# Adversarial Coordination and Public Information Design: Supplementary Appendix

**Nicolas Inostroza**     **Alessandro Pavan**

August 20, 2024

**Abstract**

This document contains the proofs of Examples 2 and 3 in the main text (Sections S1 and S2, respectively). All numbered items contain the prefix "S." Any numbered reference without the prefix "S" refers to an item in the main text.

# S1. Proof of Example 2 in Main Text

**Proof of Example 2.** The proof is in two steps. Step 1 characterizes the threshold $\theta_\sigma^*$ defining the optimal deterministic monotone rule, whereas Step 2 constructs the non-monotone policy that strictly improves over the optimal deterministic monotone one.

    **Step 1.** The primitives in this example satisfy the conditions in Theorem 2 in the main text. This means that, given any signal $s$ disclosed by any policy $\Gamma$, MARP is in threshold strategies, which in turn implies that the default outcome is monotone in $\theta$.

    Next recall that, for any default threshold $\theta \in [0, 1]$, the corresponding signal threshold $x_\sigma^*(\theta)$ is implicitly defined by $P_\sigma(x_\sigma^*(\theta) | \theta) = \theta$. Using the fact that, for any $\theta \in [-K, 1+K]$ and $x \in [\theta - \sigma, \theta + \sigma]$, $P_\sigma(x|\theta) = (x - \theta + \sigma)/2\sigma$, we have that $x_\sigma^*(\theta) = (1 + 2\sigma)\theta - \sigma$.

    For any $\hat{\theta} \in [0, 1]$, let $\Gamma^{\hat{\theta}} \equiv \{\{0, 1\}, \pi^{\hat{\theta}}\}$ be the deterministic monotone policy with cutoff $\hat{\theta}$. Next, let $V_\sigma^{\Gamma^{\hat{\theta}}}(\cdot)$ be the function defined by $V_\sigma^{\Gamma^{\hat{\theta}}}(\theta) \equiv U_\sigma^{\Gamma^{\hat{\theta}}}(x_\sigma^*(\theta), 1|x_\sigma^*(\theta))$ for any $\theta \in [\hat{\theta}/(1+2\sigma), 1]$. This function represents the expected payoff differential between investing and not investing of the marginal investor with signal $x_\sigma^*(\theta)$, when each investor invests if and only if their signal is above $x_\sigma^*(\theta)$ (and hence default occurs if, and only if, fundamentals are below $\theta$), the quality of the investors' signal is $\sigma$, and the policy $\Gamma^{\hat{\theta}}$ announces that $s = 1$, thus revealing that $\theta \geq \hat{\theta}$. Note that, for any $0 \leq \theta < \hat{\theta}/(1+2\sigma)$, $x_\sigma^*(\theta) + \sigma < \hat{\theta}$, which implies that the signal $x_\sigma^*(\theta)$ is not consistent with the event that fundamentals are above $\hat{\theta}$. Equivalently, when $\theta \geq \hat{\theta}$, the lowest possible signal that an individual may receive is $\hat{\theta} - \sigma$. When each investor invests if and only if $x > \hat{\theta} - \sigma$, default occurs if and only if $\theta \leq \hat{\theta}/(1+2\sigma)$. Hence, the lowest default threshold that is consistent with the policy $\Gamma^{\hat{\theta}}$ is $\hat{\theta}/(1+2\sigma)$. The function $V_\sigma^{\Gamma^{\hat{\theta}}}(\theta)$ is thus defined only for $\theta \in [\hat{\theta}/(1+2\sigma), 1]$.

    The cutoff $\theta_\sigma^*$ characterizing the optimal deterministic monotone policy is given by

$$\theta_\sigma^* = \inf\{\hat{\theta} \in [0, 1] : V_\sigma^{\Gamma^{\hat{\theta}}}(\theta) \geq 0 \text{ for all } \theta \in [\hat{\theta}/(1+2\sigma), 1]\}. \tag{S1}$$

**Claim S1.** *For any $\hat{\theta} \in [0, 1]$, $V_\sigma^{\Gamma^{\hat{\theta}}}(\cdot)$ has a unique minimizer. Letting*

$$\theta_\sigma^{\min}(\hat{\theta}) \equiv \arg \min_{\theta \in [\hat{\theta}/(1+2\sigma), 1]} V_\sigma^{\Gamma^{\hat{\theta}}}(\theta),$$

*we have that $\theta_\sigma^{\min}(\hat{\theta})$ satisfies $x_\sigma^*(\theta_\sigma^{\min}(\hat{\theta})) - \sigma = \hat{\theta}$.*

    **Proof of Claim S1.** Clearly, for any $\theta \in [\hat{\theta}/(1+2\sigma), \hat{\theta}]$, $V_\sigma^{\Gamma^{\hat{\theta}}}(\theta) = g$. This is because

when each investor invests if and only if $x > x_\sigma^* (\theta)$ default occurs only for fundamentals below $\theta$. Hence the announcement that $\theta > \hat{\theta}$ reveals to the marginal investor with signal $x_\sigma^* (\theta)$ that default will not occur.

Next, observe that for any $\theta \in (\hat{\theta}, (\hat{\theta} + 2\sigma)/ (1 + 2\sigma)]$, $x_\sigma^* (\theta) - \sigma < \hat{\theta}$, implying that[1]

$$V_\sigma^{\Gamma^{\hat{\theta}}} (\theta) = g - (g + |b|) \, \mathbb{P}_\sigma[\tilde{\theta} \leq \theta | \tilde{\theta} \geq \hat{\theta}; x_\sigma^* (\theta)] = g - (g + |b|) \, \frac{\theta - \hat{\theta}}{(1 + 2\sigma) \theta - \hat{\theta}},$$

which is strictly decreasing in $\theta$. Finally, note that, for any $\theta \in ((\hat{\theta} + 2\sigma)/ (1 + 2\sigma), 1]$, $x_\sigma^* (\theta) - \sigma > \hat{\theta}$, implying that

$$V_\sigma^{\Gamma^{\hat{\theta}}} (\theta) = g - (g + |b|) \, \mathbb{P}_\sigma[\tilde{\theta} \leq \theta | \tilde{\theta} \geq \hat{\theta}; x_\sigma^* (\theta)] = g + (g + |b|) \, (\theta - 1) \,,$$

which is strictly increasing in $\theta$. Hence, $V_\sigma^{\Gamma^{\hat{\theta}}} (\cdot)$ has a single minimizer over $[\hat{\theta}/(1 + 2\sigma), 1]$. The latter is equal to $\theta_\sigma^{min}(\hat{\theta}) = (\hat{\theta} + 2\sigma)/ (1 + 2\sigma)$ and is such that $x_\sigma^*(\theta_\sigma^{min}(\hat{\theta})) - \sigma = \hat{\theta}$. $\square$

Next, let $\Gamma^{\theta_\sigma^*} \equiv (\{0, 1\}, \pi^{\theta_\sigma^*})$ be the optimal deterministic monotone policy (with cut-off $\hat{\theta} = \theta_\sigma^*$). Using the characterization of $\theta_\sigma^*$ in (S1), we thus have that, under $\Gamma^{\theta_\sigma^*}$, at the point $\theta_\sigma^{min} (\theta_\sigma^*)$ at which $V_\sigma^{\Gamma^{\theta_\sigma^*}}$ reaches its minimum, $V_\sigma^{\Gamma^{\theta_\sigma^*}} (\theta_\sigma^{min} (\theta_\sigma^*)) = 0$. Using the fact that

$$V_\sigma^{\Gamma^{\theta_\sigma^*}} (\theta_\sigma^{min} (\theta_\sigma^*)) = g - (g + |b|) \, \frac{\theta_\sigma^{min} (\theta_\sigma^*) - \theta_\sigma^*}{(1 + 2\sigma) \theta_\sigma^{min} (\theta_\sigma^*) - \theta_\sigma^*},$$

we then have that $\theta_\sigma^* = (1 + 2\sigma) \frac{|b|}{g+|b|} - 2\sigma$. Next, let $\Gamma_\emptyset$ be the no-disclosure policy and note that, for any $\theta \in [0, 1]$,

$$V_\sigma^{\Gamma_\emptyset} (\theta) = g - (g + |b|) \, \mathbb{P}_\sigma[\tilde{\theta} \leq \theta | x_\sigma^* (\theta)] = g + (g + |b|) \, (\theta - 1) \,,$$

which is increasing in $\theta$ and has a unique zero at $\theta = |b|/ (g + |b|) \equiv \theta^{MS}$.

This means that, in the absence of any disclosure, under the unique rationalizable strategy profile (and hence under MARP), each agent invests if and only if $x > x_\sigma^* (\theta^{MS})$, and default occurs if and only if fundamentals are below $\theta^{MS}$. The results above then imply that the optimal deterministic policy $\Gamma^{\theta_\sigma^*}$ is defined by a threshold $\theta_\sigma^* = (1 + 2\sigma) \theta^{MS} - 2\sigma = x_\sigma^* (\theta^{MS}) - \sigma$ that coincides with the left end-point of the support of the posterior beliefs of each agent

---

[1]The notation $\mathbb{P}_\sigma \left[\tilde{\theta} \leq \theta | \tilde{\theta} \geq \hat{\theta}; x\right]$ stands for the probability that an investor with signal $x$ assigns to the event that $\tilde{\theta} \leq \theta$ when the quality of his exogenous signal is parametrized by $\sigma$ and the policy reveals that $\tilde{\theta} \geq \hat{\theta}$.

with signal $x^*_\sigma\left(\theta^{MS}\right)$. In fact, for any truncation point $\hat{\theta} < x^*_\sigma\left(\theta^{MS}\right) - \sigma$, there exists $\theta$ close to $\theta^{MS}$ such that $V^{\Gamma^{\hat{\theta}}}_\sigma\left(\theta\right) < 0$ implying that refraining from investing for all $x < x^*_\sigma\left(\theta^{MS}\right)$ is rationalizable in the continuation game following the announcement that $\theta \geq \hat{\theta}$, implying that the policy $\Gamma^{\hat{\theta}}$ fails to satisfy PCP. Similarly, for any truncation point $\hat{\theta} > x^*_\sigma(\theta^{MS}) - \sigma$, $V^{\Gamma^{\hat{\theta}}}_\sigma\left(\theta\right)$ reaches its minimum at $\theta^{min}_\sigma(\hat{\theta}) > \theta^{MS}$ and is such that $V^{\Gamma^{\hat{\theta}}}_\sigma(\theta^{min}_\sigma(\hat{\theta})) = V^{\Gamma_\emptyset}_\sigma(\theta^{min}_\sigma(\hat{\theta})) > V^{\Gamma_\emptyset}_\sigma(\theta^{MS}) = 0$, where the inequality follows from the monotonicity of $V^{\Gamma_\emptyset}_\sigma\left(\cdot\right)$. Hence, $\theta^*_\sigma = x^*_\sigma\left(\theta^{MS}\right) - \sigma$.

**Step 2.** Having characterized the optimal deterministic monotone policy $\Gamma^{\theta^*_\sigma}$, we now show that, when $\sigma$ is small, there exists another policy $\Gamma$ that also satisfies PCP and guarantees no default for a larger set of fundamentals than $\Gamma^{\theta^*_\sigma}$.

