Online Appendix to "Robust Multiplicity with a Grain of Naivité"

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This appendix contains some results not included in Heifetz and Kets [2017]. Unless stated otherwise, all references to sections, results, etcetera, are to Heifetz and Kets [2017].

I Universal type space

We show that the type space $\mathcal{T}^*$ is *universal*, in the sense that it generates all belief hierarchies. We show this by proving that every type space can be mapped into $\mathcal{T}^*$ using a belief-preserving mapping, which we define next.

Let $\mathcal{T} := \langle (T_i)_{i=1,2}, (\beta_i)_{i=1,2}, (\chi_i^T)_{i=1,2} \rangle$ and $\mathcal{Q} := \langle (Q_i)_{i=1,2}, (\lambda_i)_{i=1,2}, (\chi_i^Q)_{i=1,2} \rangle$ be type spaces. Let I_i^T be the set of indices $k \leq \infty$ such that the set T_i^k of types with index k is nonempty, and let I_i^Q be defined analogously. For each player $i = 1, 2$ and $k \in I_i^T$, let φ_i^k be a measurable function from T_i^k to Q_i^k. Define $\varphi_i := (\varphi_i^k)_{k \in I_i^T}$, and let $\varphi := (\varphi_i)_{i=1,2}$. Also, for $i = 1, 2$ and $k < \infty$, if T_i^k is nonempty, then define

$$\varphi_{-i}^{<k} : T_{-i}^{\leq k-1} \to Q_{-i}^{\leq k-1}$$

by

$$\varphi_{-i}^{<k}\left((t_{-i}^{m_{-i}})\right) := \left(\varphi_{-i}^{m_{-i}}(t_{-i}^{m_{-i}})\right)$$

where $t_{-i}^{m_{-i}} \in T_{-i}^{m_{-i}}$, $m_{-i} < k$. Note that the function $\varphi_{-i}^{<k}$ is well-defined. Let Id_Θ be the identity function on Θ.

The function φ is a *type morphism* from $\mathcal{T}$ to $\mathcal{Q}$ if for each player $i = 1, 2$, $I_i^Q \supseteq I_i^T$, and

(i) for each $k = 1, 2, \ldots$, type $t_i \in T_i^k$, and $E \in \mathscr{B}(\Theta) \otimes (Q_{-i}^{\leq k-1})$,

$$\lambda_i^k\left(\varphi_i^k(t_i)\right)(E) = \beta_i^k(t_i)\left((\mathrm{Id}_\Theta, \varphi_{-i}^{<k})^{-1}(E)\right); \tag{I.1}$$

(*ii*) for $t_i \in T_i^\infty$, $E \in \mathscr{B}(\Theta) \otimes \mathscr{B}(Q_{-i})$,

$$\lambda_i^\infty\left(\varphi_i^\infty(t_i)\right)(E) = \beta_i^\infty(t_i)\left((\mathrm{Id}_\Theta, \varphi_{-i}^\infty)^{-1}(E)\right); \tag{I.2}$$

(*iii*) for $t_i \in T_i^k$, $k = 1, 2, \ldots, \infty$, we have $\chi_i^Q(\varphi_i^k(t_i)) = \chi_i^T(t_i)$.

The mapping φ is a *type isomorphism* if $I_i^T \supseteq I_i^Q$, the inverse of φ_i is measurable for each $i = 1, 2$, and the inverse satisfies (*i*)–(*ii*).

Conditions (*i*)–(*iii*) are the analogues of the standard condition that a type morphism preserves beliefs for the case where players can have any depth of reasoning. If a type space only consists of types of infinite depth, the current definition of a type morphism reduces to the standard one. Lemma II.3 below shows that type morphisms preserve belief hierarchies [cf. Heifetz and Samet, 1998, Prop. 5.1].

Using the concept of a type morphism, we next show that modeling belief hierarchies by types is without loss of generality in the sense that every (coherent) belief hierarchy can be modeled in this way. Say that a type space $\mathcal{Q}$ is *universal* if every nonredundant[1] type space $\mathcal{T}$ can be mapped into $\mathcal{Q}$ by a unique type morphism, and its image is a model [Mertens and Zamir, 1985].

We show that $\mathcal{T}^*$ is universal. We prove this by showing that there is a unique type morphism from any type space to $\mathcal{T}^*$.

Lemma I.1. *For every type space $\mathcal{T}$, there is a unique type morphism from $\mathcal{T}$ to $\mathcal{T}^*$.*

The proof shows that each type $t_i \in T_i^k$ with index k in $\mathcal{T}$ can be mapped into a belief hierarchy in H_i^k of depth k using a so-called *hierarchy map* $h_i^{T,k}$. This implies that every type with index k generates a belief hierarchy of depth $k = 0, 1, \ldots, \infty$. With some abuse of terminology, we say that a type *has depth (of reasoning)* k if it generates a belief hierarchy of depth k. A type space $\mathcal{T}$ is *nonredundant* if for all $i = 1, 2$ and k such that T_i^k is nonempty, the hierarchy map $h_i^{T,k} : T_i^k \to H_i^k$ is one-to-one.

Proposition I.2. *The type space $\mathcal{T}^*$ is universal, and the universal space is unique (up to type isomorphism).*

Thus, the type space $\mathcal{T}^*$ contains all type spaces. The proof shows that the converse is also true: every model corresponds to a type space. Proposition I.2 implies that $\mathcal{T}^*$ generates all belief hierarchies.

[1]A type space is nonredundant if no two types generate the same belief hierarchy [Mertens and Zamir, 1985]; see below for a formal definition in our setting.

I.1 Common belief in infinite depth of reasoning

We show that the universal Harsanyi space, constructed by Mertens and Zamir [1985] and others, is a model (contained in the universal space $\mathcal{T}^*$) that is characterized by the event that players have an infinite depth of reasoning, and commonly believe that all players have an infinite depth of reasoning.

