Online Supplementary Appendix to: "A Tractable Model of Monetary Exchange with Ex-Post Heterogeneity"

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I A: Proof and Propositions

In this section we provide a detailed analysis of the dynamic programming problem in the general case with two assets. That is, as in Section 5, we assume that a household can save with money and nominal bond, and that only money can be used to finance lumpy consumption. The real rate of return of money holdings is $-\pi \leq 0$, while the real rate of return of nominal bonds is ρ. Anticipating properties of equilibrium, we assume that $\rho \in [-\pi, r)$. The analysis of the pure currency economy corresponds to the special case in which money and bond are perfect substitute, i.e., $\rho + \pi = 0$.

I.1 The Bellman equation

Let c_t flow consumption, h_t flow labor, and y_t lumpy consumption. Let z_t denote the real balances, and ω_t the real wealth of a household at time t (the sum of his real balances and of his real holdings of nominal bonds). The constraint that lumpy consumption must be finance by real balance can be written $y_t \leq z_t$. But notice that, since $\rho \geq -\pi$, the real return on bond weakly dominates that on money. Therefore we can assume without loss of generality that $z_t = y_t$. That is, the real balance of the household at time t is exactly equal to his intended lumpy consumption, and the rest of the household real wealth is invested in nominal bonds.[1]

[1] If $\rho + \pi = 0$, this is only weakly optimal. In fact, in some cases, equilibrium will require that $z_t > y_t$. However, at this stage of the analysis, we are only concerned with deriving elementary properties of the value function and of the total saving function, $\dot{\omega}_t$. Clearly, these do not depend on the particular optimal portfolio choice chosen by the household.

The Bellman equation for a household in our model is the functional equation $T[W] = W$ where:

$$T[W](\omega) = \sup \int_0^\infty e^{-(r+\alpha)t} \left\{ u\left(c_t, \bar{h} - h_t\right) + \alpha \left[U\left(y_t\right) + W\left(\omega_t - y_t\right)\right] \right\} dt, \tag{1}$$

with respect to a left-continuous plans for flow consumption, c_t, flow labor, h_t, lumpy consumption, y_t, a piecewise continuously differentiable plan for real wealth, ω_t, and subject to:

$$\omega_0 = \omega \tag{2}$$

$$0 \leq y_t \leq \omega_t, \tag{3}$$

$$\dot{\omega}_t = h_t - c_t + \rho\left(\omega_t - y_t\right) - \pi y_t + \Upsilon. \tag{4}$$

In what follows we will say that consumption flows, labor flows, and lumpy consumption are feasible if they satisfy the above stated requirements, together with the path of wealth they generate. We will also maintain the assumption that $\bar{h} + \Upsilon > 0$, i.e., when ω is small enough, a household can increase its wealth by working full time and consuming nothing. Let $\mathcal{C}$ denote the set of bounded, positive, continuous, increasing, and concave function of ω. We obtain:

Lemma I.1 *If $W \in \mathcal{C}$ then $T[W] \in \mathcal{C}$.*

Proof. Clearly, if $W \geq 0$ then $T[W] \geq 0$ since this is also true for $u(c, \ell)$ and $U(y)$. If W is bounded then one sees directly from the objective that:

$$|T[W](\omega)| \leq \frac{1}{r+\alpha}\|u\| + \frac{\alpha}{r+\alpha}\left\{\|U\| + \|W\|\right\}, \tag{5}$$

where $\|u\|$ and $\|U\|$ denote the sup norm. It thus follows that $T[W]$ is bounded. We also have that $T[W]$ is increasing if W is increasing: indeed, any plan feasible with initial condition ω_1 is also feasible with initial condition $\omega_2 \geq \omega_1$ and yield higher value since it the household is left with higher wealth at his next lumpy consumption opportunity. Finally, the concavity of $T[W]$ follows directly from the objective being concave and the graph of the constraint correspondence being convex.

The harder part of the proof is to establish that $T[W]$ is continuous. We proceed in two steps. First note that, since $T[W]$ is concave and increasing, it must be continuous over $(0, \infty)$ (see, for example, Corollary 1, Chapter 7, in Luenberger 69). To show continuity at $\omega = 0$, consider some small $\varepsilon > 0$. By working full time, $h_t = \bar{h}$, consuming nothing, and saving only in cash, the household can reach ε at time T_ε solving $\omega_{T_\varepsilon} = \varepsilon$, where $\dot{\omega}_t = \bar{h} + \Upsilon - \pi\omega_t$. Solving this ODE explicitly shows that

$$T_\varepsilon = -\frac{1}{\pi}\log\left(1 - \frac{\pi}{\bar{h} + \Upsilon}\varepsilon\right) = \frac{\varepsilon}{\bar{h} + \Upsilon} + o(\varepsilon). \tag{6}$$

Clearly, since utility flows are bounded below by zero, we must have that $T[W](0) \geq e^{-(r+\alpha)T_\varepsilon} T\left[W(\varepsilon)\right]$, that is $T[W](0)$ is greater than the value of working full time, consuming nothing, saving only in cash until T_ε, and behaving optimally thereafter. This implies in turn that:

$$0 \leq T[W](\varepsilon) - T[W](0) \leq \left(1 - e^{-(r+\alpha)T_\varepsilon}\right) T[W](\varepsilon). \tag{7}$$

Since $T[W]$ is bounded and $T_\varepsilon \to 0$ as $\varepsilon \to 0$, continuity at zero follows. ■

An important result for what follows is:

Lemma I.2 *If $W \in \mathcal{C}$, then $T[W]$ has a bounded derivative at $z = 0$:*

$$\lim_{\varepsilon \to 0} \frac{T[W](\varepsilon) - T[W](0)}{\varepsilon} \leq \frac{r+\alpha}{\bar{h} + \Upsilon}\|W\|,$$

In particular, $T[W]$ is Lipchitz with coefficient $(r+\alpha)\|W\|/\bar{h}$.

