Supplementary Material

Supplement to "Durable goods monopoly with stochastic costs"

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S1. Appendix

S1.1 *Proof of Theorem 2*

The proof of Theorem 2 is organized as follows. First, I show that the lower bound $L(x, q)$ (i.e., the value function of the optimal stopping problem (15)) is well defined for all $q \in [0, 1]$. Then I show that the monopolist's equilibrium profits are equal to $L(x, q)$ for all states (x, q).

I begin by showing that the lower bound $L(x, q)$ is well defined for all $q \in [0, 1]$. I use the following result in optimal stopping problems.

Lemma S1. *Let $h : \mathbb{R}_+ \to \mathbb{R}$ be a continuous function that is bounded on any compact subset of* $[0, \infty)$. *Then*

$$W(x) = \sup_\tau E\big[e^{-r\tau} h(x_\tau) \mid x_0 = x\big] \tag{S1}$$

is continuous in x. Moreover, the stopping time $\tau(S) = \inf\{t : x_t \in S\}$ solves (S1), where $S = \{x \in [0, \infty) : W(x) = h(x)\}$.

The proof of Lemma S1 can be found in Dayanik and Karatzas (2003).

Lemma S1 can be used to show that $L(x, q)$ is well defined for all $q \in [0, 1]$ and all x. Indeed, by Lemma B2, $L(x, q)$ is well defined for all $q \in [\alpha_3, 1]$. Consider next $q \in [\alpha_4, \alpha_3)$, and let

$$\begin{aligned} g(x, q) &= (\alpha_3 - q)\big(P(x, \alpha_3) - x\big) + L(x, \alpha_3) \\ \Rightarrow \quad L(x, q) &= \sup_\tau E\big[e^{-r\tau} g(x_\tau, q) \mid x_0 = x\big]. \end{aligned} \tag{S2}$$

Note that $g(x, q)$ is continuous and bounded on any compact subset of $[0, \infty)$ (since $P(x, \alpha_3) - x$ and $L(x, \alpha_3)$ satisfy these conditions). Therefore, by Lemma S1, $L(x, q)$ is continuous in x for all $q \in [\alpha_4, \alpha_3)$. Moreover, the stopping time $\tau(q) = \inf\{t : x_t \in S(q)\}$ solves (S2), where $S(q) = \{x \in (0, \infty) : L(x, q) = g(x, q)\}$. Repeating this argument inductively establishes that, for all k, $L(x, q)$ is continuous in x for all $q \in [\alpha_{k+1}, \alpha_k)$.

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Note that since $L(x,q)$ and $g(x,q)$ are continuous, the optimal stopping region $S(q)$ is a union of intervals. Fix $x \notin S(q)$. When $x_0 = x$, the stopping time $\tau(q)$ is equal to the first time at which x_t reaches either $\overline{z}(x) = \inf\{y \in S(q), y > x\}$ or $\underline{z}(x) = \sup\{y \in S(q), y < x\}$ (if the first set is empty, set $\overline{z}(x) = \infty$; if the second set is empty, set $\underline{z}(x) = 0$). Let $\tau_x = \inf\{t : x_t \notin (\underline{z}(x), \overline{z}(x))\}$, and note that $L(y,q) = E[e^{-r\tau_x} g(x_{\tau_x}, q) \mid x_0 = y]$ for all $y \in (\underline{z}(x), \overline{z}(x))$. By Lemma A1, for all $y \in (\underline{z}(x), \overline{z}(x))$, $L(y,q)$ solves[1]

$$rL(y,q) = \mu y L_y(y,q) + \frac{\sigma^2 y^2}{2} L_{yy}(y,q). \tag{S3}$$

The following results are the counterparts of Lemmas B3–B6 to the current setting with $n > 2$ types of consumers. Their proofs are the same as the proofs of Lemmas B3–B6, and hence are omitted for conciseness.

Lemma S2. *Let $(\{q_t\}, \mathbf{P})$ be an equilibrium.*

(i) If $x_t \le z_k$ and $q_{t^-} \in [\alpha_{k+1}, \alpha_k)$, then $q_s > q_{t^-}$ for all $s > t$ (i.e., the monopolist makes positive sales between t and $s > t$).

(ii) if $x_t > z_k$ and $q_{t^-} \in [\alpha_{k+1}, \alpha_k)$, then $q_s = q_{t^-}$ for all $s \in (t, \tau_k)$ (i.e., the monopolist does not make sales until costs reach z_k).

Lemma S3. *Let $(\{q_s\}, \mathbf{P})$ be an equilibrium. Then, for all k and all $x \in (0, z_k]$, $P(x, \cdot)$ is continuous on $[\alpha_{k+1}, \alpha_k)$.*

Lemma S4. *Let $(\{q_s\}, \mathbf{P})$ be an equilibrium and let $t \in [0, \infty)$ be such that $q_{t^-} < \alpha_2$. If $\{q_s\}$ is continuous and increasing in $[t, \tau)$ for some $\tau > t$, then there exists $u > t$ such that $P(x_t, q_t) - x_t = E_t[e^{-r(s-t)}(P(x_s, q_t) - x_s)]$ for all $s \in [t, u]$.*

Lemma S5. *Let $(\{q_s\}, \mathbf{P})$ be an equilibrium and let $\Pi(x,q)$ be the seller's profits. Let $t \in [0, \infty)$ be such that $q_{t^-} < \alpha_2$ and such that $\{q_s\}$ is continuous in $[t, \tau)$ for some $\tau > t$. Then there exists $u > t$ such that $\Pi(x_t, q_t) = E_t[e^{-r(s-t)} \Pi(x_s, q_t)]$ for all $s \in [t, u]$.*

Lemma S6. *Let $(\{q_t\}, \mathbf{P})$ be an equilibrium and let $\Pi(x,q)$ be the seller's profits. Let $t \in [0, \infty)$ be such that $q_{t^-} < \alpha_2$ and such that $\{q_s\}$ is continuous in $[t, \tau)$ for some $\tau > t$. Then there exists $\hat{\tau} > t$ such that $\{q_t\}$ is discontinuous at state $(x_{\hat{\tau}}, q_t)$; i.e., $\{q_t\}$ jumps up at this state. Moreover,*

$$\Pi(x_t, q_t) = E_t\big[e^{-r(\hat{\tau}-t)}\big(\big(P(x_{\hat{\tau}}, q_t + dq_{\hat{\tau}}) - x_{\hat{\tau}}\big)\, dq_{\hat{\tau}} + \Pi(x_{\hat{\tau}}, q_t + dq_{\hat{\tau}})\big)\big],$$

where $dq_{\hat{\tau}}$ denotes the jump of $\{q_t\}$ at state $(x_{\hat{\tau}}, q_t)$.

