Supplementary Appendix to "Friends and Enemies: A Model of Signed Network Formation"

Timo Hiller (University of Bristol)

Appendix B: Production

This section presents an alternative model specification, which allows for production. Furthermore, different from the main part of the paper, agents compete for a single prize (as in, for example, Garfinkel, 1990, Hirshleifer, 1995 and Skaperdas, 1992) and the prize consists of the total production of all agents. Each agent receives a fraction of total production, which is determined by the contest success function in ratio form. Production is assumed to incur a convex cost. We uphold our assumptions regarding link formation from the main part of the paper. Moreover, we assume that agents are homogeneous.

Again we find that all Nash equilibria obey structural balance. Note that for our results to hold, we need to assume that the parameter of the contest success function, ϕ, is smaller than one. This guarantees that agents have incentives to coordinate their actions relative to third agents. Interestingly, efficiency considerations now depend on the shape of the cost function. If the parameter of the cost function is weakly larger than 2, then the network such that all links are positive yields the (uniquely) largest sum of payoffs and the sum of production levels is maximal. Note that, as in the main part of the paper, this network can always be sustained as a Nash equilibrium. However, we show by way of an example that, if the parameter of the cost function is smaller than 2, then a Nash equilibrium may exist, such that the sum of payoffs is strictly larger than in the network where all links are positive.

Note that this model specification can be seen as a variation of König et al. (2015), where, rather than assuming a fixed network of alliances and conflicts and agents choose their fighting effort, we assume that fighting effort is fixed, while the network of alliances and conflicts is endogenous (and production is introduced).

Model Description

Let $N = \{1, 2, ..., n\}$ be the set of identical agents, with $n \geq 3$. Denote by p_i agent i's production level and define i's strategy by $\mathbf{s}_i = (p_i, g_{i,1}, g_{i,2}, ..., g_{i,i-1}, g_{i,i+1}, ..., g_{i,n})$, with $p_i \geq 0$ and, as in the main part of the paper, $g_{i,j} \in \{-1, 1\}$ for each $j \in N \setminus \{i\}$. Again agent i is said to extend a positive link to j if $g_{i,j} = 1$ and a negative link if $g_{i,j} = -1$. The set of strategies of i is denoted by S_i and the strategy space by $S = S_1 \times ... \times S_n$. The network of relationships is written as $\mathbf{g} = (\mathbf{g}_1, \mathbf{g}_2, ..., \mathbf{g}_n)$, where $\mathbf{g}_i = (g_{i,1}, g_{i,2}, ..., g_{i,i-1}, g_{i,i+1}, ..., g_{i,n})$. The undirected network $\bar{\mathbf{g}}$ is defined as in the main part of the paper. The vector of production levels is denoted by $\mathbf{p} = (p_1, ..., p_n)$.

We define the following sets: $N_i^+(\mathbf{g}) = \{j \in N \mid \bar{g}_{i,j} = 1\}$ is the set of agents to which agent i reciprocates a positive link and therefore $\bar{g}_{i,j} = 1$ in the undirected network $\bar{\mathbf{g}}$. $N_i^-(\mathbf{g}) = \{j \in N \mid \bar{g}_{i,j} = -1\}$ is the set of agents such that i extends and/or receives a negative link and therefore $\bar{g}_{i,j} = -1$. $N_i^{e-}(\mathbf{g}) = \{j \in N \mid g_{i,j} = -1\}$ is the set of agents to which agent i extends a negative link. We further define the set of agents that are negatively connected with agent i and sustain a higher number of positive links than agent i. We denote this set by $N_i^{s-}(\mathbf{g}) = \{j \in N \mid \bar{g}_{i,j} = -1 \, and \, n_i(\mathbf{g}) < n_j(\mathbf{g})\}$, where $n_i(\mathbf{g})$ is defined as in the main part of the paper. Denote the following cardinalities: $e_i(\mathbf{g}) =\mid N_i^{e-}(\mathbf{g}) \mid$ and $n_i^{s-}(\mathbf{g}) =\mid N_i^{s-}(\mathbf{g}) \mid$. The strength or power of an agent is again determined endogenously. More precisely, the strength of agent i in network $\mathbf{g}$ is given by $\eta_i(\mathbf{g}) = a + \beta n_i(\mathbf{g}) - \gamma n_i^{s-}(\mathbf{g})$, with $a > 0$, $\beta > 0$ and $\gamma > 0$. Note that the term $-\gamma n_i^{s-}(\mathbf{g})$ ensures that agents may find it profitable to extend negative links in the payoff specification presented here. We consider the following payoff function, which allows for production

