

# High Frequency Repeated Games with Costly Monitoring - Online Appendix<sup>†</sup>

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## Abstract

In the Online Appendix we provide the proofs of the main theorem of the paper.

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# 1 Proof of Lemma 4

By applying an affine transformation on the payoffs of the players we can assume w.l.o.g. that  $u(\beta) = (0, R)$  and  $u(\gamma) = (R, 0)$ , and then  $J_\eta$  is the line segment that connects  $(\eta, R - 2\eta)$  and  $(R - 2\eta, \eta)$ . We will prove that all the points on  $J_\eta$  are equilibrium payoffs.<sup>1</sup>

We construct an equilibrium in public strategies, in which the expected discounted payoff after every public history is in  $J_\eta$ . The mixed-action pair  $\alpha^n$  that the players play along the equilibrium path is either  $\beta$  or  $\gamma$ . Whenever the repeated game payoff  $x^n$  satisfies  $x_1^n < \frac{R-\eta}{2}$ , i.e., whenever  $x^n$  is in the upper half of the line segment  $J_\eta$ , the players will play  $\alpha^n = \beta$ ; otherwise they play  $\alpha^n = \gamma$ . Since the sum of payoffs of both players in both  $\beta$  and  $\gamma$  is  $R$ , while the sum of payoffs of both players in each point on  $J_\eta$  is  $R - \eta$ , the players must spend at every stage an expected amount of  $(1 - r^\Delta)\eta$  on monitoring. The expected amount spent on monitoring by Player  $i$  at stage  $n$  is  $p_i^n c_i$ . Consequently, define

$$p_i := \frac{(1-r^\Delta)\eta}{c_i}, \quad (12)$$

and instruct Player 1 (resp. Player 2) to monitor Player 2 (resp. Player 1) with probability  $p_1$  (resp.  $p_2$ ) in every stage in which the players play the mixed-action pair  $\beta$  (resp.  $\gamma$ ). Condition **(A4)** implies that  $p_i < 1$ . By Condition **(A3)** we have  $p_i > \frac{2(1-r^\Delta)}{r^\Delta \eta}$  for  $i \in \{1, 2\}$ . Due to the discussion in Section 5.2 (see Eq. (8)), a deviation of Player 2 (resp. Player 1) to an action outside the support of  $\beta_2$  (resp.  $\gamma_1$ ) is not profitable, provided it triggers a punishment at the minmax level.

We now turn to the formal definition of the proposed equilibrium. For every stage  $n = 1, 2, \dots$  if  $x_1^n < \frac{R-\eta}{2}$  then

- $\alpha^n = \beta$ : the players play the mixed-action pair  $\beta$ .
- $p_1^n = p_1$  and  $p_2^n = 0$ . That is, only Player 1 monitors and he does it with probability  $p_1$  given in (12).
- If Player 1 monitors Player 2 then  $x^{n+1}$  is given by (see Figure 8)

$$x_1^{n+1} := \frac{x_1^n + c_1}{r^\Delta}, \quad x_2^{n+1} := R - \eta - \frac{x_1^n + c_1}{r^\Delta}.$$

- If Player 1 does not monitor Player 2 then  $x^{n+1}$  is given by

$$x_1^{n+1} := \frac{x_1^n}{r^\Delta}, \quad x_2^{n+1} := R - \eta - \frac{x_1^n}{r^\Delta}.$$

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<sup>1</sup>The attentive reader will note that due to the affine transformation on payoffs we in fact prove Lemma 4 only in the case that  $u_1(\gamma) - u_1(\beta) = u_2(\beta) - u_2(\gamma)$ . This assumption simplifies the calculations and highlights the main ideas of the proof.

If  $x_1^n \geq \frac{R-\eta}{2}$  the play is defined analogously: the players play the mixed-action pair  $\gamma$ , Player 1 does not monitor Player 2, Player 2 monitors Player 1 with probability  $p_2$  given in (12), if Player 2 monitors Player 1 at stage  $n$  then

$$x_1^{n+1} := R - \eta - \frac{x_2^n + c_2}{r\Delta}, \quad x_2^{n+1} := \frac{x_2^n + c_2}{r\Delta},$$

while if Player 2 does not monitor Player 1 at stage  $n$  then

$$x_1^{n+1} := R - \eta - \frac{x_2^n}{r\Delta}, \quad x_2^{n+1} := \frac{x_2^n}{r\Delta}.$$

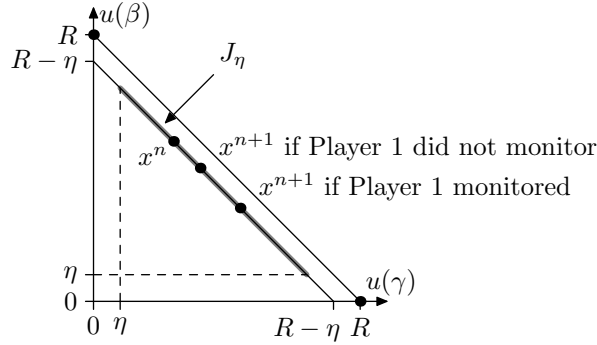


Figure 8: The construction in the proof of Lemma 4.