[1]The boundary condition at $\overline{z}(x)$ depends on whether $\overline{z}(x) < \infty$ or $\overline{z}(x) = \infty$. In the first case, $L(\overline{z}(x), q) = g(\overline{z}(x), q)$; in the second case, $\lim_{x\to\infty} L(x,q) = 0$. Similarly, the boundary condition at $\underline{z}(x)$ depends on whether $\underline{z}(x) > 0$ or $\underline{z}(x) = 0$. In the first case, $L(\underline{z}(x), q) = g(\underline{z}(x), q)$; in the second case, $\lim_{x\to 0} L(x,q) = 0$.

LEMMA S7. *Let* $(\{q_t\}, \mathbf{P})$ *be an equilibrium and let* $\Pi(x, q)$ *be the seller's profits. Let* $t \in [0, \infty)$ *be such that* $q_{t^-} < \alpha_2$ *and such that* $\{q_s\}$ *is continuous in* $[t, \tau)$ *for some* $\tau > t$. *Then* $-\Pi_q(x_s, q_s) = P(x_s, q_s) - x_s$ *for all* $s \in [t, \tau)$.

The following result, which is the counterpart of Lemma B7 to the current setting, uses Lemmas S2–S7 to establish that in any equilibrium the monopolist's profits are equal to $L(x, q)$ for all states (x, q).[2]

LEMMA S8. *Let* $(\{q_t\}, \mathbf{P})$ *be an equilibrium and let* $\Pi(x, q)$ *denote the monopolist's profits. Then* $\Pi(x, q) = L(x, q)$ *for all states* (x, q) *with* $q \in [0, 1]$.

PROOF. By the arguments in the main text, $\Pi(x, q) \geq L(x, q)$ for all states (x, q) with $q \in [\alpha_{k+1}, \alpha_k)$. I now show that $\Pi(x, q) \leq L(x, q)$ for all such states.

The proof is by induction on k. From Theorem 1, we know that the result is true for all states (x, q) with $q \geq \alpha_3$. Suppose next that the result holds for all states (x, q) with $q \in [\alpha_{\tilde{k}+1}, \alpha_{\tilde{k}})$ with $\tilde{k} \leq k - 1$. I now show that this implies that the result also holds for all states (x, q) with $q \in [\alpha_{k+1}, \alpha_k)$.

Fix a state (x, q) with $q \in [\alpha_{k+1}, \alpha_k)$ and suppose $(x_t, q_{t^-}) = (x, q)$. Note that at this state either $\{q_u\}$ jumps at t (i.e., $dq_t = q_t - q_{t^-} > 0$) or $\{q_u\}$ is continuous on $[t, s)$ for some $s > t$. In the first case, $\Pi(x_t, q_{t^-}) = (P(x_t, q_{t^-} + dq_t) - x_t)\, dq_t + \Pi(x_t, q_{t^-} + dq_t)$. In the second case, by Lemma S6 there exists $\hat{\tau} > t$ and $dq_{\hat{\tau}} > 0$ such that $\Pi(x_t, q_{t^-}) = E_t[e^{-r(\hat{\tau}-t)}((P(x_{\hat{\tau}}, q_{t^-} + dq_{\hat{\tau}}) - x_{\hat{\tau}})\, dq_{\hat{\tau}} + \Pi(x_{\hat{\tau}}, q_{t^-} + dq_{\hat{\tau}}))]$. Let

$$\tilde{\tau} = \sup\bigl\{\tau \geq t : \Pi(x_t, q_{t^-}) = E_t\bigl[e^{-r(\tau-t)}\bigl(\bigl(P(x_\tau, q_{t^-} + dq_\tau) - x_\tau\bigr)\, dq_\tau + \Pi(x_\tau, q_{t^-} + dq_\tau)\bigr)\bigr]\bigr\}.$$

Note that if $dq_{\tilde{\tau}} \geq \alpha_k - q_{t^-}$, then

$$\begin{aligned}
&\bigl(P(x_{\tilde{\tau}}, q_{t^-} + dq_{\tilde{\tau}}) - x_{\tilde{\tau}}\bigr)\, dq_{\tilde{\tau}} + \Pi(x_{\tilde{\tau}}, q_{t^-} + dq_{\tilde{\tau}}) \\
&\quad \leq \bigl(P(x_{\tilde{\tau}}, \alpha_k) - x_{\tilde{\tau}}\bigr)(\alpha_k - q_{t^-}) + \bigl(P(x_{\tilde{\tau}}, q_{t^-} + dq_{\tilde{\tau}}) - x_{\tilde{\tau}}\bigr)(dq_{\tilde{\tau}} - \alpha_k + q_{t^-}) \\
&\qquad + \Pi(x_{\tilde{\tau}}, q_{t^-} + dq_{\tilde{\tau}}) \\
&\quad \leq \bigl(P(x_{\tilde{\tau}}, \alpha_k) - x_{\tilde{\tau}}\bigr)(\alpha_k - q_{t^-}) + L(x_{\tilde{\tau}}, \alpha_k) = g(x_{\tilde{\tau}}, q_{t^-}),
\end{aligned}$$

where the first inequality follows since $P(x_{\tilde{\tau}}, q_{t^-} + dq_{\tilde{\tau}}) \leq P(x_{\tilde{\tau}}, \alpha_k)$ and the second inequality follows since, by the induction hypothesis, $L(x_{\tilde{\tau}}, \alpha_k) \geq (P(x_{\tilde{\tau}}, q_{t^-} + dq_{\tilde{\tau}}) - x_{\tilde{\tau}})(dq_{\tilde{\tau}} - \alpha_k + q_{t^-}) + \Pi(x_{\tilde{\tau}}, q_{t^-} + dq_{\tilde{\tau}})$ when $dq_{\tilde{\tau}} \geq \alpha_k - q_{t^-}$. This implies that $\Pi(x_t, q_{t^-}) \leq E_t[e^{-r(\tilde{\tau}-t)} g(x_{\tilde{\tau}}, q_{t^-})] \leq L(x_t, q_{t^-}) = \sup_\tau E[e^{-r\tau} g(x_\tau, q_{t^-})]$, and so $\Pi(x_t, q_{t^-}) = L(x_t, q_{t^-})$. The rest of the proof establishes that, indeed, $dq_{\tilde{\tau}} \geq \alpha_k - q_{t^-}$.