The universal type space for the class of Harsanyi type spaces can be constructed in a similar way as the universal type space $\mathcal{T}^*$ for type spaces that allow for finite-order reasoning. Let $\hat{Z}_i^0 := X_i \times \{\hat{z}_i^0\}$, where $\hat{z}_i^0$ is an arbitrary singleton, and define

$$\hat{\Omega}_i^0 := \Theta \times \hat{Z}_{-i}^0,$$

and

$$\hat{Z}_i^1 := \hat{Z}_i^0 \times \Delta(\hat{\Omega}_i^0).$$

For $k = 1, 2, \ldots$, assume, inductively, that we have already defined $\hat{Z}_i^\ell$ for each player $i = 1, 2$ and all $\ell \leq k$. Define

$$\hat{\Omega}_i^k := \Theta \times \hat{Z}_{-i}^k,$$

and let

$$\hat{Z}_i^{k+1} := \big\{(x_i, \mu_i^0, \ldots, \mu_i^k, \mu_i^{k+1}) \in \hat{Z}_i^k \times \Delta(\hat{\Omega}_i^k) : \text{marg}_{\hat{\Omega}_i^{k-1}} \mu_i^{k+1} = \mu_i^k\big\}.$$

The space $\hat{Z}_i$ for player i is the set of all $(x_i, \mu_i^0, \mu_i^1, \ldots)$ such that $(x_i, \mu_i^0, \mu_i^1 \ldots, \mu_i^k) \in \hat{Z}_i^k$ for all k. By standard arguments, the analogue of Lemma B.1 holds. Moreover, there is a Borel measurable function $\hat{\zeta}_i$ that assigns to each belief hierarchy $z_i \in \hat{Z}_i$ a Borel probability measure $\hat{\zeta}_i(z_i) \in \Delta(\Theta \times \hat{Z}_{-i})$ [cf. Heifetz, 1993]. If we define $\hat{\chi}_i^{MZ} : \hat{Z}_i \to X_i$ to be the projection function, we can view $\widehat{\mathcal{T}}^{MZ} := \langle (\hat{Z}_i)_{i=1,2}, (\hat{\zeta}_i)_{i=1,2}, (\hat{\chi}_i^{MZ})_{i=1,2} \rangle$ as a Harsanyi type space. As is well-known, the Harsanyi type space $\widehat{\mathcal{T}}^{MZ}$ is universal with respect to the class of Harsanyi type spaces, in the sense that every Harsanyi type space can be embedded into $\widehat{\mathcal{T}}^{MZ}$ via a unique type morphism for Harsanyi type spaces.

The Harsanyi type space $\widehat{\mathcal{T}}^{MZ}$ corresponds to a type space $\mathcal{T}^{MZ} = \langle (Z_i)_{i=1,2}, (\zeta_i^\infty)_{i=1,2}, (\chi_i^\infty)_{i=1,2} \rangle$ in our sense if we take the type set for player $i = 1, 2$ to be $Z_i = Z_i^\infty \cup \bigcup_{k=0}^\infty Z_i^k$, where $Z_i^\infty := \hat{Z}_i$ and $Z_i^k = \emptyset$ for $k < \infty$, and the belief map given by $\zeta_i^\infty := \hat{\zeta}_i$. Also, let $\chi_i^\infty(z_i) := \hat{\chi}_i^{MZ}(z_i)$ for $i = 1, 2$ and $z_i \in Z_i$. It then follows from Proposition I.2 that $\mathcal{T}^{MZ}$ can be embedded in the universal type space $\mathcal{T}^*$ via a unique type morphism. The converse clearly does not hold, as $\mathcal{T}^*$ contains types that have a finite depth of reasoning, types that assign a positive probability to types with a finite depth of reasoning, types that assign a positive probability to types that assign a positive probability to types with a finite depth of reasoning, and so on. Moreover, because the space $\mathcal{T}^{MZ}$ is nonredundant by construction, the type space $\mathcal{T}^{MZ}$ corresponds to a model in the universal type space $\mathcal{T}^*$.

We now characterize this model in terms of players' higher-order beliefs. More specifically, we show that the the model corresponding to $\mathcal{T}^{MZ}$ is characterized by the event that there is correct common belief in the event that players have an infinite depth of reasoning, that is, all players have an infinite depth of reasoning, believe that others have an infinite depth of reasoning, believe that others believe that, and so on, *ad infinitum*.

To state the result, we define the event that a player believes an event that concerns other players' signals and beliefs.[2] An *assumption* E_i about player i is a measurable subset of H_i. A *joint assumption* is a set of the form $E = \prod_{i=1,2} E_i$, where E_i is an assumption about player i.

Let $i = 1, 2$ and let $E = E_1 \times E_2$ be a joint assumption. Then, define[3]

$$B_i(E) := \big\{h_i \in H_i \setminus H_i^0 : \psi_i(h_i)(\Theta \times E_{-i}) = 1\big\}.$$

Thus, $B_i(E)$ consists of the types that *believe* E_{-i} (with probability 1). Let $B(E) := B_1(E) \times B_2(E)$. The following result is immediate:

Lemma I.3. *For each player $i = 1, 2$ and joint assumption E, we have that $B_i(E) \in \mathscr{B}(H_i)$. So, $B_i(E)$ is an assumption about player i.*

Then, we say that the joint assumption E is *(correct) common belief* at $h \in H$ if

$$h \in CB(E) := E \cap \bigcap_{\ell=0}^{\infty} [B]^{\ell}(E),$$

where $[B]^1(E) := B(E)$, and $[B]^{\ell}(E) := B_1([B]^{\ell-1}(E)) \times B_2([B]^{\ell-1}(E))$ for $\ell > 1$. It follows from Lemma I.3 that $B(E)$ and $CB(E)$ are measurable for any joint assumption E. Finally, let $E_i^{\infty} := H_i^{\infty}$ be the assumption that player i has an infinite depth of reasoning, so that E^{∞} is the joint assumption that players have an infinite depth of reasoning. We can now formally state Proposition 3.2:

Proposition 3.2. Let φ be the unique type morphism from $\mathcal{T}^{MZ}$ to the universal type space $\mathcal{T}^*$. Then,

$$\varphi_1^{\infty}(Z_1) \times \varphi_2^{\infty}(Z_2) = CB(E^{\infty}).$$

[2]We thus do not consider players' beliefs about the state of nature directly, and we implicitly assume that players know their own signal. We could consider the more general case, but the current definition is simpler, and suffices for our purposes.