Since  $D_i^n = 0$  for every  $n \in \mathbf{N}$ , it is sufficient to verify that Conditions **(C2)**–**(C6)** are satisfied. Since  $x_i \geq \eta > 0 \geq v_i$  for every  $x \in J_\eta$  and every  $i = 1, 2$ , Condition **(C2)** holds. The definition of  $p_i$  and Condition **(A1)** imply that Condition **(C3)** holds. The verification that Conditions **(C4)**–**(C6)** hold follows by simple algebraic manipulations. We provide here the verification of Condition **(C4)**. Assume then that  $x_1^n \leq \frac{R-\eta}{2}$ , so that  $\alpha^n = \beta$  and  $p_1^n = p_1$ . Since Player 1 plays a best response at  $\beta$ , we have  $u_1(a_1, \alpha_2^n) = 0$  for every  $a_1 \in \text{supp}(\alpha_1^n)$ . Since  $D_1^n = 0$ , Condition **(C4)** translates to  $x_i^n = r^\Delta x_i^{n+1} - c_i \cdot \mathbf{1}_{I_i^n}$ , which holds by the definition of  $x_1^{n+1}$ . Regarding Player 2, since he is indifferent at  $\beta$ , we have  $u_2(\alpha_1^n, a_2) = R$  for every  $a_2 \in \text{supp}(\alpha_2^n)$ . Since  $D_2^n = 0$ , Condition **(C4)** translates to

$$x_2^n = (1 - r^\Delta)R + r^\Delta \left( p_1 \left( R - \eta - \frac{x_1^n + c_1}{r\Delta} \right) + (1 - p_1) \left( R - \eta - \frac{x_1^n}{r\Delta} \right) \right).$$

Since  $x_2^n = R - \eta - x_1^n$ , after cancelling the term  $R - \eta - x_1^n$  from its both sides, this equation reduces to  $p_1 c_1 = (1 - r^\Delta)\eta$ , which holds by the definition of  $p_1$ .

## 2 Proof of Lemma 5

By applying an affine transformation on the payoffs of the players we can assume that  $R_1^{(1)} = R_2^{(2)}$  and  $u_1(\beta) = u_2(\gamma) = 0$ . We will prove the following result which implies Lemma 5.

**Lemma 9.** Let  $\beta = (\beta_1, \beta_2)$  and  $\gamma = (\gamma_1, \gamma_2)$  be two mixed-action pairs and let  $R > 0$  such that the following conditions hold:

1. Player 1 plays a best response at  $\beta$ ,  $u_1(\beta) \geq 0$ , and  $u_2(\beta_1, a_2) \geq R$  for every action  $a_2 \in \text{supp}(\beta_2)$ ;
2. Player 2 plays a best response at  $\gamma$ ,  $u_2(\gamma) \geq 0$ , and  $u_1(a_1, \gamma_2) \geq R$  for every action  $a_1 \in \text{supp}(\gamma_1)$ ;

Then the pentagon  $Q_\eta$  whose extreme points are (see Figure 9)  $(v_1 + \eta, v_2 + \eta)$ ,  $(v_1 + \eta, R - 2\eta)$ ,  $(R - 2\eta, v_2 + \eta)$ ,  $(\eta, R - 2\eta)$ , and  $(R - 2\eta, \eta)$  is a subset of  $NE(r, c_1, c_2, \Delta)$ , provided that the parameters  $r$ ,  $c_1$ ,  $c_2$ ,  $\Delta$ , and  $\eta$  satisfy Conditions **(A1)**–**(A4)**.

*Proof of Lemma 9.* Let  $\xi \in Q_\eta$ . We will construct an equilibrium with payoff  $\xi$ . The construction will be similar to the construction in the proof of Theorem 4, and will use burning-money processes. Recall that  $p_i = \frac{(1-r\Delta)\eta}{c_i}$  for  $i \in \{1, 2\}$ .

Fix a Nash equilibrium  $\alpha^*$  in the base game. The play in the first stages depends on three parameters: a payoff vector  $x \in Q_\eta$  close to  $J_\eta$  and two nonnegative integers  $k_1$  and  $k_2$ . We first describe the play in the first  $k := \max\{k_1, k_2\}$  stages, and then explain how to choose the parameters  $x$ ,  $k_1$ , and  $k_2$ .

The players play as follows:

- They play the mixed action  $\alpha^*$  for  $k$  stages.
- In the first  $k_1$  stages Player 1 monitors Player 2, and in the first  $k_2$  stages Player 2 monitors Player 1. If, for example,  $k_1 < k_2$ , then in the first  $k_1$  stages both players monitor each other, and in the following  $k_2 - k_1$  stages Player 2 monitors Player 1 while Player 1 does not monitor Player 2.
- From stage  $k + 1$  onwards the players implement an equilibrium with payoff  $x$ .

The payoff to each Player  $i$  is then  $(1 - r^{k\Delta})u_i(\alpha^*) + r^{k\Delta}x_i - (1 - r^{k_i\Delta})c$ : in the first  $k$  stages the players play  $\alpha^*$ , in the first  $k_i$  stages Player  $i$  monitors Player  $j$ , and the continuation payoff at stage  $k$  is  $x$ . We choose the parameters  $x$ ,  $k_1$ , and  $k_2$  to satisfy

$$(D1) \quad \xi_i = (1 - r^{k\Delta})u_i(\alpha^*) + r^{k\Delta}x_i - (1 - r^{k_i\Delta})c.$$

$$(D2) \quad x_i \geq \eta \text{ for } i \in \{1, 2\}.$$

$$(D3) \quad R - \eta - 2c \leq x_1 + x_2 \leq R - \eta.$$

Conditions (D2) and (D3) ensure that  $x$  is close to  $J_\eta$ : there is  $y \in J_\eta$  that dominates  $x$  and satisfies  $y_i - x_i \leq 3c$ . Fix then such  $y \in J_\eta$  and set

$$x^1 := y, \quad D^1 := y - x.$$

For every stage  $n > k$ , if  $x_1^n \leq \frac{R-\eta}{2}$ :

- $\alpha^n = \beta$ : the players play the mixed-action pair  $\beta$ .
- If  $D_i^n \geq c_i$  then  $p_i^n = 1$ : a player with a high debt monitors the other player (and burns money).
- If  $D_1^n < c_1$  then  $p_1^n = p_1$ ; if  $D_2^n < c_2$  then  $p_2^n = 0$ : Only Player 1 monitors with positive probability. Recall that Player 1 plays a best response at  $\beta$ , so that he cannot gain by deviating from  $\beta_1$ , hence he does not have to be monitored.
- If Player 1 monitors Player 2 and finds out that Player 2 played an action  $a_2 \notin \text{supp}(\beta_2)$ , then from stage  $n+1$  onwards he switches to a punishment strategy that reduces Player 2's payoff to  $v_2 + \eta$ .