Toward a contradiction, suppose that $dq_{\tilde{\tau}} = q_{\tilde{\tau}} - q_{t^-} < \alpha_k - q_{t^-}$, so that $\Pi(x_{\tilde{\tau}}, q_{t^-}) = (P(x_{\tilde{\tau}}, q_{\tilde{\tau}}) - x_{\tilde{\tau}})(q_{\tilde{\tau}} - q_{t^-}) + \Pi(x_{\tilde{\tau}}, q_{\tilde{\tau}})$. By Lemma S6, there exists $\tau' > \tilde{\tau}$ such that $\Pi(x_{\tilde{\tau}}, q_{\tilde{\tau}}) = E_{\tilde{\tau}}[e^{-r(\tau'-\tilde{\tau})}((P(x_{\tau'}, q_{\tilde{\tau}} + dq_{\tau'}) - x_{\tau'})\, dq_{\tau'} + \Pi(x_{\tau'}, q_{\tilde{\tau}} + dq_{\tau'}))]$, where $dq_{\tau'}$ denotes the jump of $\{q_t\}$ at state $(x_{\tau'}, q_{\tilde{\tau}})$. This implies that $-\Pi_q(x_{\tilde{\tau}}, q_{\tilde{\tau}}) =$

[2] I include the proof of Lemma S8, since it uses an induction argument that is not present in the proof of Lemma B7.

$E_{\tilde{\tau}}[e^{-r(\tau'-\tilde{\tau})}(P(x_{\tau'}, q_{\tilde{\tau}} + dq_{\tau'}) - x_{\tau'})]$. Alternatively, by Lemma S7 it must be that $P(x_{\tilde{\tau}}, q_{\tilde{\tau}}) - x_{\tilde{\tau}} = -\Pi_q(x_{\tilde{\tau}}, q_{\tilde{\tau}})$, and so $P(x_{\tilde{\tau}}, q_{\tilde{\tau}}) - x_{\tilde{\tau}} = E_{\tilde{\tau}}[e^{-r(\tau'-\tilde{\tau})}(P(x_{\tau'}, q_{\tilde{\tau}} + dq_{\tau'}) - x_{\tau'})]$. Since $\Pi(x_{\tilde{\tau}}, q_{t^-}) = (P(x_{\tilde{\tau}}, q_{\tilde{\tau}}) - x_{\tilde{\tau}})(q_{\tilde{\tau}} - q_{t^-}) + \Pi(x_{\tilde{\tau}}, q_{\tilde{\tau}})$ and since $\Pi(x_{\tilde{\tau}}, q_{\tilde{\tau}}) = E_{\tilde{\tau}}[e^{-r(\tau'-\tilde{\tau})}((P(x_{\tau'}, q_{\tilde{\tau}} + dq_{\tau'}) - x_{\tau'})\, dq_{\tau'} + \Pi(x_{\tilde{\tau}}, q_{\tilde{\tau}} + dq_{\tau'}))]$, it follows that

$$\Pi(x_{\tilde{\tau}}, q_{t^-}) = E_{\tilde{\tau}}\big[e^{-r(\tau'-\tilde{\tau})}\big(\big(P(x_{\tau'}, q_{\tau'}) - x_{\tau'}\big)(dq_{\tau'} + q_{\tilde{\tau}} - q_{t^-}) + \Pi(x_{\tau'}, q_{\tau'})\big)\big].$$

By the law of iterated expectations,

$$\begin{aligned}\Pi(x_t, q_{t^-}) &= E_t\big[e^{-r(\tilde{\tau}-t)}\big(\big(P(x_{\tilde{\tau}}, q_{t^-} + dq_{\tilde{\tau}}) - x_{\tilde{\tau}}\big)\, dq_{\tilde{\tau}} + \Pi(x_{\tilde{\tau}}, q_{\tilde{\tau}})\big)\big] \\ &= E_t\big[e^{-r(\tau'-t)}\big(\big(P(x_{\tau'}, q_{\tau'}) - x_{\tau'}\big)(dq_{\tau'} + dq_{\tilde{\tau}}) + \Pi(x_{\tau'}, q_{\tau'})\big)\big],\end{aligned}$$

which contradicts the definition of $\tilde{\tau}$. Hence, it must be that that $dq_{\tilde{\tau}} \geq \alpha_k - q_{t^-}$. □

By Lemma S8, in any equilibrium the monopolist's profits are equal to the lower bound $L(x, q)$. Using this, the equilibrium strategies can be constructed as in the proof of Theorem 1. For any $q \in [\alpha_{k+1}, \alpha_k)$ and any $x \in S(q)$, the monopolist sells to all consumers with valuation v_k at price $P(x, \alpha_k)$.[3] For all $x \notin S(q)$, $x > z_k$, the monopolist does not make sales. Finally, for all $x \notin S(q)$, $x \leq z_k$, the monopolist sells gradually to consumers with valuation v_k. By the same arguments as in the proof of Theorem 1, for all $x_s \notin S(q_s)$, $x_s \leq z_k$,

$$rL(x_s, q_s)\, ds = \big(P(x_s, q_s) - x_s + L_q(x_s, q_s)\big)\, dq_s + \mu x_s L_x(x_s, q_s)\, ds + \frac{\sigma^2 x_s^2}{2} L_{xx}(x_s, q_s)\, ds.$$

Comparing this equation with (S3), it follows that $P(x, q) - x = -L_q(x, q)$ for all $x \notin S(q)$, $x \leq z_k$. This pins down the prices that consumers are willing to pay for all $x \notin S(q)$, $x \leq z_k$. Note that, for all $q \in [\alpha_{k+1}, \alpha_k)$ and all $x \notin S(q)$,

$$\begin{aligned}L(x, q) &= E\big[e^{-r\tau(q)}\big[(\alpha_k - q)\big(P(x_{\tau_x}, \alpha_k) - x_{\tau(q)}\big) + L(x_{\tau(q)}, \alpha_k)\big] \mid x_0 = x\big] \\ \Rightarrow -L_q(x, q) &= P(x, q) - x = E\big[e^{-r\tau(q)}\big(P(x_{\tau(q)}, \alpha_k) - x_{\tau(q)}\big) \mid x_0 = x\big].\end{aligned} \tag{S4}$$

Last, the rate at which the monopolist sells to consumers with valuation v_k when $x_s \notin S(q_s)$, $x_s \leq z_k$ is determined by the indifference condition of v_k consumers, as in the proof of Theorem 1:

$$\frac{dq_s}{ds} = \frac{-r\big(v_k - P(x_s, q_s)\big) - \mu x_s P_x(x_s, q_s) - \frac{1}{2}\sigma^2 x_s^2 P_{xx}(x_s, q_s)}{P_q(x_s, q_s)}.$$

I end this section by showing that, for all $q \in [\alpha_{k+1}, \alpha_k)$ and all $x_0 > z_k$, the solution to (S2) is to sell to all consumers with valuation v_k the first time costs reach z_k. Recall that, for any $q \in [0, 1]$, the stopping time $\tau(q) = \inf\{t : x_t \in S(q)\}$ solves (S2), where $S(q) = \{x : L(x, q) = g(x, q)\}$. For any $q \in [0, 1]$, let $z(q) := \sup\{x \in S(q)\}$. Since $L(x, q)$ and $g(x, q)$ are continuous, $z(q) \in S(q)$. The following result shows that $z(q) = z_k$ for all $q \in [\alpha_{k+1}, \alpha_k)$.

