Online Appendix

Detailed Calculations in Examples 3 and 5

As in the proof of Proposition 1, let $\rho \equiv \frac{1}{\psi}$. Recall that, in EZ, lotteries are evaluated according to the recursion

$$V_t = \left\{ (1-\beta) C_t^{1-\rho} + \beta \left[E_t \left(V_{t+1}^{1-\alpha} \right) \right]^{\frac{1-\rho}{1-\alpha}} \right\}^{\frac{1}{1-\rho}}.$$

It can be verified that the EZ value of the stream $(x_2, x_3, x_4, ...)$ is:

$$V_2 = \left\{ (1-\beta) \cdot \left[x_2^{1-\rho} + \sum_{t=3}^{\infty} \beta^{t-2} x_t^{1-\rho} \right] \right\}^{\frac{1}{1-\rho}}.$$

In our domain, all uncertainty is resolved in period 1 and all streams have the same period-1 consumption ($x_1 = c$). So, the EZ utility of the lottery with random payments $\{\tilde{x}_t\}$ in periods $t \geq 2$ is:

$$V_1 = \left\{ (1-\beta) c^{1-\rho} + \beta \left[E_1 \left(\left\{ (1-\beta) \cdot \sum_{t=2}^{\infty} \beta^{t-2} x_t^{1-\rho} \right\}^{\frac{1-\alpha}{1-\rho}} \right) \right]^{\frac{1-\rho}{1-\alpha}} \right\}^{\frac{1}{1-\rho}}. \quad (27)$$

It is convenient to split in two cases depending on whether $\rho < 1$ or $\rho > 1$.

Case 1: $\rho < 1$

Since $x^{\frac{1}{1-\rho}}$ is a strictly increasing function of x for $\rho < 1$, it follows that preferences can be represented by

$$\tilde{V}_1 = (1-\beta) c^{1-\rho} + \beta \left[E_1 \left(\left\{ (1-\beta) \cdot \sum_{t=2}^{\infty} \beta^{t-2} x_t^{1-\rho} \right\}^{\frac{1-\alpha}{1-\rho}} \right) \right]^{\frac{1-\rho}{1-\alpha}}.$$

Since all lotteries have the same period-1 consumption c and $\beta > 0$, they are also represented by

$$\tilde{\tilde{V}}_1 = \left[E_1 \left(\left\{ (1-\beta) \cdot \sum_{t=2}^{\infty} \beta^{t-2} x_t^{1-\rho} \right\}^{\frac{1-\alpha}{1-\rho}} \right) \right]^{\frac{1-\rho}{1-\alpha}}. \quad (28)$$

There are two subcases.

Case 1a: $\alpha, \rho < 1$.

If $\alpha < 1$, so that $\frac{1-\rho}{1-\alpha} > 0$, we can raise the expression above by $\frac{1-\alpha}{1-\rho} > 0$ (which is a monotone transformation) to obtain the following equivalent representation for EZ:

$$\hat{V}_1 = E_1 \left\{ \left[(1-\beta) \cdot \sum_{t=2}^{\infty} \beta^{t-2} x_t^{1-\rho} \right]^{\frac{1-\alpha}{1-\rho}} \right\}.$$

Dividing this expression by $(1-\rho)^{\frac{1-\alpha}{1-\rho}} > 0$, we obtain

$$\hat{\hat{V}}_1 = E_1 \left\{ \left[(1-\beta) \cdot \sum_{t=2}^{\infty} \beta^{t-2} \cdot \frac{x_t^{1-\rho}}{1-\rho} \right]^{\frac{1-\alpha}{1-\rho}} \right\},$$

which is a KM representation with $\phi(z) = z^{\frac{1-\alpha}{1-\rho}}$ and $u(x) = \frac{x^{1-\rho}}{1-\rho}$. Note that ϕ is indeed increasing and its coefficient of absolute risk aversion of ϕ is $-\frac{\phi''(z)}{\phi'(z)} = \frac{1}{z} \cdot \frac{\alpha-\rho}{1-\rho}$.

Case 1b: $\rho < 1 < \alpha$.

Next, suppose $\alpha > 1 > \rho$. Applying the increasing transformation $g(z) \equiv -\frac{\left(z^{\frac{1-\alpha}{1-\rho}}\right)}{(1-\rho)^{\frac{1-\alpha}{1-\rho}}}$ to (28), we find that preferences can be represented by:

$$\hat{V} = E_1 \left\{ - \left[(1-\beta) \cdot \sum_{t=2}^{\infty} \beta^{t-2} \frac{x_t^{1-\rho}}{1-\rho} \right]^{\frac{1-\alpha}{1-\rho}} \right\},$$

giving a KM representation with $\phi(z) = -\left(z^{\frac{1-\alpha}{1-\rho}}\right)$ and $u(x) = \frac{x^{1-\rho}}{1-\rho}$. Note that ϕ is increasing (since $\frac{1-\alpha}{1-\rho} < 0$) and its coefficient of absolute risk aversion is $-\frac{\phi''(z)}{\phi'(z)} = \frac{1}{z} \cdot \frac{\alpha-\rho}{1-\rho}$.

Case 2: $\rho > 1$

We now consider the case of $\rho > 1$. Since $f(x) = x^{\frac{1}{1-\rho}}$ is a decreasing function when $\rho > 1$, it follows from (27) that preferences can be represented by

$$\tilde{V}_1 = -(1-\beta)\, c^{1-\rho} - \beta \left[E_1 \left(\left\{ (1-\beta) \cdot \sum_{t=2}^{\infty} \beta^{t-2} x_t^{1-\rho} \right\}^{\frac{1-\alpha}{1-\rho}} \right) \right]^{\frac{1-\rho}{1-\alpha}}.$$

As before, since the first term, $-(1-\beta)\, c^{1-\rho}$, is the same in all lotteries in our domain (the first-period consumption c is constant) and since $\beta > 0$ is a constant, preferences in this case can be represented by

$$-\left[E_1\left(\left\{(1-\beta)\cdot\sum_{t=2}^{\infty}\beta^{t-2}x_t^{1-\rho}\right\}^{\frac{1-\alpha}{1-\rho}}\right)\right]^{\frac{1-\rho}{1-\alpha}}. \tag{29}$$

There are two subcases: $\alpha, \rho > 1$ and $\rho > 1 > \alpha$.