$$u_i(\mathbf{g}, \mathbf{p}) = \sum_{j \in N} p_j \cdot \frac{\eta_i(\mathbf{g})^\phi}{\sum_{j \in N} \eta_j(\mathbf{g})^\phi} - \frac{1}{\alpha} \cdot p_i^\alpha - e_i(\mathbf{g})\varepsilon.$$

Relative to the main part of the paper, we abstract for simplicity from the cost of conflict and exclude the term $-c_i(\mathbf{g})\kappa$. Total production is split among all agents according to their endogenously determined strengths. Production incurs a convex cost, i.e. $\alpha > 1$. We assume that $0 < \phi < 1$, which guarantees incentives for stronger agents to coordinate their actions relative to weaker agents and is therefore crucial for our equilibrium characterization.[1] The equilibrium concept used is again pure strategy Nash Equilibrium. A strategy profile $\mathbf{s}^*$ is a pure strategy Nash Equilibrium (NE) iff

$$u_i(\mathbf{s}_i^*, \mathbf{s}_{-i}^*) \geq u_i(\mathbf{s}_i, \mathbf{s}_{-i}^*), \forall \mathbf{s}_i \in S_i, \forall i \in N.$$

We denote agent i's deviation strategy from strategy $\mathbf{s}_i$ by $\mathbf{s}_i'$ and the resulting strategy profile after a proposed deviation is denoted by $\mathbf{s}'$. If i's deviation only consists of altering links (i.e. the production level remains constant in the proposed deviation), then we simply write $\mathbf{g}_i'$ and again denote the network after proposed deviation by $\mathbf{g}'$.

[1] Note that for $\phi > 1$ one can easily construct examples for which Nash equilibria are not structurally balanced.

Analysis

We start by showing that in a fixed network, which we denote with $\boldsymbol{g}^f$, there exists a unique pure strategy Nash equilibrium in production levels, $\mathbf{p}$.

Lemma 1B: *For any fixed network $\boldsymbol{g}^f$, there exists a unique pure strategy Nash equilibrium in production levels, $\mathbf{p}$, with $p_i > 0$ $\forall i$. Furthermore, if $\eta_i(\boldsymbol{g}^f) > \eta_j(\boldsymbol{g}^f)$, then $p_i(\boldsymbol{g}^f) > p_j(\boldsymbol{g}^f)$ and $u_i(\boldsymbol{g}^f, \mathbf{p}) > u_j(\boldsymbol{g}^f, \mathbf{p})$.*

Proof. Note that each agent i's equilibrium production level, p_i, is uniquely defined since $u_i(\boldsymbol{g}^f, \mathbf{p})$ is concave in p_i and agent i's share of production is fixed for a given network $\boldsymbol{g}^f$, so that $p_i(\boldsymbol{g}^f) = \left(\eta_i(\boldsymbol{g}^f)^\phi / \sum_{j \in N} \eta_j(\boldsymbol{g}^f)^\phi\right)^{\frac{1}{\alpha-1}}$. Since $\eta_i(\boldsymbol{g}^f)^\phi / \sum_{j \in N} \eta_j(\boldsymbol{g}^f)^\phi > 0$ for all i in any network $\boldsymbol{g}^f$, it then follows directly that $p_i(\boldsymbol{g}^f) > 0$ $\forall i$ in any $\boldsymbol{g}^f$. Since we assume $\alpha > 1$, $\eta_i(\boldsymbol{g}^f) > \eta_j(\boldsymbol{g}^f)$ implies that $p_i > p_j$ also holds in any Nash equilibrium $\mathbf{p}$. Note next that if $\eta_i(\boldsymbol{g}^f) > \eta_j(\boldsymbol{g}^f)$, then $u_i(\boldsymbol{g}^f, \mathbf{p}) > u_i(\boldsymbol{g}^f, p_1, .., p_{i-1}, p_j, p_{i+1}, ...p_n) > u_j(\boldsymbol{g}^f, \mathbf{p})$. *Q.E.D.*