Proof. The upper bound follows directly from (6) and (7). The Lipchitz properties follows by concavity. ■

Lemma I.3 *The functional equation $W = T[W]$ has a unique bounded solution, and this solution belongs to $\mathcal{C}$.*

Proof. The Blackwell sufficient conditions for a contraction (see Theorem 3.3 in Stokey Lucas, 89) are satisfied for $T[W]$: monotonicity follows directly because if $W_1(\omega) \geq W_2(\omega)$ for all $\omega \geq 0$, then any feasible plan generates a higher utility with W_1 rather than W_2, implying that $T[W_1] \geq T[W_2]$. Discounting follows directly, with a modulus of contraction $\frac{\alpha}{r+\alpha} < 1$. It then follows from the Contraction Mapping Theorem (see Theorem 3.2 in Stokey Lucas, 89) that $T[W]$ has a unique bounded solution. Lemma I.1 implies that this solution belongs to $\mathcal{C}$. ■

From the arguments in the proofs above, we can derive two auxiliary results:

Corollary I.4 *The solution of the functional equation $T[W] = W$ satisfies:*

$$\begin{aligned} r\|W\| &\leq \|u\| + \alpha\|U\| \\ W'_+(0) &\leq \frac{r+\alpha}{\bar{h}+\Upsilon} W(0) \\ \lim_{\omega \to \infty} W'_-(z) &= 0. \end{aligned}$$

where $W'_-(\omega)$ and $W'_+(\omega)$ denote, respectively, the left- and the right-derivative of $W(\omega)$.

Proof. The first upper bound follows directly from using $\|T[W]\| = \|W\|$ in equation (5). The second one follows directly from Lemma I.2 and from the fact that $T[W] = W$. To show that $\lim_{z\to\infty} W'_-(\omega) = 0$ note that, by concavity, we have that $\omega W'_-(\omega) \leq W(\omega) - W(0)$. But since $W(z)$ is bounded, it follows that $\lim_{\omega\to\infty} W'_-(\omega) = 0$. ■

A useful result is that:

Lemma I.5 *The solution of the functional equation $T[W] = W$ is strictly increasing and has strictly positive left- and right-derivatives.*

Proof. Consider any $\eta > 0$ and some feasible path $\{c_t, h_t, y_t\}$ that achieves at least $W(\omega) - \eta$ starting at ω. Then, for any $\varepsilon > 0$, the value $W(\omega + \varepsilon)$ must be greater than the value of following $\{c_t, h_t, y_t + \varepsilon e^{-\pi t}\}$. That is, one keeps the same plan for flow consumption and labor and save the extra ε initial wealth in cash. The extra saving in cash allows to consume an extra $\varepsilon e^{-\pi t}$ at the first lumpy consumption opportunity. Plugging this into the Bellman equation, we obtain that:

$$\begin{aligned} W(\omega + \varepsilon) &\geq W(\omega) - \eta + \int_0^\infty \alpha e^{-(r+\alpha)t} \left[U\left(y_t + \varepsilon e^{-\pi t}\right) - U\left(y_t\right) \right] \, dt \\ &\geq W(\omega) - \eta + \int_0^T \alpha e^{-(r+\alpha)t} \left[U\left(y_t + \varepsilon e^{-\pi t}\right) - U\left(y_t\right) \right] \, dt, \end{aligned} \tag{8}$$

for any arbitrary horizon T. Now fix some horizon T. It is clear from (4) that the wealth of a household is bounded above by the wealth obtained at time T by consuming nothing, working full time, and saving only in bond for the entire interval. Let us denote this upper bound by $\bar{\omega}$. Given the concavity of $U(y)$, the inequality $y_t < \bar{\omega}$ provides a lower bound for the integral on the right-hand side of (8):

$$W(\omega + \varepsilon) \geq W(\omega) - \eta + \int_0^T \alpha e^{-(r+\alpha)t} \left[U\left(\bar{\omega} + \varepsilon e^{-\pi t}\right) - U\left(\bar{\omega}\right) \right] \, dt.$$

Taking the limit $\eta \to 0$ we obtain that $W(\omega + \varepsilon) > W(\omega)$, i.e., the value function is strictly increasing. It must have strictly positive left- and right-derivatives because it is concave. ■

I.2 The Hamilton Jacobi Bellman Equation

First, let us note that, by standard arguments:

Lemma I.6 *The solution of the functional equation $T[W] = W$ satisfies the maximum principle. For any ω and δ,*

$$W(\omega) = \sup \left\{ \int_0^\delta \left\{ u\left(c_t, \bar{h} - h_t\right) + \alpha \left[U(y_t) + W(\omega_t - y_t) \right] \right\} e^{-(r+\alpha)t} \, dt + e^{-(r+\alpha)\delta} W\left(\omega_\delta\right) \right\}$$

with respect to feasible consumption flow, labor flows, and lumpy consumption.

Our main result is:

Proposition I.7 *For all $\omega \geq 0$ and all $\lambda \in [W'_+(\omega), W'_-(\omega)]$:*

$$(r + \alpha) W(\omega) \leq \sup \left\{ u\left(c, \bar{h} - h\right) + \alpha \left[U(y) + W(\omega - y) \right] + \lambda \left[h - c + \rho\left(\omega - y\right) - \pi y + \Upsilon \right] \right\}, \tag{9}$$

with respect to $c \geq 0$, $h \in \left[0, \bar{h}\right]$, $y \in [0, \omega]$, and with the convention that $W'_-(0) = +\infty$. Moreover, if $W'_+(\omega) = W'_-(\omega)$, then (9) holds with equality.

Notice that, at $\omega = 0$, $W'_-(\omega) = \infty$ so that λ is only restricted to be larger than $W'_+(0)$, and so can be chosen to be arbitrarily large. One can show that this is equivalent to letting $\lambda = W'_+(0)$ and imposing the constraint that the saving function is positive.[2]

Proposition I.7 is a version of the statement that the value function is a viscosity solution of the Hamilton-Jacobi-Bellman equation (HJB). To prove this result we cannot directly apply existing theorems, because these usually assume that the rate of change of the state variable is bounded (for example Assumption A1 in Chapter 3 of Bardi and Capuzzo-Dolcetta 1997). This assumption fails in our model, since a household can in principle choose arbitrarily large consumption flows and so deplete its wealth balance very quickly. Another difference with standard theorems is that we consider an optimization problem for which the agent is making some of its decision at Poisson arrival times, so that the "flow reward" depends on the function whose smoothness we seek to establish, $W(\omega)$.