Let $\sigma^\# \equiv \frac{\theta^{MS}}{2(1-\theta^{MS})} > 0$. For any $\sigma \in (0, \sigma^\#)$, $\theta^*_\sigma = (1+2\sigma)\theta^{MS} - 2\sigma > 0$. For any $\sigma, \delta, \gamma > 0$ small, let $\theta''_\sigma(\delta, \gamma) \equiv x^*_\sigma\left(\theta^{MS} - \delta\right) - \sigma = (1+2\sigma)(\theta^{MS} - \delta) - 2\sigma$ and $\theta'_\sigma(\delta, \gamma) \equiv \theta''_\sigma(\delta, \gamma) - \gamma$. Note that, for any $\sigma \in (0, \sigma^\#)$, $\delta > 0$ and $\gamma > 0$ can be chosen so that $0 < \theta'_\sigma(\delta, \gamma) < \theta''_\sigma(\delta, \gamma) < \theta^*_\sigma$.

Consider the non-monotone deterministic policy $\Gamma_{\delta,\gamma} \equiv \{\{0,1\}, \pi_{\delta,\gamma}\}$ given by

$$\pi_{\delta,\gamma}\left(1|\theta\right) \equiv \mathbf{1}\left\{\theta \in \left[\theta'_\sigma(\delta, \gamma), \theta''_\sigma(\delta, \gamma)\right] \cup \left[\theta^*_\sigma, \infty\right)\right\}.$$

We show that, for any $\sigma \in (0, \sigma^\#)$, there exit $\delta, \gamma > 0$ such that (i) $0 \leq \theta'_\sigma(\delta, \gamma) < \theta''_\sigma(\delta, \gamma) < \theta^*_\sigma$, and (ii) $V^{\Gamma_{\delta,\gamma}}_\sigma\left(\theta\right) \geq 0$ for all $\theta > \theta'_\sigma(\delta, \gamma)/(1 + 2\sigma)$, with $V^{\Gamma_{\delta,\gamma}}_\sigma\left(\theta\right) = 0$ only for $\theta = \theta^{MS}$.[2]

First observe that, for any $\sigma \in (0, \sigma^\#)$, $\delta \in \left(0, \theta^{MS} - \frac{2\sigma}{1+2\sigma}\right)$ and

$$0 < \gamma \leq (1 + 2\sigma)\left(\theta^{MS} - \delta\right) - 2\sigma \equiv R_0\left(\delta, \theta^{MS}, \sigma\right)$$

guarantee that $0 \leq \theta'_\sigma(\delta, \gamma) < \theta''_\sigma(\delta, \gamma) < \theta^*_\sigma$.[3]

Next note that, for any $(\sigma, \delta, \gamma)$ with $\sigma \in (0, \sigma^\#)$, $\delta \in \left(0, \theta^{MS} - 2\sigma/(1 + 2\sigma)\right)$ and $0 < \gamma \leq R_0\left(\delta, \theta^{MS}, \sigma\right)$, $V^{\Gamma_{\delta,\gamma}}_\sigma\left(\theta\right) = V^{\Gamma^{\theta^*_\sigma}}_\sigma\left(\theta\right)$ for all $\theta \in \left[\theta^{MS} - \delta, 1\right]$. Indeed, for any $\theta \in \left[\theta^{MS} - \delta, 1\right]$, $x^*_\sigma\left(\theta\right) - \sigma > \theta''_\sigma(\delta, \gamma)$ implying that the the posterior beliefs of the marginal investor with signal $x^*_\sigma\left(\theta\right)$ under the policy $\Gamma_{\delta,\gamma}$ coincide with those under the policy $\Gamma^{\theta^*_\sigma}$.

---

[2]Consistently with the notation above, $V^{\Gamma_{\delta,\gamma}}_\sigma\left(\theta\right)$ is the expected payoff of the marginal investor with signal $x^*_\sigma(\theta)$ when the policy $\Gamma_{\delta,\gamma}$ announces that $s = 1$ and the quality of the agents' exogenous signals is parametrized by $\sigma$. For any $\theta < \theta'_\sigma(\delta, \gamma)/(1 + 2\sigma)$, $x^*_\sigma(\theta) + \sigma < \theta'$, which implies that the signal $x^*_\sigma(\theta)$ is not consistent with the event that fundamentals are above $\theta'_\sigma(\delta, \gamma)$. Equivalently, because the lowest signal that is consistent with $\theta \in [\theta'_\sigma(\delta, \gamma), \theta''_\sigma(\delta, \gamma)] \cup [\theta^*_\sigma, \infty)$ is $\theta'_\sigma(\delta, \gamma) - \sigma$, the lowest default threshold is $\theta'_\sigma(\delta, \gamma)/(1+2\sigma)$.

[3]Observe that $\sigma \in (0, \sigma^\#)$ implies that $\theta^{MS} - 2\sigma/(1 + 2\sigma) > 0$. In turn, $\delta \in \left(0, \theta^{MS} - 2\sigma/(1 + 2\sigma)\right)$ implies that $0 < \theta''_\sigma(\delta, \gamma) < \theta^*_\sigma$ and that $R_0\left(\delta, \theta^{MS}, \sigma\right) > 0$. Finally, that $0 < \gamma \leq R_0\left(\delta, \theta^{MS}, \sigma\right)$ implies that $0 \leq \theta'_\sigma(\delta, \gamma) < \theta''_\sigma(\delta, \gamma)$.

Let $\theta_\sigma^\sharp(\delta,\gamma)$ be such that $x_\sigma^*\left(\theta_\sigma^\sharp(\delta,\gamma)\right) - \sigma = \theta_\sigma'(\delta,\gamma)$. Dropping the arguments of $\theta_\sigma^\sharp(\delta,\gamma)$, $\theta_\sigma'(\delta,\gamma)$ and $\theta_\sigma''(\delta,\gamma)$ to ease the notation, we have that

$$\theta' = \theta'' - \gamma = x_\sigma^*\left(\theta^{MS} - \delta\right) - \sigma - \gamma = (1 + 2\sigma)\left(\theta^{MS} - \delta\right) - 2\sigma - \gamma.$$

From the definition of $\hat{\theta}$ we have that $x_\sigma^*(\hat{\theta}) - \sigma = (1+2\sigma)\theta^\sharp - 2\sigma = \theta'$. Combining the above two results we obtain that $\theta^\sharp = \theta^{MS} - \delta - \gamma/(1+2\sigma)$. Fixing $\sigma \in (0, \sigma^\#)$, note that, for $\delta, \gamma > 0$ small, $\theta^\sharp \geq \theta_\sigma^*$. Specifically, for any $\sigma \in (0, \sigma^\#)$ and any $0 < \delta < 2\sigma\left(1 - \theta^{MS}\right)$, $\theta^\sharp \geq \theta_\sigma^*$ if and only if

$$\gamma \leq (1 + 2\sigma)\left(2\sigma\left(1 - \theta^{MS}\right) - \delta\right) \equiv R_1\left(\delta, \theta^{MS}, \sigma\right).$$

Next, observe that, for any $\theta \in \left[\theta^\sharp, \theta^{MS} - \delta\right)$,

$$\begin{aligned}
V_\sigma^{\Gamma_{\delta,\gamma}}(\theta) &= g - (g + |b|)\, \mathbb{P}_\sigma[\tilde{\theta} \leq \theta | \tilde{\theta} \in [x_\sigma^*(\theta) - \sigma, \theta''] \cup [\theta_\sigma^*, \infty); x_\sigma^*(\theta)] \\
&= g - (g + |b|)\left(\theta'' - \theta_\sigma^* + 2\sigma(1 - \theta)\right) / \left(\theta'' - \theta_\sigma^* + 2\sigma\right),
\end{aligned}$$

which is strictly increasing in $\theta$. Similarly, for any $\theta \in \left[\theta_\sigma^*, \theta^\sharp\right)$,

$$\begin{aligned}
V_\sigma^{\Gamma_{\delta,\gamma}}(\theta) &= g - (g + |b|)\, \mathbb{P}_\sigma[\tilde{\theta} \leq \theta | \tilde{\theta} \in [\theta', \theta''] \cup [\theta_\sigma^*, \infty); x_\sigma^*(\theta)] \\
&= g - (g + |b|)\frac{\theta - \theta_\sigma^* + \gamma}{x_\sigma^*(\theta) + \sigma - \theta_\sigma^* + \gamma} = g - (g + |b|)\frac{\theta - \theta_\sigma^* + \gamma}{(1 + 2\sigma)\theta - \theta_\sigma^* + \gamma},
\end{aligned}$$

which is strictly deceasing for any $\gamma \leq \theta_\sigma^*$. Note that $\theta' \geq 0$ requires that $\gamma \leq \theta_\sigma^*$. Next, note that, for $\theta \in [\theta'', \theta_\sigma^*)$,

$$\begin{aligned}
V_\sigma^{\Gamma_{\delta,\gamma}}(\theta) &= g - (g + |b|)\, \mathbb{P}_\sigma[\tilde{\theta} \leq \theta | \tilde{\theta} \in [\theta', \theta''] \cup [\theta_\sigma^*, \infty); x_\sigma^*(\theta)] \\
&= g - (g + |b|)\frac{\gamma}{x_\sigma^*(\theta) + \sigma - \theta_\sigma^* + \gamma} = g - (g + |b|)\frac{\gamma}{(1 + 2\sigma)\theta - \theta_\sigma^* + \gamma},
\end{aligned}$$

and, therefore, $V_\sigma^{\Gamma_{\delta,\gamma}}(\cdot)$ is increasing over the range $[\theta'', \theta_\sigma^*)$. Finally, for $\theta \in [\theta', \theta'')$, we have that

$$\begin{aligned}
V_\sigma^{\Gamma_{\delta,\gamma}}(\theta) &= g - (g + |b|)\, \mathbb{P}_\sigma[\tilde{\theta} \leq \theta | \tilde{\theta} \in [\theta', \theta''] \cup [\theta_\sigma^*, \infty); x_\sigma^*(\theta)] \\
&= g - (g + |b|)\frac{\theta - \theta'}{x_\sigma^*(\theta) + \sigma - \theta_\sigma^* + \gamma} = g - (g + |b|)\frac{\theta - \theta'}{(1 + 2\sigma)\theta - \theta_\sigma^* + \gamma}.
\end{aligned}$$

Hence $V_\sigma^{\Gamma_{\delta,\gamma}}(\cdot)$ is decreasing over $[\theta', \theta'')$ if $(1 + 2\sigma)\theta' = x_\sigma^*(\theta') + \sigma > \theta_\sigma^*$. Using the fact that $\theta' = \theta'' - \gamma$, together with the fact that $\theta'' = x_\sigma^*\left(\theta^{MS} - \delta\right) - \sigma$ and $\theta_\sigma^* = (1 + 2\sigma)\theta^{MS} - 2\sigma$,

5

we have that $(1 + 2\sigma)\theta' > \theta^*_\sigma$ if

$$\gamma \quad < \quad 2\sigma \left[(1 + 2\sigma)\theta^{MS} - 2\sigma\right] / (1 + 2\sigma) - (1 + 2\sigma)\delta \equiv R_2\left(\delta, \theta^{MS}, \sigma\right).$$

Lastly, observe that, for any $\theta \in [\theta'/(1 + 2\sigma), \theta']$, $V_\sigma^{\Gamma_{\delta,\gamma}}(\theta) = g$.