[3]We define the belief operator for the universal space $\mathcal{T}^*$, but the definition can clearly be extended to arbitrary type spaces.

II Proofs

II.1 Proof of Lemma I.1

Let $\mathcal{T} = \langle (T_i)_{i=1,2}, (\beta_i^k)_{i=1,2,k\in I_i^T}, (\chi_i)_{i=1,2}\rangle$ be a type space. Given a collection of functions $f_\lambda : V_\lambda \to W_\lambda$, we define the induced functions $f : V \to W$ and $f_{-\lambda} : V_{-\lambda} \to W_{-\lambda}$, $\lambda \in \Lambda$, by $f(v) := (f_\lambda(v_\lambda))_{\lambda\in\Lambda}$ and $f_{-\lambda}(v_{-\lambda}) := (f_\lambda(v_\ell))_{\ell\in\Lambda\setminus\{\lambda\}}$.

To construct a type morphism from the types in $\mathcal{T}$ to the types in the space $\mathcal{T}^*$, we first construct a collection of functions that maps each type into the associated hierarchy of beliefs (Step 1). Step 2 establishes that these mappings define a type morphism from $\mathcal{T}$ to $\mathcal{T}^*$. Step 3 then shows that this type morphism is unique.

Step 1: From types to belief hierarchies

Each type induces a belief hierarchy, as we show now. The mapping from types to belief hierarchies is standard [e.g., Mertens and Zamir, 1985], except that we take into account that hierarchies may have a finite depth.

We define a collection of mappings. Lemma II.1 below shows that these functions are well-defined. For $i = 1, 2$, if $T_i^0 \neq \emptyset$, let $h_i^{T,0,0} : T_i^0 \to H_i^0$ be defined by

$$h_i^{T,0,0}(t_i) = (\chi_i(t_i), h_i^{*,0}).$$

Clearly, $h_i^{T,0,0}(T_i^0) \subseteq H_i^0$. Also, $h_i^{T,0,0}$ is measurable.

Similarly, if T_i^1 is nonempty, define $h_i^{T,1,0} : T_i^1 \to \widetilde{H}_i^0$ by

$$h_i^{T,1,0}(t_i) = (\chi_i(t_i), \tilde{\mu}_i^0).$$

Again, it is easy to see that $h_i^{T,1,0}(T_i^1) \subseteq \widetilde{H}_i^0$, and that $h_i^{T,1,0}$ is measurable. If T_i^0 is nonempty, define the function $h_i^{T,<1,0} : T_i^0 \to H_i^0$ by

$$h_i^{T,<1,0}(t_i) := h_i^{T,0,0}(t_i).$$

Again, $h_i^{T,<1,0}(T_i^0) \subseteq H_i^0$, and $h_i^{T,<1,0}$ is measurable. Finally, define the function $h_i^{T,1,1} : T_i^1 \to H_i^1$ by

$$h_i^{T,1,1}(t_i) := \left(h_i^{T,1,0}(t_i), \beta_i^1(t_i) \circ \left(\mathrm{Id}_\Theta, h_{-i}^{T,<1,0}\right)^{-1}\right),$$

where Id_Θ is the identity function on Θ. It is easy to verify that $h_i^{T,1,1}(T_i^i) \subseteq H_i^1$. Since an image measure $\mu \circ f^{-1}$ induced by a Borel probability measure μ and a measurable function f from a metrizable space into a metrizable space is measurable, the function $h_i^{T,1,1}$ is measurable.

Fix $k = 1, 2, \ldots$, and let $\ell = 0, \ldots, k-1$. Suppose, inductively, that the mappings $h_i^{T,m,\ell}$ have been defined for $m = 0, 1 \ldots, k$ whenever the relevant domain is nonempty. If $T_i^{\leq k} = \bigcup_{m=0}^k T_i^m$ is nonempty, then define

$$h_i^{T,<k+1,\ell} : T_i^{\leq k} \to \widetilde{H}_i^{\leq \ell}$$

by

$$\forall m=0,1,\ldots,k, \quad t_i \in T_i^m : \qquad h_i^{T,<k+1,\ell}(t_i) := \begin{cases} h_i^{T,m,\ell}(t_i) & \text{if } m>\ell; \\ h_i^{T,m,m}(t_i) & \text{if } m\leq \ell. \end{cases}$$

Also, for $k>0$, let

$$h_i^{T,<k+1,k} : T_i^{\leq k} \to H_i^{\leq k}$$

be defined by

$$\forall m=0,1,\ldots,k, t_i \in T_i^m : \qquad h_i^{T,<k+1,k}(t_i) := h_i^{T,m,m}(t_i).$$

Then, if $T_i^{k+1} \neq \emptyset$, let $h_i^{T,k+1,0} : T_i^{k+1} \to \widetilde{H}_i^0$ be defined by

$$h_i^{T,k+1,0}(t_i) := (\chi_i(t_i), t_i^{*,0}),$$

as before, and for $\ell = 1,\ldots,k$, define $h_i^{T,k+1,\ell} : T_i^{k+1} \to \widetilde{H}_i^\ell$ by

$$h_i^{T,k+1,\ell}(t_i) := \left(h_i^{T,k+1,\ell-1}(t_i), \beta_i^{k+1}(t_i) \circ \left(\mathrm{Id}_\Theta, h_{-i}^{T,<k+1,\ell-1}\right)^{-1}\right).$$

Finally, define $h_i^{T,k+1,k+1} : T_i^{k+1} \to H_i^{k+1}$ by

$$h_i^{T,k+1,k+1}(t_i) := \left(h_i^{T,k+1,k}(t_i), \beta_i^{k+1}(t_i) \circ \left(\mathrm{Id}_\Theta, h_{-i}^{T,<k+1,k}\right)^{-1}\right).$$

The next lemma states that these functions are well-defined:

Lemma II.1. *Let $i=1,2$ and $k=0,1,\ldots$.*

(a) If T_i^k is nonempty, then $h_i^{T,k,\ell}$ is well-defined and measurable for $\ell = 0,1,\ldots,k$.