Case 2a: $\alpha, \rho > 1$

Suppose first $\alpha, \rho > 1$, so that $\frac{1-\rho}{1-\alpha} > 0$. Applying the increasing transformation $f(x) = x^{\frac{1-\alpha}{1-\rho}}$, we find that preferences can also be represented by

$$-\left[E_1\left(\left\{(1-\beta)\cdot\sum_{t=2}^{\infty}\beta^{t-2}x_t^{1-\rho}\right\}^{\frac{1-\alpha}{1-\rho}}\right)\right].$$

Dividing by the constant $(\rho-1)^{\frac{1-\alpha}{1-\rho}} > 0$, establishes that preferences can be represented by

$$E_1\left\{-\left[-(1-\beta)\cdot\sum_{t=2}^{\infty}\beta^{t-2}\frac{x_t^{1-\rho}}{1-\rho}\right]^{\frac{1-\alpha}{1-\rho}}\right\},$$

which is a KM representation with $\phi(z) = -(-z)^{\frac{1-\alpha}{1-\rho}}$ and $u(x) = \frac{x^{1-\rho}}{1-\rho}$. Note that ϕ is increasing (since $\frac{1-\rho}{1-\alpha} > 0$) and the coefficient of absolute risk aversion is $-\frac{\phi''(z)}{\phi'(z)} = \frac{1}{z}\cdot\frac{\alpha-\rho}{1-\rho}$.

Case 2b: $\rho > 1 > \alpha$

Since $\frac{1-\rho}{1-\alpha} < 0$, it follows from (29) that preferences can be represented by

$$E_1\left(\left\{(1-\beta)\cdot\sum_{t=2}^{\infty}\beta^{t-2}x_t^{1-\rho}\right\}^{\frac{1-\alpha}{1-\rho}}\right).$$

Dividing this expression by $(\rho-1)^{\frac{1-\alpha}{1-\rho}} > 0$, we obtain

$$E_1\left\{\left[-(1-\beta)\cdot\sum_{t=2}^{\infty}\beta^{t-2}\frac{x_t^{1-\rho}}{1-\rho}\right]^{\frac{1-\alpha}{1-\rho}}\right\},$$

which is a KM representation with $\phi(z) = (-z)^{\frac{1-\alpha}{1-\rho}}$ and $u(x) = \frac{x^{1-\rho}}{1-\rho}$. Again, the coefficient of absolute risk aversion is $-\frac{\phi''(z)}{\phi'(z)} = \frac{1}{z}\cdot\frac{\alpha-\rho}{1-\rho}$.

SI conditions for EZ using the KM representation

From Proposition 3, a sufficient condition for SI is that ϕ is more convex than $\underline{\phi}(z) = \log(z - \underline{u})$ and more concave than $\bar{\phi}(z) \equiv -\log(\bar{u} - z)$. As calculated previously, the coefficients of relative risk aversion of ϕ equals $-\frac{\phi''(z)}{\phi'(z)} = \frac{1}{z} \cdot \frac{\alpha-\rho}{1-\rho}$, whereas:

$$-\frac{\bar{\phi}''(z)}{\bar{\phi}'(z)} = -\frac{1}{\bar{u} - z} \quad \text{and} \quad -\frac{\underline{\phi}''(z)}{\underline{\phi}'(z)} = \frac{1}{z - \underline{u}}.$$

Therefore, the sufficient condition for SI from from Proposition 3 is

$$-\frac{1}{\bar{u} - z} \leq \frac{1}{z} \cdot \frac{\alpha - \rho}{1 - \rho} \leq \frac{1}{z - \underline{u}} \tag{30}$$

for all $z \in u(\mathbb{R}_+)$, where $\bar{u} \equiv \sup\{u(c): \; c \in \mathbb{R}_+\}$ and $\underline{u} \equiv \inf\{u(c): \; c \in \mathbb{R}_+\}$.

Note that when $\rho < 1$, we have $u(\mathbb{R}_+) = [0, +\infty)$, so that $\bar{u} = +\infty$ and $\underline{u} = 0$. Then, condition (30) becomes

$$0 \leq \frac{\alpha - \rho}{1 - \rho} \leq 1 \iff \rho \leq \alpha \leq 1.$$

When, instead, $\rho > 1$, we have $u(\mathbb{R}_+) = (-\infty, 0]$, so that $\bar{u} = 0$ and $\underline{u} = -\infty$. Then, condition (30) becomes

$$0 \leq \frac{\alpha - \rho}{1 - \rho} \leq 1 \iff \rho \geq \alpha \geq 1.$$

Noting that since in EZ $\alpha \neq 1$, these are the same as the necessary and sufficient conditions from Proposition 1.

Detailed Calculations in Examples 4 and 6

Recall that the Risk Sensitive preferences of Hansen and Sargent (HS) admit the following recursive representation:

$$V_t = u(x_t) - \frac{\beta}{k} \log\left[E_t\left(e^{-kV_{t+1}}\right)\right].$$

In our setting, all lotteries have the same consumption in period 0 and all uncertainty is resolved in period 1. Since consumption is deterministic after the realization of uncertainty at the start of period 1, we have:

$$V_t = u(x_t) + \beta V_{t+1}$$

for all $t \geq 1$. It can be verified that the following expression solves this equation:

$$V_1 = \sum_{t=1}^{\infty} \beta^{t-1} u(x_t).$$

Taking expectations in period 0 (before uncertainty is resolved), we obtain the following expression:

$$V_0 = u(x_0) - \frac{\beta}{k} \log \left[E_0 \left(e^{-k \sum_{t=1}^{\infty} \beta^{t-1} u(x_t)} \right) \right].$$

Since all lotteries have the same consumption in period 0 in the domain we consider, we can omit the period-0 consumption. Moreover, since $\frac{\beta}{\kappa} > 0$ is a constant and the logarithm function is strictly increasing, HS preferences over lotteries in our domain can be also represented by:

$$\tilde{V}_0 = E_0 \left(-e^{-k \sum_{t=1}^{\infty} \beta^{t-1} u(x_t)} \right) = E_0 \left(-e^{-\kappa(1-\beta) \sum_{t=1}^{\infty} \beta^{t-1} u(x_t)} \right),$$

where $\kappa \equiv \frac{k}{1-\beta}$. This coincides with the KM representation for $\phi(z) \equiv -\exp\left(-\frac{kz}{1-\beta}\right)$.