Lemma 2B: *In any NE $\mathbf{s} = (\mathbf{g}, \mathbf{p})$, there does not exist a pair of agents i and j such that $g_{i,j} = g_{j,i} = -1$ for all $i, j \in N$.*

Proof. Assume there exists a pair of agents i and j, which extend negative links to each other. This cannot be part of any Nash equilibrium, since a deviation in the form of extending a positive link to the respective other agent (while keeping the production level constant) increases payoffs by ε. *Q.E.D.*

Lemma 3B: *In any NE $\mathbf{s} = (\mathbf{g}, \mathbf{p})$, if $\bar{g}_{i,j} = -1$ with $\eta_i(\mathbf{g}) < \eta_j(\mathbf{g})$, then $g_{i,j} = 1$.*

Proof. Assume, contrary to the above, that $\bar{g}_{i,j} = -1$ with $g_{i,j} = -1$, $g_{j,i} = 1$ and $\eta_i(\mathbf{g}) < \eta_j(\mathbf{g})$. Then agent i can profitably deviate with $g_i + g_{i,j}^+$, yielding $\bar{g}'_{i,j} = 1$, while keeping the production level, p_i, constant. Denote the network after proposed deviation with $\mathbf{g}'$ and note that $\eta_i(\mathbf{g}') = \eta_i(\mathbf{g}) + \beta + \gamma$ and $\eta_j(\mathbf{g}') = \eta_j(\mathbf{g}) + \beta$. Note that since $\eta_i(\mathbf{g}) < \eta_j(\mathbf{g})$ and $0 < \phi < 1$, there exists a pair of real numbers $x, y \in \mathbb{R}$ with $x > y > 1$ such that $x \cdot \eta_i(\mathbf{g})^\phi = (\eta_i(\mathbf{g}) + \beta + \gamma)^\phi$ and $y \cdot \eta_j(\mathbf{g})^\phi = (\eta_j(\mathbf{g}) + \beta)^\phi$. To show that $u_i(\mathbf{g}{+}\mathrm{g}_{i,j}^+, \mathbf{p}) - u_i(\mathbf{g}, \mathbf{p}) > 0$ holds, it is sufficient to show that

$$\frac{(\eta_i(\mathbf{g})+\beta+\gamma)^\phi}{(\eta_i(\mathbf{g})+\beta+\gamma)^\phi+(\eta_j(\mathbf{g})+\beta)^\phi+\sum_{l\in N\setminus\{i,j\}}\eta_l(\mathbf{g})^\phi} > \frac{\eta_i(\mathbf{g})^\phi}{\eta_i(\mathbf{g})^\phi+\eta_j(\mathbf{g})^\phi+\sum_{l\in N\setminus\{i,j\}}\eta_l(\mathbf{g})^\phi}.$$

With the above we can now write

$$\frac{(\eta_i(\mathbf{g})+\beta+\gamma)^\phi}{(\eta_i(\mathbf{g})+\beta+\gamma)^\phi+(\eta_j(\mathbf{g})+\beta)^\phi+\sum_{l\in N\setminus\{i,j\}}\eta_l(\mathbf{g})^\phi} = \frac{x\cdot\eta_i(\mathbf{g})^\phi}{x\cdot\eta_i(\mathbf{g})^\phi+y\cdot\eta_j(\mathbf{g})^\phi+\sum_{l\in N\setminus\{i,j\}}\eta_l(\mathbf{g})^\phi} >$$

$$\frac{x\cdot\eta_i(\mathbf{g})^\phi}{x\cdot\eta_i(\mathbf{g})^\phi+x\cdot\eta_j(\mathbf{g})^\phi+x\cdot\sum_{l\in N\setminus\{i,j\}}\eta_l(\mathbf{g})^\phi} = \frac{\eta_i(\mathbf{g})^\phi}{\eta_i(\mathbf{g})^\phi+\eta_j(\mathbf{g})^\phi+\sum_{l\in N\setminus\{i,j\}}\eta_l(\mathbf{g})^\phi}.$$