(b) If $T_i^{\leq k}$ is nonempty, then $h_i^{T,<k+1,\ell}$ is well-defined and measurable for $\ell = 0,1,\ldots,k$.

Proof. We start with some preliminary observations. Let $Y = \bigcup_{\lambda\in\Lambda} Y^\lambda$ be a countable union of topological spaces, endowed with the sum topology. By standard results, for $B \in \mathscr{B}(Y)$ and $\lambda \in \Lambda$, we have that $B \cap Y^\lambda \in \mathscr{B}(Y^\lambda)$. Also, for $B^\lambda \in \mathscr{B}(Y^\lambda)$, $\lambda \in \Lambda$, we have $B^\lambda \in \mathscr{B}(Y)$. Finally, if Y and W are Polish, then $\mathscr{B}(Y \times W) = \mathscr{B}(Y) \otimes \mathscr{B}(W)$. We will make use of these results without mention.

We are now ready to prove Lemma II.1. The proof is by induction. As noted above, the functions $h_i^{T,0,0}$, $h_i^{T,1,0}$, and $h_i^{T,1,1}$ are well-defined and measurable (as is $h_i^{T,<1,0}$) for every player i (whenever the respective domains are nonempty). Let $k=1,2,\ldots$. Suppose that the functions $h_i^{T,k,\ell}$ and $h_i^{T,k,k}$ are well-defined and measurable whenever T_i^k is nonempty. It suffices to show that:

(i) The function $h_i^{T,<k+1,\ell}$ is well-defined and measurable for $\ell = 0,1,\ldots,k$.

(ii) The function $h_i^{T,k+1,\ell}$ is well-defined and measurable for $\ell = 0, 1, \ldots, k+1$.

To prove (i), first note that $T_i^{\leq k}$ is nonempty whenever T_i^k is nonempty. It follows directly from the induction hypothesis that $h_i^{T,<k+1,\ell}$ and $h_i^{T,<k+1,k}$ are well-defined for $\ell = 0, 1, \ldots, k-1$, i.e.,

$$h_i^{T,<k+1,\ell}\big(T_i^{\leq k}\big) \subseteq \widetilde{H}_i^{\leq \ell}, \qquad \text{and} \qquad h_i^{T,<k+1,k}\big(T_i^{\leq k}\big) \subseteq H_i^{\leq k}.$$

To show that $h_i^{T,<k+1,k}$ is measurable, let $B \in \mathscr{B}(H_i^{\leq k})$. Then,

$$\begin{aligned}\big(h_i^{T,<k+1,k}\big)^{-1}(B) &= \big\{t_i \in T_i^{\leq k} : h_i^{T,<k+1,k}(t_i) \in B\big\} \\ &= \bigcup_{m=0}^{k} \big\{t_i \in T_i^m : h_i^{T,m,m}(t_i) \in B \cap H_i^m\big\}.\end{aligned}$$

Hence, it suffices to show that for all $\ell = 0, \ldots k$,

$$\big\{t_i \in T_i^\ell : h_i^{T,\ell,\ell}(t_i) \in B \cap H_i^\ell\big\} \in \mathscr{B}\big(T_i^{\leq k}\big). \tag{II.1}$$

By our earlier observations, we have that $B \cap H_i^\ell \in \mathscr{B}(H_i^\ell)$. It then follows from the measurability of $h_i^{T,\ell,\ell}$ that

$$\big\{t_i \in T_i^\ell : h_i^{T,\ell,\ell}(t_i) \in B \cap H_i^\ell\big\} \in \mathscr{B}\big(T_i^\ell\big),$$

and (II.1) follows. The proof that $h_i^{T,<k+1,\ell}$ is measurable for $\ell = 0, \ldots, k-1$ is similar and thus omitted.

The proof of (ii) consists of two parts. We first show that $h_i^{T,k+1,\ell}$ and $h_i^{T,k+1,k+1}$ are well-defined for $\ell = 0, 1, \ldots, k$ whenever T_i^{k+1} is nonempty. That is, suppose T_i^{k+1} is nonempty. Then,

$$h_i^{T,k+1,\ell}\big(T_i^{k+1}\big) \subseteq \widetilde{H}_i^\ell \qquad \text{and} \qquad h_i^{T,k+1,k+1}\big(T_i^{k+1}\big) \subseteq H_i^{k+1}.$$

Clearly, $h_i^{T,k+1,0}(T_i^{k+1}) \subseteq \widetilde{H}_i^0$. Let $\ell = 1, \ldots, k-1$, and suppose $h_i^{T,k+1,\ell-1}\big(T_i^{k+1}\big) \subseteq \widetilde{H}_i^{\ell-1}$. We show that $h_i^{T,k+1,\ell}\big(T_i^{k+1}\big) \subseteq \widetilde{H}_i^\ell$. From the induction hypothesis and (i) it follows that $h_{-i}^{T,<k+1,\ell-1}$ is well-defined and measurable (recall condition (d) in the definition of a type space). Using the induction hypothesis, we have that for all $t_i \in T_i^{k+1}$,

$$h_i^{T,k+1,\ell}(t_i) = \Big(h_i^{T,k+1,\ell-1}(t_i), \beta_i^{k+1}(t_i) \circ \big(\mathrm{Id}_\Theta, h_{-i}^{T,<k+1,\ell-1}\big)^{-1}\Big) \in \widetilde{H}_i^\ell \times \Delta(\Theta \times \widetilde{H}_{-i}^{\leq \ell-1}).$$