Preliminary results. To adapt the standard proof, we first establish that, in fact, depleting money balance very quickly cannot be optimal. To see this, consider an agent who consumes at a very high rate during a time interval of length δ, in such a way that its wealth decrease by $k \times \delta$, for some very large k. The utility gain would be bounded by $\|u\|\delta$ but the continuation value would decrease by an amount that is approximately equal to $W'_-(\omega) \times k\delta$. If k is very large, the net utility must be negative. Formally, we show that:

Lemma I.8 *For all $\omega > 0$ and all $\theta > 0$, there is some $k > 0$ such that, for all $\delta > 0$ and any feasible controls c_t h_t and y_t starting at $\omega_0 = \omega$, $\omega_\delta \leq \omega - k\delta$ implies that*

$$W(\omega) > \theta\delta + \int_0^\delta \left\{u\left(c_t, \bar{h} - h_t\right) + \alpha\left[U(y_t) + W(\omega_t - y_t)\right]\right\} e^{-(r+\alpha)t}\, dt + W\left(\omega_\delta\right) e^{-(r+\alpha)\delta}.$$

Proof. Consider any feasible control starting at $\omega_0 = \omega$:

$$\begin{aligned} &W\left(\omega\right) - \int_0^\delta \left\{u\left(c_t, \bar{h} - h_t\right) + \alpha\left[U\left(y_t\right) + W(\omega_t - y_t)\right]\right\} e^{-(r+\alpha)t}\, dt - W\left(\omega_\delta\right) e^{-(r+\alpha)\delta} \\ \geq &W\left(\omega\right) - W\left(\omega_\delta\right) - \delta\left[\|u\| + \alpha\|U\| + \alpha\|W\|\right] \geq \lambda\left(\omega - \omega_\delta\right) - \delta\left[\|u\| + \alpha\|U\| + \alpha\|W\|\right], \end{aligned}$$

for any $\lambda \in \left[W'_+(\omega), W'_-(\omega)\right]$, since W is concave. By Lemma I.5, $\lambda > 0$ and so the result follows by choosing some $k > (\theta + \|u\| + \alpha\|U\| + \alpha\|W\|)/\lambda$. ■

Next, we establish an equi-continuity property for all optimal controls that satisfy $\omega_\delta > \omega_0 - k\delta$.

Lemma I.9 *Consider any $k > 0$ and any $\varepsilon > 0$. Then, there exists some $\delta > 0$ such that, for any control over $[0, \delta]$, starting at ω_0:*

$$\omega_\delta \geq \omega_0 - k\delta \;\Rightarrow\; |\omega_t - \omega_0| \leq \varepsilon \text{ for all } t \in [0, \delta].$$

[2] To see this, write $\lambda = W'_+(0) + \mu$ for some $\mu \geq 0$. The right-side of (9) is greater than the left-side for all $\lambda \geq W'_+(0)$ if and only if the infimum of the right-side of (9) with respect to $\mu \geq 0$ is greater than the left-side. By the saddle point theorem, this infimum is equal to the maximized value of the HJB subject to the constraint that the saving function is positive.

Proof. Since $\rho + \pi \geq 0$, the law of motion for ω_t implies that

$$\dot{\omega}_t \leq \bar{h} + \Upsilon + \gamma \omega_t \tag{10}$$

for some $\gamma > |\rho|$. Direct integration of (10) over $[0, t]$ gives:

$$\omega_t \leq \omega_0 e^{\gamma t} + \frac{\bar{h} + \Upsilon}{\gamma} \left(e^{\gamma t} - 1 \right) \Rightarrow \omega_t - \omega_0 \leq \left(\frac{\bar{h} + \Upsilon}{\gamma} + \omega_0 \right) \left(e^{\gamma t} - 1 \right).$$

Since the right-side is continuous and equal to zero at $t = 0$, it follows that there exists $\delta > 0$ such that $\omega_t - \omega_0 \leq \varepsilon$ for all $t \in [0, \delta]$.

Next we show that $\omega_0 - \omega_t \geq -\varepsilon$. To do so, we integrate (10) over $[t, \delta]$ instead. We obtain:

$$\begin{aligned} &\omega_t \geq \omega_\delta e^{-\gamma(\delta - t)} - \frac{\bar{h} + \Upsilon}{\gamma} \left[1 - e^{-\gamma(\delta - t)} \right] \\ \Rightarrow\ &\omega_t - \omega_0 \geq \left(\omega_\delta - \omega_0 \right) e^{-\gamma(\delta - t)} - \left[\frac{\bar{h} + \Upsilon}{\gamma} + \omega_0 \right] \left[1 - e^{-\gamma(\delta - t)} \right] \geq -k\delta - \left[\frac{\bar{h} + \Upsilon}{\gamma} + \omega_0 \right] \left[1 - e^{-\gamma \delta} \right]. \end{aligned}$$

Again, since the right-hand side is continuous in δ and equal to zero at $\delta = 0$, we obtain that, if we choose $\delta > 0$ small enough, $\omega_t - \omega_0 \geq -\varepsilon$ for all $t \in [0, \delta]$. ■

Proof of the inequality in Proposition I.7. Towards a contradiction, suppose that there is some $\lambda \in \left[W'_+(\omega), W'_-(\omega) \right]$, and some $\theta > 0$ such that:

$$(r + \alpha) W(\omega) > \theta + \sup \left\{ u(c, \bar{h} - h) + \alpha \left[U(y) + W(\omega - y) \right] + \lambda \left[h - c + \rho (\omega - y) - \pi y + \Upsilon \right] \right\} \tag{11}$$

with respect to $c \geq 0$, $h \in [0, \bar{h}]$ and $y \in [0, \omega]$. By continuity, there exists $\varepsilon > 0$ such that this inequality holds for all $\hat{\omega}$ such that $|\omega - \hat{\omega}| \leq \varepsilon$. Given this $\varepsilon > 0$ and the k constructed in Lemma I.8, pick δ according to Lemma I.9 so that $|\omega_t - \omega| \leq \varepsilon$ for all $t \in [0, \delta]$. This implies that, for any feasible control such that $\omega_\delta \geq \omega - k\delta$:

$$(r + \alpha) W(\omega_t) > \theta + u\left(c_t, \bar{h} - h_t \right) + \alpha \left[U(y_t) + W(\omega_t - y_t) \right] + \lambda \left[h_t - c_t + \rho (\omega_t - y_t) - \pi y_t + \Upsilon \right]$$

for all $t \in [0, \delta]$. Now let $\varphi(\hat{\omega}) = W(\omega) + \lambda(\hat{\omega} - \omega)$. By construction, $W(\omega) = \varphi(\omega)$ and by concavity, $\varphi(\hat{\omega}) \geq W(\hat{\omega})$ for all $\hat{\omega}$. In particular, if $\omega = 0$, then this inequality holds for all $\lambda \geq W'_+(0)$. Therefore:

$$\begin{aligned} &\varphi(\omega_\delta) e^{-(r+\alpha)\delta} - W(\omega_\delta) e^{-(r+\alpha)\delta} \geq \varphi(\omega) - W(\omega) = 0 \\ \Leftrightarrow\ &W(\omega) \geq -\int_0^\delta \frac{d}{dt} \left[\varphi(\omega_t) e^{-(r+\alpha)t} \right] dt + W(\omega_\delta) e^{-(r+\alpha)\delta} \\ \Leftrightarrow\ &W(\omega) \geq \int_0^\delta \left\{ (r + \alpha) \varphi(\omega_t) - \varphi'(\omega_t) \dot{\omega}_t \right\} e^{-(r+\alpha)t} dt + W(\omega_\delta) e^{-(r+\alpha)\delta}. \end{aligned}$$

But $\varphi(\omega_t) \geq W(\omega_t)$, $\varphi'(\omega) = \lambda$, and $\dot{\omega}_t = h_t - c_t + \rho(\omega_t - y_t) - \pi y_t + \Upsilon$. Plugging these into the above, substituting in inequality (11), we obtain:

$$W(\omega) \geq \theta\delta + \int_0^\delta \left\{u\left(c_t, \bar{h} - h_t\right) + \alpha\left[U(y_t) + W(\omega_t - y_t)\right]\right\} e^{-(r+\alpha)t}\, dt + W(\omega_\delta)\, e^{-(r+\alpha)\delta}.$$

for any control such that $\omega_\delta \geq \omega - k\delta$. If $\omega > 0$, Lemma I.8 shows that this inequality also holds for any control such that $\omega_\delta \leq \omega - k\delta$. Thus, we can take the suppremum over all feasible controls, and we obtain a contradiction of the maximum principle.

Proof of the equality in Proposition I.7, when the value function is differentiable. In this case the standard proof applies. If $W'_+(\omega) = W'_-(\omega)$, then the value function is differentiable and so $W'(\omega)$ is an element of its sub-differential. Thus, by Lemma 1.7 in Bardi and Capuzzo-Dolcetta (1997) there exists a continuously differentiable $\varphi(\hat{\omega})$ such that $\varphi(\omega) = W(\omega)$ and $\varphi(\hat{\omega}) \leq W(\hat{\omega})$ in a neighborhood of ω. This implies in particular that $\varphi'(\omega) = W'(\omega)$. Thus, all we need to show is that the reverse inequality of (9) holds with $\lambda = \varphi'(\omega)$. Towards a contradiction, assume that there is some $\theta > 0$ and some $(\hat{c}, \hat{h}, \hat{y})$ such that:

$$(r+\alpha)W(\omega) + \theta < u(\hat{c}, \bar{h} - \hat{h}) + \alpha\left[U(\hat{y}) + W(\omega - \hat{y})\right] + \varphi'(\omega)\left[\hat{h} - \hat{c} + \rho(\omega - \hat{y}) - \pi\hat{y} + \Upsilon\right]. \quad (12)$$

Consider the control $c_t = \hat{c}$ and $h_t = \hat{h}$ and $y_t = \min\{\omega_t, \hat{y}\}$. Then by continuity there is some small enough δ such that the inequality continues to hold for all $t \in [0, \delta]$ and $\varphi(\omega_t) \leq W(\omega_t)$. Using the definition of $\varphi(x)$, we also have that:

$$\begin{aligned}
& 0 = W(\omega) - \varphi(\omega) \leq W(\omega_\delta)\, e^{-(r+\alpha)\delta} - \varphi(\omega_\delta)\, e^{-(r+\alpha)\delta} \\
\Leftrightarrow\quad & W(\omega) \leq -\int_0^\delta \frac{d}{dt}\left[\varphi(\omega_t)\, e^{-(r+\alpha)t}\right] dt + W(\omega_\delta)\, e^{-(r+\alpha)\delta} \\
\Leftrightarrow\quad & W(\omega) \leq \int_0^\delta \left[(r+\alpha)\varphi(\omega_t) - \varphi'(\omega_t)\,\dot{\omega}_t\right] e^{-(r+\alpha)t}\, dt + W(\omega_\delta)\, e^{-(r+\alpha)\delta}.
\end{aligned}$$

Now note that $\varphi(\omega_t) \leq W(\omega_t)$, that $\dot{\omega}_t = h_t - c_t + \rho(\omega_t - y_t) - \pi y_t + \Upsilon$, and substitute in (12) to obtain:

$$W(\omega) \leq \int_0^\delta \left\{-\theta + u\left(c_t, \bar{h} - h_t\right) + \alpha\left[U(y_t) + W(\omega_t - y_t)\right]\right\} e^{-(r+\alpha)t}\, dt + W(\omega_\delta)\, e^{-(r+\alpha)\delta},$$

which contradicts the Maximum Principle of Lemma I.6 .

I.3 The derivatives of the value function

In this section we derive a number of results regarding the derivative of the value function. In particular, we show that the value function is continuously differentiable, and twice differentiable almost everywhere over $(0, \infty)$. The key implication of differentiability is that the HJB (9) holds with equality. This leads to simple characterizations of the policy functions.