We thus have that the function $V_\sigma^{\Gamma_{\delta,\gamma}}$ is such that (1) $V_\sigma^{\Gamma_{\delta,\gamma}}(\theta) \geq 0$ for all $\theta \geq \theta'/(1 + 2\sigma)$, and (2) $V_\sigma^{\Gamma_{\delta,\gamma}}(\theta) = 0$ only if $\theta = \theta^{MS}$, if and only if the following conditions hold: (a) $V_\sigma^{\Gamma_{\delta,\gamma}}(\theta^\sharp) > 0$, and (b) $V_\sigma^{\Gamma_{\delta,\gamma}}(\theta'') > 0$. Requiring that $V_\sigma^{\Gamma_{\delta,\gamma}}(\theta^\sharp) > 0$ is equivalent to

$$g - (g + |b|)\left(\theta^\sharp - \theta^*_\sigma + \gamma\right) / \left(x^*_\sigma\left(\theta^\sharp\right) + \sigma - \theta^*_\sigma + \gamma\right) \quad > \quad 0$$
$$\Leftrightarrow \theta^{MS}\left(\theta^*_\sigma - \gamma\right) - \left((1 + 2\sigma)\theta^{MS} - 2\sigma\right)\theta^\sharp \quad > \quad 0.$$

Recall that $\theta^*_\sigma = (1 + 2\sigma)\theta^{MS} - 2\sigma$. Using the fact that $\theta^\sharp = \theta^{MS} - \delta - \frac{\gamma}{1 + 2\sigma}$, we conclude that a sufficient condition for $V_\sigma^{\Gamma_{\delta,\gamma}}(\theta^\sharp) > 0$ is that

$$\left(\theta^{MS} - \delta - \gamma/(1 + 2\sigma)\right)\theta^*_\sigma \quad < \quad \theta^{MS}\left(\theta^*_\sigma - \gamma\right)$$
$$\Leftrightarrow \gamma \quad < \quad \delta(1 + 2\sigma)\left((1 + 2\sigma)\theta^{MS} - 2\sigma\right) / (2\sigma) \equiv R_3\left(\delta, \theta^{MS}, \sigma\right).$$

Next, observe that $V_\sigma^{\Gamma_{\delta,\gamma}}(\theta'') > 0$ is equivalent to

$$\gamma \quad < \quad \left(1 - \theta^{MS}\right)\left((1 + 2\sigma)\theta'' - \theta^*_\sigma + \gamma\right)$$
$$\Leftrightarrow \gamma \quad < \quad \left(\frac{1 - \theta^{MS}}{\theta^{MS}}\right)\left((1 + 2\sigma)\left[(1 + 2\sigma)\left(\theta^{MS} - \delta\right) - 2\sigma\right] - (1 + 2\sigma)\theta^{MS} + 2\sigma\right) \equiv R_4\left(\delta, \theta^{MS}, \sigma\right).$$

We conclude that, for any $\sigma \in (0, \sigma^\#)$, (i) $0 \leq \theta'_\sigma(\delta, \gamma) < \theta''_\sigma(\delta, \gamma) < \theta^*_\sigma$, and (ii) $V_\sigma^{\Gamma_{\delta,\gamma}}(\theta) \geq 0$ for all $\theta > \theta'_\sigma(\delta, \gamma)/(1 + 2\sigma)$, with $V_\sigma^{\Gamma_{\delta,\gamma}}(\theta) = 0$ only for $\theta = \theta^{MS}$, if

$$0 < \delta < \min\left\{\theta^{MS} - \frac{2\sigma}{1 + 2\sigma}, 2\sigma\left(1 - \theta^{MS}\right), \frac{2\sigma\left[(1 + 2\sigma)\theta^{MS} - 2\sigma\right]}{(1 + 2\sigma)^2}, \frac{2\sigma}{1 + 2\sigma}\left[\theta^{MS} - \frac{2\sigma}{1 + 2\sigma}\right]\right\} \equiv \varsigma\left(\theta^{MS}, \sigma\right)$$

and $0 < \gamma < \min_{0 \leq i \leq 4} R_i\left(\delta, \theta^{MS}, \sigma\right)$. Note that $\sigma < \sigma^\#$ implies that $\varsigma\left(\theta^{MS}, \sigma\right) > 0$, whereas $\delta < \varsigma\left(\theta^{MS}, \sigma\right)$ implies that $\min_{0 \leq i \leq 4} R_i\left(\delta, \theta^{MS}, \sigma\right) > 0$. Finally note that, for any $\sigma \in (0, \sigma^\#)$, and any $\theta \geq \theta'_\sigma(\delta, \gamma)$, the payoff $V_\sigma^{\Gamma_{\delta,\gamma}}(\theta)$ is continuous in the threshold $\theta^*_\sigma$. Hence there exists a policy $\Gamma$ whose rule $\pi$ is given by $\pi(1|\theta) \equiv \mathbf{1}\{\theta \in [\theta'_\sigma(\delta, \gamma), \theta''_\sigma(\delta, \gamma)] \cup [\theta^*_\sigma + \varepsilon, \infty)\}$ with $\varepsilon > 0$ arbitrarily small, such that $\Gamma$ strictly improves over $\Gamma^{\theta^*_\sigma}$ and is such that $V_\sigma^\Gamma(\theta) > 0$ for all $\theta > \theta'_\sigma(\delta, \gamma)/(1 + 2\sigma)$, implying that $\Gamma$ satisfies PCP. Q.E.D.

6

# S2. Proof of Example 3 in Main Text

**Preliminaries.** For any $\theta \in (0,1)$, any $\sigma \in \mathbb{R}_+$, let $x_\sigma^*(\theta)$ be the critical signal threshold such that, when all agents invest for $x > x_\sigma^*(\theta)$ and refrain from investing for $x < x_\sigma^*(\theta)$, default occurs if and only if the fundamentals are below $\theta$. Note that, in this example $x_\sigma^*(\theta) \equiv \theta + \sigma\Phi^{-1}(\theta)$, where $\Phi$ is the cdf of the standard Normal distribution and $\phi$ its density. Also let $x_\sigma^*(0) \equiv -\infty$ and $x_\sigma^*(1) \equiv +\infty$. For any $(\theta_0, \hat{\theta}, \sigma) \in (0,1) \times \mathbb{R} \times \mathbb{R}_+$, let $\psi(\theta_0, \hat{\theta}, \sigma)$ denote the payoff from investing of an investor with private signal $x_\sigma^*(\theta_0)$, when default occurs if and only if $\theta \leq \theta_0$, the policy reveals that $\theta \geq \hat{\theta}$, and the precision of private information is $\sigma^{-2}$. Then let $\hat{\sigma} \equiv \inf\{\sigma \in \mathbb{R}_+ : \psi(\theta_0, 0, \sigma) > 0 \text{ all } \theta_0 \in (0,1)\}$ if $\{\sigma \in \mathbb{R}_+ : \psi(\theta_0, 0, \sigma) > 0 \text{ all } \theta_0 \in (0,1)\} \neq \emptyset$ and else $\hat{\sigma} = +\infty$.[4] Then let $\Psi(\sigma) \equiv \inf_{\theta_0 \in (0,1)} \psi(\theta_0, 0, \sigma)$ and note that $\lim_{\sigma \to 0^+} \Psi(\sigma) < 0$, implying that $\hat{\sigma} > 0$. For any $\sigma \in \mathbb{R}_+$ for which $\psi(\theta_0, 0, \sigma) > 0$ for all $\theta_0 \in (0,1)$, the policy maker can avoid default for every $\theta > 0$ by using the monotone rule $\pi(\theta) = \mathbf{1}\{\theta > 0\}$. This case is uninteresting. Hereafter, we thus confine attention to the case in which $\sigma < \hat{\sigma}$.

Let $U_\sigma^\Gamma(x, 1|x)$ denote the payoff from investing of an agent with signal $x$ who expects all other agents to invest if and only if their signal exceeds $x$, when the precision of private information is $\sigma^{-2}$, and the policy $\Gamma$ announces that $s = 1$. Also let

$$U_\sigma^\Gamma(x_\sigma^*(0), 1|x_\sigma^*(0)) \equiv \lim_{x \to -\infty} U_\sigma^\Gamma(x, 1|x)$$

and

$$U_\sigma^\Gamma(x_\sigma^*(1), 1|x_\sigma^*(1)) \equiv \lim_{x \to +\infty} U_\sigma^\Gamma(x, 1|x).$$

Now let $\mathbb{G}_\sigma$ denote the set of deterministic binary policies $\Gamma = (\{0,1\}, \pi)$ such that, $\pi(\theta) = 0$ for all $\theta \leq 0$, $\pi(\theta) = 1$ for all $\theta > 1$ and $U_\sigma^\Gamma(x, 1|x) \geq 0$ for all $x \in \mathbb{R}$.[5] From the proofs of Theorems 1 and 2, observe that, given any $\sigma$, any deterministic binary policy $\Gamma$ satisfying PCP and such that $\pi(\theta) = 0$ for all $\theta \leq 0$ and $\pi(\theta) = 1$ for all $\theta > 1$ belongs in $\mathbb{G}_\sigma$. However, $\mathbb{G}_\sigma$ contains also policies that do not satisfy PCP.[6]

---

[4]Recall that, when the announcement that $s = 1$ reveals that $\theta \geq 0$, the unique rationalizable profile features all agents investing, irrespective of $x$, if and only if $\psi(\theta_0, 0, \sigma) > 0$ for all $\theta_0 \in (0,1)$.

[5]We let $\pi(\theta) = 1$ (alternatively, $\pi(\theta) = 0$) denote the degenerate lottery assigning measure 1 to $s = 1$ (alternatively, $s = 0$).

[6]These are those for which there exists $x$ such that $U_\sigma^\Gamma(x, 1|x) = 0$; when $\Gamma$ announces $s = 1$, in addition to

**Proof Structure.** The proof is in four steps. Step 1 establishes that, when $\sigma$ is small, under any policy $\Gamma = (\{0,1\}, \pi) \in \mathbb{G}_\sigma$, any interval $(\theta', \theta''] \subset (0, \theta^{MS}]$ receiving a pass grade (i.e., such that $\pi(\theta) = 1$ for all $\theta \in (\theta', \theta'']$) has a sufficiently small Lebesgue measure, with the measure vanishing as $\sigma \to 0^+$.

Step 2 then considers an *auxiliary game* $G_\sigma$ in which the agents play less aggressively than under MARP. Namely, $G_\sigma$ is the game in which (i) the policy maker's choice set is $\mathbb{G}_\sigma$ and (ii) given *any* policy $\Gamma \in \mathbb{G}_\sigma$, all agents invest after receiving the signal $s = 1$ and refrain from investing after receiving the signal $s = 0$.[7] We show that, when $\sigma$ is small, given any policy $\Gamma \in \mathbb{G}_\sigma$ that gives a fail grade to an interval $(\theta', \theta''] \subseteq (\underline{\theta}, \theta^{MS}]$ of large Lebesgue measure, there exists another policy $\Gamma^\# \in \mathbb{G}_\sigma$ that gives a pass grade to a $F$-positive measure subset of $(\theta', \theta'']$, has a mesh smaller than $\Gamma$, and is such that, when agents play as in $G_\sigma$, the probability of default under $\Gamma^\#$ is strictly smaller than under $\Gamma$.