[3] For all $x \in S(q)$, the equilibrium strategy of all buyers $i \in [q, \alpha_k]$ is $P(x, i) = P(x, \alpha_k)$.

Lemma S9. *For all integers $k \in \{2, \dots, n\}$ and all $q \in [\alpha_{k+1}, \alpha_k)$, $z(q) = z_k$.*

Before proceeding to its proof, note that Lemma S9 implies that the monopolist sells to the different types of consumers at the efficient time when $x_0 > z_n$: for any integer $k \in \{1, \dots, n\}$, the monopolist sells to all consumers with valuation v_k the first time costs reach z_k at price $P(z_k, \alpha_k)$. Moreover, for $k \in \{2, \dots, n\}$ and for all $x \geq z_{k-1}$, $P(z_k, \alpha_k) = v_k - E[e^{-r\tau_{k-1}}(v_k - P(x_{\tau_{k-1}}, \alpha_{k-1})) \mid x_0 = z_k]$. The following corollary summarizes this.

Corollary S1. *When $x_0 > z_n$, the monopolist sells to the different types of consumers at the efficient time.*

Proof of Lemma S9. To prove the lemma, I use the following claim.

Claim 1. *Fix $k \in \{2, \dots, n\}$ and $q \in [\alpha_{k+1}, \alpha_k)$. Then, if $x \in S(q)$ and $x \leq z_{k-1}$, it must be that $x \in S(\alpha_k)$.*

Claim 1 (whose proof can be found below) implies that if $x \leq z_m < z_{k-1}$ for some $m < k-1$ and $x \in S(q)$ for $q \in [\alpha_{k+1}, \alpha_k)$, then $x \in S(\alpha_{\tilde{k}})$ for all integers $\tilde{k} \in \{m+1, \dots, k\}$.[4]

I now prove the lemma. The proof is by induction. Note first that, by Lemma B2, the result is true for $k = 2$. Suppose next that the result is true for all $\tilde{k} = 2, \dots, k-1$. I now show that this implies that the result is also true for k. Fix $q \in [\alpha_{k+1}, \alpha_k)$. Note that for all $x > z_{k-1}$,

$$P(x, \alpha_k) = v_k - E\big[e^{-r\tau_{k-1}}\big(v_k - P(x_{\tau_{k-1}}, \alpha_{k-1})\big) \mid x_0 = x\big],$$

where the equality follows since, by the induction hypothesis, when the state is (x, α_k) with $x > z_{k-1}$ the monopolist waits until time τ_{k-1} and at this point sells to all v_{k-1} consumers at price $P(x_{\tau_{k-1}}, \alpha_{k-1})$. Suppose first that $z(q) > z_k$. This implies that

$$\begin{aligned}
L\big(z(q), q\big) &= (\alpha_k - q)\big(P\big(z(q), \alpha_k\big) - z(q)\big) + L\big(z(q), \alpha_k\big) \\
&= (\alpha_k - q)\big(v_k - z(q)\big) - (\alpha_k - q)E\big[e^{-r\tau_{k-1}}\big(v_k - P(x_{\tau_{k-1}}, \alpha_{k-1})\big) \mid x_0 = z(q)\big] \\
&\quad + L\big(z(q), \alpha_k\big) \\
&< (\alpha_k - q)E\big[e^{-r\tau_k}(v_k - x_{\tau_k}) \mid x_0 = z(q)\big] \\
&\quad - (\alpha_k - q)E\big[e^{-r\tau_{k-1}}\big(v_k - P(x_{\tau_{k-1}}, \alpha_{k-1})\big) \mid x_0 = z(q)\big] + L\big(z(q), \alpha_k\big) \\
&= E\big[e^{-r\tau_k}(\alpha_k - q)\big(P(x_{\tau_k}, \alpha_k) - x_{\tau_k}\big) \mid x_0 = z(q)\big] + L\big(z(q), \alpha_k\big),
\end{aligned} \tag{S5}$$

where the strict inequality follows from Lemma 1 and the last equality follows since, for all $x > z_{k-1}$ and all stopping times $\tau < \tau_{k-1}$,

$$\begin{aligned}
&E\big[e^{-r\tau}\big(P(x_\tau, \alpha_k) - x_\tau\big) \mid x_0 = x\big] \\
&= E\big[e^{-r\tau}\big(v_k - x_\tau - E\big[e^{-r(\tau_{k-1} - \tau)}\big(v_k - P(x_{\tau_{k-1}}, \alpha_{k-1})\big) \mid x_\tau\big]\big) \mid x_0 = x\big] \\
&= E\big[e^{-r\tau}(v_k - x_\tau) \mid x_0 = x\big] - E\big[e^{-r\tau_{k-1}}(v_k - P(x_{\tau_{k-1}}, \alpha_{k-1}) \mid x_0 = x\big].
\end{aligned} \tag{S6}$$

[4] Proof. The statement follows directly from Claim 1 for $\tilde{k} = k$. Suppose next that the statement is true for $\tilde{k} = m' + 1, \dots, k$, with $m' \geq m + 1$. Since $x \leq z_m \leq z_{m'-1}$ and $x \in S(\alpha_{m'+1})$, Claim 1 implies that $x \in S(\alpha_{m'})$.