SI conditions for HS using the KM representation

The coefficient of absolute risk aversion of ϕ equals:

$$-\frac{\phi''(z)}{\phi'(z)} = \frac{k}{1-\beta}.$$

Since $-\frac{\phi''(z)}{\phi'(z)} > 0$, the sufficient conditions from Proposition 3 hold if and only if ϕ is less concave than $\underline{\phi}$. Recall that the coefficient of absolute risk aversion of $\underline{\phi}$ equals:

$$-\frac{\underline{\phi}''(x)}{\underline{\phi}'(x)} = \frac{1}{x - \underline{u}}.$$

Therefore, the sufficient conditions from Proposition 3 hold if and only if:

$$\frac{k}{1-\beta} \leq \frac{1}{x - \underline{u}} \quad \forall x \in u(X) \iff \bar{u} - \underline{u} \leq \frac{1-\beta}{k}, \tag{31}$$

where $\bar{u} \equiv \sup\{u(x)\}_{x \in C}$ and $\underline{u} \equiv \inf\{u(x)\}_{x \in C}$.

Contrast (31) with the necessary and sufficient condition from Proposition 2:

$$\bar{u} - \underline{u} \leq -\frac{\log(\beta)}{\beta} \frac{1}{k}. \tag{32}$$

We claim that the sufficient condition from Proposition 3 is strictly weaker than the necessary and sufficient condition from Proposition 2, so there exist preferences that satisfy SI but do not satisfy the sufficient condition from Proposition 3. To establish this, we need to show that the bound in (32) is higher than the bound in (31):

$$-\frac{\log(\beta)}{\beta}\frac{1}{k} > \frac{1-\beta}{k} \iff \beta^2 - \beta - \log(\beta) > 0.$$

We claim that this inequality holds for all $\beta \in [0, 1)$. To see this first note that at $\beta = 1$, the LHS equals 0 so both bounds coincide. Moreover the derivative is negative for all $\beta \in [0, 1)$:

$$2\beta - 1 - \frac{1}{\beta} < 0 \iff \beta^2 - \frac{\beta}{2} - \frac{1}{2} < 0,$$

which is true since the expression on the LHS is an upward facing parabola with roots $-\frac{1}{2}$ and $+1$.

Proof of Proposition 1 (detailed calculations)

Recall that with EZ, lotteries are evaluated according to

$$V_t = \{(1-\beta)x_t^{1-\rho} + \beta[\mathbb{E}_t(V_{t+1}^{1-\alpha})]^{\frac{1-\rho}{1-\alpha}}\}^{\frac{1}{1-\rho}}. \tag{33}$$

Substitution verifies that the value of a constant stream that pays c is c:

$$V_0 = \{(1-\beta)c^{1-\rho} + \beta c^{1-\rho}\}^{\frac{1}{1-\rho}} = c.$$

Next, consider a stream that pays $(\underbrace{\overbrace{x}^{1}, x, ..., \overbrace{x}^{t}}_{t}, \overbrace{c}^{t+1}, c, c, ...)$. By the previous expression, the continuation value at $t + 1$ is c. Using the expression in (33), we obtain:

$$V_t = \{(1-\beta)x^{1-\rho} + \beta c^{1-\rho}\}^{\frac{1}{1-\rho}}.$$

Substituting this expression for V_{t-1}, gives:

$$V_{t-1} = \{(1-\beta)x^{1-\rho} + \beta V_t^{1-\rho}\}^{\frac{1}{1-\rho}} = \{(1-\beta)(1+\beta)x^{1-\rho} + \beta^2 c^{1-\rho}\}^{\frac{1}{1-\rho}}.$$

Substituting recursively s times, gives the following expression:

$$V_{t-s} = \{(1-\beta)x^{1-\rho}(1 + \beta + \beta^2 + ... + \beta^s) + \beta^{s+1}c^{1-\rho}\}^{\frac{1}{1-\rho}}.$$

In particular, taking $s = t - 1$, gives value of the stream:

$$V_1 = \{(1-\beta)x^{1-\rho}(1+\beta+\beta^2+...+\beta^{t-1})+\beta^t c^{1-\rho}\}^{\frac{1}{1-\rho}} = \{(1-\beta^t)x^{1-\rho}+\beta^t c^{1-\rho}\}^{\frac{1}{1-\rho}}. \quad (34)$$

Next, consider the stream $(\overbrace{c}^{1}, ..., \overbrace{c}^{\tau-1}, \underbrace{\overbrace{x}^{\tau}, x, ..., \overbrace{x}^{\tau+t-1}}_{t}, c, c, ...)$. Note that the stream starting at τ is the same as the one evaluated in the previous parargaph. Therefore, by the previous calculations, we have

$$V_\tau = \{(1-\beta^t)x^{1-\rho} + \beta^t c^{1-\rho}\}^{\frac{1}{1-\rho}}.$$

Using the expression in (33), we obtain the value in period $\tau - 1$:

$$V_{\tau-1} = [(1-\beta)c^{1-\rho} + \beta V_\tau^{1-\rho}]^{\frac{1}{1-\rho}} = [c^{1-\rho} + \beta(1-\beta^t)(x^{1-\rho} - c^{1-\rho})]^{\frac{1}{1-\rho}}.$$