Step 3 then combines the results from Steps 1 and 2 to show that, when $\sigma$ is small, given any policy $\Gamma \in \mathbb{G}_\sigma$ for which the mesh $M(\Gamma)$ of $(0, \theta^{MS}]$ is larger than $\varepsilon$, there exists another policy $\Gamma' \in \mathbb{G}_\sigma$ with a mesh $M(\Gamma')$ smaller than $\varepsilon$ such that, when agents play as in $G_\sigma$, the probability of default is strictly smaller under $\Gamma'$ than under $\Gamma$. Starting from $\Gamma' \in \mathbb{G}_\sigma$ one can then construct a "nearby" policy $\Gamma^* \in \mathbb{G}_\sigma$ such that the probability of default under $\Gamma^*$ is arbitrarily close to that under $\Gamma'$ (and hence strictly smaller than under the original policy $\Gamma$) and such that $U_\sigma^{\Gamma^*}(x, 1|x) > 0$ for all $x$. The last property implies that $\Gamma^*$ satisfies PCP also when agents play according to MARP. The policy $\Gamma^*$ thus strictly improves upon $\Gamma$ also in the original game, as claimed in the main text.

Finally, step 4 closes the proof by showing how to construct the function $\mathcal{E}$ relating the noise $\sigma$ in the agents' exogenous private information to the bound $\mathcal{E}(\sigma)$ on the mesh of the policies.

**Step 1.** We start with the following result:

**Lemma S2-A.** *For any $\varepsilon \in \mathbb{R}_{++}$, there exists $\sigma(\varepsilon) \in \mathbb{R}_{++}$ such that, for any $\sigma \in (0, \sigma(\varepsilon)]$, the following is true: for any policy $\Gamma = (\{0,1\}, \pi) \in \mathbb{G}_\sigma$ and any cell $(\theta', \theta''] \in D^\Gamma$ with $|\theta'' - \theta'| > \varepsilon$, necessarily $\pi(\theta) = 0$.*

---

the rationalizable profile under which all agents invest, there also exists a rationalizable profile under which each agent invests if and only if his signal exceeds $x$.

[7]The agent's behavior is consistent with MARP only for those $\Gamma \in \mathbb{G}_\sigma$ for which, for all $x$, $U_\sigma^\Gamma(x, 1|x) > 0$.

8

**Proof of Lemma S2-A.** We first show (Property S2-A below) that, for any $\sigma > 0$, if the policy maker were to replace $\Gamma$ with the cutoff policy $\Gamma^{\theta'}$, then for any $\theta \leq \theta''$, $U_\sigma^{\Gamma^{\theta'}}(x_\sigma^*(\theta), 1 | x_\sigma^*(\theta)) \geq U_\sigma^{\Gamma}(x_\sigma^*(\theta), 1 | x_\sigma^*(\theta))$.[8] Next, we show (Property S2-B below) that, for any $\theta > \theta'$, as $\sigma$ goes to zero, $U_\sigma^{\Gamma^{\theta'}}(x_\sigma^*(\theta), 1 | x_\sigma^*(\theta))$ converges uniformly to $\int_0^1 u(\theta, A) \mathrm{d}A$. Because $\int_0^1 u(\theta, A) \mathrm{d}A < 0$ for $\theta < \theta^{MS}$, the above two properties imply that, for $\sigma$ small, $U_\sigma^{\Gamma}(x_\sigma^*(\theta), 1 | x_\sigma^*(\theta) < 0$ for some $\theta \in (\theta', \theta'']$, and hence that $\Gamma \notin \mathbb{G}_\sigma$. The result in the lemma then follows by contrapositive.

**Property S2-A.** *For any policy $\Gamma = (\{0,1\}, \pi) \in \mathbb{G}_\sigma$ and any cell $(\theta', \theta''] \in D^\Gamma$ such that $\pi(\theta) = 1$ for all $\theta \in (\theta', \theta'']$, $U_\sigma^{\Gamma}(x_\sigma^*(\theta), 1 | x_\sigma^*(\theta)) \leq U_\sigma^{\Gamma^{\theta'}}(x_\sigma^*(\theta), 1 | x_\sigma^*(\theta))$ for all $\theta \leq \theta''$.*

**Proof of Property S2-A.** The proof follows from Results S2-A-1 and S2-A-2 below.

**Result S2-A-1.** *Pick any policy $\Gamma = (\{0,1\}, \pi) \in \mathbb{G}_\sigma$. Given the partition $D^\Gamma \equiv \left\{ d_i = \left( \underline{\theta}_i, \bar{\theta}_i \right] : i = 1, ..., N \right\}$ of $(0, \theta^{MS}]$ induced by $\Gamma$, take any cell $d_i = \left( \underline{\theta}_i, \bar{\theta}_i \right]$ for which $\pi(\theta) = 1$ for all $\theta \in d_i$. Let $\Gamma_L^i = \{\{0,1\}, \pi_L^i\} \in \mathbb{G}_\sigma$ be the policy constructed as follows: (a) $\pi_L^i(\theta) = 0$ for all $\theta \leq \underline{\theta}_i$; and (b) $\pi_L^i(\theta) = \pi(\theta)$ for all $\theta > \underline{\theta}_i$. Then, for all $\theta \in [0,1]$, $U_\sigma^{\Gamma_L^i}(x_\sigma^*(\theta), 1 | x_\sigma^*(\theta)) \geq U_\sigma^{\Gamma}(x_\sigma^*(\theta), 1 | x_\sigma^*(\theta))$.*

**Proof of Result S2-A-1.** Note that, under the new policy, $\pi_L^i(\theta) = \pi(\theta) \times \mathbf{1}\{\theta > \underline{\theta}_i\}$. The posterior beliefs $\Lambda_\sigma^{\Gamma_L^i}(\cdot | x, 1)$ about $\theta$ of an agent with exogenous signal $x$ and endogenous signal $s = 1$ under the new policy $\Gamma_L^i$ thus dominate, in the FOSD sense, the analogous beliefs $\Lambda_\sigma^\Gamma(\cdot | x, 1)$ under the original policy $\Gamma$.[9] The result then follows from the fact that, given any default threshold $\theta$, the payoff differential from investing when the fundamentals are equal to $\tilde{\theta}$ and default occurs if and only if $\tilde{\theta} \leq \theta$ is nondecreasing in $\tilde{\theta}$. *End of Proof of Result S2-A-1.*

**Result S2-A-2.** *Pick any policy $\Gamma = \{\{0,1\}, \pi\} \in \mathbb{G}_\sigma$. Given the partition $D^\Gamma \equiv \left\{ d_i = \left( \underline{\theta}_i, \bar{\theta}_i \right] : i = 1, ..., N \right\}$ of $(0, \theta^{MS}]$ induced by $\Gamma$, take any cell $d_i = \left( \underline{\theta}_i, \bar{\theta}_i \right]$, $i \geq 2$, for which $\pi(\theta) = 1$ for all $\theta \in d_i$. Let $\Gamma_R^i = \{\{0,1\}, \pi_R^i\} \in \mathbb{G}_\sigma$ be the policy constructed from $\Gamma$ as follows: (a) $\pi_R^i(\theta) = \pi(\theta)$ for all $\theta \leq \underline{\theta}_i$; and (b) $\pi_R^i(\theta) = 1$ for all $\theta > \underline{\theta}_i$. Then, for all $\theta \leq \bar{\theta}_i$, $U_\sigma^{\Gamma_R^i}(x_\sigma^*(\theta), 1 | x_\sigma^*(\theta)) \geq U_\sigma^{\Gamma}(x_\sigma^*(\theta), 1 | x_\sigma^*(\theta))$.*

**Proof of Result S2-A-2.** Let $\Theta^1 \equiv \{\theta \in \Theta : \pi(\theta) = 1\}$ and $\Theta_i^0 \equiv \{\theta \in (\underline{\theta}_i, 1] : \pi(\theta) = 0\}$.

---

[8]For any $\hat{\theta} \in [0,1]$, $\Gamma^{\hat{\theta}} = (\{0,1\}, \pi^{\hat{\theta}})$ is the deterministic monotone policy with cut-off $\hat{\theta}$.

[9]No matter the shape of the beliefs $\Lambda_\sigma^\Gamma(\cdot | x, 1)$, the announcement that $\theta > \underline{\theta}_i$ is always "good news" in the sense of Milgrom (1981) and hence $\Lambda_\sigma^{\Gamma_L^i}(\cdot | x, 1) \succ_{FOSD} \Lambda_\sigma^\Gamma(\cdot | x, 1)$.

For any $\theta^\# \leq \bar{\theta}_i$, and any $x$,

$$
\begin{aligned}
\Lambda_\sigma^{\Gamma_R^i}\left(\theta^\#|x,1\right) &= \mathbb{P}[\theta \leq \theta^\#|x, \theta \in (\Theta^1 \cup \Theta_i^0)] = \frac{\mathbb{P}[\theta \leq \theta^\# \wedge \theta \in (\Theta^1 \cup \Theta_i^0)|x]}{\mathbb{P}[\theta \in (\Theta^1 \cup \Theta_i^0)|x]} \\
&= \frac{\mathbb{P}[\theta \leq \theta^\# \wedge \theta \in \Theta^1|x]}{\mathbb{P}[\theta \in (\Theta^1 \cup \Theta_i^0)|x]} + \frac{\mathbb{P}[\theta \leq \theta^\# \wedge \theta \in \Theta_i^0|x]}{\mathbb{P}[\theta \in (\Theta^1 \cup \Theta_i^0)|x]} = \frac{\mathbb{P}[\theta \leq \theta^\# \wedge \theta \in \Theta^1|x]}{\mathbb{P}[\theta \in (\Theta^1 \cup \Theta_i^0)|x]} \\
&\leq \mathbb{P}[\theta \leq \theta^\#|x, \theta \in \Theta^1] = \Lambda_\sigma^\Gamma(\theta^\#|x,1).
\end{aligned}
$$

The first equality follows from the fact that, under the new policy $\Gamma_R^i$, the signal $s = 1$ carries the same information as the announcement that $\theta \in (\Theta^1 \cup \Theta_i^0)$. The third equality follows from the fact that $\Theta^1 \cap \Theta_i^0 = \emptyset$. The fourth equality follows from the fact that $\Theta_i^0$ contains only fundamentals above $\bar{\theta}_i$ and that $\theta^\# \leq \bar{\theta}_i$. The inequality follows from the fact that $\mathbb{P}[\theta \in (\Theta^1 \cup \Theta_i^0)|x] \geq \mathbb{P}[\theta \in \Theta^1|x]$ along with the definition of conditional probability. The last equality follows from the fact that, under the original policy $\Gamma$, the signal $s = 1$ carries the same information as the announcement that $\theta \in \Theta^1$. Given the above inequality, and the fact that $b < 0 < g$, we then have that, for any $\theta \leq \bar{\theta}_i$,

$$
\begin{aligned}
U_\sigma^\Gamma(x_\sigma^*(\theta), 1|x_\sigma^*(\theta)) &= b \cdot \Lambda_\sigma^\Gamma(\theta|x_\sigma^*(\theta), 1) + g \cdot \left(1 - \Lambda_\sigma^\Gamma(\theta|x_\sigma^*(\theta), 1)\right) \\
&\leq b \cdot \Lambda_\sigma^{\Gamma_R^i}(\theta|x_\sigma^*(\theta), 1) + g \cdot \left(1 - \Lambda_\sigma^{\Gamma_R^i}(\theta|x_\sigma^*(\theta), 1)\right) = U_\sigma^{\Gamma_R^i}(x_\sigma^*(\theta), 1|x_\sigma^*(\theta)).
\end{aligned}
$$