In case  $x_1^n > \frac{R-\eta}{2}$  the play is analogous: the players play the mixed-action pair  $\gamma$ , a player with a debt of at least  $c_i$  monitors the other with probability 1; if Player 1's (resp. Player 2's) debt is lower than  $c_1$  (resp.  $c_2$ ), then he does not monitor Player 2 (resp. monitors Player 2 with probability  $p_2$ ); and if Player 2 monitors Player 1 and finds out that Player 1 played an action outside the support of  $\gamma_1$ , then he switches to a minmax strategy against Player 1.

It is left to define the processes  $(x^n)_{n \in \mathbf{N}}$  and  $(D^n)_{n \in \mathbf{N}}$  so that (a) the discounted payoff will be  $x$ , and (b) no player will have an incentive to deviate. We will define these two processes recursively. Suppose that  $x^n \in J_\eta$  and  $D^n \in \mathbb{R}_+^2$  have already been defined, and assume that  $x_i^n - D_i^n \geq v_i + \eta$  for  $i = 1, 2$ . If  $x_1^n \leq \frac{R-\eta}{2}$ , define  $y^n, z^n, w^n \in J_\eta$  as follows (see Figure 9):

$$\begin{aligned} w_1^n &:= \frac{x_1^n}{r^\Delta} + \eta(1 - r^\Delta) = \frac{R - x_2^n}{r^\Delta} - \eta, & w_2^n &:= \frac{x_2^n - (1 - r^\Delta)R}{r^\Delta}, \\ y_1^n &:= \frac{x_1^n + c}{r^\Delta}, & y_2^n &:= R - \eta - \frac{x_1^n + c}{r^\Delta}, \\ z_1^n &:= \frac{x_1^n}{r^\Delta}, & z_2^n &:= R - \eta - \frac{x_1^n}{r^\Delta}. \end{aligned} \tag{13}$$

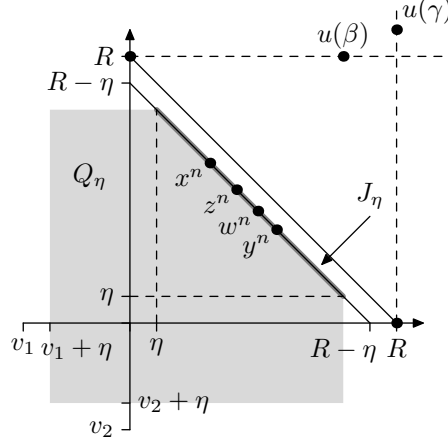


Figure 9: The construction in the proof of Lemma 4.

The payoff vectors  $y^n$  and  $z^n$  will be the continuation payoffs when Player 1's debt is lower than  $c_1$ . These quantities are similar to  $x^{n+1}$  in the proof of Lemma 4, where  $y^n$  (resp.  $z^n$ ) was the continuation payoff when Player 1 monitored (resp. did not monitor) Player 2.  $w^n$  will be the continuation payoff when Player 1's debt is higher than  $c_1$ . Note that these three vectors are on  $J_\eta$ .

When  $x_1^n > \frac{R-\eta}{2}$ , the roles of the two players are exchanged: they play the mixed-action pair  $\gamma$ , Player 2 monitors Player 1 with positive probability, and the continuation payoffs  $w^n$ ,  $y^n$ , and  $z^n$  are defined analogously.

The continuation payoff  $x^{n+1}$  and the debt  $D^{n+1}$  are given by the following table. For the sake of convenience we provide the quantity  $r^\Delta D_i^{n+1}$  for  $i = 1, 2$ . When Player 1 monitors Player 2, the action  $a_2$  that Player 2 plays at stage  $n$  is common knowledge, and  $D^{n+1}$  can depend on it. Recall that the event that Player  $i$  monitors Player  $j$  at stage  $n$  is denoted by  $I_i^n$ .

	If $x_1^n \leq \frac{R-\eta}{2}$ and ...	$x^{n+1}$	$r^\Delta D_1^{n+1}$	$r^\Delta D_2^{n+1}$
1	$D_1^n \geq c_1, D_2^n \geq c_2$	$w^n$	$D_1^n - c_1 + (1 - r^\Delta)u_1(\beta) + (1 - r^\Delta)\eta$	$D_2^n - c_2 + (1 - r^\Delta)(u_2(\beta_1, a_2) - R)$
2	$D_1^n \geq c_1, D_2^n < c_2$	$w^n$	$D_1^n - c_1 + (1 - r^\Delta)u_1(\beta) + (1 - r^\Delta)\eta$	$D_2^n + (1 - r^\Delta)(u_2(\beta_1, a_2) - R)$
3	$D_1^n < c_1, I_1^n, D_2^n \geq c_2$	$y^n$	$D_1^n + (1 - r^\Delta)u_1(\beta)$	$D_2^n - c_2 + \frac{c_1}{\eta}(u_2(\beta_1, a_2) - R)$
4	$D_1^n < c_1, I_1^n, D_2^n < c_2$	$y^n$	$D_1^n + (1 - r^\Delta)u_1(\beta)$	$D_2^n + \frac{c_1}{\eta}(u_2(\beta_1, a_2) - R)$
5	$D_1^n < c_1, \neg I_1^n, D_2^n \geq c_2$	$z^n$	$D_1^n + (1 - r^\Delta)u_1(\beta)$	$D_2^n - c_2 + (1 - r^\Delta)(u_2(\beta) - R)$
6	$D_1^n < c_1, \neg I_1^n, D_2^n < c_2$	$z^n$	$D_1^n + (1 - r^\Delta)u_1(\beta)$	$D_2^n + (1 - r^\Delta)(u_2(\beta) - R)$

Figure 10: The continuation payoff and the debt.

We explain below the intuition behind the definition of the burning-money process for Player 1.