Substituting recursively s times, gives

$$V_{\tau-s} = \{c^{1-\rho} + \beta^s(1-\beta^t)(x^{1-\rho} - c^{1-\rho})\}^{\frac{1}{1-\rho}}.$$

Taking $s = \tau - 1$ gives

$$V_1 = \{c^{1-\rho} + \beta^{\tau-1}(1-\beta^t)(x^{1-\rho} - c^{1-\rho})\}^{\frac{1}{1-\rho}}. \quad (35)$$

Let c_0 be an arbitrary but fixed consumption in period 0. We are interested in the lottery that pays either

$$(\overbrace{c_0}^{0}, \underbrace{\overbrace{x}^{1}, x, ..., \overbrace{x}^{t}}_{t}, \overbrace{c}^{t+1}, c, c, ...)$$

or

$$(\overbrace{c_0}^{0}, \overbrace{c}^{1}, ..., \overbrace{c}^{\tau-1}, \underbrace{\overbrace{y}^{\tau}, y, ..., \overbrace{y}^{\tau+t-1}}_{t}, c, c, ...)$$

with 50-50 chance each. From the recursion in (33), the value of this lottery is:

$$V_0 = \left\{(1-\beta)c_0^{1-\rho} + \beta[\mathbb{E}_0(V_1^{1-\alpha})]^{\frac{1-\rho}{1-\alpha}}\right\}^{\frac{1}{1-\rho}}.$$

Using expressions in (34) and (35), we obtain

$$\mathbb{E}_0(V_1^{1-\alpha}) = \frac{\{(1-\beta^t)x^{1-\rho} + \beta^t c^{1-\rho}\}^{\frac{1-\alpha}{1-\rho}} + \{c^{1-\rho} + \beta^{\tau-1}(1-\beta^t)\,(x^{1-\rho} - c^{1-\rho})\}^{\frac{1-\alpha}{1-\rho}}}{2}.$$

Substituting in the expression for V_0, gives

$$V_0 = \left\{ (1-\beta)c_0^{1-\rho} + \beta \left[\frac{\begin{array}{l} \{(1-\beta^t)x^{1-\rho} + \beta^t c^{1-\rho}\}^{\frac{1-\alpha}{1-\rho}} \\ + \{c^{1-\rho} + \beta^{\tau-1}(1-\beta^t)\left(y^{1-\rho} - c^{1-\rho}\right)\}^{\frac{1-\alpha}{1-\rho}} \end{array}}{2} \right]^{\frac{1-\rho}{1-\alpha}} \right\}^{\frac{1}{1-\rho}}.$$

Using this formula, we can write the condition for Stochastic Impatience in EZ as:

$$\left\{ (1-\beta)c_0^{1-\rho} + \beta \left[\frac{\begin{array}{l} \{(1-\beta^t)x^{1-\rho} + \beta^t c^{1-\rho}\}^{\frac{1-\alpha}{1-\rho}} \\ + \{c^{1-\rho} + \beta^{\tau-1}(1-\beta^t)\left(y^{1-\rho} - c^{1-\rho}\right)\}^{\frac{1-\alpha}{1-\rho}} \end{array}}{2} \right]^{\frac{1-\rho}{1-\alpha}} \right\}^{\frac{1}{1-\rho}}$$

$$\geq$$

$$\left\{ (1-\beta)c_0^{1-\rho} + \beta \left[\frac{\begin{array}{l} \{(1-\beta^t)y^{1-\rho} + \beta^t c^{1-\rho}\}^{\frac{1-\alpha}{1-\rho}} \\ + \{c^{1-\rho} + \beta^{\tau-1}(1-\beta^t)\left(x^{1-\rho} - c^{1-\rho}\right)\}^{\frac{1-\alpha}{1-\rho}} \end{array}}{2} \right]^{\frac{1-\rho}{1-\alpha}} \right\}^{\frac{1}{1-\rho}}$$

for all $t \in \mathbb{N}$ all $\tau \in \{2, 3, ...\}$ and all $x, y, c \in \mathbb{R}_+$ with $x > y$. Letting $\tilde{\tau} \equiv \tau - 1$, we can rewrite this condition as:

$$\left\{ (1-\beta)c_0^{1-\rho} + \beta \left[\frac{\begin{array}{l} \{(1-\beta^t)x^{1-\rho} + \beta^t c^{1-\rho}\}^{\frac{1-\alpha}{1-\rho}} \\ + \left\{c^{1-\rho} + \beta^{\tilde{\tau}}(1-\beta^t)\left(y^{1-\rho} - c^{1-\rho}\right)\right\}^{\frac{1-\alpha}{1-\rho}} \end{array}}{2} \right]^{\frac{1-\rho}{1-\alpha}} \right\}^{\frac{1}{1-\rho}}$$

$$\geq$$

$$\left\{ (1-\beta)c_0^{1-\rho} + \beta \left[\frac{\begin{array}{l} \{(1-\beta^t)y^{1-\rho} + \beta^t c^{1-\rho}\}^{\frac{1-\alpha}{1-\rho}} \\ + \left\{c^{1-\rho} + \beta^{\tilde{\tau}}(1-\beta^t)\left(x^{1-\rho} - c^{1-\rho}\right)\right\}^{\frac{1-\alpha}{1-\rho}} \end{array}}{2} \right]^{\frac{1-\rho}{1-\alpha}} \right\}^{\frac{1}{1-\rho}}$$

for all $t, \tilde{\tau} \in \mathbb{N}$ and all $x, y, c \in \mathbb{R}_+$ with $x > y$.