Therefore, proposed deviation is profitable. *Q.E.D.*

Proposition 1B: *In any NE* $\mathbf{s} = (\mathbf{g}, \mathbf{p})$*, if* $\eta_i(\mathbf{g}) = \eta_j(\mathbf{g})$*, then* $\bar{g}_{i,j} = 1$ *and if* $\eta_i(\mathbf{g}) \neq \eta_j(\mathbf{g})$*, then* $\bar{g}_{i,j} = -1$*.*

Proof. We start with the first part of the statement.

Step 1: *In any NE* $\mathbf{s} = (\mathbf{g}, \mathbf{p})$*, if* $\eta_i(\mathbf{g}) = \eta_j(\mathbf{g})$*, then* $\bar{g}_{i,j} = 1$*.*

Assume, contrary to the above, that there exists a pair of agents i and j, such that $\eta_i(\mathbf{g}) = \eta_j(\mathbf{g})$ and $\bar{g}_{i,j} = -1$. Assume w.l.o.g. that i extends the negative link, i.e. $g_{i,j} = -1$. This cannot be part of any Nash equilibrium, since agent i can profitably deviate by extending a positive link to j, while keeping p_i constant. Note that $\eta_i(\mathbf{g}') = \eta_i(\mathbf{g}) + \beta$ and $\eta_j(\mathbf{g}') = \eta_j(\mathbf{g}) + \beta$ holds. Marginal payoffs of proposed deviation can be written as

$u_i(\mathbf{g}+\mathrm{g}^{+}_{i,j}, \mathbf{p}) - u_i(\mathbf{g}, \mathbf{p}) =$

$$= (\textstyle\sum_{j \in N} p_j) \cdot \left(\frac{(\eta_i(\mathbf{g})+\beta)^{\phi}}{(\eta_i(\mathbf{g})+\beta)^{\phi}+(\eta_j(\mathbf{g})+\beta)^{\phi}+\sum_{l \in N\setminus\{i,j\}} \eta_l(\mathbf{g})^{\phi}} - \frac{\eta_i(\mathbf{g})^{\phi}}{\eta_i(\mathbf{g})^{\phi}+\eta_j(\mathbf{g})^{\phi}+\sum_{l \in N\setminus\{i,j\}} \eta_l(\mathbf{g})^{\phi}} \right) + \varepsilon.$$

To show that $u_i(\mathbf{g}+\mathrm{g}^{+}_{i,j}, \mathbf{p}) - u_i(\mathbf{g}, \mathbf{p}) > 0$, it is sufficient to show that

$$\frac{(\eta_i(\mathbf{g})+\beta)^{\phi}}{(\eta_i(\mathbf{g})+\beta)^{\phi}+(\eta_j(\mathbf{g})+\beta)^{\phi}+\sum_{l \in N\setminus\{i,j\}} \eta_l(\mathbf{g})^{\phi}} > \frac{\eta_i(\mathbf{g})^{\phi}}{\eta_i(\mathbf{g})^{\phi}+\eta_j(\mathbf{g})^{\phi}+\sum_{l \in N\setminus\{i,j\}} \eta_l(\mathbf{g})^{\phi}}$$

holds. Note that since $\eta_i(\mathbf{g}) = \eta_j(\mathbf{g})$, there exists a $x \in \mathbb{R}$ with $x > 1$ such that $x \cdot \eta_i(\mathbf{g})^{\phi} = x \cdot \eta_j(\mathbf{g})^{\phi} = (\eta_i(\mathbf{g}) + \beta)^{\phi} = (\eta_j(\mathbf{g}) + \beta)^{\phi}$ and we an therefore write

