Online Appendix to "Equilibria in symmetric games: Theory and Applications"

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A.1 A counting rule

If asymmetric equilibria exist, the set of asymmetric equilibria which are permutations of each other with respect to $\{1, ..., N\}$ form an equivalence class within the set of all asymmetric equilibria. I refer to a class of equivalent asymmetric equilibria simply as an equivalent asymmetric equilibrium. Consider a symmetric game, where the C^1-vector field ∇F satisfies: i) ∇F has only regular zeroes and ii) ∇F points inwards on the boundary of S^N. Let $\mathcal{I}^s$ denote the sum of the indices of all symmetric equilibria with respect to ∇F.

If a symmetric game satisfies the above Index conditions, and we can identify a zero with index -1, then there are multiple equilibria, but we cannot decide without further information whether there are multiple symmetric equilibria or a single symmetric equilibrium but asymmetric equilibria. As the following theorem implies, it is *not* sufficient, in general, to restrict the candidate zeroes to the symmetric points to exclude the possibility of asymmetric equilibria: It can be the case that there is a single symmetric equilibrium with index $+1$, but several asymmetric equilibria, such that the index sum equals 1.[1]

Proposition 6 *Consider a symmetric game, where ∇F satisfies the above Index conditions.*

(a) *If $\mathcal{I}^s = 1$ and there are asymmetric equilibria, then there is more than one equivalent asymmetric equilibrium. If especially $N = 2$ then there is an even number of equivalent asymmetric equilibria.*

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[1]Proposition 6 is obviously related to the well-known "oddness" of the number of equilibria in Index games.

(b) If $\mathcal{I}^s \neq 1$ then asymmetric equilibria exist. For $N = 2$:

(i) if $\mathcal{I}^s = 3 + 4z$ for $z \in \mathbb{Z}$ then there is an odd number of equivalent asymmetric equilibria

(ii) if $\mathcal{I}^s = 5+4z$ for $z \in \mathbb{Z}\setminus\{-1\}$ then there is an even number of equivalent asymmetric equilibria

Proof: Let $\omega \geq 1$ be the (necessarily odd) number of symmetric equilibria. Hence $\mathcal{I}^s$ must be a number from $\{\pm 1, \pm 3, \pm 5, ..., \pm\omega\}$. Further, if $\mathcal{I}^a$ denotes the index sum of all asymmetric equilibria, we must have $\mathcal{I}^s + \mathcal{I}^a = 1$. Note that all asymmetric equilibria in a given equivalence class have the same index. If $\mathcal{I}^s = 1$ but there are asymmetric equilibria, then $\mathcal{I}^a = 0$, which requires the existence of at least two equivalent asymmetric equilibria. If $\mathcal{I}^s \neq 1$ we must have $\mathcal{I}^a \neq 0$, which implies the existence of asymmetric equilibria. To see the rest set $N = 2$ and note that, if asymmetric equilibria exist, there are exactly two asymmetric equilibria within an equivalence class. Let n_- denote the number of equivalence classes with index -1, and n_+ those with index $+1$. Then $n_+ - n_- = \frac{1-\mathcal{I}^s}{2}$. If $\mathcal{I}^s$ is a number $3+4z$, the RHS of this equation is an odd number. Hence either n_- or n_+ must be odd and the other number must be even or zero. Consequently, $n_- + n_+$ must be odd. For $\mathcal{I}^s = 5 + 4z$ with $z \in \mathbb{Z}\setminus\{-1\}$ the RHS must be even and hence $n_1 + n_2$ must also be even. Finally, if $\mathcal{I}^s = 1$ then $n_- = n_+ = n$. For $n > 0$ this implies $n_- + n_+ = 2n$, which is even. ■

A.2 Symmetric equilibria: Additional results

From theorem 1 further sufficient conditions for the existence of a single symmetric equilibrium can be deduced.