First, suppose $\rho < 1$. The condition becomes

$$\left[\{(1-\beta^t)x^{1-\rho} + \beta^t c^{1-\rho}\}^{\frac{1-\alpha}{1-\rho}} + \left\{c^{1-\rho} + \beta^{\tilde{\tau}}(1-\beta^t)\left(y^{1-\rho} - c^{1-\rho}\right)\right\}^{\frac{1-\alpha}{1-\rho}}\right]^{\frac{1-\rho}{1-\alpha}}$$

$$\geq$$

$$\left[\{(1-\beta^t)y^{1-\rho} + \beta^t c^{1-\rho}\}^{\frac{1-\alpha}{1-\rho}} + \left\{c^{1-\rho} + \beta^{\tilde{\tau}}(1-\beta^t)\left(x^{1-\rho} - c^{1-\rho}\right)\right\}^{\frac{1-\alpha}{1-\rho}}\right]^{\frac{1-\rho}{1-\alpha}}.$$

Next, suppose $\rho > 1$. The condition becomes

$$\left[\{(1-\beta^t)x^{1-\rho} + \beta^t c^{1-\rho}\}^{\frac{1-\alpha}{1-\rho}} + \left\{c^{1-\rho} + \beta^{\tilde{\tau}}(1-\beta^t)\left(y^{1-\rho} - c^{1-\rho}\right)\right\}^{\frac{1-\alpha}{1-\rho}}\right]^{\frac{1-\rho}{1-\alpha}}$$

$$\leq$$

$$\left[\{(1-\beta^t)y^{1-\rho} + \beta^t c^{1-\rho}\}^{\frac{1-\alpha}{1-\rho}} + \left\{c^{1-\rho} + \beta^{\tilde{\tau}}(1-\beta^t)\left(x^{1-\rho} - c^{1-\rho}\right)\right\}^{\frac{1-\alpha}{1-\rho}}\right]^{\frac{1-\rho}{1-\alpha}}.$$

Note that c_0 does not enter this expressions, so the period-0 consumption does not affect the conditions for Stochastic Impatience.

It is straightforward to see that (by homotheticity) we can take $c = 1$ without loss of generality (express $x \equiv \lambda_x c$ and $y \equiv \lambda_y c$ for $\lambda_x, \lambda_y \in (0, +\infty)$, then note that $c^{1-\rho}$ cancels out in all expressions). So the conditions become

$$\left[\{(1-\beta^t)x^{1-\rho} + \beta^t\}^{\frac{1-\alpha}{1-\rho}} + \left\{1 + \beta^{\tilde{\tau}}(1-\beta^t)\left(y^{1-\rho} - 1\right)\right\}^{\frac{1-\alpha}{1-\rho}}\right]^{\frac{1-\rho}{1-\alpha}}$$

$$\geq$$

$$\left[\{(1-\beta^t)y^{1-\rho} + \beta^t\}^{\frac{1-\alpha}{1-\rho}} + \left\{1 + \beta^{\tilde{\tau}}(1-\beta^t)\left(x^{1-\rho} - 1\right)\right\}^{\frac{1-\alpha}{1-\rho}}\right]^{\frac{1-\rho}{1-\alpha}}$$

if $\rho < 1$, and

$$\left[\{(1-\beta^t)x^{1-\rho} + \beta^t\}^{\frac{1-\alpha}{1-\rho}} + \left\{1 + \beta^{\tilde{\tau}}(1-\beta^t)\left(y^{1-\rho} - 1\right)\right\}^{\frac{1-\alpha}{1-\rho}}\right]^{\frac{1-\rho}{1-\alpha}}$$

$$\leq$$

$$\left[\{(1-\beta^t)y^{1-\rho} + \beta^t\}^{\frac{1-\alpha}{1-\rho}} + \left\{1 + \beta^{\tilde{\tau}}(1-\beta^t)\left(x^{1-\rho} - 1\right)\right\}^{\frac{1-\alpha}{1-\rho}}\right]^{\frac{1-\rho}{1-\alpha}}$$

if $\rho > 1$.

There are 4 cases.

Case 1: $\alpha, \rho < 1$.

Here, the condition becomes

$$\left\{(1-\beta^t)x^{1-\rho}+\beta^t\right\}^{\frac{1-\alpha}{1-\rho}} - \left\{1+\beta^{\tilde{\tau}}(1-\beta^t)\left(x^{1-\rho}-1\right)\right\}^{\frac{1-\alpha}{1-\rho}}$$

$$\geq$$

$$\left\{(1-\beta^t)y^{1-\rho}+\beta^t\right\}^{\frac{1-\alpha}{1-\rho}} - \left\{1+\beta^{\tilde{\tau}}(1-\beta^t)\left(y^{1-\rho}-1\right)\right\}^{\frac{1-\alpha}{1-\rho}}$$

for all $x > y$ and all $t, \tilde{\tau}$. This holds iff

$$\frac{d}{dz}\left\{\left\{(1-\beta^t)z^{1-\rho}+\beta^t\right\}^{\frac{1-\alpha}{1-\rho}} - \left\{1+\beta^{\tilde{\tau}}(1-\beta^t)\left(z^{1-\rho}-1\right)\right\}^{\frac{1-\alpha}{1-\rho}}\right\} \geq 0$$

for all $z \in \mathbb{R}_+$.

Case 2: $\alpha, \rho > 1$.

$$\left\{(1-\beta^t)x^{1-\rho}+\beta^t\right\}^{\frac{1-\alpha}{1-\rho}} - \left\{1+\beta^{\tilde{\tau}}(1-\beta^t)\left(x^{1-\rho}-1\right)\right\}^{\frac{1-\alpha}{1-\rho}}$$

$$\leq$$

$$\left\{(1-\beta^t)y^{1-\rho}+\beta^t\right\}^{\frac{1-\alpha}{1-\rho}} - \left\{1+\beta^{\tilde{\tau}}(1-\beta^t)\left(y^{1-\rho}-1\right)\right\}^{\frac{1-\alpha}{1-\rho}}$$

for all $x > y$ and all $t, \tilde{\tau}$. This holds iff

$$\frac{d}{dz}\left\{\left\{(1-\beta^t)x^{1-\rho}+\beta^t\right\}^{\frac{1-\alpha}{1-\rho}} - \left\{1+\beta^{\tilde{\tau}}(1-\beta^t)\left(x^{1-\rho}-1\right)\right\}^{\frac{1-\alpha}{1-\rho}}\right\} \leq 0$$

for all $z \in \mathbb{R}_+$.