*End of Proof of Result S2-A-2.*

Property S2-A follows from Results S2-A-1 and S2-A-2, by taking the cell $d_i = (\theta', \theta'']$. $\square$

Now, fix $\varepsilon \in (0, \theta^{MS})$. For any $\theta^* \in [0, \theta^{MS} - \varepsilon]$, let $\Gamma^{\theta^*}$ be the monotone rule with cut-off $\theta^*$. For any $\theta^* \in [0, \theta^{MS} - \varepsilon]$, any $\sigma \in \mathbb{R}_{++}$, let

$$
H_\sigma(\theta^*; \varepsilon) \equiv \inf_{\theta \in [\theta^*, \theta^* + \varepsilon]} U_\sigma^{\Gamma^{\theta^*}}(x_\sigma^*(\theta), 1|x_\sigma^*(\theta)).
$$

Note that $U_\sigma^{\Gamma^{\theta^*}}(x_\sigma^*(\theta), 1|x_\sigma^*(\theta))$ is continuous in $(\theta^*, \theta, \sigma)$ over $[0, 1]^2 \times (0, \hat{\sigma}]$. From Berge's Maximum Theorem, $H_\sigma(\theta^*; \varepsilon)$ is thus continuous in $(\theta^*, \sigma)$ over $[0, \theta^{MS} - \varepsilon] \times (0, \hat{\sigma}]$.

For all $\theta^* \in [0, \theta^{MS} - \varepsilon]$, all $\theta \in (\theta^*, \theta^* + \varepsilon]$, $\lim_{\sigma \to 0^+} U_\sigma^{\Gamma^{\theta^*}}(x_\sigma^*(\theta), 1|x_\sigma^*(\theta)) = \int_0^1 u(\theta, A)\mathrm{d}A$. Because $\int_0^1 u(\theta, A)\mathrm{d}A$ is strictly increasing in $\theta$ and equal to zero at $\theta = \theta^{MS}$, for any $\theta^* \in [0, \theta^{MS} - \varepsilon]$, $H_{0^+}(\theta^*; \varepsilon) \equiv \lim_{\sigma \to 0^+} H_\sigma(\theta^*; \varepsilon) = \lim_{\sigma \to 0^+} \lim_{\theta \to \theta^{*+}} U_\sigma^{\Gamma^{\theta^*}}(x_\sigma^*(\theta), 1|x_\sigma^*(\theta)) = \int_0^1 u(\theta^*, A)\mathrm{d}A$. We show next that $H_\sigma(\cdot; \varepsilon)$ converges uniformly to the limit function $H_{0^+}(\cdot; \varepsilon)$ over $[0, \theta^{MS} - \varepsilon]$.

**Property S2-B.** *Fix $\varepsilon \in (0, \theta^{MS})$. For any $\epsilon < \varepsilon$, there exists $\sigma'(\epsilon) > 0$ such that, for*

*any $\sigma \leq \sigma'(\epsilon)$, and any $\theta^* \in [0, \theta^{MS} - \varepsilon]$, $|H_\sigma(\theta^*; \varepsilon) - H_{0+}(\theta^*; \varepsilon)| < \epsilon$.*

**Proof of Property S2-B.** The limit function $H_{0+}(\cdot; \varepsilon)$ is *uniformly* continuous over $[0, \theta^{MS} - \varepsilon]$. As a consequence, there exists $\delta > 0$ such that for any $\theta, \tilde{\theta} \in [0, \theta^{MS} - \varepsilon]$, with $|\tilde{\theta} - \theta| \leq \delta$, necessarily $|H_{0+}(\tilde{\theta}; \varepsilon) - H_{0+}(\theta; \varepsilon)| < \epsilon/2$. Next, let $D_\delta \equiv \{(\underline{\theta}_i, \bar{\theta}_i] : i = 1, ..., N\}$, $N \in \mathbb{N}$, be any interval partition of $(0, \theta^{MS} - \varepsilon]$ with the property that every cell $(\underline{\theta}_i, \bar{\theta}_i] \in D_\delta$ is such that $|\bar{\theta}_i - \underline{\theta}_i| \leq \delta$. For any $i = 1, ..., N$, any $\sigma > 0$, let $\hat{\theta}_\sigma^i \equiv \sup\{\arg\max_{\theta \in [\underline{\theta}_i, \bar{\theta}_i]} H_\sigma(\theta; \varepsilon)\}$. That $H_\sigma(\theta; \varepsilon)$ is continuous in $(\sigma, \theta)$ implies that the hypothesis of Berge's Maximum Theorem hold and, hence, the correspondence $\arg\max_{\theta \in [\underline{\theta}_i, \bar{\theta}_i]} H_\sigma(\theta; \varepsilon)$ is compact-valued and upper semi-continuous in $\sigma$. As a result, for any $\sigma > 0$, $\hat{\theta}_\sigma^i = \max\{\arg\max_{\theta \in [\underline{\theta}_i, \bar{\theta}_i]} H_\sigma(\theta; \varepsilon)\}$. Moreover, $\lim_{\sigma \to 0+} H_\sigma(\hat{\theta}_\sigma^i; \varepsilon) = H_{0+}(\hat{\theta}_{0+}^i; \varepsilon)$, where $\hat{\theta}_{0+}^i \equiv \lim_{\sigma \to 0+} \hat{\theta}_\sigma^i$.

For any $\theta^* \in [0, \theta^{MS} - \varepsilon]$, let $(\underline{\theta}_j, \bar{\theta}_j] \in D_\delta$ be the partition cell containing $\theta^*$. Then,

$$H_\sigma(\theta^*; \varepsilon) - H_{0+}(\theta^*; \varepsilon) \leq H_\sigma(\hat{\theta}_\sigma^j; \varepsilon) - H_{0+}(\theta^*; \varepsilon)$$
$$= H_\sigma(\hat{\theta}_\sigma^j; \varepsilon) - H_{0+}(\hat{\theta}_{0+}^j; \varepsilon) + H_{0+}(\hat{\theta}_{0+}^j; \varepsilon) - H_{0+}(\theta^*; \varepsilon) < H_\sigma(\hat{\theta}_\sigma^j; \varepsilon) - H_{0+}(\hat{\theta}_{0+}^j; \varepsilon) + \epsilon/2 < \epsilon$$

for all $\sigma < \bar{\sigma}_j(\epsilon)$, for some $\bar{\sigma}_j(\epsilon) > 0$. The first inequality is by definition of $\hat{\theta}_\sigma^i$. The second inequality follows from the fact that $|\hat{\theta}_{0+}^j - \theta^*| < \delta$. The last inequality follows from the fact that $\lim_{\sigma \to 0+} H_\sigma(\hat{\theta}_\sigma^j) = H_{0+}(\hat{\theta}_{0+}^j)$. Similar arguments imply that $H_\sigma(\theta^*; \varepsilon) - H_{0+}(\theta^*; \varepsilon) > -\epsilon$ for all $\sigma < \underline{\sigma}_j(\epsilon)$, for some $\underline{\sigma}_j(\epsilon) > 0$.

Now let $\sigma'(\epsilon) \equiv \min\{\min_{i \in N}\{\bar{\sigma}_i(\epsilon)\}, \min_{i \in N}\{\underline{\sigma}_i(\epsilon)\}\}$. For any $\sigma \leq \sigma'(\epsilon)$, and any $\theta^* \in [0, \theta^{MS} - \varepsilon]$, we thus have that $|H_\sigma(\theta^*; \varepsilon) - H_{0+}(\theta^*; \varepsilon)| < \epsilon$, thus proving that $H_\sigma(\cdot; \varepsilon)$ converges uniformly to $H_{0+}(\cdot; \varepsilon)$ as $\sigma \to 0^+$. This completes the proof of Property S2-B. $\square$

Next, given $\varepsilon \in (0, \theta^{MS})$, pick an arbitrary $\eta \in (\int_0^1 u(\theta^{MS} - \varepsilon, A)\mathrm{d}A, 0)$. Because $H_{0+}(\theta^*; \varepsilon) \leq \eta$ for all $\theta^* \in [0, \theta^{MS} - \varepsilon]$, and because $H_\sigma(\cdot; \varepsilon)$ converges uniformly to $H_{0+}(\cdot; \varepsilon)$, there exists $\sigma(\varepsilon) > 0$ such that, for any $\sigma < (\varepsilon)$, and any $\theta^* \in [0, \theta^{MS} - \varepsilon]$, $H_\sigma(\theta^*; \varepsilon) \leq \eta < 0$. Therefore, for any $\sigma < \sigma(\varepsilon)$, and any deterministic monotone policy $\Gamma^{\theta^*}$ with cut-off $\theta^* \in [0, \theta^{MS} - \varepsilon]$, there exists $\theta \in [\theta^*, \theta^* + \varepsilon]$ such that $U_\sigma^{\Gamma^{\theta^*}}(x_\sigma^*(\theta), 1|x_\sigma^*(\theta)) \leq \eta$.

Together, Properties S2-A and S2-B then imply that, for any $\sigma < \sigma(\varepsilon)$, and any policy $\Gamma$ such that $\pi(\theta) = 1$ for all $\theta \in (\theta', \theta'']$ for some $(\theta', \theta''] \in D^\Gamma$ with $|\theta'' - \theta'| > \varepsilon$, necessarily $U_\sigma^\Gamma(x_\sigma^*(\theta), 1|x_\sigma^*(\theta)) < 0$ for some $\theta \in (\theta', \theta'']$. Hence $\Gamma \notin \mathbb{G}_\sigma$. The claim in Lemma S2-A then follows by contrapositive. This completes the proof of Lemma S2-A. $\blacksquare$

11

**Step 2.** Next, we show that, for any policy $\Gamma = (\{0,1\}, \pi) \in \mathbb{G}_\sigma$ that gives a fail grade to an interval $(\theta', \theta''] \subseteq (0, \theta^{MS}]$ of large Lebesgue measure, there exists another policy $\Gamma^{\#} \in \mathbb{G}_\sigma$ with a mesh $M(\Gamma^{\#}) < M(\Gamma)$ such that, when agents play as in $G_\sigma$, the probability of default under $\Gamma^{\#}$ is strictly smaller than under $\Gamma$. The result follows from Lemmas S2-B, S2-C and S2-D below.