$$\frac{(\eta_i(\mathbf{g})+\beta)^{\phi}}{(\eta_i(\mathbf{g})+\beta)^{\phi}+(\eta_j(\mathbf{g})+\beta)^{\phi}+\sum_{l \in N\setminus\{i,j\}} \eta_l(\mathbf{g})^{\phi}} = \frac{x \cdot \eta_i(\mathbf{g})^{\phi}}{x \cdot \eta_i(\mathbf{g})^{\phi}+x \cdot \eta_j(\mathbf{g})^{\phi}+\sum_{l \in N\setminus\{i,j\}} \eta_l(\mathbf{g})^{\phi}} >$$

$$\frac{x \cdot \eta_i(\mathbf{g})^{\phi}}{x \cdot \eta_i(\mathbf{g})^{\phi}+x \cdot \eta_j(\mathbf{g})^{\phi}+x \cdot \sum_{l \in N\setminus\{i,j\}} \eta_l(\mathbf{g})^{\phi}} = \frac{\eta_i(\mathbf{g})^{\phi}}{\eta_i(\mathbf{g})^{\phi}+\eta_j(\mathbf{g})^{\phi}+\sum_{l \in N\setminus\{i,j\}} \eta_l(\mathbf{g})^{\phi}}.$$

Therefore, proposed deviation is profitable. That is, in any Nash equilibrium, if $\eta_i(\mathbf{g}) = \eta_j(\mathbf{g})$, then $\bar{g}_{i,j} = 1$. Note that we therefore know that if $\eta_i(\mathbf{g}) = \eta_j(\mathbf{g})$ $\forall i, j \in N$, then all links in $\mathbf{g}$ are positive. Next, assume that there exists a pair of agents such that $\eta_i(\mathbf{g}) \neq \eta_j(\mathbf{g})$. The remaining part of the proof uses an induction argument and we start by proving the base case in three steps (Step 2 to Step 4 of the proof).

Base Case: *In any NE* $\mathbf{s} = (\mathbf{g}, \mathbf{p})$*,* $\bar{g}_{i,j} = 1$ $\forall i, j \in P^m(\mathbf{g})$ *and* $\bar{g}_{i,k} = -1$ $\forall i \in P^m(\mathbf{g})$ *and* $\forall k \notin P^m(\mathbf{g})$*.*

Step 2: *In any NE* $\mathbf{s} = (\mathbf{g}, \mathbf{p})$*,* $N_i^{+}(\mathbf{g})\setminus\{j\} = N_j^{+}(\mathbf{g})\setminus\{i\}$ *and* $N_i^{-}(\mathbf{g}) = N_j^{-}(\mathbf{g})$ $\forall i, j \in P^m(\mathbf{g})$*.*

The statement holds trivially for $\mid P^m(\mathbf{g}) \mid = 1$. Assume $\mid P^m(\mathbf{g}) \mid \geq 2$ and, contrary to the above, that $\exists i, j \in P^m(\mathbf{g}) : N_i^{+}(\mathbf{g})\setminus\{j\} \neq N_j^{+}(\mathbf{g})\setminus\{i\}$ (and therefore $N_i^{-}(\mathbf{g}) \neq N_j^{-}(\mathbf{g})$).

That is, there exists a pair of agents $i, j \in P^m(\mathbf{g})$, such that their respective sets of friends and enemies are different. Note that from the first part of the proof we know that $\bar{g}_{i,j} = 1$ $\forall i, j \in P^m(\mathbf{g})$. From Lemma 2B we further know that agents not in $P^m(\mathbf{g})$ extend positive links to all agents in $P^m(\mathbf{g})$, i.e. $g_{k,i} = 1 \; \forall k \notin P^m(\mathbf{g})$ and $\forall i \in P^m(\mathbf{g})$. Therefore, for $i, j \in P^m(\mathbf{g}) : N_i^+(\mathbf{g})\backslash\{j\} \neq N_j^+(\mathbf{g})\backslash\{i\}$ (and therefore $N_i^-(\mathbf{g}) \neq N_j^-(\mathbf{g})$) to hold, it must be that i and j play different strategies relative to third agents, which we denote with $g_{i\backslash j} \neq g_{j\backslash i}$. That is, there must exist a pair of agents k and l, such that $\bar{g}_{i,k} = -1$ and $\bar{g}_{i,l} = 1$, while $\bar{g}_{j,k} = 1$ and $\bar{g}_{j,l} = -1$. Without loss of generality assume that $\eta_k(\mathbf{g}) \geq \eta_l(\mathbf{g})$. Agent i can then profitably deviate with $g_i + g_{i,k}^+ + g_{i,l}^-$. To see this, note that we can write the marginal payoffs of proposed deviation as