If $\ell = 1$, then we are done. If $\ell = 2, 3, \ldots, k$, we need to show that a player's higher-order beliefs are coherent, i.e., for each $t_i \in T_i^{k+1}$,

$$\operatorname{marg}_{\widetilde{\Omega}_i^{\ell-2}} \beta_i^{k+1}(t_i) \circ \big(\mathrm{Id}_\Theta, h_{-i}^{T,<k+1,\ell-1}\big)^{-1} = \beta_i^{k+1}(t_i) \circ \big(\mathrm{Id}_\Theta, h_{-i}^{T,<k+1,\ell-2}\big)^{-1}.$$

Fix $E \in \mathscr{B}(\widetilde{\Omega}_i^{\ell-2})$. Then, using the extended definition of the marginal,

$$\begin{aligned}
& \operatorname{marg}_{\widetilde{\Omega}_i^{\ell-2}} \beta_i^{k+1}(t_i) \circ \left(\mathrm{Id}_\Theta, h_{-i}^{T,<k+1,\ell-1}\right)^{-1}(E) \\
= \;& \beta_i^{k+1}(t_i) \circ \left(\mathrm{Id}_\Theta, h_{-i}^{T,<k+1,\ell-1}\right)^{-1}\big(\{(\theta, x_{-i}, \mu_{-i}^0, \dots, \mu_{-i}^{\ell-2}, \mu_{-i}^{\ell-1}) \in \Theta \times \widetilde{H}_{-i}^{\leq \ell-1} : \\
& \qquad (\theta, x_{-i}, \mu_{-i}^0, \dots, \mu_{-i}^{\ell-2}) \in E\}\big) + \beta_i^{k+1}(t_i) \circ \left(\mathrm{Id}_\Theta, h_{-i}^{T,<k+1,\ell-1}\right)^{-1}\big(E \cap \widetilde{\Omega}_i^{\ell-2}\big) \\
= \;& \beta_i^{k+1}(t_i) \circ \left(\mathrm{Id}_\Theta, h_{-i}^{T,<k+1,\ell-2}\right)^{-1}(E),
\end{aligned}$$

so that $h_i^{T,k+1,\ell}(t_i) \in \widetilde{H}_i^\ell$ for $\ell = 2, 3, \dots, k$. A similar argument shows that $h_i^{T,k+1,k+1}(t_i) \in H_i^{k+1}$.

Next, we show that $h_i^{T,k+1,\ell}$ is measurable, where $\ell = 0, 1, \dots, k+1$. For $\ell = 0$, this is immediate. So let $\ell = 1, 2, \dots, k+1$, and suppose the claim is true for $\ell - 1$. It then follows directly from the induction hypothesis and (i) that the claim is true for ℓ (recall that the image measure induced by a measurable function from a metrizable space into a metrizable space is measurable). □

For future reference, it will be convenient to define $G_i^k := H_i^k \cup \widetilde{H}_i^k$ to be the set of kth-order belief hierarchies. Then, the functions $h_i^{T,n,k}$, $n \geq k$, together define a map $g_i^{T,k}$ that maps each type into a kth-order belief hierarchy (i.e., $g_i^{T,k}(t_i) = h_i^{T,n,k}(t_i)$ for $t_i \in T_i^n$.

For $i = 1, 2$ and $k < \infty$ such that T_i^k is nonempty, define $h_i^{T,k} : T_i^k \to H_i^k$ by:

$$h_i^{T,k}(t_i) := \Big(h_i^{T,k,0}(t_i), \beta_i^k(t_i) \circ \left(\mathrm{Id}_\Theta, h_{-i}^{T,<k,0}\right)^{-1}, \beta_i^k(t_i) \circ \left(\mathrm{Id}_\Theta, h_{-i}^{T,<k,1}\right)^{-1}, \dots, \\ \beta_i^k(t_i) \circ \left(\mathrm{Id}_\Theta, h_{-i}^{T,<k,k-1}\right)^{-1}\Big),$$

i.e., $h_i^{T,k}(t_i)$ is the belief hierarchy (of depth k) induced by t_i. It follows directly from the above that $h_i^{T,k}$ is well-defined and measurable.

We next define a collection of functions that will be used to obtain the belief hierarchies of infinite depth. For $i = 1, 2$, if T_i^∞ is nonempty, let $h_i^{T,\infty,0} : T_i^\infty \to \widetilde{H}_i^0$ be defined as before. For $\ell = 1, 2, \dots$, assume that the function $h_i^{T,\infty,\ell-1} : T_i^\infty \to \widetilde{H}_i^{\ell-1}$ has been defined and is measurable. Define the function $h_i^{T,\leq\infty,\ell-1} : T_i^\infty \cup \bigcup_{m=0}^\infty T_i^m \to \widetilde{H}_i^{\leq \ell-1}$ by

$$\forall m = \infty, 0, 1, \dots, t_i \in T_i^m : \qquad h_i^{T,\leq\infty,\ell-1}(t_i) = \begin{cases} h_i^{T,m,\ell-1}(t_i) & \text{if } m > \ell - 1; \\ h_i^{T,m,m}(t_i) & \text{if } m \leq \ell - 1; \end{cases}$$

Also, define $h_i^{T,\infty,\ell} : T_i^\infty \to \widetilde{H}_i^\ell$ by

$$h_i^{T,\infty,\ell}(t_i) := \Big(h_i^{T,\infty,\ell-1}(t_i), \beta_i^\infty(t_i) \circ \left(\mathrm{Id}_\Theta, h_{-i}^{T,\leq\infty,\ell-1}\right)^{-1}\Big).$$

Again, these functions are well-defined:

Lemma II.2. *Let $i = 1, 2$.*

(a) If T_i^∞ is nonempty, then $h_i^{T,\infty,\ell}$ is well-defined and measurable for $\ell = 0, 1, \ldots$.