Case 3: $\alpha > 1 > \rho$.

$$\left\{(1-\beta^t)x^{1-\rho}+\beta^t\right\}^{\frac{1-\alpha}{1-\rho}} - \left\{1+\beta^{\tilde{\tau}}(1-\beta^t)\left(x^{1-\rho}-1\right)\right\}^{\frac{1-\alpha}{1-\rho}}$$

$$\leq$$

$$\left\{(1-\beta^t)y^{1-\rho}+\beta^t\right\}^{\frac{1-\alpha}{1-\rho}} - \left\{1+\beta^{\tilde{\tau}}(1-\beta^t)\left(y^{1-\rho}-1\right)\right\}^{\frac{1-\alpha}{1-\rho}}$$

for all $x > y$ and all $t, \tilde{\tau}$. This holds iff

$$\frac{d}{dz}\left\{\left\{(1-\beta^t)z^{1-\rho}+\beta^t\right\}^{\frac{1-\alpha}{1-\rho}} - \left\{1+\beta^{\tilde{\tau}}(1-\beta^t)\left(z^{1-\rho}-1\right)\right\}^{\frac{1-\alpha}{1-\rho}}\right\} \leq 0$$

for all $z \in \mathbb{R}_+$.

Case 4: $\alpha < 1 < \rho$**.**

$$\left\{(1-\beta^t)x^{1-\rho}+\beta^t\right\}^{\frac{1-\alpha}{1-\rho}} - \left\{1+\beta^{\tilde{\tau}}(1-\beta^t)\left(x^{1-\rho}-1\right)\right\}^{\frac{1-\alpha}{1-\rho}}$$
$$\geq$$
$$\left\{(1-\beta^t)y^{1-\rho}+\beta^t\right\}^{\frac{1-\alpha}{1-\rho}} - \left\{1+\beta^{\tilde{\tau}}(1-\beta^t)\left(y^{1-\rho}-1\right)\right\}^{\frac{1-\alpha}{1-\rho}}$$

for all $x > y$ and all $t, \tilde{\tau}$. This holds iff

$$\frac{d}{dz}\left\{\left\{(1-\beta^t)z^{1-\rho}+\beta^t\right\}^{\frac{1-\alpha}{1-\rho}} - \left\{1+\beta^{\tilde{\tau}}(1-\beta^t)\left(z^{1-\rho}-1\right)\right\}^{\frac{1-\alpha}{1-\rho}}\right\} \geq 0$$

for all $z \in \mathbb{R}_+$.

To combine all cases, let

$$\Phi(z) \equiv \left\{(1-\beta^t)z^{1-\rho}+\beta^t\right\}^{\frac{1-\alpha}{1-\rho}} - \left\{1+\beta^{\tilde{\tau}}(1-\beta^t)\left(z^{1-\rho}-1\right)\right\}^{\frac{1-\alpha}{1-\rho}}.$$

We have shown that Stochastic Impatience requires $\Phi'(z) \geq 0$ if either $\alpha, \rho < 1$ or $\alpha < 1 < \rho$, and $\Phi'(z) \leq 0$ if either $\alpha, \rho > 1$ or $\alpha > 1 > \rho$. That is, Stochastic Impatience holds if and only if:

- $\Phi'(z) \geq 0$ for all z if $\alpha < 1$
- $\Phi'(z) \leq 0$ for all z if $\alpha > 1$

But note that

$$\Phi'(z) = (1-\alpha)\,(1-\beta^t)z^{-\rho}\left\{\begin{array}{l}\left\{(1-\beta^t)z^{1-\rho}+\beta^t\right\}^{\frac{\rho-\alpha}{1-\rho}} \\ -\left\{1+\beta^{\tilde{\tau}}(1-\beta^t)\,(z^{1-\rho}-1)\right\}^{\frac{\rho-\alpha}{1-\rho}}\beta^{\tilde{\tau}}\end{array}\right\}.$$

Moreover, $(1-\beta^t)z^{-\rho} > 0$ for all $z \in \mathbb{R}_+$. Combining the two cases for α, we find that Stochastic Impatience holds if and only if:

$$\left\{(1-\beta^t)z^{1-\rho}+\beta^t\right\}^{\frac{\rho-\alpha}{1-\rho}} - \left\{1+\beta^{\tilde{\tau}}(1-\beta^t)\left(z^{1-\rho}-1\right)\right\}^{\frac{\rho-\alpha}{1-\rho}}\beta^{\tilde{\tau}} \geq 0.$$

We have therefore shown the following lemma:

Lemma 7. *Stochastic Impatience holds if and only if*

$$\left\{(1-\beta^t)z^{1-\rho}+\beta^t\right\}^{\frac{\rho-\alpha}{1-\rho}} \geq \beta^{\tilde{\tau}}\left\{1+\beta^{\tilde{\tau}}(1-\beta^t)\left(z^{1-\rho}-1\right)\right\}^{\frac{\rho-\alpha}{1-\rho}}$$

for all $t, \tilde{\tau}$ *and all* $z \in \mathbb{R}_+$.

Now we need to verify when this condition holds.