I.3.1 Preliminary results

For what follows it will be useful to study the following penalized problem, for any $(\omega, \lambda) \in [0, \infty) \times (0, \infty)$:

$$H(\omega, \lambda) \equiv \sup \left\{ u\left(c, \bar{h} - h\right) + \alpha \left[U(y) + W(\omega - y)\right] + \lambda \left[h - c + \rho\left(\omega - y\right) - \pi y + \Upsilon\right] \right\}$$
$$X(\omega, \lambda) \equiv \arg\max \left\{ u\left(c, \bar{h} - h\right) + \alpha \left[U(y) + W(\omega - y)\right] + \lambda \left[h - c + \rho\left(\omega - y\right) - \pi y + \Upsilon\right] \right\}$$

with respect to $c \geq 0$, $h \in [0, \bar{h}]$ and $y \in [0, \omega]$. We obtain two Lemmas about this penalized problem.

Lemma I.10 *Under both SI and linear preferences, the maximized objective, $H(\omega, \lambda)$ is continuous, concave in ω and convex in λ. The the maximum correspondence, $X(\omega, \lambda)$, is compact-valued, upper hemi continuous, non-empty, and convex.*

Proof. We have that $u(c, \bar{h} - h) + \lambda(h - c) \leq \|u\| + \lambda\bar{h} - \lambda c$, which is strictly negative for all $c > \|u\|/\lambda + \bar{h}$. Therefore, any consumption $c > \|u\|/\lambda + \bar{h}$ is dominated by $c = 0$. This implies that we can restrict attention to consumption choices such that $c \leq \|u\|/\lambda + \bar{h}$. Thus an application of the Theorem of the Maximum (see Theorem 3.6 in Stokey and Lucas, 89) shows that the maximum correspondence is compact valued, upper hemi continuous and non empty. The convexity of the set $X(\omega, \lambda)$ follows because the objective is concave and the constrained set convex. The maximized objective is convex in λ because it is the upper envelope of affine functions of λ. It is concave in ω because the objective is concave and, holding λ fixed, the graph of the constraint correspondence is convex. ■

Next, we discuss properties of the problem when the household has SI preferences:

Lemma I.11 *Under SI preferences*

- *The optimal consumption choice, $c(\lambda)$, is strictly decreasing, continuous, and satisfies* $\lim_{\lambda \to 0} c(\lambda) = \infty$, $\lim_{\lambda \to \infty} c(\lambda) = 0$.
- *There is some $\bar{\lambda} > 0$ such that, for all $\lambda \in [0, \bar{\lambda}]$, the optimal labor choice is $h(\lambda) = 0$. For all $\lambda > \bar{\lambda}$, the optimal labor choice is strictly positive, strictly increasing, continuous, with* $\lim_{\lambda \to \bar{\lambda}} h(\lambda) = 0$. *Moreover,* $\lim_{\lambda \to \infty} h(\lambda) = \bar{h}$.
- *The consumption and labor flows, $[c(\lambda), h(\lambda)]$, are continuously differentiable over $(0, \infty)$ except perhaps at $\bar{\lambda}$ where they have left and right-hand side derivatives.*
- *The maximized objective, $H(\omega, \lambda)$, is strictly convex in λ.*

Proof. Given Inada conditions, we must have $c(\lambda) > 0$ and $h(\lambda) < \bar{h}$. The necessary and sufficient first-order conditions are

$$u_c \left[c(\lambda), \bar{h} - h(\lambda)\right] = \lambda$$
$$u_\ell \left[c(\lambda), \bar{h} - h(\lambda)\right] \geq \lambda \text{ with `` = '' if } h(\lambda) > 0.$$

Let $\hat{c}(\lambda)$ denote the solution of $u_c\left[\hat{c}(\lambda), \bar{h}\right] = \lambda$. By strict concavity, it follows that $c(\lambda)$ is strictly decreasing, and using the Inada conditions that $\lim_{\lambda \to 0} \hat{c}(\lambda) = \infty$ and $\lim_{\lambda \to \infty} \hat{c}(\lambda) = 0$. The first-order condition implies that $h(\lambda) = 0$ if and only if $c(\lambda) = \hat{c}(\lambda)$ and

$$u_\ell\left[\hat{c}(\lambda), \bar{h}\right] - \lambda \geq 0.$$

Since $u_{c,\ell}(c, \ell) \geq 0$ and since $c(\lambda)$ is decreasing, the left-hand side is a strictly decreasing function of λ, which is positive when $\lambda \to 0$ and negative when $\lambda \to \infty$. Thus, there exists $\bar{\lambda}$ such that $h(\lambda) = \bar{h}$ if and only if $\lambda \leq \bar{\lambda}$. When $\lambda \in (0, \bar{\lambda})$, the first-order conditions hold with equality, and a direct application of the Implicit Function Theorem implies, after some calculations, that

$$\begin{aligned} c'(\lambda) &= \frac{1}{u_{c,c}} < 0 \\ h'(\lambda) &= 0, \end{aligned}$$

while, for $\lambda \in (\bar{\lambda}, \infty)$:

$$\begin{aligned} c'(\lambda) &= \frac{u_{\ell,\ell} - u_{c,\ell}}{u_{c,c}u_{\ell,\ell} - u_{c,\ell}^2} < 0 \\ h'(\lambda) &= \frac{u_{c,\ell} - u_{c,c}}{u_{c,c}u_{\ell,\ell} - u_{c,\ell}^2} > 0, \end{aligned}$$

where all second derivatives above are evaluated at $[c(\lambda), h(\lambda)]$ and where we used that both c and ℓ are normal goods, which implies that $u_{\ell,\ell} - u_{c,\ell} < 0$. This shows that the consumption and labor flows are, respectively, strictly increasing and decreasing, and continuously differentiable except perhaps at $\bar{\lambda}$ where they have left- and right-derivatives.

To show that $\lim_{\lambda \to \infty} c(\lambda) = 0$, note that $\lambda = u_c\left[c(\lambda), \bar{h} - h(\lambda)\right] \leq u_c\left[c(\lambda), \bar{h}\right]$ since $u_{c,\ell} \geq 0$. Therefore, $\lim_{\lambda \to \infty} u_c\left[c(\lambda), \bar{h}\right] = \infty$ and the result follows from the Inada conditions. Similarly, for $\lambda > \bar{\lambda}$, we have $\lambda = u_\ell\left[c(\lambda), \bar{h} - h(\lambda)\right] \leq u_\ell\left[c(\bar{\lambda}), \bar{h} - h(\lambda)\right]$. Therefore, $\lim_{\lambda \to \infty} u_\ell\left[c(\bar{\lambda}), \bar{h} - h(\lambda)\right] = \infty$, and the result follow from the Inada conditions.