- Whenever a player monitors the other to burn money (i.e.,  $D_i^n \geq c_i$ , which implies  $p_i^n = 1$ ), his debt decreases by  $c_i$ . For instance, in lines 1 and 2, the first part of  $r^\Delta D_1^{n+1}$  is  $D_1^n - c_i$ .
- When  $D_1^n \geq c_i$ , as in lines 1 and 2, the last part of  $r^\Delta D_1^{n+1}$  is  $(1 - r^\Delta)\eta$ . The reason for adding this term is that the construction assumes that the stage payoff is  $(0, R)$  or  $(R, 0)$ , so that the sum of stage payoffs of the two players is  $R$ . However, the sum of payoffs in all points in  $J_\eta$  is  $R - \eta$ . Since both the continuation payoff and the current payoff should be on  $J_\eta$ , we need to add  $(1 - r^\Delta)\eta$  to the debt of the players.
- Whenever  $D_1^n \geq c_1$ , the continuation payoff  $w^{n+1}$  is defined to satisfy Condition **(C6)** for Player 1. Indeed, using the definition of  $w^n$  and  $D^n$ ,

$$\begin{aligned}
& (1 - r^\Delta)u_1(\beta) - c_1 + r^\Delta w_1 - r^\Delta D_1^{n+1} \\
&= (1 - r^\Delta)u_1(\beta) - c_1 + r^\Delta \left( \frac{x_1^n}{r^\Delta} + \eta(1 - r^\Delta) \right) - (D_1^n - c_1 + (1 - r^\Delta)u_1(\beta) + (1 - r^\Delta)\eta) \\
&= x_1^n - D_1^n.
\end{aligned}$$

- Whenever Player 1 decides randomly whether or not to monitor Player 2 (lines 3–6 in the table in Figure 10), the continuation payoff  $x^{n+1}$  is given as in the proof of Lemma 4. In case Player 1 monitors Player 2, the continuation payoff is  $y^n$  and otherwise it is  $z^n$ . The continuation payoffs were chosen to ensure that Condition **(C4)** holds. We verify this for line 3.

$$\begin{aligned}
& (1 - r^\Delta)u_1(\beta) + r^\Delta y_1 - r^\Delta D_1^{n+1} \\
&= (1 - r^\Delta)u_1(\beta) - c_1 + r^\Delta y_1 - [D_1^n + (1 - r^\Delta)u_1(\beta)] \\
&= (1 - r^\Delta)u_1(\beta) - c_1 + r^\Delta \left[ \frac{x_1^n + c_1}{r^\Delta} \right] - [D_1^n + (1 - r^\Delta)u_1(\beta)] \\
&= x_1^n - D_1^n.
\end{aligned}$$

We now verify that this definition satisfies the conditions listed in Section 5.3. In the first  $k$  stages no player can profit by deviating, and so we need to verify these conditions only from stage  $k$  onwards. We first check that Condition **(C2)** holds. One can verify that if  $D_i^n \geq c_i$  then  $x_i^{n+1} - D_i^{n+1} \geq x_i^n - D_i^n$ , and the result follows by induction. Otherwise,  $D_i^{n+1} \leq \frac{c_i}{r^\Delta}$ . When  $x_1^n \leq \frac{R\eta}{2}$ , we have  $y_1^n, z_1^n \geq x_1^n \geq \eta$ , and therefore  $x_1^{n+1} - D_1^{n+1} \geq 0$ . Since  $x_2^n \leq \frac{R}{2}$ , it follows that  $x_2^{n+1} - D_2^{n+1} \geq 0$ . The definition of  $p_i$  together with Condition **(A3)** imply that Condition **(C3)** is satisfied. The verification that Conditions **(C4)**–**(C6)** hold amounts to substituting the quantities defined above in the relevant equations, as illustrated above.  $\square$

### 3 Proof of Lemma 6

We denote  $u(a) = (A, B)$ , and distinguish between four cases that are handled separately (see Figure 11):

Case 1:  $u_1(a) \geq t_1^2$  and  $u_2(a) \geq t_2^1$ .

Case 2:  $u_1(a) < t_1^1$  and  $u_2(a) > t_2^1$ .

Case 3:  $u_2(a) < t_2^2$  and  $u_1(a) > t_1^2$ , which is analogous to Case 2.

Case 4: Cases 1–3 do not hold.

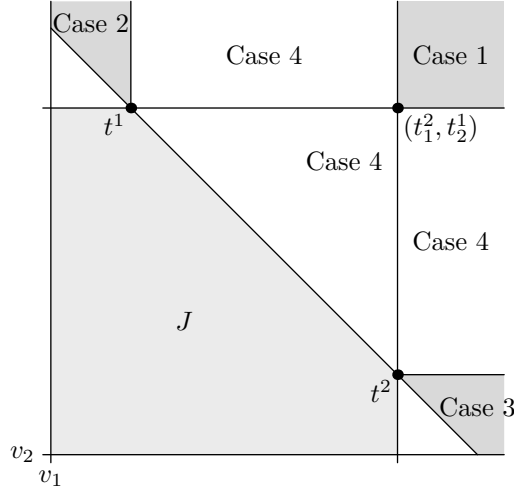


Figure 11: The four cases in the proof of Lemma 5.

The construction in this section will not employ burning-money processes. Rather, we use a recursive construction: we identify a set  $J_1$  of payoff vectors, which can be arbitrarily close to  $J'$ , and for every payoff vector  $g \in J_1$  we define a one-shot auxiliary game in which (a) the payoffs are the stage-payoff in the base game plus a continuation payoff; (b) the continuation payoff, which depends on the players' choices, are in  $J \cup J_1$ ; and (c) there is an equilibrium whose payoff is  $g$  and in which each player monitors the other with probability  $p$  that satisfies Eq. (8).