**Lemma S2-B.** *For any $\Gamma = (\{0,1\}, \pi) \in \mathbb{G}_\sigma$ such that $\inf_{\theta \in [0,1]} U_\sigma^\Gamma (x_\sigma^* (\theta), 1 | x_\sigma^* (\theta)) > 0$, there exists another policy $\hat{\Gamma} = (\{0,1\}, \hat{\pi}) \in \mathbb{G}_\sigma$, with $M(\hat{\Gamma}) \leq M(\Gamma)$, such that, in the auxiliary game $G_\sigma$, the probability of default under $\hat{\Gamma}$ is strictly smaller than under $\Gamma$.*

**Proof of Lemma S2-B.** That $\inf_{\theta \in [0,1]} U_\sigma^\Gamma (x_\sigma^* (\theta), 1 | x_\sigma^* (\theta)) > 0$ implies that, starting from $\Gamma = (\{0,1\}, \pi)$, one can construct another policy $\hat{\Gamma} = (\{0,1\}, \hat{\pi})$ sufficiently close to $\Gamma$ (in the $L_1$ norm) and such that $\hat{\pi}(\theta) \geq \pi(\theta)$ for all $\theta$, with the inequality strict over some positive $F$-measure set $(\tilde{\theta}', \tilde{\theta}'') \subseteq (0,1]$, and such that (a) $\hat{\pi}(\theta) = 0$ for all $\theta \leq 0$, (b) $\hat{\pi}(\theta) = 1$ for all $\theta > 1$, (c) $U_\sigma^{\hat{\Gamma}}(x, 1 | x) \geq 0$ all $x$, and (d) $M(\hat{\Gamma}) \leq M(\Gamma)$. By definition of $\mathbb{G}_\sigma$, $\hat{\Gamma} \in \mathbb{G}_\sigma$. That, in the auxiliary game $G_\sigma$, the probability of default under $\hat{\Gamma}$ is strictly smaller than under $\Gamma$, then follows from the fact that all agents invest when they receive the signal $s = 1$. This completes the proof of Lemma S2-B. ∎

For any $\sigma > 0$, and any policy $\Gamma = (\{0,1\}, \pi) \in \mathbb{G}_\sigma$, $U_\sigma^\Gamma (x_\sigma^* (\cdot), 1 | x_\sigma^* (\cdot))$ is continuous over $[0,1]$. Hence $\inf_{\theta \in [0,1]} U_\sigma^\Gamma (x_\sigma^* (\theta), 1 | x_\sigma^* (\theta)) = \min_{\theta \in [0,1]} U_\sigma^\Gamma (x_\sigma^* (\theta), 1 | x_\sigma^* (\theta))$.

**Lemma S2-C.** *Let $\Gamma = (\{0,1\}, \pi) \in \mathbb{G}_\sigma$ be such that $\min_{\theta \in [0,1]} U_\sigma^\Gamma (x_\sigma^* (\theta), 1 | x_\sigma^* (\theta)) = 0$. For any $\theta_\sigma^\sharp \in \arg \min_{\theta \in [0,1]} U_\sigma^\Gamma (x_\sigma^* (\theta), 1 | x_\sigma^* (\theta))$, there exists $\gamma_\sigma^\Gamma > 0$ such that $\pi(\theta) = 1$ for $F-$almost all $\theta \in (\theta_\sigma^\sharp - \gamma_\sigma^\Gamma, \theta_\sigma^\sharp)$.*

**Proof of Lemma S2-C.** The proof is by contraposition. Suppose there exists $\delta > 0$ such that $\pi(\theta) = 0$ for $F$-almost all $\theta \in (\theta_\sigma^\sharp - \delta, \theta_\sigma^\sharp)$. Observe that the sign of

$$U_\sigma^\Gamma \left( x_\sigma^* \left( \theta_\sigma^\sharp - \delta \right), 1 | x_\sigma^* \left( \theta_\sigma^\sharp - \delta \right) \right)$$

is the same as the sign of

$$b \int_{-\infty}^{\theta_\sigma^\sharp - \delta} \phi \left( \left( x_\sigma^* \left( \theta_\sigma^\sharp - \delta \right) - \theta \right) / \sigma \right) \pi(\theta) \, \mathrm{d}F(\theta) + g \int_{\theta_\sigma^\sharp - \delta}^{+\infty} \phi \left( \left( x_\sigma^* \left( \theta_\sigma^\sharp - \delta \right) - \theta \right) / \sigma \right) \pi(\theta) \, \mathrm{d}F(\theta).$$

Next observe that

$$
\begin{aligned}
0 &= U_\sigma^\Gamma\left(x_\sigma^*\left(\theta_\sigma^\sharp\right), 1 \middle| x_\sigma^*\left(\theta_\sigma^\sharp\right)\right) \int_{-\infty}^{+\infty} \phi\left(\left(x_\sigma^*\left(\theta_\sigma^\sharp\right) - \theta\right)/\sigma\right) \pi\left(\theta\right) \mathrm{d}F(\theta) \\
&= \int_{-\infty}^{\infty} \left(b\mathbf{1}\left\{\theta \le \theta_\sigma^\sharp\right\} + g\mathbf{1}\left\{\theta > \theta_\sigma^\sharp\right\}\right) \phi\left(\left(x_\sigma^*\left(\theta_\sigma^\sharp\right) - \theta\right)/\sigma\right) \pi\left(\theta\right) \mathrm{d}F(\theta) \\
&> \int_{-\infty}^{\infty} \left(b\mathbf{1}\left\{\theta \le \theta_\sigma^\sharp\right\} + g\mathbf{1}\left\{\theta > \theta_\sigma^\sharp\right\}\right) \phi\left(\left(x_\sigma^*\left(\theta_\sigma^\sharp - \delta\right) - \theta\right)/\sigma\right) \pi\left(\theta\right) \mathrm{d}F(\theta) \\
&= \int_{-\infty}^{\infty} \left(b\mathbf{1}\left\{\theta \le \theta_\sigma^\sharp - \delta\right\} + g\mathbf{1}\left\{\theta > \theta_\sigma^\sharp - \delta\right\}\right) \phi\left(\left(x_\sigma^*\left(\theta_\sigma^\sharp - \delta\right) - \theta\right)/\sigma\right) \pi\left(\theta\right) \mathrm{d}F(\theta) \\
&= U_\sigma^\Gamma\left(x_\sigma^*\left(\theta_\sigma^\sharp - \delta\right), 1 \middle| x_\sigma^*\left(\theta_\sigma^\sharp - \delta\right)\right) \int_{-\infty}^{+\infty} \phi\left(\left(x_\sigma^*\left(\theta_\sigma^\sharp - \delta\right) - \theta\right)/\sigma\right) \pi\left(\theta\right) \mathrm{d}F(\theta)
\end{aligned}
$$

The first equality follows from the assumptions of the lemma. The second equality follows from the definition of the function $U_\sigma^\Gamma\left(x_\sigma^*\left(\theta_\sigma^\sharp\right), 1 \middle| x_\sigma^*\left(\theta_\sigma^\sharp\right)\right)$. The inequality follows from the monotonicity of $x_\sigma^*\left(\cdot\right)$, the fact that $\phi\left((x - \theta)/\sigma\right)$ is log-supermodular in $(x, \theta)$, and Property SCB in the proof of Theorem 2 in the main text. The third equality follows from the fact that $\pi\left(\theta\right) = 0$ for $F$-almost all $\theta \in \left(\theta_\sigma^\sharp - \delta, \theta_\sigma^\sharp\right)$. The last equality follows from the definition of the function $U_\sigma^\Gamma(x_\sigma^*(\theta_\sigma^\sharp - \delta), 1 | x_\sigma^*(\theta_\sigma^\sharp - \delta))$. Hence, $U_\sigma^\Gamma(x_\sigma^*(\theta_\sigma^\sharp - \delta), 1 | x_\sigma^*(\theta_\sigma^\sharp - \delta)) < 0$, thus contradicting the assumption that $\Gamma \in \mathbb{G}_\sigma$. This completes the proof of Lemma S2-C. $\blacksquare$

**Lemma S2-D.** *For any $\varepsilon > 0$, there exists $\sigma^\#(\varepsilon) \in (0, \hat{\sigma})$ such that, for any $\sigma \in (0, \sigma^\#(\varepsilon)]$, and any policy $\Gamma = (\{0, 1\}, \pi) \in \mathbb{G}_\sigma$ for which there exists $(\theta', \theta''] \in D^\Gamma$ such that (a) $|\theta'' - \theta'| > \varepsilon$ and (b) $\pi(\theta) = 0$ for all $\theta \in (\theta', \theta'']$, there exists another policy $\Gamma^\# = \left(\{0, 1\}, \pi^\#\right) \in \mathbb{G}_\sigma$, with $M(\Gamma^\#) \le M(\Gamma)$, such that, in the auxiliary game $G_\sigma$, the probability of default under $\Gamma^\#$ is strictly smaller than under $\Gamma$.*

**Proof of Lemma S2-D.** For any $\theta \in (0, 1)$, $\lim_{\sigma \to 0^+} x_\sigma^*\left(\theta\right) \equiv x_{0+}^*\left(\theta\right) = \theta$. Furthermore, for any $\varepsilon \in (0, \min\{\theta^{MS}, 1 - \theta^{MS}\})$, the function $x_{0+}^* : \left[\frac{\varepsilon}{4}, 1 - \frac{\varepsilon}{4}\right] \to \mathbb{R}$ is uniformly continuous. Hence, for any $\delta < \varepsilon/4$, there exists $\tilde{\sigma}\left(\delta\right) > 0$ such that, for any $\sigma \in (0, \tilde{\sigma}(\delta)]$, and any $\theta \in \left[\frac{\varepsilon}{4}, 1 - \frac{\varepsilon}{4}\right]$, we have that $|x_\sigma^*\left(\theta\right) - \theta| \le \delta$.[10] In turn, this implies that, for any $\varepsilon > 0$ small, there exists $\sigma^\#(\varepsilon) \in (0, \hat{\sigma}]$ such that, for any $\sigma \in (0, \sigma^\#(\varepsilon)]$, and any $(\theta', \theta''] \in D^\Gamma$ such that $|\theta'' - \theta'| > \varepsilon$, we have that, for any $\theta \in [\theta'', 1 - \frac{\varepsilon}{4}]$, $|\theta - x_\sigma^*\left(\theta\right)| < |(\theta' + \theta'')/2 - x_\sigma^*\left(\theta\right)|$. Likewise, for any $\theta \in [\varepsilon/4, \theta']$, and any $\hat{\theta} \ge \theta''$, we have that $|\theta - x_\sigma^*\left(\theta\right)| < |x_\sigma^*\left(\theta\right) - \hat{\theta}|$ when

---

[10] The proof for the existence of a sequence $\left\{x_{\sigma_n}^*(\cdot)\right\}_n$ with domain $\left[\frac{\varepsilon}{4}, 1 - \frac{\varepsilon}{4}\right]$ converging uniformly to its limit function $x_{0+}^*(\cdot)$ follows from the same arguments that establish the uniform convergence of $\left\{H_{\sigma_n}(\cdot)\right\}_n$ to $H_{0+}(\cdot)$ in Step 1.

$\sigma \in (0, \sigma^{\#}(\varepsilon)]$.