Finally, we show that the maximized objective is strictly convex. The maximized objective can be written as the sum of two functions of λ, $H(\lambda, \omega) = H_1(\lambda) + H_2(\omega, \lambda)$, where

$$\begin{aligned} H_1(\lambda) &\equiv \max_{c,h} \left\{u(c, \bar{h} - h) + \lambda(h - c)\right\} \\ H_2(\omega, \lambda) &\equiv \max_{y \in [0,\omega]} \left\{\alpha\left[U(y) + W(\omega - y)\right] + \lambda\left[\rho(\omega - y) - \pi\omega + \Upsilon\right]\right\}. \end{aligned}$$

Both H_1 and H_2 are convex in λ, since they are the upper envelope of affine functions of λ. To show strict convexity, it is sufficient to show that $H_1(\lambda)$ is strictly convex. This follows because, by an application of the envelope theorem:

$$\frac{\partial H_1}{\partial \lambda} = h(\lambda) - c(\lambda),$$

which is strictly increasing. ■

I.3.2 The first derivative of the value function

Lemma I.12 *Suppose that (9) holds with equality for some $\omega > 0$ and some $\lambda \in [W'_+(\omega), W'_-(\omega)]$. Suppose in addition that there exists some $(c, h, y) \in X(\omega, \lambda)$ such that $h - c + \rho(\omega - y) - \pi y + \Upsilon = 0$. Then the value function is differentiable at ω with*

$$W'(\omega) = \frac{\alpha U'(y)}{r + \alpha + \pi}.$$

Proof. Since the value function is concave, it is differentiable almost everywhere. Consider, then, any $\hat{\omega}$ near ω such that W is differentiable at $\hat{\omega}$. Notice that, for $\hat{\omega}$ close enough to ω, c, h and $y + \hat{\omega} - \omega$ is feasible for the optimization problem defining $H(\hat{\omega}, \lambda)$. Therefore:

$$\begin{aligned}(r+\alpha)W(\hat{\omega}) &\geq u(c, \bar{h} - h) + \alpha\left[U\left(y + \hat{\omega} - \omega\right) + W\left(\omega - y\right)\right] \\ &\quad + W'(\hat{\omega})\left[h - c + \rho\left(\omega - y\right) - \pi\left(y + \hat{\omega} - \omega\right) + \Upsilon\right] \\ &\geq u(c, \bar{h} - h) + \alpha\left[U\left(y + \hat{\omega} - \omega\right) + W\left(\omega - y\right)\right] - \pi W'(\hat{\omega})\left[\hat{\omega} - \omega\right] \\ &\geq u(c, \bar{h} - h) + \alpha\left[U\left(y + \hat{\omega} - \omega\right) + W\left(\omega - y\right)\right] - \pi\left[W(\hat{\omega}) - W(\omega)\right],\end{aligned}$$

where the second line follows from our maintained assumption that $h - c + \rho\left(\omega - y\right) - \pi y + \Upsilon = 0$, and the third line follows from the concavity of $W(\omega)$. At ω, this inequality holds with equality. Subtracting the equality at ω from the inequality at $\hat{\omega}$, we obtain:

$$\left(r + \alpha + \pi\right)\left[W(\hat{\omega}) - W(\omega)\right] \geq \alpha\left[U\left(y + \hat{\omega} - \omega\right) + W\left(\omega - y\right)\right].$$

After dividing by $\hat{\omega} - \omega > 0$ and letting $\hat{\omega} \downarrow \omega$, we obtain that:

$$W'_+(\omega) \geq \frac{\alpha U'\left(y\right)}{r + \alpha + \pi}.$$

Dividing by $\hat{\omega} - \omega < 0$ change the direction of the inequality. Letting $\hat{\omega} \uparrow \omega$, we obtain

$$W'_-(\omega) \leq \frac{\alpha U'\left(y\right)}{r + \alpha + \pi}.$$

Since, by concavity, $W'_+(\omega) \leq W'_-(\omega)$, we obtain that

$$W'_-(\omega) = W'_+(\omega) = \frac{\alpha U'\left(y\right)}{r + \alpha + \pi}$$

as claimed. ∎

Proposition I.13 *The value function is continuously differentiable over* $[0, \infty)$.

Proof. Suppose that there is some $\omega \in (0, \infty)$ such that $W'_+(\omega) < W'_-(\omega)$. Since W is concave, it is differentiable almost everywhere, and so there exists an increasing sequence $\underline{\omega}_n < \omega$ such that $\underline{\omega}_n \to \omega$ and W is differentiable at $\underline{\omega}_n$. Note that, by concavity, $W'(\underline{\omega}_n)$ is decreasing and positive and so has a limit $\overline{\lambda}$. Going to the limit in the Hamilton-Jacobi-Bellman equation of Proposition

I.7, we obtain $(r+\alpha)W(\omega) = H\left(\omega, \bar{\lambda}\right)$, where $H(\omega,\lambda)$ is the function on the right-hand side of the HJB equation. Likewise, there exists a decreasing sequence $\bar{\omega}_n > \omega$ such that $\bar{\omega}_n \to \omega$ and $W(\omega)$ is differentiable at $\bar{\omega}_n$. By concavity, $W'(\bar{\omega}_n)$ is increasing and bounded by $W'_+(0)$, which is finite by Corollary I.4, and so has a limit $\underline{\lambda}$. Going to the limit in the HJB equation, we obtain this time $(r+\alpha)W(\omega) = H\left(\omega, \underline{\lambda}\right)$. We know from Lemma I.10 that $H(\omega,\lambda)$ is convex in λ. Hence it follows that, for all $\lambda \in [\underline{\lambda}, \overline{\lambda}]$, $(r+\alpha)W(\omega) \geq H\left(z, \lambda\right)$. By concavity, $\underline{\lambda} \leq W'_+(\omega) \leq W'_-(\omega) \leq \overline{\lambda}$. Together with the inequality shown in Proposition I.7, this implies that the HJB equation must hold with equality, that is:

$$(r+\alpha)W(\omega) = H(\omega,\lambda)$$

for all $\lambda \in \left[W'_+(\omega), W'_-(\omega)\right]$. With SI preferences, we have reached a contradiction because we know from Lemma I.11 that $H(\omega,\lambda)$ is in fact strictly convex in λ.