#### 4 Case 1: $u_1(a) \geq t_1^2$ and $u_2(a) \geq t_2^1$ .

Roughly, we prove that all the points in the triangle whose extreme points are  $t^1$ ,  $t^2$ , and  $(t_1^2, t_2^1)$  are in  $E(r, c, \Delta)$ , provided that  $c_1$ ,  $c_2$ , and  $\Delta$  are sufficiently small. Fix  $\eta < \min\{\frac{t_1^2 - u_1(\beta^*)}{7}, \frac{t_2^1 - u_2(\gamma^*)}{7}\}$ . Denote by  $-\alpha$  the slope of the line segment  $[t^1, t^2]$ . Set (see



Figure 12)

$$w^1 := (t_1^1 + \eta, t_2^1 - 2\alpha\eta), \quad w^2 := (t_1^2 - 2\eta, t_2^2 + \alpha\eta), \quad (14)$$

$$s^1 := (t_1^1 + 2\eta, t_2^1 - 3\alpha\eta), \quad s^2 := (t_1^2 - 3\eta, t_2^2 + 2\alpha\eta), \quad (15)$$

$$\widehat{z} := (t_1^2 - 4\eta, t_2^1 - 4\alpha\eta). \quad (16)$$

Since  $J$  is an asymptotic set of Nash equilibrium payoffs, Lemma 5 implies that all the points in the pentagon  $J_0$  whose extreme points are  $(v_1, v_2)$ ,  $(v_1, w_2^1)$ ,  $(w_1^2, v_2)$ ,  $w^1$ , and  $w^2$  are in  $E(r, c_1, c_2, \Delta)$ , provided that  $c_1$ ,  $c_2$ , and  $\Delta$  are sufficiently small. Denote the slope of the line segment  $[s^1, \widehat{z}]$  by  $-d := -\frac{\alpha\eta}{t_1^2 - u_1(\beta^*) - 6\eta}$ , and the slope of the line segment  $[\widehat{z}, s^2]$  by  $-e := -\frac{t_2^1 - u_2(\gamma^*) - 6\alpha\eta}{\eta}$  (see Figure 12).

By the choice of  $\eta$ ,

$$\mathbf{E0}: s_1^1 < \widehat{z}_1 \text{ and } s_2^2 < \widehat{z}_2,$$

so that  $e > d > 0$ .

Assume that  $c_1$ ,  $c_2$ , and  $\Delta$  are sufficiently small to satisfy Conditions **(A1)**–**(A4)**, as well as the following conditions for  $i \in \{1, 2\}$ :

$$\mathbf{E1}: 1 - r^\Delta < c_i < \frac{\eta}{6}.$$

$$\mathbf{E2}: \frac{1}{\eta}c_i < \frac{1}{4}Be, \frac{1}{4}Ad < \frac{\eta}{2(1-r^\Delta)}.$$

$$\mathbf{E3}: \frac{2c_i}{dr^\Delta}, \frac{2ec_i}{r^\Delta} < \eta.$$

$$\mathbf{E4}: r^\Delta > \frac{1}{2} \text{ and } 4(1 - r^\Delta) < \eta.$$

We will show that all points in the triangle  $J_1$ , whose extreme points are  $s^1$ ,  $s^2$ , and  $\widehat{z}$ , are in  $E(r, c_1, c_2, \Delta)$ .

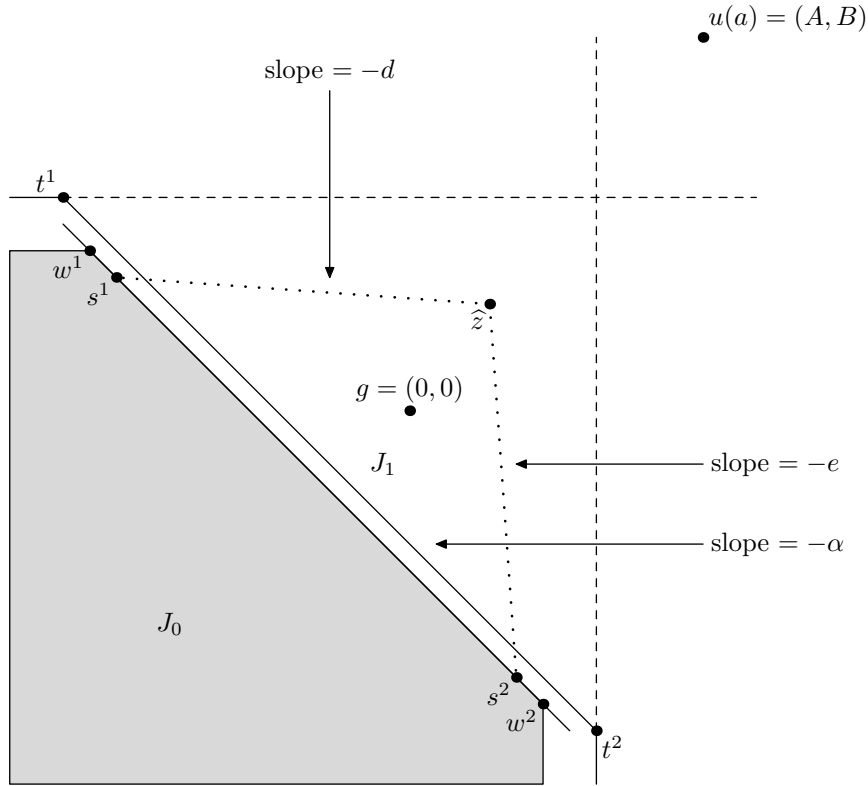


Figure 12: Case 1.

Fix a point  $g$  in the triangle  $J_1$ , and for the calculations below add a constant to the payoff so that  $g = (0, 0)$ . We now describe a  $2 \times 2$  one-shot auxiliary game  $G(\zeta_1, \zeta_2, x, y)$  whose payoffs depend on four positive real numbers  $\zeta_1, \zeta_2, x$ , and  $y$ , and an equilibrium in that game that yields the payoff  $(0, 0)$ .

- Each player has two actions, “Monitor” and “Don’t Monitor”.
- The payoff function is given by the table in Figure 13, in which at each entry, Player 1’s payoff appears at the top and Player 2’s payoff at the bottom.