(b) The function $h_i^{T,\leq\infty,\ell}$ is well-defined and measurable for $\ell = 0, 1, \ldots$.

The proof is similar to that of Lemma II.1, and thus omitted. Define $h_i^{T,\infty} : T_i^\infty \to H_i^\infty$ by:

$$h_i^{T,\infty}(t_i) := \big(h_i^{T,\infty,0}(t_i), \beta_i^\infty(t_i) \circ \big(\mathrm{Id}_\Theta, h_{-i}^{T,\leq\infty,0}\big)^{-1}, \beta_i^\infty(t_i) \circ \big(\mathrm{Id}_\Theta, h_{-i}^{T,\leq\infty,1}\big)^{-1}, \ldots\big).$$

That is, $h_i^{T,\infty}(t_i)$ is the belief hierarchy (of infinite depth) induced by t_i. By the above, $h_i^{T,\infty}$ is well-defined and measurable.

Each type generates a well-defined belief hierarchy, and a type with index k corresponds to a belief hierarchy of depth k. Define $h_i^T : T_i \to H_i$ to be the mapping that maps each type into a belief hierarchy (i.e., $h_i^T(t_i) = h_i^{T,k}(t_i)$ if $t_i \in T_i^k$, $k \leq \infty$).

We next define a type morphism from an arbitrary type space $\mathcal{T}$ to $\mathcal{T}^*$, using the mappings defined in Step 1.

Step 2: Constructing a type morphism
Recall that I_i^T is the set of indices $k = 0, 1, \ldots, \infty$ such that T_i^k is nonempty. For $i = 1, 2$, define $\varphi_i := (\varphi_i^k)_{k \in I_i^T}$ as follows. If $k \in I_i^T$ is finite, then define $\varphi_i^k : T_i^k \to H_i^k$ by:

$$\varphi_i^k(t_i) := h_i^{T,k}(t_i).$$

If T_i^∞ is nonempty, then define $\varphi_i^\infty : T_i^\infty \to H_i^\infty$ by:

$$\varphi_i^\infty(t_i) := h_i^{T,\infty}(t_i).$$

We show that $\varphi = (\varphi_i)_{i=1,2}$ is a type morphism. By Lemmas II.1 and II.2, the functions φ_i^k, $i = 1, 2$, $k \in I_i^T$, are well-defined and measurable. Also, for each $t_i \in H_i^k$, we have that $\chi_i^*(\varphi_i^k(t_i)) = \chi_i(t_i)$, that is, signals are preserved.

It remains to show that the mappings preserve higher-order beliefs. To show this, let $i = 1, 2$ and suppose there is $k < \infty$ such that $T_i^k \neq \emptyset$. We need to show that for each $t_i \in T_i^k$ and $E \in \mathscr{B}(\Theta) \otimes \mathscr{B}(H_{-i}^{\leq k-1})$,

$$\psi_i^k\left(\varphi_i^k(t_i)\right)(E) = \beta_i^k(t_i)\left(\left(\mathrm{Id}_\Theta, \varphi_{-i}^{<k}\right)^{-1}(E)\right).$$

Let $t_i \in T_i^k$. Using that $\mathcal{T}^*$ is canonical, we obtain

$$\begin{aligned}
\psi_i^k\left(\varphi_i^k(t_i)\right)(E) &= \psi_i^k\Big(h_i^{T,k,0}(t_i), \beta_i^k \circ (\mathrm{Id}_\Theta, h_{-i}^{T,<k,0})^{-1}, \ldots, \beta_i^k \circ (\mathrm{Id}_\Theta, h_{-i}^{T,<k,k-1})^{-1}\Big)(E) \\
&= \beta_i^k(t_i)\big((\mathrm{Id}_\Theta, h_{-i}^{T,<k,k-1})^{-1}(E)\big).
\end{aligned}$$

Next suppose that $T_i^\infty \neq \emptyset$, and let $t_i \in T_i^\infty$. We need to show that for each $E \in \mathscr{B}(\Theta) \otimes \mathscr{B}(H_{-i})$,

$$\psi_i^\infty\left(\varphi_i^\infty(t_i)\right)(E) = \beta_i^\infty(t_i)\left(\left(\mathrm{Id}_\Theta, \varphi_{-i}^\infty\right)^{-1}(E)\right).$$

Let $t_i \in T_i^\infty$. Again using that the belief maps in $\mathcal{T}^*$ are canonical, we have

$$\begin{aligned}\psi_i^\infty\left(\varphi_i^\infty(t_i)\right)(E) &= \psi_i^\infty\left(h_i^{T,\infty,0}(t_i), \beta_i^\infty \circ (\mathrm{Id}_\Theta, h_{-i}^{T,\leq\infty,0})^{-1}, \dots\right)(E) \\ &= \beta_i^\infty(t_i)\big((\mathrm{Id}_\Theta, h_{-i}^{T,\infty})^{-1}(E)\big).\end{aligned}$$

It follows that φ is a type morphism.

Step 3: There is a unique type morphism from any type space to $\mathcal{T}^*$

We show that for any type space $\mathcal{T}$, there is a unique type morphism from $\mathcal{T}$ to $\mathcal{T}^*$. The proof uses the following lemmas. Lemma II.3 shows that type morphisms preserve belief hierarchies [cf. Heifetz and Samet, 1998, Prop. 5.1]:

Lemma II.3. *Fix arbitrary type spaces $\mathcal{T}$ and $\mathcal{Q}$, and let φ be a type morphism from $\mathcal{T}$ to $\mathcal{Q}$. Then, for each $i = 1, 2$,*

(a) if T_i^k is nonempty, where $k < \infty$, then $h_i^{Q,k}(\varphi_i^k(t_i)) = h_i^{T,k}(t_i)$;

(b) if T_i^∞ is nonempty, then $h_i^{Q,\infty}(\varphi_i^\infty(t_i)) = h_i^{T,\infty}(t_i)$.