With linear preference, we need a little more work before reaching a contradiction. First, we notice that, since $\underline{\lambda} < \overline{\lambda}$, there exist $\lambda_1 < \lambda_2$, both in $[\underline{\lambda}, \overline{\lambda}]$, such that $1 \notin [\lambda_1, \lambda_2]$. Suppose for example that $\lambda_2 < 1$ (the argument when $\lambda_1 > 1$ is symmetric). Then, for all $\lambda \in [\lambda_1, \lambda_2]$, the optimal choice of consumption and labor flow in the HJB is $c = 0$ and $h = \bar{h}$. Hence:

$$H(\omega,\lambda) = \sup_{y \in [0,\omega]} \left\{\alpha\left[U(y) + W(\omega - y)\right] + \lambda\left[\bar{h} + \rho\left(\omega - y\right) - \pi y + \Upsilon\right]\right\} \tag{13}$$

with respect to $y \in [0,\omega]$. An application of the envelope theorem (Theorem 1 in Milgrom and Segal, 2002) implies that

$$H(\omega,\lambda_2) - H(\omega,\lambda_1) = \int_{\lambda_1}^{\lambda_2} \left\{\bar{h} + \rho\left[\omega - y(\lambda)\right] - \pi y(\lambda) + \Upsilon\right\} \, d\lambda,$$

where $y(\lambda)$ is the solution of the optimization program on the right-side of (13). Notice that, by strict concavity, $y(\lambda)$ is indeed uniquely defined. By the Theorem of the Maximum (Theorem 3.6 in Stokey and Lucas, 1989), it is continuous in λ. Now since $H(\omega,\lambda_2) = H(\omega,\lambda_1)$, it thus follows that there evidently exists some $\lambda \in [\lambda_1, \lambda_2]$ such that the integrand is equal to zero, that is $\bar{h} + \rho\left[\omega - y(\lambda)\right] - \pi y(\lambda) + \Upsilon = 0$. But we have reached a contradiction because, according to Lemma I.12, this implies that the value function is differentiable at ω.

Finally, we need to establish that the value function is continuously differentiable. For this, take any $\omega \geq 0$. By concavity, it follows that $W'(\omega) \geq \lim_{\hat{\omega})\downarrow\omega} W'(\hat{\omega}) \equiv W'(\omega_+)$. To obtain the reverse inequality, take any $\omega < \hat{\omega} < \tilde{\omega}$. By concavity, we have that

$$W(\hat{\omega}) \geq \frac{W(\tilde{\omega}) - W(\hat{\omega})}{\tilde{\omega} - \hat{\omega}} \Rightarrow W'(\omega_+) \geq \frac{W(\tilde{\omega}) - W(\omega)}{\tilde{\omega} - \omega} \Rightarrow W'(\omega_+) \geq W'(\omega),$$

where the first inequality follows by letting $\hat{\omega} \downarrow \omega$ and the second inequality follows by letting $\tilde{\omega} \downarrow \omega$. This shows that $W'(\omega_+) = W'(\omega)$. Proceeding similarly to the left of any $\omega > 0$, we obtain that $W'(\omega_-) = W'(\omega)$. ■

Because the value function is continuously differentiable we obtain that:

Corollary I.14 *For all $\omega \geq 0$*

$$(r+\alpha)W(\omega) = \max\left\{u(c, \bar{h}-h) + \alpha\left[U(y) + W(\omega - y)\right] + W'(\omega)\left[h - c + \rho(\omega - y) - \pi y + \Upsilon\right]\right\},$$

with respect to $c \geq 0$, $h \in [0, \bar{h}]$ and $y \in [0, \omega]$. In particular, at $\omega = 0$, the equality holds with or without imposing the positive saving constraint that $h - c + \Upsilon \geq 0$.

Proof. The equality when $\omega > 0$ follows directly from Proposition I.7 and the finding that the value function is differentiable. All we need to show is that the equality holds at $\omega = 0$, with and without imposing the positive saving function constraint. To see this, recall $W'(\omega)$ is continuous and that, according to Lemma I.10, $H(\omega, \lambda)$ is continuous in (ω, λ). Therefore, it follows that $H[\omega, W'(\omega)]$ is continuous in $\omega \geq 0$. Going to the limit as $\omega \to 0$ leads that $(r+\alpha)W(0) = H[0, W'(0)]$, i.e., it is equal to the value of the optimization program on the right-side of HJB without imposing the positive saving constraint. By Proposition I.7, the $(r+\alpha)W(0) \leq \inf_{\lambda \geq W'(0)} H(0, \lambda)$. It thus follows that $\inf_{\lambda \geq W'(0)} H(0, \lambda) = H[0, W'(0)]$. As argued earlier in Footnote 2, taking the infimum over $\lambda \geq W'(0)$ is equivalent to setting $\lambda = W'(0)$ and imposing the constraint that the saving function is positive. ■

Next we show that:

Proposition I.15 *The derivative of the value function, $W'(\omega)$, is strictly decreasing.*

Proof. Suppose that there exists some $a_0 < b_0$ such that $W'(a_0) = W'(b_0) = \lambda_0$. Then, since $W'(z)$ is decreasing, $W'(\omega) = \lambda_0$ for all $\omega \in [a_0, b_0]$. Plugging this back into the HJB we obtain that:

$$(r+\alpha)W(\omega) = \sup\left\{u(c, \bar{h}-h) + \alpha\left[U(y) + W(\omega - y)\right] + \lambda_0\left[h - c + \rho(\omega - y) - \pi y + \Upsilon\right]\right\}$$

with respect to $c \geq 0$, $h \in [0, \bar{h}]$ and $y \in [0, \omega]$.