		Player 2	
		Don’t Monitor	Monitor
Player 1	Don’t Monitor	$(1 - r^\Delta)A - r^\Delta x$ $(1 - r^\Delta)B - r^\Delta y$	$(1 - r^\Delta)A - r^\Delta \zeta_1$ $(1 - r^\Delta)B + r^\Delta d \zeta_1 - c_2$
	Monitor	$(1 - r^\Delta)A + r^\Delta \frac{\zeta_2}{e} - c_1$ $(1 - r^\Delta)B - r^\Delta \zeta_2$	$(1 - r^\Delta)A - c_1$ $(1 - r^\Delta)B - c_2$

Figure 13: The game  $G(\zeta_1, \zeta_2, x, y)$ .

The payoff is calculated as if, in the original repeated game with costly observation, the players play the pure action pair  $a$ , each player chooses whether to monitor the other player, and the continuation payoffs, which depend on the identity of the players who chose to monitor, are given by the matrix in Figure 14.

		Player 2	
		Don't Monitor	Monitor
Player 1	Don't Monitor	$-x, -y$	$-\zeta_1, d\zeta_1$
	Monitor	$\frac{\zeta_2}{e}, -\zeta_2$	$0, 0$

Figure 14: The continuation payoffs that underlie the game  $G(\zeta_1, \zeta_2, x, y)$ .

Because the slopes of the line segments that define  $J_1$  are  $-d$  and  $-e$ , the vectors  $(\frac{\zeta_2}{e}, -\zeta_2)$  and  $(-\zeta_1, d\zeta_1)$  are in  $J_0 \cup J_1$ , provided that  $\zeta_1$  and  $\zeta_2$  are sufficiently small. Recall that  $g = (0, 0)$  is in  $J_1$ . We will have to ensure that  $x, y \in [0, \eta]$  so that  $(-x, -y)$  is in  $J_0 \cup J_1$ .

Set

$$p := \frac{2(1-r^\Delta)}{r^\Delta \eta}. \quad (17)$$

We will find positive numbers  $\zeta_1$ ,  $\zeta_2$ ,  $x$ , and  $y$  such that the pair of strategies in which each player monitors the other with probability  $p$  is an equilibrium of  $G(\zeta_1, \zeta_2, x, y)$  with payoff  $(0, 0)$ . By Eq. (8), this will imply that in the repeated game no player can profit by a deviation to an action that he is supposed to play with probability 0, provided such a deviation leads to a punishment at the maxmin level. By solving the indifference conditions of the players we obtain that if

$$\zeta_2 = \frac{e(c_1 - (1 - r^\Delta)A)}{r^\Delta(1 - p)}, \quad (18)$$

$$\zeta_1 = \frac{c_2 - (1 - r^\Delta)B}{dr^\Delta(1 - p)}, \quad (19)$$

$$x = \frac{(1 - r^\Delta)Ad(1 - p) - p(c_2 - (1 - r^\Delta)B)}{d(1 - p)^2}, \quad (20)$$

$$y = \frac{(1 - r^\Delta)B(1 - p) - pe(c_1 - (1 - r^\Delta)A)}{(1 - p)^2}, \quad (21)$$

then having both players monitor each other with probability  $p$  is an equilibrium of  $G(\zeta_1, \zeta_2, x, y)$ .

Condition **(E1)** implies that  $\zeta_1$  and  $\zeta_2$  are positive; Conditions **(E2)** and **(E4)** imply that  $x$  and  $y$  are positive; Conditions **(E3)** and **(E4)** imply that  $\zeta_1$  and  $\zeta_2$  are smaller than  $\eta$ ; and Condition **(E2)** implies that  $x$  and  $y$  are smaller than  $\eta$ . This concludes the proof for Case 1.

## 5 Case 2: $u_1(a) < t_1^2$ and $u_2(a) > t_2^1$ .

The proof in this case is similar to the proof in Case 1, with a different definition of  $\widehat{z}$ , and the calculations are slightly more cumbersome. Recall that  $u(a) = (A, B)$ , so that in this case  $A < t_1^1$  and  $B > t_2^1$  (see Figure 15). The slope of the line segment  $[u(a), t^2]$  is  $\frac{t_2^2 - B}{t_1^2 - A} < 0$ . Fix  $\eta > 0$  sufficiently small to satisfy  $\eta < -\frac{t_2^2 - B}{t_1^2 - A}$ , and define the points  $w^1, w^2, s^1$ , and  $s^2$  as in Case 1 (see Eqs. (14)–(15)). Set

$$e := -\frac{t_2^2 - B}{t_1^2 - A} - \eta > 0.$$

Consider the line with slope  $-e$  that passes through  $s^2$ , and let  $\widehat{z}$  be the point on this line that satisfies  $\widehat{z}_2 = t_2^1 - 4\eta$ . Let  $d$  be the slope of the line segment  $[s^1, \widehat{z}]$ . Then  $0 < d < e < \infty$ . Suppose that **(E2)**–**(E4)** hold for the  $d$  and  $e$  defined here, as well as the following two conditions:

**E5:**  $ec_1 < \frac{1}{8}$ ,  $e(1 - r^\Delta)(-A) < \frac{1}{8}$ , and  $c_2 < 4d\eta$ .

**E6:**  $d(-A)\eta^2 > 8e(c_2 + dc_1)$ .

Denote by  $J_1$  the triangle whose extreme points are  $s^1, s^2$ , and  $\widehat{z}$ . We will prove that all the points in the triangle  $J_1$  are in  $E(r, c_1, c_2, \Delta)$ , provided that  $c_1, c_2$ , and  $\Delta$  are sufficiently small.