$$u_i(\mathbf{g}+\mathrm{g}_{i,k}^+ + \mathrm{g}_{i,l}^-, \mathbf{p}) - u_i(\mathbf{g}, \mathbf{p}) =$$

$$= \left(\sum_{j\in N} p_j\right) \cdot \left(\frac{\eta_i(\mathbf{g})^\phi}{(\eta_k(\mathbf{g})+\beta+\gamma)^\phi + (\eta_l(\mathbf{g})-\beta-\gamma)^\phi + \sum_{j\in N\backslash\{k,l\}} \eta_j(\mathbf{g})^\phi} - \frac{\eta_i(\mathbf{g})^\phi}{\eta_k(\mathbf{g})^\phi + \eta_l(\mathbf{g})^\phi + \sum_{j\in N\backslash\{k,l\}} \eta_j(\mathbf{g})^\phi}\right).$$

The expression is strictly positive, due to $0 < \phi < 1$ and $\eta_k(\mathbf{g}) \geq \eta_l(\mathbf{g})$ and proposed deviation is therefore profitable.

Step 3: *In any NE* $\mathbf{s} = (\mathbf{g}, \mathbf{p})$, $\bar{g}_{i,k} = -1 \; \forall i \in P^m(\mathbf{g})$ *and* $\forall k \in P^{m-1}(\mathbf{g})$.

Assume to the contrary that there exists an agent $k \in P^{m-1}(\mathbf{g})$ such that $\bar{g}_{i,k} = 1$ for some $i \in P^m(\mathbf{g})$. From the previous step we know that then $\bar{g}_{i,k} = 1 \; \forall i \in P^m(\mathbf{g})$. Note that $k \in P^{m-1}(\mathbf{g})$ and therefore $N_k^-(\mathbf{g}) \neq \emptyset$. From the first part of the proof we know that $\bar{g}_{j,k} = 1 \; \forall j, k \in P^{m-1}(\mathbf{g})$. By Lemma 2B we further know that $g_{h,k} = 1 \; \forall h \in P^{m-x}(\mathbf{g})$ and $\forall x \in \mathbb{N} : x \geq 2$. Therefore, $g_{l,k} = 1 \; \forall l \in N\backslash\{k\}$. We can now discern two cases. For $g_{k\backslash i} \neq g_{i\backslash k}$ we can use the same argument as in Step 2 to show that either k or i (or both) can profitably deviate. If, on the other hand, $g_{k\backslash i} = g_{i\backslash k}$ holds, then we reach an immediate contradiction, as $\eta_k(\mathbf{g}) = \eta_i(\mathbf{g})$ for $k \in P^{m-1}(\mathbf{g})$ and $i \in P^m(\mathbf{g})$.

Step 4: *In any NE* $\mathbf{s} = (\mathbf{g}, \mathbf{p})$, $\bar{g}_{i,k} = -1 \; \forall i \in P^m(\mathbf{g})$ *and* $\forall k \notin P^m(\mathbf{g})$.