Note next that, by the induction hypothesis, $L(x, \alpha_k) = E[e^{-r\tau_{k-1}} g(x_{\tau_{k-1}}, \alpha_k) \mid x_0 = x]$ for all $x > z_{k-1}$. Therefore, by the law of iterated expectations, for all $x > z_{k-1}$ and all stopping times $\tau < \tau_{k-1}$,

$$\begin{aligned} E\big[e^{-r\tau} L(x_\tau, \alpha_k) \mid x_0 = x\big] &= E\big[e^{-r\tau} E\big[e^{-r(\tau_{k-1}-\tau)} g(x_{\tau_{k-1}}, \alpha_k) \mid x_\tau\big] \mid x_0 = x\big] \\ &= E\big[e^{-r\tau_{k-1}} g(x_{\tau_{k-1}}, \alpha_k) \mid x_0 = x\big] = L(x, \alpha_k). \end{aligned} \tag{S7}$$

Combining this with the inequality in (S5), it follows that

$$L\big(z(q), q\big) < E\big[e^{-r\tau_k}\big[(\alpha_k - q)\big(P(x_{\tau_k}, \alpha_k) - x_{\tau_k}\big) + L(x_{\tau_k}, \alpha_k)\big] \mid x_0 = z(q)\big],$$

a contradiction. Hence, it must be that $z(q) \leq z_k$.

Suppose next that $z(q) \in [z_{k-1}, z_k)$. This implies that whenever $x_0 > z(q)$, $\tau(q) = \inf\{t : x_t \in S(q)\} = \inf\{t : x_t = z(q)\}$. Therefore,

$$\begin{aligned} L(z_k, q) &= E\big[e^{-r\tau(q)}\big[\big((\alpha_k - q)\big(P(x_{\tau(q)}, \alpha_k) - x_{\tau(q)}\big) + L(x_{\tau(q)}, \alpha_k)\big)\big] \mid x_0 = z_k\big] \\ &= E\big[e^{-r\tau(q)}(\alpha_k - q)(v_k - x_{\tau(q)}) \mid x_0 = z_k\big] \\ &\quad - (\alpha_k - q)E\big[e^{-r\tau_{k-1}}\big(v_k - P(x_{\tau_{k-1}}, \alpha_{k-1})\big) \mid x_0 = z_k\big] + L(z_k, \alpha_k) \\ &< (\alpha_k - q)\big((v_k - z_k) - E\big[e^{-r\tau_{k-1}}\big(v_k - P(x_{\tau_{k-1}}, \alpha_{k-1})\big) \mid x_0 = z_k\big]\big) + L(z_k, \alpha_k) \\ &= (\alpha_k - q)\big(P(z_k, \alpha_k) - z_k\big) + L(z_k, \alpha_k) = g(z_k, q), \end{aligned}$$

where the second equality uses (S6) and (S7) and the strict inequality follows from Lemma 1. This cannot be, since $L(z_k, q) = \sup_\tau E[e^{-r\tau} g(x_\tau, q) \mid x_0 = z_k]$. Hence, $z(q) \notin [z_{k-1}, z_k)$.

Finally, I show that $z(q) \notin [0, z_{k-1}]$. Letting $z_0 := 0$, suppose that $z(q) \in (z_{m-1}, z_m]$ for some $m \leq k-1$ (for the case of $m = 1$, suppose $z(q) \in [0, z_1] = [z_0, z_1]$). Note that by Claim 1 and the paragraph that follows the claim, it follows that $z(q) \in S(\alpha_{\tilde{k}})$ for all integers $\tilde{k} \in \{m+1, \ldots, k\}$. Since $z(q) \in S(\alpha_k)$, at state $(z(q), \alpha_k)$ the monopolist sells to all consumers with valuation v_{k-1} immediately at price $P(z(q), \alpha_{k-1})$, and so $P(z(q), \alpha_k) = P(z(q), \alpha_{k-1})$. If $k - 1 \geq m + 1$, then $z(q) \in S(\alpha_{k-1})$. Therefore, at state $(z(q), \alpha_{k-1})$ the monopolist sells to all consumers with valuation v_{k-2} immediately at price $P(z(q), \alpha_{k-2})$, and so $P(z(q), \alpha_k) = P(z(q), \alpha_{k-1}) = P(z(q), \alpha_{k-2})$. Continuing in this way, it follows that $P(z(q), \alpha_k) = P(z(q), \alpha_m)$.

Since $z(q) \in S(\alpha_k)$, it follows that $L(z(q), \alpha_k) = (\alpha_{k-1} - \alpha_k)(P(z(q), \alpha_{k-1}) - z(q)) + L(z(q), \alpha_{k-1})$. Therefore,

$$\begin{aligned} L\big(z(q), q\big) &= (\alpha_k - q)\big(P\big(z(q), \alpha_k\big) - z(q)\big) + L\big(z(q), \alpha_k\big) \\ &= (\alpha_k - q)\big(P\big(z(q), \alpha_k\big) - z(q)\big) + (\alpha_{k-1} - \alpha_k)\big(P\big(z(q), \alpha_{k-1}\big) - z(q)\big) \\ &\quad + L\big(z(q), \alpha_{k-1}\big) \\ &= (\alpha_{k-1} - q)\big(P\big(z(q), \alpha_{k-1}\big) - z(q)\big) + L\big(z(q), \alpha_{k-1}\big), \end{aligned}$$

where the last equality follows since $P(z(q), \alpha_k) = P(z(q), \alpha_{k-1}) = P(z(q), \alpha_m)$. If $k - 1 \geq m + 1$, then $L(z(q), \alpha_{k-1}) = (\alpha_{k-2} - \alpha_{k-1})(P(z(q), \alpha_{k-2}) - z(q)) + L(z(q), \alpha_{k-2})$.

Moreover, in this case, $P(z(q), \alpha_{k-1}) = P(z(q), \alpha_{k-2})$ and so $L(z(q), q) = (\alpha_{k-2} - q)(P(z(q), \alpha_{k-2}) - z(q)) + L(z(q), \alpha_{k-2})$. Continuing in this way, it follows that $L(z(q), q) = (\alpha_m - q)(P(z(q), \alpha_m) - z(q)) + L(z(q), \alpha_m)$.