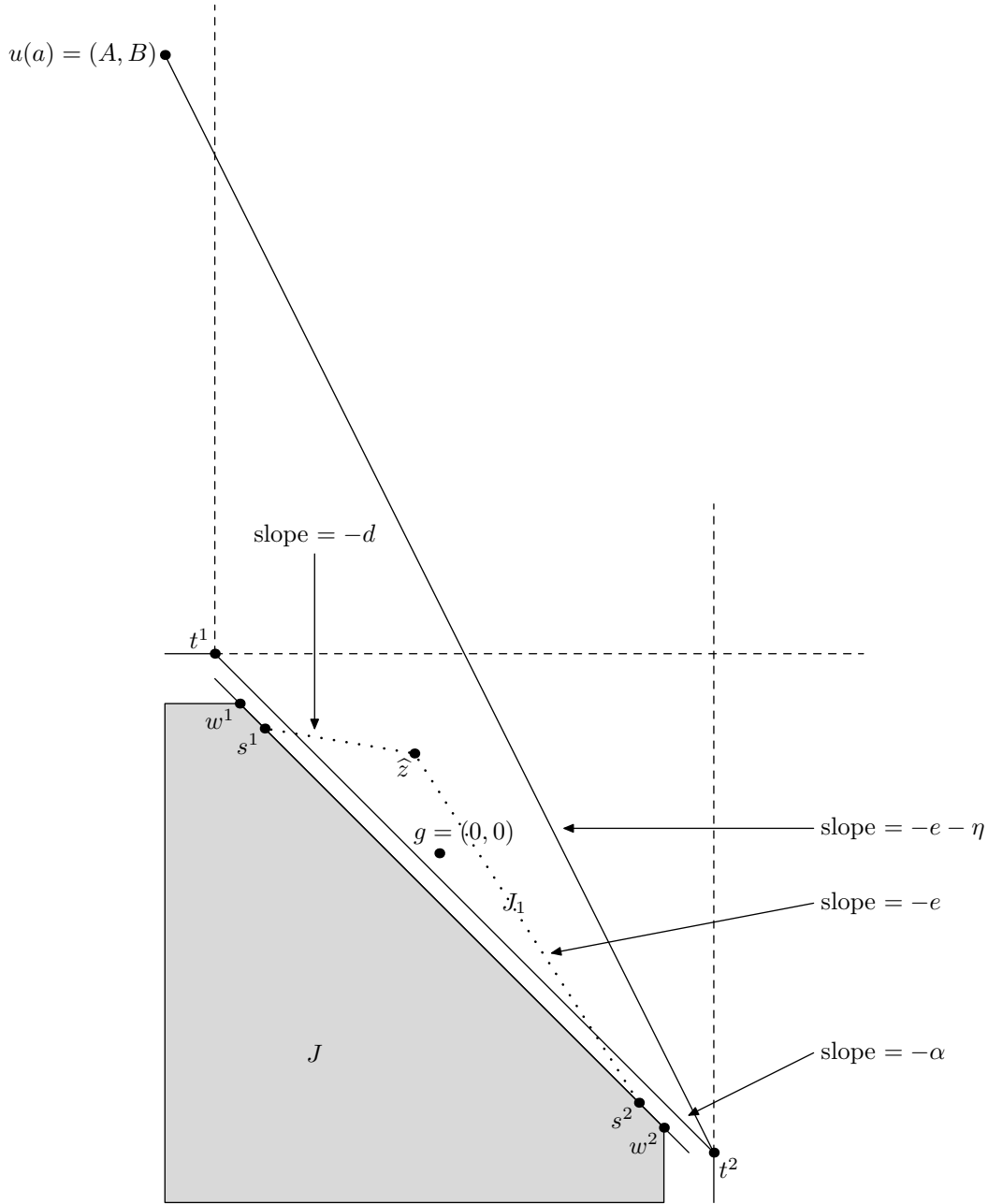


Figure 15: The setup in Case 2.

Fix a point  $g \in J_1$  and for the calculation below add a constant to the payoffs so that  $g = (0, 0)$ . Because  $g = (0, 0)$  is below the line segment  $[u(a), t^2]$ ,

$$\frac{B}{-A} > e + \eta. \quad (22)$$

Consider the  $2 \times 2$  one-shot auxiliary game  $G(\zeta_1, \zeta_2, x, y)$  that is defined in Figure 14. Having each player monitor the other with probability  $p = \frac{3(1-r\Delta)}{r\Delta\eta}$  is an equilibrium in the

game  $G(\zeta_1, \zeta_2, x, y)$  that yields payoff  $(0, 0)$ , where  $\zeta_1$ ,  $\zeta_2$ ,  $x$ , and  $y$  are given by (18)–(21). Because  $A$  is negative,  $x$  is negative as well.

Condition **(E1)** implies that  $\zeta_1$  is positive and together with Condition **(E3)** it implies that it is less than  $\eta$ . Plainly  $\zeta_2$  is positive, and Conditions **(E1)** and **(E5)** imply that it is less than  $\eta$ . Condition **(E2)** implies that  $y$  is positive and Conditions **(E1)** and **(E4)** implies that it is less than  $\eta$ . As mentioned above,  $x$  is negative, and Conditions **(E1)** and **(E5)** imply that it is larger than  $-\eta$ .

To complete the proof we need to show that  $(-x, y)$  lies in  $J_0 \cup J_1$ . To this end we show that  $\frac{-y}{-x} > -e$ . By Eqs. (20)–(21), this inequality reduces to

$$(1 - r^\Delta)dB(1 - p) - ped(c_1 - (1 - r^\Delta)A) > pe(c_2 - (1 - r^\Delta)B) - (1 - r^\Delta)Ade(1 - p).$$

Since  $p = \frac{3(1-r^\Delta)}{r^\Delta\eta}$ , we can divide all terms by  $(1 - r^\Delta)$ , so that this inequality is equivalent to

$$d(1 - p)(B + Ae) + pe(dA + B) > \frac{2e(c_2 + dc_1)}{r^\Delta\eta},$$

which holds by Condition **(E6)**.

## 6 Case 4

To solve Case 4 we use Case 1 and jointly controlled lotteries. Specifically, by Case 1, the grey area in Figure 16 is in  $NE(r, c_1, c_2, \Delta)$ , provided that  $c_1$ ,  $c_2$ , and  $\Delta$  are sufficiently small.