If there are only two sets of agents with different numbers of positive links, then we are done by Step 3. Assume that there are at least three such sets and that there exists a pair of agents $i \in P^m(\mathbf{g})$ and $k \notin P^m(\mathbf{g}) \cup P^{m-1}(\mathbf{g})$ such that $\bar{g}_{i,k} = 1$. Recall that from Step 2 we know that agents in $P^m(\mathbf{g})$ are positively connected, while agents in $P^m(\mathbf{g})$ extend negative directed links to all agents $l \in P^{m-1}(\mathbf{g})$. Note further that $\eta_l(\mathbf{g}) > \eta_k(\mathbf{g}) \; \forall l \in P^{m-1}(\mathbf{g})$. Agent i can then profitably deviate with the following strategy $\mathbf{g}_i + g_{i,l}^+ + g_{i,k}^-$. Proposed deviation is profitable by an analogous argument to the one used in Step 2.

Define the super set $\tilde{P}^r(\mathbf{g}) = \cup_{i=m-r}^{i=m} P^i(\mathbf{g})$. Note that $\tilde{P}^0(\mathbf{g}) = P^m(\mathbf{g})$.

Inductive Step: *In any NE* $\mathbf{s} = (\mathbf{g}, \mathbf{p})$*, if* $\bar{g}_{i,j} = 1 \; \forall i, j \in P^{m-x}(\mathbf{g})$ *and* $\bar{g}_{i,k} = -1$ $\forall i \in P^{m-x}(\mathbf{g})$ *and* $\forall k \notin P^{m-x}(\mathbf{g})$ *holds* $\forall x \in \mathbb{N} : 0 \leq x \leq r$*, then* $\bar{g}_{i,j} = 1 \; \forall i, j \in P^{m-(r+1)}(\mathbf{g})$ *and* $\bar{g}_{i,k} = -1 \; \forall i \in P^{m-(r+1)}(\mathbf{g})$ *and* $\forall k \notin P^{m-(r+1)}(\mathbf{g})$.

In Step 4 we showed that $\bar{g}_{i,j} = 1 \; \forall i, j \in P^m(\mathbf{g})$ and $\bar{g}_{i,k} = -1 \; \forall i \in P^m(\mathbf{g})$ and $\forall k \notin P^m(\mathbf{g})$. Assume the statement holds for all sets $P^{m-x}(\mathbf{g})$ with $x \in \mathbb{N} : 0 \leq x \leq r$. From Lemma 3B we know that $g_{i,k} = -1$ and $g_{k,i} = 1 \; \forall i \in \tilde{P}^r(\mathbf{g})$ and $\forall k \notin \tilde{P}^r(\mathbf{g})$, while from Lemma 2B we know that in any Nash equilibrium there does not exist a pair of agents i and k such that $g_{i,k} = g_{k,i} = -1$. We can now use an argument analogous to the one used in Step 2 to Step 4 of the base case, relabeling $P^m(\mathbf{g})$ with $P^{m-(r+1)}(\mathbf{g})$ and $P^{m-1}(\mathbf{g})$ with $P^{m-(r+2)}(\mathbf{g})$, to establish the above result. *Q.E.D.*

To simplify notation, we write $A_i(\mathbf{g}) = \frac{\eta_i(\mathbf{g})^\phi}{\sum_{j \in N} \eta_j(\mathbf{g})^\phi}$.

Proposition 2B: *There exists a NE* $\mathbf{s} = (\mathbf{g}, \mathbf{p})$ *such that all agents are friends.*

Proof. If all links are positive, then a relevant deviation of agent i consists of extending negative links to some subset of agents $N \setminus \{i\}$ and adjusting the production level. It is easy to see that then $A_i(\mathbf{g}') < A_i(\mathbf{g})$ and no profitable deviation exists. *Q.E.D.*

In Proposition 3B we compare the sum of payoffs and sum of production levels across different networks, where production levels are assumed to be the Nash equilibrium production levels for the corresponding network. That is, we write $p_i(\mathbf{g})$ for agent i's equilibrium production level given $\mathbf{g}$ and, likewise, denote the Nash equilibrium vector of effort levels for given network $\mathbf{g}$ with $\mathbf{p}(\mathbf{g})$. We show that when the parameter of the cost function α is larger or equal to 2, then the network where all links are positive, which we denote with $\mathbf{g}^+$, yields the uniquely largest sum of payoffs and the (weakly) largest sum of production levels. Note that $\mathbf{g}^+$ can be sustained as a Nash equilibrium.

