of bundles, the probability that this set is affordable is the same in L and L'. Hence, the induced allocation rule must be the same. ∎

2 Proof of Proposition 4

Recall that a spot mechanism $\mathbf{x}$ is characterized by a GLC with parameters $(\boldsymbol{\alpha}, G)$ where $x_{\pi(h)}(\pi) = G(\min_{m=1,\ldots,h-1} \alpha_{\pi(m)}) - G(\min_{m=1,\ldots,h} \alpha_{\pi(m)})$ for every π and $h = 1, \ldots, |\mathbf{O}|$. In addition, we know that there exists non-linear $\mathbf{p} = (p^t)_{t=1,\ldots,T}$ where $p^t = (p_i^t)_{i \in O_t}$ for each $t = 1, ..., T$ satisfying

$$\alpha_{\mathbf{o}} = \sum_{t=1}^{T} p_{o_t}^t$$

for each $\mathbf{o} = (o_1, ..., o_T) \in \mathbf{O}$. We say that $(\boldsymbol{\alpha}, G, \mathbf{p})$ corresponds to spot mechanism $\mathbf{x}$.

Lemma 5. *Take a sequence* $\mathbf{x}^n \to \mathbf{x}$ *where, for each* n, $\mathbf{x}^n$ *is a spot mechanism. Further, assume that the corresponding sequence* $(\boldsymbol{\alpha}^n, G^n, \mathbf{p}^n)$ *converges to* $(\boldsymbol{\alpha}, G, \mathbf{p})$. *We must have that* $\mathbf{x}$ *is a spot mechanism and* $(\boldsymbol{\alpha}, G, \mathbf{p})$ *corresponds to* $\mathbf{x}$.

PROOF. Since for each n :

$$x^n_{\pi(h)}(\pi) = G^n(\min_{m=1,\ldots,h-1} \alpha^n_{\pi(m)}) - G^n(\min_{m=1,\ldots,h} \alpha^n_{\pi(m)})$$

for every π and $h = 1, \ldots, |\mathbf{O}|$, the same must hold as well in the limit, i.e.,

$$x_{\pi(h)}(\pi) = G(\min_{m=1,\ldots,h-1} \alpha_{\pi(m)}) - G(\min_{m=1,\ldots,h} \alpha_{\pi(m)})$$

for every π and $h = 1, \ldots, |\mathbf{O}|$.

In addition, since for each n, for each $t = 1, ..., T$, we have

$$\alpha^n_{\mathbf{o}} = \sum_{t=1}^{T} p^{t,n}_{o_t}$$

for each $\mathbf{o} = (o_1, ..., o_T) \in \mathbf{O}$, the same must hold as well in the limit, i.e.,

$$\alpha_{\mathbf{o}} = \sum_{t=1}^{T} p^{t}_{o_t}.$$

Hence, $\mathbf{x}$ is a spot mechanism and $(\boldsymbol{\alpha}, G, \mathbf{p})$ corresponds to $\mathbf{x}$, as claimed. ∎

PROOF OF PROPOSITION 4.

"$\Longrightarrow$" Assume that $\mathbf{x}$ is robustly OE and IC at F. Pick a sequence $F_n \to F$ where F_n has full-support. Because, $\mathbf{x}$ is robustly OE and IC at F, we know that there is a sequence $\{\mathbf{x}_n\}$ such that $\mathbf{x}_n \to \mathbf{x}$ and $\mathbf{x}_n$ is OE and IC at F_n for each n. Note that, by Theorem 1, this implies that $\mathbf{x}_n$ is a spot mechanism for each n. Let $(\boldsymbol{\alpha}^n, G^n, \mathbf{p}^n)$ correspond to $\mathbf{x}_n$ for each n. Note that $\boldsymbol{\alpha}^n$ and $\mathbf{p}^n$ clearly lie in a (sequentially) compact set. In addition, the space of probability measures over the compact set $[0, 1]$ is sequentially compact in the topology of weak convergence of measures. So G^n also lies in a sequentially compact set.[5] Thus, taking a subsequence if necessary, we can assume that $(\boldsymbol{\alpha}^n, G^n, \mathbf{p}^n) \to (\boldsymbol{\alpha}, G, \mathbf{p})$. By Lemma 5, $\mathbf{x}$ is a spot mechanism.

"$\Longleftarrow$" Assume that $\mathbf{x}$ is a spot mechanism. By Theorem 1 (and the observation that Theorem 1 "$\Longleftarrow$" holds without the full-support assumption—see Footnote 58), $\mathbf{x}$ is OE and IC for all distributions F'. Now, fix any sequence of distributions $F_n \to F$ and let $\mathbf{x}_n$ be the

[5]This comes from Prokhorov's Theorem and the observation that probability measures in our space are all supported on the same compact set and so the space is automatically tight.

constant sequence equal to $\mathbf{x}$ for all n. By the previous observation, $\mathbf{x}_n$ is OE and IC at F_n for each n. Trivially, $\mathbf{x}_n \to \mathbf{x}$. Hence, $\mathbf{x}$ is robustly OE and IC at F. ∎

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