Proof. Here we show (a); the proof that (b) holds is similar and is thus omitted. The claim clearly holds for $k = 0$. Let $k = 1, 2, \dots$, and suppose the claim is true for $m = 0, 1, \dots, k-1$. Again, for each $i = 1, 2$ such that $T_i^k \neq \emptyset$, it is easy to see that $h_i^{Q,k,0}(\varphi_i^k(t_i)) = h_i^{T,k,0}(t_i)$ for every $t_i \in T_i^k$, where $h_i^{Q,k,0}$ is defined analogously to $h_i^{T,k,0}$ (recall that $I_i^Q \supseteq I_i^T$, so that $h_i^{Q,k,0}$ is well-defined). Let $\ell = 1, \dots, k$ and suppose that

$$h_i^{Q,k,m}\big(\varphi_i^k(t_i)\big) = h_i^{T,k,m}(t_i)$$

for every $t_i \in T_i^k$ and $m \leq \ell - 1$. Denoting the belief maps for player i in $\mathcal{Q}$ by λ_i^k, where $k \in I_i^Q$, we can use condition (I.1) to obtain

$$\begin{aligned}\lambda_i^k\big(\varphi_i^k(t_i)\big) \circ \big(\mathrm{Id}_\Theta, h_{-i}^{Q,<k,\ell-1}\big)^{-1} &= \beta_i^k(t_i) \circ \big(\mathrm{Id}_\Theta, \varphi_{-i}^{<k}\big)^{-1} \circ \big(\mathrm{Id}_\Theta, h_{-i}^{Q,<k,\ell-1}\big)^{-1} \\ &= \beta_i^k(t_i) \circ \big(\mathrm{Id}_\Theta, h_{-i}^{Q,<k,\ell-1} \circ \varphi_{-i}^{<k}\big)^{-1} \\ &= \beta_i^k(t_i) \circ \big(\mathrm{Id}_\Theta, h_{-i}^{T,<k,\ell-1}\big)^{-1},\end{aligned}$$

where the last line uses the induction hypothesis. Again using the induction hypothesis, we obtain

$$\begin{aligned}h_i^{Q,k,\ell}\big(\varphi_i^k(t_i)\big) &= \big(h_i^{Q,k-1,\ell}\big(\varphi_i^k(t_i)\big), \lambda_i^k\big(\varphi_i^k(t_i)\big) \circ \big(\mathrm{Id}_\Theta, h_{-i}^{Q,<k,\ell-1}\big)^{-1}\big) \\ &= \big(h_i^{T,k,\ell-1}(t_i), \beta_i^k(t_i) \circ \big(\mathrm{Id}_\Theta, h_{-i}^{T,<k,\ell-1}\big)^{-1}\big) \\ &= h_i^{T,k,\ell}(t_i),\end{aligned}$$

for every $t_i \in T_i^k$. $\square$

Lemma II.4. *Let $i = 1, 2$ and $k = 0, 1, \ldots$ or $k = \infty$. Then $h_i^{T^*,k} : H_i^k \to H_i^k$ is the identity function.*

The proof of Lemma II.4 follows directly from Lemma B.2 and Proposition B.3.

To show that φ is the unique type morphism from $\mathcal{T}$ to $\mathcal{T}^*$, suppose that $\tilde{\varphi}$ is a type morphism from $\mathcal{T}$ to $\mathcal{T}^*$. Then, it follows from Lemma II.3 that for every $i = 1, 2$ and $k \in I_i^T$,

$$h_i^{T,k}\big(\tilde{\varphi}_i^k(t_i)\big) = h_i^{T,k}(t_i).$$

But by Lemma II.4,

$$h_i^{T,k}\big(\tilde{\varphi}_i^k(t_i)\big) = \tilde{\varphi}_i^k(t_i),$$

so that $\tilde{\varphi}_i^k(t_i) = h_i^{T,k}(t_i)$. The result then follows by noting that $\varphi_i^k = h_i^k$.

To summarize: Step 2 shows that for any type space $\mathcal{T}$, there is a type morphism from $\mathcal{T}$ to $\mathcal{T}^*$, using the functions defined in Step 1. Step 3 shows that this type morphism is unique. Hence, $\mathcal{T}^*$ is universal. By a similar argument as in the proof of Proposition 3.5 of Heifetz and Samet [1998], there is at most one universal space, up to type isomorphism. $\square$

II.2 Proof of Proposition I.2

The proof follows directly from the following lemma:

Lemma II.5. *Suppose $\mathcal{T}$ is a type space, and suppose φ is a type morphism from $\mathcal{T}$ to $\mathcal{T}^*$. If $\mathcal{T}$ is nonredundant, then, for all $i = 1, 2$ and $t_i \in T_i \setminus T_i^0$,*

$$\psi_i\big(\varphi_i^{\kappa(t_i)}(t_i)\big)\Big(\Theta \times \big\{h_{-i} \in H_{-i} : h_{-i} = \varphi_{-i}^{\kappa(t_{-i})}(t_{-i}) \text{ for some } t_{-i} \in T_{-i}\big\}\Big) = 1,$$

where $\kappa(t_{-i}) = k$ for $t_{-i} \in T_{-i}^k$. Conversely, if $H_i' \subseteq H_i$, $i = 1, 2$, is such that

$$\operatorname{supp} \psi_i(h_i) \subseteq \Theta \times H_{-i}'$$

for all $i = 1, 2$ and $h_i \in H_i' \setminus H_i^0$, then there is a type space $\mathcal{T}$ and a type morphism φ from $\mathcal{T}$ to $\mathcal{T}^*$ such that for all players i,

$$H_i' = \{h_i \in H_i : h_i = \varphi_i^{\kappa(t_i)}(t_i) \text{ for some } t_i \in T_i\}.$$