Case 1: $\alpha > 1 > \rho$**.**

Taking $t \to \infty$, Stochastic Impatience becomes

$$z^{\rho-\alpha} \geq \left\{1 + \beta^{\tilde{\tau}}\left(z^{1-\rho} - 1\right)\right\}^{\frac{\rho-\alpha}{1-\rho}} \beta^{\tilde{\tau}}$$

Since $\rho - \alpha < 0$, the condition becomes

$$z \leq \left\{1 + \beta^{\tilde{\tau}}\left(z^{1-\rho} - 1\right)\right\}^{\frac{1}{1-\rho}} \beta^{\frac{\tilde{\tau}}{\rho-\alpha}}$$

$$\iff \left[1 - \beta^{\left(\frac{1-\rho}{\rho-\alpha}+1\right)\tilde{\tau}}\right] \leq \beta^{\frac{1-\rho}{\rho-\alpha}\tilde{\tau}} \frac{1 - \beta^{\tilde{\tau}}}{z^{1-\rho}}.$$

Note that the RHS converges to zero as $z \nearrow +\infty$ and the LHS is bounded away from zero since

$$1 > \beta^{\left(\frac{1-\rho}{\rho-\alpha}+1\right)\tilde{\tau}} \iff \frac{\alpha - 1}{\alpha - \rho} > 0.$$

Therefore, Stochastic Impatience fails in this case.

Case 2: $\alpha > \rho > 1$**.**

Here, we can rearrange the Stochastic Impatience condition as:

$$\left\{(1-\beta^t)z^{1-\rho} + \beta^t\right\}^{\frac{\rho-\alpha}{1-\rho}} \geq \beta^{\tilde{\tau}} \left\{1 + \beta^{\tilde{\tau}}(1-\beta^t)\left(z^{1-\rho} - 1\right)\right\}^{\frac{\rho-\alpha}{1-\rho}}$$

for all $t, \tilde{\tau}$ and all $z \in \mathbb{R}_+$. Take $t \to \infty$, so the condition becomes

$$z^{\rho-\alpha} \geq \beta^{\tilde{\tau}} \left\{1 + \beta^{\tilde{\tau}}\left(z^{1-\rho} - 1\right)\right\}^{\frac{\rho-\alpha}{1-\rho}}$$

$$\iff 1 - \beta^{\tilde{\tau}\frac{1-\rho}{\rho-\alpha}+\tau} \geq \beta^{\tilde{\tau}\frac{1-\rho}{\rho-\alpha}} \left(1 - \beta^{\tilde{\tau}}\right) z^{\rho-1}.$$

Taking $z \nearrow \infty$, we find that the RHS converges to $+\infty$, violating Stochastic Impatience.

Case 3: $1 > \alpha \geq \rho$**.**

Here, we can rearrange the Stochastic Impatience condition as:

$$(1-\beta^t)z^{1-\rho} + \beta^t \leq \beta^{\tilde{\tau}\frac{1-\rho}{\rho-\alpha}} \left\{1 + \beta^{\tilde{\tau}}(1-\beta^t)\left(z^{1-\rho} - 1\right)\right\}$$

for all $t, \tilde{\tau}$ and all $z \in \mathbb{R}_+$. Rearrange this condition as:

$$\left[1 - \beta^{\tilde{\tau}\left(\frac{1-\rho}{\rho-\alpha}+1\right)}\right] z^{1-\rho} \leq \frac{\beta^{\tilde{\tau}\frac{1-\rho}{\rho-\alpha}} - \beta^t}{1 - \beta^t} - \beta^{\tilde{\tau}\left(\frac{1-\rho}{\rho-\alpha}+1\right)}$$

Note that

$$1-\beta^{\tilde{\tau}\left(\frac{1-\rho}{\rho-\alpha}+1\right)}<0 \iff \frac{1-\alpha}{\rho-\alpha}<0,$$

which is true.

Note that $\frac{\beta^{\tilde{\tau}\frac{1-\rho}{\rho-\alpha}}-\beta^t}{1-\beta^t}$ is decreasing in t whenever $\beta^{\tilde{\tau}\frac{1-\rho}{\rho-\alpha}}>1$, which is true since $\frac{1-\rho}{\rho-\alpha}<0$. Thus, Stochastic Impatience holds if and only if the condition above holds for $t=\infty$. Take $t\to+\infty$, so it becomes:

$$\left[1-\beta^{\tilde{\tau}\left(\frac{1-\rho}{\rho-\alpha}+1\right)}\right]z^{1-\rho}\le\beta^{\tilde{\tau}\frac{1-\rho}{\rho-\alpha}}-\beta^{\tilde{\tau}\left(\frac{1-\rho}{\rho-\alpha}+1\right)}.$$

This is true if and only if

$$\beta^{\tau\left(\frac{1-\rho}{\rho-\alpha}+1\right)}-\beta^{\tau\frac{1-\rho}{\rho-\alpha}}\le 0 \iff \beta\le 1,$$

verifying that Stochastic Impatience holds.

Case 4: $1<\alpha\le\rho$**.**

Recall the condition for Stochastic Impatience to hold:

$$\left\{(1-\beta^t)z^{1-\rho}+\beta^t\right\}^{\frac{\rho-\alpha}{1-\rho}}\ge\beta^{\tilde{\tau}}\left\{1+\beta^{\tilde{\tau}}(1-\beta^t)\left(z^{1-\rho}-1\right)\right\}^{\frac{\rho-\alpha}{1-\rho}}$$

for all $t,\tilde{\tau}$ and all $z\in\mathbb{R}_+$.