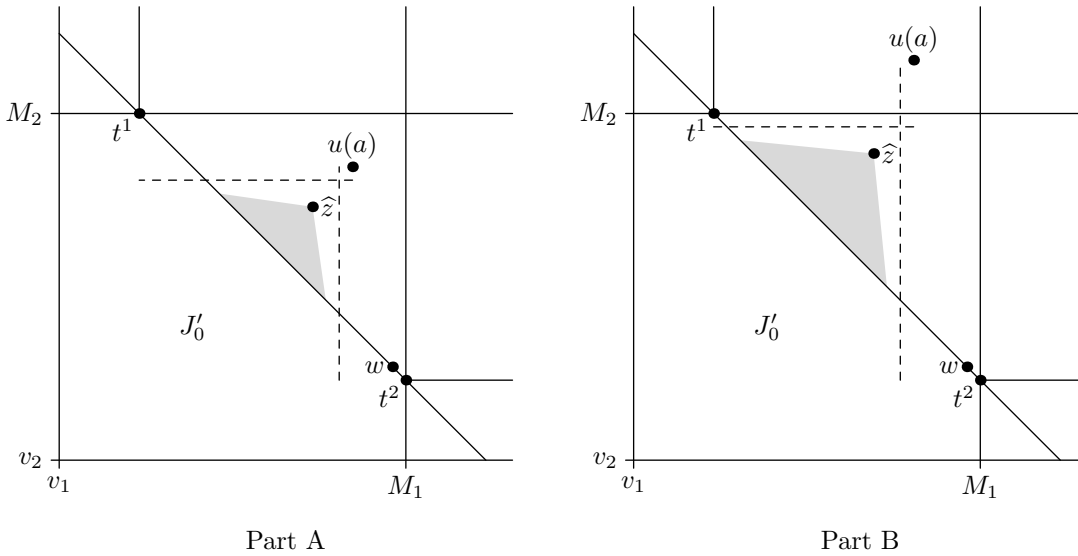


Figure 16: The Proof in Case 4.

To complete Case 4 we will show that the set  $NE(r, c_1, c_2, \Delta)$  is almost convex. Indeed, suppose that at the first stage the players jointly choose between, say, implementing as an equilibrium payoff a vector close to  $\widehat{z}$  or a vector close to  $t^2$ . This is done as follows:

- Let  $a_1, a'_1 \in A_1$  and  $a_2, a'_2 \in A_2$  be two distinct actions of the two players. At the first stage Player 1 (resp. Player 2) chooses either  $a_1$  or  $a'_1$  (resp.  $a_2$  or  $a'_2$ ) with equal probabilities, and both players monitor each other.
- If at the first stage one of the players fails to monitor the other, or fails to play one of the designated actions, both players switch to punishment strategies.
- Otherwise, according to the realized action pair at the first stage, the players implement from the second stage onwards one of the following payoff vectors as an equilibrium payoff, where  $w = t^2 + (-\eta, \eta)$ .

Action pair	Player 1's payoff	Player 2's payoff
$(a_1, a_2)$	$\widehat{z}_1 - (1 - r^\Delta)u_1(a_1, a_2)$	$\widehat{z}_2 - (1 - r^\Delta)u_2(a_1, a_2)$
$(a_1, a'_2)$	$w_1 - (1 - r^\Delta)u_1(a_1, a'_2)$	$w_2 - (1 - r^\Delta)u_2(a_1, a'_2)$
$(a'_1, a_2)$	$w_1 - (1 - r^\Delta)u_1(a'_1, a_2)$	$w_2 - (1 - r^\Delta)u_2(a'_1, a_2)$
$(a'_1, a'_2)$	$\widehat{z}_1 - (1 - r^\Delta)u_1(a'_1, a'_2)$	$\widehat{z}_2 - (1 - r^\Delta)u_2(a'_1, a'_2)$

## 7 Public Perfect Equilibria

In the construction of Nash equilibria we used threats of punishment. In this section we modify the proof of Lemma 5 so that the implementation of the vector  $\xi := (v_1 + \eta, v_2 + \eta)$  will not involve noncredible threats. As is common in the literature, the implementation of a credible punishment is accomplished by having the players lower their payoffs for a fixed number of stages and returning to the equilibrium play afterwards.

The implementation of  $\xi$  in the proof of Lemma 5 includes a first phase that lasts  $k$  stages, in which the players follow an equilibrium  $\alpha^*$  of the base game and partially monitor each other. We change only the implementation of this phase.

Suppose w.l.o.g. that  $k_1 \leq k_2$ , so that  $k = k_2$ . Thus, Player 1 monitors Player 2 in the first  $k_1$  stages, and Player 2 monitors Player 1 in the first  $k_2$  stages.

- In the first  $k$  stages Player 1 plays a minmax mixed action  $\beta_1$ .
- In the first  $k_1$  stages Player 2 plays a minmax mixed action  $\beta_2$ .
- In the following  $k_2 - k_1$  stages Player 2 plays a best response  $\gamma_2$  against  $\beta_1$ .

If no deviation occurs, the expected payoff to Player 1 in the first  $k$  stages, given the public history, is

$$\delta_1 := \sum_{n=1}^{k_1} (1 - r^\Delta) r^{(n-1)\Delta} u_1(a_1^n, a_2^n) + \sum_{n=k_1+1}^{k_2} (1 - r^\Delta) r^{(n-1)\Delta} u_1(a_1^n, \gamma_2) - (1 - r^{k_1\Delta})c,$$

and the expected payoff to Player 2 in the first  $k$  stages, given the public history, is

$$\delta_2 := \sum_{n=1}^{k_1} (1 - r^\Delta) r^{(n-1)\Delta} u_2(a_1^n, a_2^n) + \sum_{n=k_1+1}^{k_2} (1 - r^\Delta) r^{(n-1)\Delta} u_2(a_1^n, \gamma_2) - (1 - r^{k_2\Delta})c.$$

The continuation payoff  $x$  will be a random variable that satisfies

$$\xi = \delta + r^{k_2\Delta}x,$$

where  $\delta = (\delta_1, \delta_2)$ . Provided  $\Delta$  is small,  $x$  is in  $Q_\eta$  and satisfies  $d(x, J_\eta) \leq 2\eta$ .

Whenever a deviation is observed, the players restart implementing  $\xi$  with the above construction. It is left to the reader for verify that the construction is indeed a public perfect equilibrium.