Fix $x > z(q)$ and note that $\tau(q) = \inf\{t : x_t = S(q)\} = \inf\{t : x_t = z(q)\}$ whenever $x_0 = x$. Therefore, by the arguments in the previous paragraph, for all $x > z(q)$,

$$L(x, q) = E\big[e^{-r\tau(q)}\big[\big((\alpha_m - q)\big(P(x_{\tau(q)}, \alpha_m) - x_{\tau(q)}\big) + L(x_{\tau(q)}, \alpha_m)\big)\big] \mid x_0 = x\big].$$

Note further that, by the induction hypothesis, for all $x > z_m$,

$$\begin{aligned} P(x, \alpha_m) &= v_m - E\big[e^{-r\tau_{m-1}}\big(v_m - P(x_{\tau_{m-1}}, \alpha_{m-1})\big) \mid x_0 = x\big], \\ L(x, \alpha_m) &= E\big[e^{-r\tau_{m-1}} g(x_{\tau_{m-1}}, \alpha_m) \mid x_0 = x\big]. \end{aligned}$$

Applying the law of iterated expectations, for all $x > z(q) \geq z_{m-1}$,

$$\begin{aligned} &E\big[e^{-r\tau(q)}\big[(\alpha_m - q)\big(P(x_{\tau(q)}, \alpha_m) - x_{\tau(q)}\big) + L(x_{\tau(q)}, \alpha_m)\big] \mid x_0 = x\big] \\ &\quad = E\big[e^{-r\tau(q)}(\alpha_m - q)(v_m - x_{\tau(q)}) \mid x_0 = x\big] \\ &\qquad - E\big[e^{-r\tau_{m-1}}(\alpha_m - q)\big(v_m - P(x_{\tau_{m-1}}, \alpha_{m-1})\big) \mid x_0 = x\big] + L(x, \alpha_m). \end{aligned}$$

Therefore,

$$\begin{aligned} L(z_m, q) &= E\big[e^{-r\tau(q)}\big[\big((\alpha_m - q)\big(P(x_{\tau(q)}, \alpha_m) - x_{\tau(q)}\big) + L(x_{\tau(q)}, \alpha_m)\big)\big] \mid x_0 = z_m\big] \\ &= E\big[e^{-r\tau(q)}(\alpha_m - q)(v_m - x_{\tau(q)}) \mid x_0 = z_m\big] \\ &\quad - E\big[e^{-r\tau_{m-1}}(\alpha_m - q)\big(v_m - P(x_{\tau_{m-1}}, \alpha_{m-1})\big) \mid x_0 = z_m\big] + L(z_m, \alpha_m) \\ &< (\alpha_m - q)(v_m - z_m) - (\alpha_m - q)E\big[e^{-r\tau_{m-1}}\big(v_m - P(x_{\tau_{m-1}}, \alpha_{m-1})\big) \mid x_0 = z_m\big] \\ &\quad + L(z_m, \alpha_m) \\ &= (\alpha_m - q)\big(P(z_m, \alpha_m) - z_m\big) + L(z_m, \alpha_m), \end{aligned}$$

where the strict inequality follows from Lemma 1. But this is a contradiction, since $L(z_m, q) \geq (\alpha_m - q)(P(z_m, \alpha_m) - z_m) + L(z_m, \alpha_m)$.[5] Hence, $z(q) \notin (z_{m-1}, z_m]$. Combining all these arguments, it follows that $z(q) = z_k$. □

Proof of Claim 1. Fix $x \leq z_{k-1}$ with $x \notin S(\alpha_k)$. Then, in equilibrium, at state (x, α_k) the monopolist sells to consumers with valuation v_{k-1} gradually over time. By (S4),

$$P(x, \alpha_k) - x = -L_q(x, \alpha_k) = E\big[e^{-r\tau(\alpha_k)}\big(P(x_{\tau(\alpha_k)}, \alpha_{k-1}) - x_{\tau(\alpha_k)}\big) \mid x_0 = x\big].$$

[5] Indeed, note that for all $q \in [\alpha_{k+1}, \alpha_k)$,

$$\begin{aligned} L(x, q) &\geq (\alpha_k - q)\big(P(x, \alpha_k) - x\big) + L(x, \alpha_k) \\ &\geq (\alpha_k - q)\big(P(x, \alpha_k) - x\big) + (\alpha_{k-1} - \alpha_k)\big(P(x, \alpha_{k-1}) - x\big) + L(x, \alpha_{k-1}) \\ &\geq (\alpha_{k-1} - q)\big(P(x, \alpha_{k-1}) - x\big) + L(x, \alpha_{k-1}), \end{aligned}$$

where the last inequality follows since $P(x, \alpha_k) \geq P(x, \alpha_{k-1})$. Repeating this argument inductively, it follows that $L(x, q) \geq (\alpha_m - q)(P(x, \alpha_m) - x) + L(x, \alpha_m)$ for all $m \leq k - 1$.

Therefore, for all $q \in [\alpha_{k+1}, \alpha_k)$ and all $x \le z_{k-1}$, $x \notin S(\alpha_k)$,

$$\begin{aligned} g(x,q) &= (\alpha_k - q)\big(P(x,\alpha_k) - x\big) + L(x,\alpha_k) \\ &= E\big[e^{-r\tau(\alpha_k)}\big[(\alpha_k - q)\big(P(x_{\tau(\alpha_k)},\alpha_{k-1}) - x_{\tau(\alpha_k)}\big) + L(x_{\tau(\alpha_k)},\alpha_k)\big] \mid x_0 = x\big] \\ &\le \sup_{\tau} E\big[e^{-r\tau} g(x_\tau, q) \mid x_0 = x\big]. \end{aligned}$$

Hence, stopping when $x_t = x$ is dominated by waiting and stopping at time $\tau(\alpha_k)$, and therefore $x \notin S(q)$. □

S1.2 *The discrete-time game*

This section studies the discrete-time version of the model in the paper. The main goal is to show that in any subgame perfect equilibrium of this game, the strategies of the buyers must satisfy condition (iii) in Definition 1. For conciseness, I focus on the case in which there are two types of buyers, as in Section 5. I stress, however, that these results generalize to settings with any (finite) number of types.

As in the main text, a monopolist faces a continuum of consumers indexed by $i \in [0,1]$. For each $i \in [0,1]$, let $f(i)$ denote the valuation of consumer i. There are two types of buyers: high types with valuation v_2 and low types with valuation $v_1 \in (0, v_2)$. Let $\alpha \in (0,1)$ be the fraction of high types in the market, so $f(i) = v_2$ for all $i \in [0,\alpha]$ and $f(i) = v_1$ for all $i \in (\alpha, 1]$.

Time is discrete. Let $T(\Delta) = 0, \Delta, 2\Delta, \ldots$ be the set of times at which players take actions, with Δ measuring the time period. At each time $t \in T(\Delta)$ the monopolist announces a price $p \in \mathbb{R}_+$. All consumers who have not purchased already simultaneously choose whether to buy at this price or wait. All players have perfect recall of the history of the game. Moreover, all players in the game are expected utility maximizers, and have a common discount factor $\delta = e^{-r\Delta}$. The monopolist's marginal cost of production evolves as (1), with $\mu < r$ and $\sigma > 0$. The seller's cost is publicly observable. Note that costs evolve continuously over time, but the monopolist can only announce a price and make sales at times $t \in T(\Delta)$. Therefore, as $\Delta \to 0$, costs become more persistent across periods.