Proposition 3B: *If* $\alpha \geq 2$*, then i)* $\sum_{i \in N} u_i(\mathbf{g}^+, \mathbf{p}(\mathbf{g}^+)) > \sum_{i \in N} u_i(\mathbf{g}, \mathbf{p}(\mathbf{g})) \; \forall \mathbf{g} \neq \mathbf{g}^+$ *and ii)* $\sum_{i \in N} p_i(\mathbf{g}^+) \geq \sum_{i \in N} p_i(\mathbf{g}) \; \forall \mathbf{g} \neq \mathbf{g}^+$.

Proof. Note first that from the first order conditions we know that for a given fixed network $\mathbf{g}$, the corresponding Nash equilibrium effort levels are given by $p_i = A_i(\mathbf{g})^{\frac{1}{\alpha-1}}$. Note that in $\mathbf{g}^+$, $A_i(\mathbf{g}^+) = \frac{1}{n} \; \forall i \in N$ and therefore $p_i = (\frac{1}{n})^{\frac{1}{\alpha-1}} \; \forall i \in N$. Note further that $\sum_{i \in N} A_i(\mathbf{g}) = 1 \; \forall \mathbf{g}$. We can write the sum of gross payoffs, which we denote with $\pi(\mathbf{g}, \mathbf{p})$, as

$$\pi(\mathbf{g}, \mathbf{p}) = \sum_{i \in N} A_i(\mathbf{g})^{\frac{1}{\alpha-1}} - \frac{1}{\alpha} \sum_{i \in N} A_i(\mathbf{g})^{\frac{\alpha}{\alpha-1}}.$$

If $\alpha \geq 2$, then $A_i(\mathbf{g})^{\frac{1}{\alpha-1}}$ is (weakly) concave, while $A_i(\mathbf{g})^{\frac{\alpha}{\alpha-1}}$ is strictly convex. Since $\sum_{i\in N} A_i(\mathbf{g}) = 1\ \forall \mathbf{g}$ and $A_i(\mathbf{g}^+) = \frac{1}{n}\ \forall i \in N$, it follows directly that the sum of gross payoffs is maximal in $\mathbf{g}^+$. Since linking cost are also strictly lower than in any other network, the sum of payoffs is uniquely largest in the network where all links are positive. Similarly, since $p_i = A_i(\mathbf{g})^{\frac{1}{\alpha-1}}$ is (weakly) concave, $\sum_{i\in N} A_i(\mathbf{g}) = 1$ and $A_i(\mathbf{g}^+) = \frac{1}{n}\ \forall i \in N$, the sum of production levels is (weakly) largest in $\mathbf{g}^+$. *Q.E.D.*

If, however, α is strictly smaller than 2, then a network other than $\mathbf{g}^+$ may yield a strictly larger sum of payoffs. We provide an example below.

Example: (Production). Assume $n = 10$, $a = 1$, $\alpha = 1.9$, $\beta = 1$, $\gamma = 0.1$, $\phi = 0.1$ and $\varepsilon = 0.001$, then one can show that 9 agents ganging up on one agent can be sustained as a Nash equilibrium. Total welfare is then given by approximately 1.4, while in the network where all links are positive total welfare is approximately 0.7.[2]

References

[1] Garfinkel, Michelle R. (1990), Arming as a strategic investment in a cooperative equilibrium, *American Economic Review*, 80:50-68.

[2] Hirshleifer, Jack (1995) Anarchy and its breakdown, *Journal of Political Economy*, 103:26-52.

[3] König, Michael, Dominic Rohner, Mathias Thoenig, Fabrizio Zilibotti (2015) Networks in Conflict: Theory and Evidence from the Great War of Africa, mimeo.

[4] Skaperdas, Stergios (1992) Cooperation, conflict, and power in the absence of property rights, *American Economic Review*, 82:720-739.

[2]The calculations were executed in Mathematica and are available from the author upon request.