Proof. Let $\mathcal{T}$ be a type space. We first prove the first claim. Let $i = 1, 2$. We need to show that the subset $\big\{h_{-i} \in H_{-i} : h_{-i} = \varphi_{-i}^{\kappa(t_{-i})}(t_{-i}) \text{ for some } t_{-i} \in T_{-i}\big\}$ is measurable. Because $\mathcal{T}$

is nonredundant, the function φ_{-i} is injective, and it follows from the results of Purves [1966] that

$$\{h_{-i} \in H_{-i} : h_{-i} = \varphi_{-i}^{\kappa(t_{-i})}(t_{-i}) \text{ for some } t_{-i} \in T_{-i}\} = \bigcup_{k \in I_{-i}^{T}} \varphi_{-i}^{k}(T_{-i}^{k}) \in \mathscr{B}(H_{-i}).$$

Hence, $\Theta \times \{h_{-i} \in H_{-i} : h_{-i} = \varphi_{-i}^{\kappa(t_{-i})}(t_{-i}) \text{ for some } t_{-i} \in T_{-i}\}$ is indeed an event in $\mathscr{B}(\Theta) \otimes \mathscr{B}(H_{-i})$. The result now follows directly from the definition of a type morphism.

The proof of the second claim is immediate: for each $i = 1, 2$, define $T_i := H_i'$; and for each $h_i \in H_i'$ of depth k, $k = 0, 1, \ldots, \infty$, define $\beta_i^k(h_i) := \psi_i^k(h_i)$, and let $\chi_i(h_i)$ be the projection of h_i on X_i. $\square$

II.3 Proof of Proposition 3.2

Clearly, $\varphi_i^\infty(z_i) \in H_i^\infty$ for all $i = 1, 2$ and $z_i \in Z_i$. Hence,

$$\big\{h_{-i} \in H_{-i} : h_{-i} = \varphi_{-i}^{\kappa(z_{-i})}(z_{-i}) \text{ for some } z_{-i} \in Z_{-i}\big\} \subseteq E^\infty.$$

The type structure $\mathcal{T}^{MZ}$ is nonredundant by construction, so that for $i = 1, 2$ and $z_i \in Z_i$,

$$\psi_i\big(\varphi_i^\infty(z_i)\big)\big(\Theta \times \big\{h_{-i} \in H_{-i} : h_{-i} = \varphi_{-i}^{\kappa(z_{-i})}(z_{-i}) \text{ for some } z_{-i} \in Z_{-i}\big\}\big) = 1$$

and it follows that

$$\big\{h_{-i} \in H_{-i} : h_{-i} = \varphi_{-i}^{\kappa(z_{-i})}(z_{-i}) \text{ for some } z_{-i} \in Z_{-i}\big\} \subseteq CB(E^\infty).$$

To prove the reverse inclusion, it is sufficient to show that for each $i = 1, 2$, there is $Y_i^\infty \subseteq Z_i^\infty$ such that

$$\varphi_i^\infty(Y_i^\infty) = \text{proj}_{H_i}\big(CB(E^\infty)\big),$$

where proj_V is the projection function into a space V. To show this, we construct a map $\hat{f}$ from $CB(E^\infty)$ to $\hat{Z}_{-i}$. First note that $CB(E^\infty) \subseteq H_{-i}^\infty$. For a hierarchy profile $h \in CB(E^\infty)$ (where $h_j = (x_j, \mu_j^0, \mu_j^1, \ldots)$)and player $i = 1, 2$, let $\hat{f}_i^0(x_i, \mu_i^0) := (x_i, \hat{z}_i^0)$. For $k = 1, 2, \ldots$, suppose $\hat{f}_{-i}^{k-1} : \text{proj}_{\widetilde{H}_{-i}^{k-1}}(CB(E^\infty)) \to \hat{Z}_{-i}^{k-1}$ has been defined for all $j = 1, 2$. For $(x_j, \mu_j^0, \mu_j^1, \ldots)_{j=1,2} \in CB(E^\infty)$ and $i = 1, 2$, define

$$\hat{f}_i^k(x_i, \mu_i^0, \ldots, \mu_i^k) := \big(\hat{f}_i^{k-1}(x_i, \mu_i^0, \ldots, \mu_i^{k-1}), \mu_i^k \circ \big(\text{Id}_\Theta, \hat{f}_{-i}^{k-1}\big)^{-1}\big).$$

It is easy to check that $\hat{f}_i^k$ is well-defined, given that the beliefs specified by the belief hierarchies in $CB(E^\infty)$ are coherent. Then, for each $x_j, \mu_j^0, \mu_j^1, \ldots)_{j=1,2} \in CB(E^\infty)$, define

$$\hat{f}((h_i)_{i=1,2}) := (x_i, \hat{z}_i^0, \mu_i^1 \circ (\text{Id}_\Theta, \hat{f}_{-i}^0)^{-1}, \ldots)_{i=1,2}.$$

Again, it is easy to verify that $\hat{f}(CB(E^\infty)) \subseteq \prod_{i=1,2} \hat{Z}_i$, so that the set $\text{proj}_{\hat{Z}_i}\hat{f}(CB(E^\infty))$ corresponds to a subset Y_i^∞ of $Z_i^\infty = \hat{Z}_i$. Given that there is a unique type morphism φ from $\mathcal{T}^{MZ}$ to $\mathcal{T}^*$, we have that $\varphi_i^\infty(Y_i^\infty) = \text{proj}_{H_i}(CB(E^\infty))$, and the result follows. □

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