Since $\frac{\rho-\alpha}{1-\rho}<0$, we can rewrite this condition as

$$(1-\beta^t)z^{1-\rho}\left(1-\beta^{\tilde{\tau}\frac{1-\rho}{\rho-\alpha}+\tilde{\tau}}\right)\le\beta^{\tilde{\tau}\frac{1-\rho}{\rho-\alpha}}-\beta^{\tilde{\tau}\frac{1-\rho}{\rho-\alpha}+\tilde{\tau}}(1-\beta^t)-\beta^t.$$

Note that the LHS is negative since $(1-\beta^t)z^{1-\rho}>0$ and

$$1-\beta^{\tilde{\tau}\frac{1-\rho}{\rho-\alpha}+\tilde{\tau}}<0 \iff \frac{1-\alpha}{\rho-\alpha}<0,$$

which is true. Note also that the RHS is positive:

$$\beta^{\tilde{\tau}\frac{1-\rho}{\rho-\alpha}}-\beta^{\tilde{\tau}\frac{1-\rho}{\rho-\alpha}+\tilde{\tau}}(1-\beta^t)-\beta^t>0$$

$$\iff \beta^{\tilde{\tau}\frac{1-\rho}{\rho-\alpha}}\left(1-\beta^{\tilde{\tau}}\right)>\beta^t\left(1-\beta^{\tilde{\tau}\frac{1-\rho}{\rho-\alpha}+\tilde{\tau}}\right),$$

but

$$\beta^{\tilde{\tau}\frac{1-\rho}{\rho-\alpha}}\left(1-\beta^{\tilde{\tau}}\right)>0>\beta^t\overbrace{\left(1-\beta^{\tilde{\tau}\frac{1-\rho}{\rho-\alpha}+\tilde{\tau}}\right)}^{<0\text{ by our previous calculations}}.$$

This establishes that Stochastic Impatience holds.

Case 5: $\alpha < \rho < 1$.

Recall the condition for Stochastic Impatience to hold:

$$\left\{(1-\beta^t)z^{1-\rho}+\beta^t\right\}^{\frac{\rho-\alpha}{1-\rho}} \geq \beta^{\tilde{\tau}}\left\{1+\beta^{\tilde{\tau}}(1-\beta^t)\left(z^{1-\rho}-1\right)\right\}^{\frac{\rho-\alpha}{1-\rho}}$$

for all $t, \tilde{\tau}$ and all $z \in \mathbb{R}_+$. Since $\frac{\rho-\alpha}{1-\rho} > 0$, we can rewrite this condition as

$$(1-\beta^t)\left(1-\beta^{\tilde{\tau}\frac{1-\rho}{\rho-\alpha}+\tilde{\tau}}\right)z^{1-\rho} \geq \beta^{\tilde{\tau}\frac{1-\rho}{\rho-\alpha}} - \beta^{\tilde{\tau}\frac{1-\rho}{\rho-\alpha}+\tilde{\tau}}(1-\beta^t) - \beta^t.$$

Recall that

$$1-\beta^{\tilde{\tau}\frac{1-\rho}{\rho-\alpha}+\tilde{\tau}} > 0 \iff \frac{1-\alpha}{\rho-\alpha} > 0,$$

which is true here. Therefore, the LHS is positive. Because $\rho < 1$, the condition holds if and only if it holds as $z \searrow 0$. Since

$$\lim_{z\searrow 0}(1-\beta^t)z^{1-\rho}\left(1-\beta^{\tilde{\tau}\frac{1-\rho}{\rho-\alpha}+\tilde{\tau}}\right) = 0,$$

Stochastic Impatience holds if and only if

$$\beta^{\tilde{\tau}\frac{1-\rho}{\rho-\alpha}} - \beta^{\tilde{\tau}\frac{1-\rho}{\rho-\alpha}+\tilde{\tau}}(1-\beta^t) - \beta^t \leq 0$$

for all $t, \tilde{\tau}$. Rearrange this inequality as

$$\beta^{\tilde{\tau}\frac{1-\rho}{\rho-\alpha}} - \beta^{\tilde{\tau}\frac{1-\rho}{\rho-\alpha}+\tilde{\tau}} \leq \beta^t\underbrace{\left(1-\beta^{\tilde{\tau}\frac{1-\rho}{\rho-\alpha}+\tilde{\tau}}\right)}_{+}.$$

Since the RHS is decreasing in t, it holds for all t if and only if it holds as $t \nearrow +\infty$. Thus, Stochastic Impatience holds if and only if

$$\beta^{\tilde{\tau}\frac{1-\rho}{\rho-\alpha}} - \beta^{\tilde{\tau}\frac{1-\rho}{\rho-\alpha}+\tilde{\tau}} \leq 0 \iff \tilde{\tau}\frac{1-\rho}{\rho-\alpha} \geq \tilde{\tau}\frac{1-\rho}{\rho-\alpha}+\tilde{\tau},$$

which is false. Therefore, Stochastic Impatience fails.

Case 6: $\alpha < 1 < \rho$.

Since $\frac{\rho-\alpha}{1-\rho} < 0$, the condition for Stochastic Impatience to hold becomes

$$(1-\beta^t)z^{1-\rho}+\beta^t \leq \beta^{\tilde{\tau}\frac{1-\rho}{\rho-\alpha}} + \beta^{\tilde{\tau}\frac{1-\rho}{\rho-\alpha}+\tilde{\tau}}(1-\beta^t)\left(z^{1-\rho}-1\right)$$

for all $t, \tilde{\tau}$ and all $z \in \mathbb{R}_+$. Rearrange it as

$$(1-\beta^t)\left(1-\beta^{\tilde{\tau}\frac{1-\rho}{\rho-\alpha}+\tilde{\tau}}\right)z^{1-\rho} \leq \beta^{\tilde{\tau}\frac{1-\rho}{\rho-\alpha}} - \beta^{\tilde{\tau}\frac{1-\rho}{\rho-\alpha}+\tilde{\tau}}(1-\beta^t) - \beta^t.$$

Recall that

$$1 - \beta^{\tilde{\tau}\frac{1-\rho}{\rho-\alpha}+\tilde{\tau}} > 0 \iff \frac{1-\alpha}{\rho-\alpha} > 0,$$

which is true here. Therefore, the LHS is positive and decreasing in z. It follows that Stochastic Impatience holds if and only if the condition holds as $z \searrow 0$. Since

$$\lim_{z\searrow 0}(1-\beta^{t})\left(1-\beta^{\tilde{\tau}\frac{1-\rho}{\rho-\alpha}+\tilde{\tau}}\right)z^{1-\rho} = +\infty,$$

Stochastic Impatience fails in this case.

Combining all cases, Stochastic Impatience holds if and only if either $1 > \alpha \geq \rho$ or $1 < \alpha \leq \rho$.