Corollary 3 *If at least one of the next two statements is satisfied, there exists only one symmetric equilibrium. i) $\tilde{J}(x_1)$ has a dominant negative diagonal if $x_1 \in Cr^s$. ii) There is a matrix norm $\|\cdot\|$ such that $\|\partial\tilde{\varphi}(x_1)\| < 1$ if $x_1 \in Cr^s$.*

Proof: Consider the decomposition $\tilde{J} = \tilde{A} + \tilde{B}$, where $\tilde{A}$ is a diagonal matrix with $\frac{\partial\tilde{\Pi}_i(x_1,x_1)}{\partial x_{1i}}$ as its ii-th entry. Hence $Det(-\tilde{J}(x_1)) > 0 \Leftrightarrow Det\left(I + \tilde{A}^{-1}\tilde{B}\right) > 0 \Leftrightarrow \prod_{i=1}^{k}(1+\tilde{\lambda}_i) > 0$, where $\tilde{\lambda}$ is eigenvalue of $\tilde{A}^{-1}\tilde{B}$. But diagonal dominance of $\tilde{J}$ implies that every row sum of the absolute values of the entries of $\tilde{A}^{-1}\tilde{B}$ must be strictly smaller than one, which by a standard result of matrix analysis implies the spectral radius of $\tilde{A}^{-1}\tilde{B}$ to be less than one (Horn and Johnson (1985)), and the claim follows from theorem 1. Similarly, ii) implies iii) of theorem 1 as the spectral radius of $\partial\tilde{\varphi}(x_1)$ is bounded from above by any matrix norm. ■

A.3 Inexistence of asymmetric equilibria with boundary solutions

Theorem 4 can be extended to the case, where $\varphi(x_{-1}) \in \partial S$ is possible under the assumptions from section 2. To see how this can be done let $k = N = 2$, and suppose that $\varphi_2(x_2^0) = \bar{S}_2$ for some $x_2^0 \in S$, but $\varphi_1(x_2^0) \in Int(S_1)$. Now consider the following two systems of equation:

$$I) \quad \Pi_1(\hat{x}_{11}, \bar{S}_2, x_2^0) = 0 \qquad\qquad II) \quad \begin{array}{l} \Pi_1(\tilde{x}_{11}, \tilde{x}_{12}, x_2^0) = 0 \\ \Pi_2(\tilde{x}_{11}, \tilde{x}_{12}, x_2^0) = 0 \end{array}$$

As $\varphi_1(x_2^0) \in Int(S_1)$ our assumptions on Π imply that, for fixed $x_{12} = \bar{S}_2$, equation I) implicitly defines a local C^1-function $\hat{\varphi}_1(x_2)$, with $\hat{\varphi}_1(x_2^0) = \varphi_1(x_2^0)$.

The technical difficulty that $\varphi_2(x_2^0) \in \partial S_2$ potentially[2] imposes, is that II) can have a local C^1-solution $(\tilde{x}_{11}, \tilde{x}_{12})$, with $\tilde{x}_{11} = \tilde{\varphi}_1(x_2)$ around x_2^0, but both $\hat{\varphi}_1(x_2) \neq \tilde{\varphi}_1(x_2)$ as well as $\partial\hat{\varphi}_1(x_2) \neq \partial\tilde{\varphi}_1(x_2)$ are possible. If II) has a solution both $\hat{\varphi}_1(x_2), \tilde{\varphi}_1(x_2)$ are local C^1-functions around x_2^0, and $\varphi_1(x_2) = \hat{\varphi}_1(x_2)$ or $\varphi_1(x_2) = \tilde{\varphi}_1(x_2)$ around x_2^0. Together with the previous result, this shows that $\varphi_1(x_2)$ may not be differentiable at or around[3] x_2^0 despite that $\varphi_1(x_2^0) \in Int(S_1)$.