A strategy for the monopolist specifies at each time $t \in T(\Delta)$ a price to charge as a function of the history. A strategy for a consumer specifies at each time the set of prices she will accept as a function of the history (provided she has not previously made a purchase). I focus on the subgame perfect equilibria (SPE) of this game.[6] [7]

Lemma S10. *In any SPE and after any history, all buyers accept a price equal to v_1, regardless of the current level of costs.*

[6]As usual in durable goods monopoly games, I restrict attention to SPE in which actions are constant on histories in which prices are the same and the sets of agents accepting at each point in time differ by sets of measure zero; see Gul et al. (1986) for a discussion of this assumption.

[7]The existence of SPE can be shown by generalizing arguments in Gul et al. (1986).

Proof. Fix a SPE and let $p(x)$ be the supremum of prices accepted by all consumers after any history such that current costs are x. Let $\underline{p} := \inf_{x\in\mathbb{R}_+} p(x)$. Note first that $\underline{p} \leq v_1$, since buyers with valuation v_1 never accept a price larger than their valuation. Suppose by contradiction that the lemma is not true, so $\underline{p} < v_1$. Note that the monopolist would never charge a price lower than $\underline{p}$. Consider the offer $p = (1-\delta)v_1 + \delta\underline{p} > \underline{p}$. Note that every buyer would accept a price of $p - \epsilon$ for any $\epsilon > 0$, since the price in the future will never be lower than $\underline{p}$. Moreover, $p - \epsilon > \underline{p}$ for ϵ small enough. This implies that there exists a cost level x such that $p - \epsilon > p(x)$, which contradicts the fact that $p(x)$ is the supremum of prices accepted by all consumers after any history such that current costs are x. Thus, $\underline{p} = v_1$. □

An immediate Corollary of Lemma S10 is that, in any SPE, consumers with valuation v_1 accept a price equal to v_1; that is, condition (4) holds in any SPE. The next result shows that condition (5) also holds in any SPE of the game.

Consider the optimal stopping problem $\sup_{\tau\in\overline{T}(\Delta)} E[e^{-r\tau}(v_1 - x_\tau) \mid x_0 = x]$, where $\overline{T}(\Delta)$ is the set of stopping times taking values on $T(\Delta)$. The solution to this problem is to stop the first time costs fall below some level z_1^Δ. For all $s \in T(\Delta)$, let $\tau_1^\Delta(s) = \inf\{t \in T(\Delta), t > s : x_t \leq z_1^\Delta\}$. Let $\tau_1^\Delta = \tau_1^\Delta(0)$. Note that, in any SPE, if all remaining high type consumers buy at time $s \in T(\Delta)$ and leave the market, the monopolist will then wait until time $\tau_1^\Delta(s)$ and charge a price of v_1 (which all low type buyers accept). For all $x > 0$, let $P^\Delta(x) = v_2 - E[e^{-r\tau_1^\Delta}(v_2 - v_1) \mid x_0 = x]$.

Lemma S11. *In any SPE and after any history, all buyers with valuation v_2 accept a price equal to $P^\Delta(x)$ if the current cost level is x.*

Proof. Fix a SPE and let $p_2(x)$ be the supremum of the prices that all buyers $i \in [0, \alpha]$ accept after any history if current costs are x. I first show that $p_2(x) \leq P^\Delta(x)$. To see this, note that by definition of $p_2(x)$, all buyers $i \in [0, \alpha]$ who remain in the market will buy if the seller charges a price $p_2(x)$. By our discussion above, the monopolist will then sell to all low types at a price v_1 the first time costs fall below z_1^Δ. The utility that a high type buyer gets by not purchasing at price $p_2(x)$ and waiting until the monopolist serves low types is $E[e^{-r\tau_1^\Delta}(v_2 - v_1) \mid x_0 = x]$. Therefore, for all buyers to be willing to purchase at price $p_2(x)$, it must be that $v_2 - p_2(x) \geq E[e^{-r\tau_1^\Delta}(v_2 - v_1) \mid x_0 = x]$, or $p_2(x) \leq P^\Delta(x)$.

I now complete the proof by showing that $p_2(x) \geq P^\Delta(x)$. To see this, let $\underline{p}(x) = p_2(x)$ if $x > z_1^\Delta$ and $\underline{p}(x) = v_1$ if $x \leq z_1^\Delta$. Note that the monopolist will never charge a price lower than $\underline{p}(x)$ if current costs are x. Moreover, all high type consumers will accept this price. As a first step to prove the inequality, I show that $p_2(x) \geq v_2 - E[e^{-r\Delta}(v_2 - \underline{p}(x_{t+\Delta})) \mid x_t = x]$. Suppose by contradiction that this is not true. Then some high type consumers would reject a price of $p^\epsilon(x) = v_2 - E[e^{-r\Delta}(v_2 - \underline{p}(x_{t+\Delta})) \mid x_t = x] - \epsilon$ for ϵ small enough (i.e., $p^\epsilon(x) > p_2(x)$ for ϵ small enough). Note that the lowest possible price that the seller would charge next period is $\underline{p}(x_{t+\Delta})$, and that this price would be accepted by all high types. This implies that the continuation utility of high types from rejecting today's price is bounded above by $E[e^{-r\Delta}(v_2 - \underline{p}(x_{t+\Delta})) \mid x_t = x]$. But this in turn implies that all high type consumers should accept a price of $v_2 - E[e^{-r\Delta}(v_2 - \underline{p}(x_{t+\Delta})) \mid x_t = x] - \epsilon > p_2(x)$, a contradiction to the fact that $p_2(x)$ is the supremum over all prices that all high types accept when costs are equal to x. Hence, $p_2(x) \geq v_2 - E[e^{-r\Delta}(v_2 - \underline{p}(x_{t+\Delta})) \mid x_t = x]$.