Next, pick any policy $\Gamma = (\{0, 1\}, \pi) \in \mathbb{G}_\sigma$ for which there exists $d \equiv (\theta', \theta''] \in D^\Gamma$ such that (a) $|\theta'' - \theta'| > \varepsilon$ and (b) $\pi(\theta) = 0$ for all $\theta \in (\theta', \theta'']$. If $\min_{\theta \in [0,1]} U_\sigma^\Gamma (x_\sigma^*(\theta), 1|x_\sigma^*(\theta)) > 0$, the result follows directly from Lemma S2-B. Thus assume that $\min_{\theta \in [0,1]} U_\sigma^\Gamma (x_\sigma^*(\theta), 1|x_\sigma^*(\theta)) = 0$.

Suppose that $\min_{\theta \in [\theta'', 1]} U_\sigma^\Gamma (x_\sigma^*(\theta), 0|x_\sigma^*(\theta)) > 0$. By Lemma S2-C, $U_\sigma^\Gamma (x_\sigma^*(\theta), 1|x_\sigma^*(\theta)) > 0$ for all $\theta \in (\theta', \theta'']$. Hence, $\min_{\theta \in [\theta', 1]} U_\sigma^\Gamma (x_\sigma^*(\theta), 1|x_\sigma^*(\theta)) > 0$.

Below we show that, starting from $\Gamma$, we can then construct a policy $\Gamma^\eta \in \mathbb{G}_\sigma$, with $M(\Gamma^\eta) \leq M(\Gamma)$ such that, when agents play as in $G_\sigma$, the probability of default under $\Gamma^\eta$ is strictly smaller than under $\Gamma$. $\Gamma^\eta$ is obtained from $\Gamma$ by giving a pass grade to a positive-measure interval of types in the middle of $(\theta', \theta'']$. Formally, take $\eta \in (0, (\theta'' - \theta')/2)$ and let $\Gamma^\eta = (\{0, 1\}, \pi^\eta)$ be the policy whose rule $\pi^\eta$ is given by (a) $\pi^\eta(\theta) = \pi(\theta)$ for all $\theta \notin [(\theta' + \theta'')/2, (\theta' + \theta'')/2 + \eta]$, and (b) $\pi^\eta (\theta) = 1$ for all $\theta \in [(\theta' + \theta'')/2, (\theta' + \theta'')/2 + \eta]$. Below we show that $U^{\Gamma^\eta} (x_\sigma^*(\theta), 1|x_\sigma^*(\theta)) \geq 0$ for all $\theta \in [0, 1]$. To see this, let $\Theta^1 \equiv \{\theta \in \Theta : \pi(\theta) = 1\}$ be the collection of fundamentals receiving a pass grade under the original policy $\Gamma$. For any $\theta \in [0, \theta']$, and any $x$,

$$\Lambda_\sigma^{\Gamma^\eta} (\theta|x, 1) = \mathbb{P}[\tilde{\theta} \leq \theta|x, \tilde{\theta} \in (\Theta^1 \cup [(\theta' + \theta'')/2, (\theta' + \theta'')/2 + \eta])]$$
$$= \frac{\mathbb{P}[\tilde{\theta} \leq \theta \wedge \tilde{\theta} \in \Theta^1|x]}{\mathbb{P}[\tilde{\theta} \in (\Theta^1 \cup [(\theta' + \theta'')/2, (\theta' + \theta'')/2 + \eta])|x]} \leq \mathbb{P}[\tilde{\theta} \leq \theta|x, \tilde{\theta} \in \Theta^1] = \Lambda_\sigma^\Gamma(\theta|x, 1).$$

The first equality follows from the fact that, under $\Gamma^\eta$, the signal $s = 1$ carries the same information as the announcement that $\tilde{\theta} \in (\Theta^1 \cup [(\theta' + \theta'')/2, (\theta' + \theta'')/2 + \eta])$. The inequality follows from the fact that $\mathbb{P}\left[\tilde{\theta} \in (\Theta^1 \cup [(\theta' + \theta'')/2, (\theta' + \theta'')/2 + \eta]) |x\right] > \mathbb{P}\left[\tilde{\theta} \in \Theta^1|x\right]$. The last equality follows from fact that, under the original policy $\Gamma$, the signal $s = 1$ carries the same information as the announcement that $\tilde{\theta} \in \Theta^1$.

Given the above inequality, and the fact that, $b < 0 < g$, we then have that, for any $\theta \in [0, \theta']$,

$$U_\sigma^\Gamma(x_\sigma^*(\theta), 1|x_\sigma^*(\theta)) = b \cdot \Lambda_\sigma^\Gamma (\theta|x_\sigma^*(\theta), 1) + g \cdot \left[1 - \Lambda_\sigma^\Gamma (\theta|x_\sigma^*(\theta), 1)\right]$$
$$\leq b \cdot \Lambda_\sigma^{\Gamma^\eta} (\theta|x_\sigma^*(\theta), 1) + g \cdot \left[1 - \Lambda_\sigma^{\Gamma^\eta} (\theta|x_\sigma^*(\theta), 1)\right] = U_\sigma^{\Gamma^\eta}(x_\sigma^*(\theta), 1|x_\sigma^*(\theta)).$$

Hence $U^{\Gamma^\eta} (x_\sigma^*(\theta), 1|x_\sigma^*(\theta)) \geq 0$, all $\theta \leq \theta'$. That $\min_{\theta \in [\theta', 1]} U_\sigma^\Gamma (x_\sigma^*(\theta), 1|x_\sigma^*(\theta)) > 0$, along with the continuity of $U_\sigma^{\Gamma^\eta} (x_\sigma^*(\theta), 1|x_\sigma^*(\theta))$ in $\eta$ implies that $\min_{\theta \in [0,1]} U_\sigma^{\Gamma^\eta} (x_\sigma^*(\theta), 1|x_\sigma^*(\theta)) \geq 0$ for $\eta$ small. Hence $\Gamma^\eta \in \mathbb{G}_\sigma$.

14

Next, consider the more interesting case in which $\min_{\theta \in [\theta'',1]} U_\sigma^\Gamma(x_\sigma^*(\theta), 0|x_\sigma^*(\theta)) = 0$. Let $\theta_\sigma^\# \equiv \inf\{\theta \geq \theta'' : U_\sigma^\Gamma(x_\sigma^*(\theta), 1|x_\sigma^*(\theta)) = 0\}$. An implication of Lemma S2-C is that that $\theta_\sigma^\# > \theta''$. Also let $(\theta''', \theta'''']  \subset [0,1]$ be the first interval to the immediate right of $(\theta', \theta'']$ such that $\pi(\theta) = 1$ for all $\theta \in (\theta''', \theta'''']$ and let $\hat{\theta} = \min\{\theta'''', \theta_\sigma^\#\}$.[11]

Now, pick $\xi > 0$ small and let $\delta(\xi)$ be implicitly defined by

$$F((\theta' + \theta'')/2 + \xi) - F((\theta' + \theta'')/2) = F((\theta''' + \hat{\theta})/2 + \delta(\xi)) - F((\theta''' + \hat{\theta})/2). \qquad \text{(S2)}$$

Consider the policy $\Gamma^\xi = (\{0,1\}, \pi^\xi)$ defined by (a) $\pi^\xi(\theta) = \pi(\theta)$ for all $\theta \notin [(\theta' + \theta'')/2, (\theta' + \theta'')/2 + \xi] \cup [(\theta''' + \hat{\theta})/2, (\theta''' + \hat{\theta})/2 + \delta(\xi)]$, (b) $\pi^\xi(\theta) = 1$ for all $\theta \in [(\theta' + \theta'')/2, (\theta' + \theta'')/2 + \xi]$, and (c) $\pi^\xi(\theta) = 0$ for all $\theta \in [(\theta''' + \hat{\theta})/2, (\theta''' + \hat{\theta})/2 + \delta(\xi)]$. Below we establish that, when $\xi > 0$ is small, such a policy is such that $\min_{\theta \in [0,1]} U^{\Gamma^\xi}(x_\sigma^*(\theta), 1|x_\sigma^*(\theta)) > 0$ and hence $\Gamma^\xi \in \mathbb{G}_\sigma$. To see this, for any arbitrary policy $\tilde{\Gamma} = \{\{0,1\}, \tilde{\pi}\}$, any $\theta \in [0,1]$, let

$$V_\sigma^{\tilde{\Gamma}}(\theta) \equiv U_\sigma^{\tilde{\Gamma}}(x_\sigma^*(\theta), 1|x_\sigma^*(\theta)) \, p_\sigma^{\tilde{\Gamma}}(x_\sigma^*(\theta), 1),$$

where, for any $x$, $p_\sigma^{\tilde{\Gamma}}(x,1) \equiv \int_\Theta \tilde{\pi}(\theta) p_\sigma(x|\theta) \mathrm{d}F(\theta)$, with $p_\sigma(x|\theta) \equiv \frac{1}{\sigma}\phi((x - \theta)/\sigma)$.

By definition of $\theta_\sigma^\#$, we must have that, for all $\theta$, $0 = V_\sigma^\Gamma(\theta_\sigma^\#) \leq V_\sigma^\Gamma(\theta)$. Next, for any $\xi > 0$, define $\varphi_R(\xi) \equiv \min_{\theta \in [\theta'',1]} V_\sigma^{\Gamma^\xi}(\theta)$. Let $\overline{u}(\tilde{\theta}, \theta) \equiv g\mathbf{1}\{\tilde{\theta} > \theta\} + b\mathbf{1}\{\tilde{\theta} \leq \theta\}$ and note that, for any $\theta$,

$$V_\sigma^{\Gamma^\xi}(\theta) = V_\sigma^\Gamma(\theta) + \int_{(\theta' + \theta'')/2}^{(\theta' + \theta'')/2 + \xi} \overline{u}(\tilde{\theta}, \theta) p_\sigma(x_\sigma^*(\theta)|\tilde{\theta}) \mathrm{d}F(\tilde{\theta}) - \int_{(\theta''' + \hat{\theta})/2}^{(\theta''' + \hat{\theta})/2 + \delta(\xi)} \overline{u}(\tilde{\theta}, \theta) p_\sigma(x_\sigma^*(\theta)|\tilde{\theta}) \mathrm{d}F(\tilde{\theta}).$$

Using the envelope theorem, we have that, for any $\theta_\sigma^\xi \in \arg\min_{\theta \in [\theta'',1]} V_\sigma^{\Gamma^\xi}(\theta)$,

$$\begin{aligned}
\varphi_R'(\xi) &= f\left((\theta' + \theta'')/2 + \xi\right) \overline{u}\left((\theta' + \theta'')/2 + \xi, \theta_\sigma^\xi\right) p_\sigma\left(x_\sigma^*(\theta_\sigma^\xi)|(\theta' + \theta'')/2 + \xi\right) \\
&\quad - f\left((\theta''' + \hat{\theta})/2 + \delta(\xi)\right) \overline{u}((\theta''' + \hat{\theta})/2 + \delta(\xi), \theta_\sigma^\xi) p_\sigma(x_\sigma^*(\theta_\sigma^\xi)|(\theta''' + \hat{\theta})/2 + \delta(\xi)) \delta'(\xi) \\
&= f\left((\theta' + \theta'')/2 + \xi\right) [\overline{u}((\theta' + \theta'')/2 + \xi, \theta_\sigma^\xi) p_\sigma(x_\sigma^*(\theta_\sigma^\xi)|(\theta' + \theta'')/2 + \xi) \\
&\quad - \overline{u}((\theta''' + \hat{\theta})/2 + \delta(\xi), \theta_\sigma^\xi) p_\sigma(x_\sigma^*(\theta_\sigma^\xi)|(\theta''' + \hat{\theta})/2 + \delta(\xi))],
\end{aligned}$$

where the second equality uses the implicit function theorem applied to (S2) to obtain that

---

[11]The existence of such an interval follows from the fact that $\pi(\theta) = 1$ in a left neighborhood of $\theta_\sigma^\#$ by virtue of Lemma S2-C. Also observe that, when $\theta'' < \theta^{MS}$, such an interval is adjacent to $(\theta', \theta'']$ and hence $\theta''' = \theta''$.