With this insight we can adapt the proof of theorem 4 to obtain a similar condition as (4). To see how, let $x_2 \neq x_2'$ and let $\psi(t) \equiv \varphi(x_2 + t(x_2' - x_2))$, $\hat{\psi}_1(t) \equiv \hat{\varphi}_1(x_2 + t(x_2' - x_2))$ and $\tilde{\psi}_1(t) \equiv \tilde{\varphi}_1(x_2 + t(x_2' - x_2))$ for $t \in [0, 1]$ and assume[4] that $\psi_2(t) = \bar{S}_2$ for some t.

[2]If the point $(\varphi_1(x_2^0), \bar{S}_2)$ is not a solution of II), then $\varphi_1(x_2)$ is implicitly defined by I) as a C^1-function around x_2^0. In sloppy terms this means that the boundary solution $\varphi_2(x_2) = \bar{S}_2$ is "strict", and the following problem does not emerge.

[3]If $k > 1$ and there are boundary solutions non-differentiable points need not be locally isolated.

[4]The following argument can easily be adjusted to capture the case where $\psi_2(t) = 0$ may also occur.

Suppose that $A_0 \equiv \psi_1(0) > \psi_1(1) \equiv A_1$. We want to show that there is $t \in (0,1)$ such that either $\hat{\psi}_1'(t) \leq \psi_1(1) - \psi_1(0)$ or $\tilde{\psi}_1'(t) \leq \psi_1(1) - \psi_1(0)$. By contradiction, assume that $\hat{\psi}_1'(t) > \psi_1(1) - \psi_1(0)$ and $\tilde{\psi}_1'(t) > \psi_1(1) - \psi_1(0)$ whenever these objects exist. Geometrically, this means that the functions $\hat{\psi}_1, \tilde{\psi}_1$ are less step (perhaps even increasing) than the line connecting A_0 and A_1. That is, for any $t_0 \in (0,1)$ there is a perfect interval $\mathbb{B} = (t_0 - \varepsilon, t_0 + \varepsilon)$ such that $\hat{\psi}_1$ or $\tilde{\psi}_1$ are moving away from the line connecting A_0 and A_1 as t increases on $\mathbb{B}$ whenever these functions are well-defined at t_0.

The fact that $\psi_1(t_0)$ corresponds either to $\hat{\psi}_1(t_0)$ or to $\tilde{\psi}_1(t_0)$ whenever $\psi_1(t_0) \in Int(S_1)$ then implies that if $\psi_1(t_0) \in Int(S_1)$, the function $\psi_1(t)$ must always be moving away from the line connecting A_0 with A_1, which, by continuity, makes $\psi_1(1) = A_1$ impossible, contradiction.

The consequence of this argument is that if $\varphi(x_2) \in \partial S$ can occur, we must apply the reasoning in the proof of theorem 4 to the function $\hat{\varphi}_1(x_2)$ as well. In practice this means that we have to determine the slopes in condition (4) not only by applying the IFT to the system II) (this is sufficient if we know that best-replies are always interior) by also by applying the IFT to the FOC with boundary points. For example, applying the IFT to $\Pi_1(x_{11}, x_{12}, x_2) = 0$, where $x_{12} = 0$ or $x_{12} = \bar{S}_2$ are held fixed, gives the slopes

$$\hat{\alpha}_1 = \frac{\partial x_{11}(\bar{S}_2, x_{21}, x_{22})}{\partial x_{21}}, \hat{\beta}_1 = \frac{\partial x_{11}(\bar{S}_2, x_{21}, x_{22})}{\partial x_{22}}, \hat{\alpha}_2 = \frac{\partial x_{11}(0, x_{21}, x_{22})}{\partial x_{21}}, \hat{\beta}_2 = \frac{\partial x_{11}(0, x_{21}, x_{22})}{\partial x_{22}}$$

The same argument applied to Π_2 gives four additional slopes $\hat{\gamma}_1, \hat{\gamma}_2, \hat{\delta}_1, \hat{\delta}_2$. Now, working through the same steps as in the proof of theorem 4 shows that if the statement in (4) *additionally* holds for *any* combination of these new slopes (where we replace α by $\hat{\alpha}_1$, $\beta = \hat{\beta}_2$...) evaluated at all $x_2, x_2' \in S$, this is sufficient to rule out the possibility of asymmetric equilibria in the game.