Recall that $\tau_1^\Delta = \tau_1^\Delta(0) = \inf\{t \in T(\Delta), t > 0 : x_t \leq z_1^\Delta\}$. For all $t \in T(\Delta)$, let $F^\Delta(t, x) = \text{Prob}(\tau_1^\Delta = t \mid x_0 = x)$ and note that $F^\Delta(0, x) = 0$. It then follows that

$$
\begin{aligned}
p_2(x) &\geq v_2 - E\big[e^{-r\Delta}\big(v_2 - \underline{p}(x_\Delta)\big) \mid x_0 = x\big] \\
&= v_2 - e^{-r\Delta} F^\Delta(\Delta, x)(v_2 - v_1) \\
&\quad - \big(1 - F^\Delta(\Delta, x)\big) E\big[e^{-r\Delta}\big(v_2 - p_2(x_\Delta)\big) \mid x_0 = x, \tau_1^\Delta > \Delta\big],
\end{aligned}
$$

where the equality follows since $\underline{p}(x) = v_1$ for all $x \leq z_1^\Delta$ and $\underline{p}(x) = p_2(x)$ for all $x > z_1^\Delta$. Using the fact that $p_2(x) \geq v_2 - E[e^{-r\Delta}(v_2 - \underline{p}(x_\Delta)) \mid x_0 = x]$ repeatedly in (S8), it follows that $p_2(x) \geq v_2 - \sum_{k=1}^{\infty} e^{-rk\Delta}(v_2 - v_1) F^\Delta(k\Delta, x) = v_2 - E[e^{-\tau_1^\Delta}(v_2 - v_1) \mid x_0 = x]$, where the last equality follows since $F^\Delta(t, x) = \text{Prob}(\tau_1^\Delta = t \mid x_0 = x)$ and since $F^\Delta(0, x) = 0$. □

Lemma S11 shows that condition (5) holds in any SPE of this discrete-time game with two types of buyers: all buyers $i \in [0, \alpha]$ accept a price that leaves them indifferent between buying at that price or waiting and buying at the time low type consumers buy; in particular, consumer α accepts such a price.

Lemmas S10 and S11 together establish that condition (iii) in Definition 1 holds in any SPE of this discrete-time game with two types of consumers. If there were three types of consumers, with valuations $v_3 > v_2 > v_1$, then the monopolist would only serve v_2 consumers when costs are below some cutoff z_2^Δ. Letting $P_2^\Delta(x)$ denote the price at which the monopolist first sells to consumers with valuation v_2 (when costs are x), arguments identical to those in Lemma S11 can be used to show that, in any SPE, all consumers with valuation v_3 accept a price equal to $P_3^\Delta(x) = v_3 - E[e^{-r\tau_2^\Delta}(v_3 - P_2^\Delta(x_{\tau_2^\Delta})) \mid x_0 = x]$, where $\tau_2^\Delta = \inf\{t \in T(\Delta), t > 0 : x_t \leq z_2^\Delta\}$. Hence, condition (5) also holds in any SPE of this discrete-time game with three types of buyers. Repeating this argument, one can show that condition (5) holds in any SPE of this game with any finite number of consumer types. Moreover, the arguments in Lemma S10 do not rely on there being only two types of buyers, so condition (4) also holds in any SPE of this game with any number of consumer types.

S1.3 *Full commitment*

In this section, I solve for the full commitment strategy of the monopolist when there are two types of buyers in the market. In the full commitment problem, the monopolist chooses a path of prices $\{p_t\}$ at time $t = 0$.[8] Given a path of prices $\{p_t\}$, consumer i makes her purchase at the earliest stopping time that solves $\sup_\tau E[e^{-r\tau}(f(i) - p_\tau)]$. Hence, with two types of buyers, there will be (at most) two times of sale: the (random) time $\hat{\tau}_1$ at which low types buy and the (random) time $\hat{\tau}_2$ at which high types buy. Moreover,

[8] Price $\{p_t\}$ must be an $\mathcal{F}_t$-progressively measurable process.

high types will buy weakly earlier than low types, so $\hat{\tau}_2 \le \hat{\tau}_1$ with probability 1. Note that by choosing the path of prices, the monopolist effectively chooses the times $\hat{\tau}_1$ and $\hat{\tau}_2$ at which the different consumers buy.

Note next that it is optimal for the monopolist to charge a price of v_1 to low type buyers. Given this, the highest price that the monopolist can charge high type buyers is given by $p(x_t) = v_2 - E[e^{-r(\hat{\tau}_1 - t)}(v_2 - v_1) \mid x_t]$. Therefore, the optimal strategy of the monopolist boils down to optimally choosing the times $\hat{\tau}_1$ and $\hat{\tau}_2$ at which the different consumers buy. That is, the monopolist's full commitment profits $\Pi^{FC}(x)$ are given by

$$\begin{aligned}\Pi^{FC}(x) &= \sup_{\hat{\tau}_1, \hat{\tau}_2} \alpha E\big[e^{-r\hat{\tau}_2}\big(p(x_{\hat{\tau}_2}) - x_{\hat{\tau}_2}\big) \mid x_0 = x\big] + (1-\alpha)E\big[e^{-r\hat{\tau}_1}(v_1 - x_{\hat{\tau}_1}) \mid x_0 = x\big] \\ &= \sup_{\hat{\tau}_1, \hat{\tau}_2} \alpha E\big[e^{-r\hat{\tau}_2}(v_2 - x_{\hat{\tau}_2}) \mid x_0 = x\big] + (1-\alpha)E\left[e^{-r\hat{\tau}_1}\left(\frac{v_1 - \alpha v_2}{1-\alpha} - x_{\hat{\tau}_1}\right) \Bigm| x_0 = x\right],\end{aligned}$$

where the equality follows from using $p(x_{\hat{\tau}_2}) = v_2 - E[e^{-r(\hat{\tau}_1 - \hat{\tau}_2)}(v_2 - v_1) \mid x_{\hat{\tau}_2}]$. Note the solution to the problem above involves choosing $\hat{\tau}_2$ to maximize the first term and choosing $\hat{\tau}_1$ separately to maximize the second term. Moreover, by Lemma 1, $\hat{\tau}_2 = \tau_2 = \inf\{t : x_t \le z_2\}$. Finally, note that the second term is always negative if $v_1 \le \alpha v_2$, so in this case the optimal strategy for the monopolist is to set $\hat{\tau}_1 = \infty$; that is, never to sell to low types. In this case, $\Pi^{FC}(x) = \sup_\tau \alpha E[e^{-r\tau}(v_2 - x_\tau) \mid x_0 = x]$. Otherwise, if $v_1 > \alpha v_2$, one can use arguments similar to those in the proof of Lemma 1 to show that it is optimal to set $\hat{\tau}_1 = \inf\{t : x_t \le -\lambda_N \hat{v}/(1-\lambda_N)\}$, where $\hat{v} = (v_1 - \alpha v_2)/(1-\alpha)$.

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