$\delta'(\xi) = f\left((\theta' + \theta'')/2 + \xi\right) / f((\theta''' + \hat{\theta})/2 + \delta(\xi))$. As a consequence,

$$\lim_{\xi \to 0^+} \varphi'_R(\xi) = f\left((\theta' + \theta'')/2\right) [\overline{u}\left((\theta' + \theta'')/2, \theta_\sigma^\#\right) p_\sigma \left(x_\sigma^*(\theta_\sigma^\#)|(\theta' + \theta'')/2\right) \tag{S3}$$
$$- \overline{u}((\theta''' + \hat{\theta})/2, \theta_\sigma^\#)p_\sigma(x_\sigma^*(\theta_\sigma^\#)|(\theta''' + \hat{\theta})/2)].$$

That $\sigma < \sigma^\#(\varepsilon)$ implies that $|x_\sigma^*\left(\theta_\sigma^\#\right) - (\theta''' + \hat{\theta})/2| < |x_\sigma^*\left(\theta_\sigma^\#\right) - (\theta' + \theta'')/2|$. That $p_\sigma(x|\theta)$ is single-peaked in turn implies that $p_\sigma(x_\sigma^*(\theta_\sigma^\#)|(\theta' + \theta'')/2) < p_\sigma(x_\sigma^*(\theta_\sigma^\#)|(\theta''' + \hat{\theta})/2)$ and hence that

$$\overline{u}\left((\theta' + \theta'')/2, \theta_\sigma^\#\right) p_\sigma \left(x_\sigma^*\left(\theta_\sigma^\#\right)|(\theta' + \theta'')/2\right) - \overline{u}\left((\theta''' + \hat{\theta})/2, \theta_\sigma^\#\right) p_\sigma \left(x_\sigma^*\left(\theta_\sigma^\#\right)|(\theta''' + \hat{\theta})/2\right)$$
$$= b \cdot \left(p_\sigma \left(x_\sigma^*\left(\theta_\sigma^\#\right)|(\theta' + \theta'')/2\right) - p_\sigma \left(x_\sigma^*\left(\theta_\sigma^\#\right)|(\theta''' + \hat{\theta})/2\right)\right) > 0.$$

Thus, $\lim_{\xi \to 0^+} \varphi'_R(\xi) > 0$. By continuity of $U_\sigma^{\Gamma^\xi}\left(x_\sigma^*(\theta), 1|x_\sigma^*(\theta)\right)$ in $\xi$, we then have that, for $\xi > 0$ small, $\min_{\theta \in [\theta'', 1]} U_\sigma^{\Gamma^\xi}\left(x_\sigma^*(\theta), 1|x_\sigma^*(\theta)\right) > 0$.

Next, we prove that, under the policy $\Gamma^\xi$, $\min_{\theta \in [0, \theta'']} U_\sigma^{\Gamma^\xi}\left(x_\sigma^*(\theta), 1|x_\sigma^*(\theta)\right) > 0$. For any $\xi > 0$, define $\varphi_L(\xi) \equiv \min_{\theta \in [0, \theta']} V_\sigma^{\Gamma^\xi}(\theta)$. Arguments similar to those used above to compute $\lim_{\xi \to 0^+} \varphi'_R(\xi)$ imply that, for any $\theta_\sigma^{\#\#} \in \arg\min_{\theta \in [0, \theta']} V_\sigma^\Gamma(\theta)$, when $\sigma \leq \sigma^\#(\varepsilon)$,

$$\lim_{\xi \to 0^+} \varphi'_L(\xi) = f\left((\theta' + \theta'')/2\right) [\overline{u}((\theta' + \theta'')/2, \theta_\sigma^{\#\#})p_\sigma(x_\sigma^*(\theta_\sigma^{\#\#})|(\theta' + \theta'')/2)$$
$$- \overline{u}((\theta''' + \hat{\theta})/2, \theta_\sigma^{\#\#})p_\sigma(x_\sigma^*(\theta_\sigma^{\#\#})|(\theta''' + \hat{\theta})/2)]$$
$$= f\left((\theta' + \theta'')/2\right) g\left[p_\sigma(x_\sigma^*(\theta_\sigma^{\#\#})|(\theta' + \theta'')/2) - p_\sigma(x_\sigma^*(\theta_\sigma^{\#\#})|(\theta''' + \hat{\theta})/2))\right] > 0.$$

The first equality follows from steps analogous to those used to establish (S3). The second equality follows from the fact that, by assumption $\theta_\sigma^{\#\#} \leq \theta'$. The inequality is a consequence of the fact that, for $\sigma \leq \sigma^\#(\varepsilon)$, $|x_\sigma^*\left(\theta_\sigma^{\#\#}\right) - (\theta' + \theta'')/2| < |x_\sigma^*\left(\theta_\sigma^{\#\#}\right) - (\theta''' + \hat{\theta})/2)|$, which, together with the fact that the noise distribution is single-peaked, implies that

$$p_\sigma(x_\sigma^*(\theta_\sigma^{\#\#})|(\theta' + \theta'')/2) > p_\sigma(x_\sigma^*(\theta_\sigma^{\#\#})|(\theta''' + \hat{\theta})/2).$$

Hence, for $\xi > 0$ small, $U_\sigma^{\Gamma^\xi}\left(x_\sigma^*(\theta), 1|x_\sigma^*(\theta)\right) > 0$ for all $\theta \in [0, \theta']$. Furthermore, by Lemma S2-C, $U_\sigma^\Gamma\left(x_\sigma^*(\theta), 1|x_\sigma^*(\theta)\right) > 0$ for all $\theta \in (\theta', \theta'']$. Hence, provided that $\xi$ is small, the continuity of $U_\sigma^{\Gamma^\xi}\left(x_\sigma^*(\theta), 1|x_\sigma^*(\theta)\right)$ in $\xi$ implies that $U_\sigma^{\Gamma^\xi}\left(x_\sigma^*(\theta), 1|x_\sigma^*(\theta)\right) > 0$ also for $\theta \in (\theta', \theta'']$. Combining all the properties above, we thus conclude that, for $\xi > 0$ small, $\min_{\theta \in [0, 1]} U_\sigma^{\Gamma^\xi}\left(x_\sigma^*(\theta), 1|x_\sigma^*(\theta)\right) > 0$. Hence $\Gamma^\xi \in \mathbb{G}_\sigma$.

By construction, $M(\Gamma^\xi) < M(\Gamma)$. Furthermore, when agents play according to $G_\sigma$, the

16

probability of default under $\Gamma^\xi$ is the same as under $\Gamma$. Lemma S2-B then implies that, starting from $\Gamma^\xi$, one can construct a policy $\Gamma^\# \in \mathbb{G}_\sigma$, close to $\Gamma^\xi$ in the $L_1$ norm, such that (1) $M(\Gamma^\#) \leq M(\Gamma^\xi)$ and (2), when agents play according to $G_\sigma$, the probability of default under $\Gamma^\#$ is strictly smaller than under $\Gamma$. This completes the proof of Lemma S2-D. ∎

**Step 3.** Steps 1 and 2 imply that there exists a function $\bar\sigma : \left(0, \min\{\theta^{MS}, 1 - \theta^{MS}\}\right) \to \mathbb{R}_{++}$ , with $\bar\sigma(\varepsilon) \leq \min\{\sigma(\varepsilon), \sigma^\#(\varepsilon)\}$ for all $\varepsilon \in \left(0, \min\{\theta^{MS}, 1 - \theta^{MS}\}\right)$ and with $\bar\sigma(\varepsilon) \to 0^+$ as $\varepsilon \to 0^+$, such that the following is true: For any $\varepsilon \in \left(0, \min\{\theta^{MS}, 1 - \theta^{MS}\}\right)$, any $\sigma \in (0, \bar\sigma(\varepsilon)]$, and any policy $\Gamma = (\{0, 1\}, \pi) \in \mathbb{G}_\sigma$ with $M(\Gamma) > \varepsilon$, there exists another policy $\Gamma' = (\{0, 1\}, \pi') \in \mathbb{G}_\sigma$ with $M(\Gamma') \leq \varepsilon$ such that, when the agents play as in the auxiliary game $G_\sigma$, the probability of default under $\Gamma'$ is strictly smaller than under $\Gamma$.[12]

Furthermore, the arguments establishing Lemma S2-D reveal that the policy $\Gamma'$ can be constructed so that $U_\sigma^{\Gamma'}(x, 1|x) > 0$ for all $x$. The policy $\Gamma'$ thus satisfies PCP also when agents play according to MARP. The claim in the Example then follows by taking $\Gamma^* = \Gamma'$ with $\Gamma'$ satisfying the above properties.

**Step 4**. We now complete the proof by showing how to construct the function $\mathcal{E}$ in the example. Let $(\varepsilon_n)$ be a non-increasing sequence satisfying $\lim_{n\to\infty} \varepsilon_n = 0$. For each $n \in \mathbb{N}$, then let $\sigma_n = \bar\sigma(\varepsilon_n)$, with the function $\bar\sigma(\cdot)$ as defined in Step 3. The results in Steps 1-3 above imply that, given $(\varepsilon_n, \sigma_n)$, there exist strictly decreasing subsequences $(\tilde\varepsilon_n)$ and $(\tilde\sigma_n)$ satisfying $\lim_{n\to\infty} \tilde\varepsilon_n = \lim_{n\to\infty} \tilde\sigma_n = 0$ such that, for any $n \in \mathbb{N}$, the conclusions in Step 3 hold for $\varepsilon = \tilde\varepsilon_n$ and $\bar\sigma(\varepsilon_n) = \tilde\sigma_n$. Then let $\bar\sigma = \tilde\sigma_0 > 0$ and $\mathcal{E} : (0, \bar\sigma] \to \mathbb{R}_+$ be the function defined by $\mathcal{E}(\sigma) = \varepsilon_n$ for all $\sigma \in (\sigma_{n+1}, \sigma_n]$. The result in the example then follows from Steps 1-3, by letting $\mathcal{E}(\cdot)$ be the function so constructed. Q.E.D.

# References

Milgrom, P. R., 1981, "Good news and bad news: Representation theorems and applications," *The Bell Journal of Economics*, 380-391.

---

[12]Observe that the thresholds $\sigma(\varepsilon)$ and $\sigma^\#(\varepsilon)$ identified in Steps 1 and 2 above are invariant to the initial policy $\Gamma$. The same arguments used to arrive at a policy $\Gamma^\#$ with mesh $M(\Gamma^\#) < M(\Gamma)$ can then be iterated till one arrives at a policy $\Gamma'$ with mesh $M(\Gamma') \leq \varepsilon$.