A.4 Uniqueness in almost symmetric games

A natural question is whether uniqueness in a symmetric game is a property that extends, at least, to almost symmetric games, i.e., games where the ex-ante asymmetries are small. The answer to this question is yes (for $k \geq 1$), provided that the symmetric equilibrium is regular.[5]

Proposition 7 *Suppose that the joint best-reply satisfies $\phi(\cdot,\cdot) \in C\left(S^N \times \mathcal{P}^N, S^N\right)$ and consider a symmetric game $\Gamma(c)$ with a unique, symmetric and regular equilibrium $x^* \in Int(S^N)$. Then $\exists\ \delta > 0$ such that $\Gamma(c')$ has a unique equilibrium for any $c' \in \mathbb{B}(c, \delta)$.*

See section 4.3 for the notation. The proof of this proposition builds on the following lemma.

Lemma 4 *Suppose that $\phi(\cdot,\cdot) \in C\left(S^N \times \mathcal{P}^N, S^N\right)$ and consider a symmetric game $\Gamma(c_0)$, $c_0 \in \mathcal{P}^N$. Suppose that (x^n) is a sequence of FPs, i.e., $\phi(x^n, c^n) = x^n$. If $(x^n, c^n) \to (x_0, c_0)$, then x_0 is an equilibrium of $\Gamma(c_0)$.*

Proof: Define $z(x,c) \equiv \phi(x,c) - x$ and note that x is a FP of ϕ iff $z(x,c) = 0$. As $(x^n, c^n) \to (x_0, c_0)$ continuity of z implies $\lim_{n\to\infty} z(x^n, c^n) = z(x_0, c_0)$. But $z(x^n, c^n) = z^n \to 0$ implies that $z(x_0, c_0) = 0$. ■

Proof of proposition 7:

As $x^* \in Int(S^N)$ is regular, $\nabla F(x^*, c) = 0$, and $\nabla F(\cdot,\cdot)$ is continuously differentiable around (x^*, c), the IFT asserts that for any c' in some neighborhood $U \subset \mathcal{P}^N$ of c the equation system $\nabla F(x, c') = 0$ has a locally unique solution $x = h(c')$, where $h \in C^1(U, V)$ and $V \subset S^N$ is a neighborhood of x^*, which shows existence and local uniqueness of an equilibrium for parameters $c \in U$. Let $E(c)$ denote the set of equilibria of the game with parameter vector c. To see global uniqueness, suppose by contradiction that for every $\delta > 0\ \exists\ c^n \in \mathbb{B}(c, \delta)$ such that $E(c^n)$ is multi-valued. Hence there is a sequence (c^n) with $\lim_{n\to\infty} c^n = c$ such that $E(c^n)$ is multi-valued for any $n \in \mathbb{N}$. Consequently, we can find two sequences $(x^n), (y^n)$ with $x^n \neq y^n$ and $\phi(x^n, c^n) = x^n$, $\phi(y^n, c^n) = y^n$ and $x^n \to x^*$. Define $z(x,c) \equiv \phi(x,c) - x$. Because $z(\cdot,\cdot)$ is continuous the set $z^{-1}(\{0\}) \subset S^N \times \mathcal{P}^N$ is compact. As (y^n, c^n) is a sequence in $z^{-1}(\{0\})$ there is a convergent subsequence (y^{n_t}, c^{n_t}), hence also $y^{n_t} \to y^*$. But then lemma 4 and the fact that x^* is unique imply $y^* = x^*$, which by the regularity of x^* means that there is a T such that $y^{n_t} = x^{n_t}$ for all $t \geq T$, a contradiction. ■

[5] Regular means $Det(J(x)) \neq 0$, where $J(x)$ is the Jacobian of $\nabla F(x)$.

References

HORN, R. A. AND C. R. JOHNSON (1985): *Matrix Analysis*, Cambridge University Press.