This implies that $a_0 \geq \bar{\omega}$, where $\bar{\omega}$ solves $\alpha U'(\bar{\omega}) = (r + \pi + \alpha)W'(0)$. Indeed, one easily sees by taking first-order condition that, for all $\omega \leq \bar{\omega}$, $y = \omega$ solves the lumpy consumption problem, i.e. it maximizes $\alpha\left[U(y) + W(\omega - y)\right] - (\rho + \pi)W'(\omega)y$. Hence, if $a_0 < \bar{\omega}$, then for all $\omega \in [a_0, \bar{\omega}]$, the value function satisfies:

$$(r+\alpha)W(\omega) = \sup\left\{u(c, \bar{h}-h) + \alpha\left[U(\omega) + W(0)\right] + \lambda_0\left[h - c - \pi\omega + \Upsilon\right]\right\},$$

with respect to $c \geq 0$ and $h \in [0, \bar{h}]$. Since $U(\omega)$ is strictly concave, then $W(\omega)$ would be strictly concave as well, contradicting the premise that $W'(\omega) = \lambda_0$ for all $\omega \in [a_0, b_0]$.

The first-order condition with respect to y is:

$$\alpha U'\left[y(\omega)\right] - \alpha W'\left[\omega - y(\omega)\right] - \lambda_0(\rho + \pi) - \psi = 0,$$

where ψ is the multiplier for the constraint $y \leq \omega$. An application of the enveloppe theorem (for example Corollary 5 in Milgrom Segal, 2002) shows that

$$(r+\alpha)W'(\omega) = (r+\alpha)\lambda_0 = \alpha W'\left[\omega - y(\omega)\right] + \rho\lambda_0 + \psi.$$

Substituting in the first-order condition to eliminate ψ, we obtain

$$(r+\alpha+\pi)\lambda_0 = \alpha U'(y),$$

which implies that y is constant and equal to $y_0 = (U')^{-1}\left(1+\frac{r+\pi}{\alpha}\right)$ for all $\omega \in [a_0, b_0]$. This implies in turn that the constraint $y \leq \omega$ is slack for all $\omega \in (a_0, b_0]$, that $\psi = 0$ and, from the envelope condition, that:

$$W'(\omega - y_0) = \left(1+\frac{r-\rho}{\alpha}\right)\lambda_0 \equiv \lambda_1,$$

for all $\omega \in (a_0, b_0]$. Clearly, this remains true by continuity at $\omega = a_0$. Thus, we have found a new interval, $[a_1, b_1]$, where $a_1 = a_0 - y_0$ and $b_1 = b_0 - y_0$, such that $W'(\omega) = \lambda_1$. As before $a_1 \geq \bar{\omega}$. By induction, we obtain a decreasing sequence $a_k \geq \bar{\omega}$ such that

$$W'(a_k) = \left(1+\frac{r-\rho}{\alpha}\right)^k \lambda_0.$$

But we know from Lemma I.4 that $W'(z)$ is bounded above. Since $\rho < r$, we have reached a contradiction. ■

I.3.3 The second derivative of the value function

We now offer two propositions about the second derivative of the value function. First, we can characterize the second derivative of the HJB whenever it exists.

Proposition I.16 *The value function is twice differentiable almost everywhere. Whenever $W''(z)$ exists:*

$$(r+\alpha+\pi)W'(\omega) \geq \alpha U'(y) + W''(\omega)\left[h - c + \rho(\omega - y) - \pi z + \Upsilon\right]$$

for all $(c, h, y) \in X[\omega, W'(\omega)]$, and with an equality if $\omega > 0$.

Proof. Since $W'(\omega)$ is decreasing, it is differentiable almost everywhere (see for instance Royden, Chapter 5, Theorem 2). To obtain the relationship shown above, consider some $\omega \geq 0$ and any $(c, h, y) \in X[\omega, W'(\omega)]$. Assume that $W'(\omega)$ admits a right-derivative, that is:

$$W''(\omega_+) = \lim_{\hat{\omega}\downarrow\omega} \frac{W'(\hat{\omega}) - W'(\omega)}{\hat{\omega} - \omega},$$

exists. Then, for all $\hat{\omega} \geq \omega$ and close enough to ω, c, h, and $\hat{y} = y + \hat{\omega} - \omega$ is feasible for the HJB at $\hat{\omega}$. Evaluating the right-hand side of the HJB at this feasible choice, we obtain:

$$(r+\alpha)W(\hat{\omega}) \geq u(c, \bar{h} - h) + \alpha\left[U(y + \hat{\omega} - \omega) + W(\omega - y)\right] + W'(\hat{\omega})\left[h - c + \rho(\omega - y) - \pi(y + \hat{\omega} - \omega)\right],$$

with an equality when $\hat{\omega} = \omega$. Subtracting the equality at ω from the inequality at $\hat{\omega}$, we obtain that:

$$\begin{aligned}(r+\alpha)\left[W(\hat{\omega}) - W(\omega)\right] \geq & \alpha\left[U(y+\hat{\omega}-\omega) - U(y)\right] \\ & + \left[W'(\hat{\omega}) - W'(\omega)\right]\left[h - c + \rho(\omega - y) - \pi y + \Upsilon\right] - \pi W'(\hat{\omega})(\hat{\omega} - \omega).\end{aligned}$$

Dividing by $\hat{\omega} - \omega > 0$ and letting $\hat{\omega} \to \omega$, we obtain the inequality of the proposition. If $\omega > 0$ and $W'(\omega)$ admits a left-derivative at ω, then repeating the same steps with $\hat{\omega} < \omega$ leads to the reverse inequality. ■

Finally, the next proposition derives simple sufficient conditions for the value function to be locally twice continuously differentiable:

Proposition I.17 *Consider any ω such that: (i) $\omega > 0$, (ii) the saving function is not zero and (iii) if preferences are linear, $W'(\omega) \neq 1$. Then $W(\omega)$ is twice continuously differentiable in a neighborhood of ω.*

Proof. Consider any such ω and let $\lambda \equiv W'(\omega)$. With SI preference, $X(\hat{\omega}, \hat{\lambda})$ is always singled valued. With linear preference, given our maintained assumption that $W'(\omega) = \lambda \neq 1$, $X(\hat{\omega}, \hat{\lambda})$ is single-valued in a neighborhood of (ω, λ). Since $\omega > 0$, there exists $\tilde{y}$ such that $0 < \tilde{y} < \hat{\omega}$ for all $(\hat{\omega}, \hat{\lambda})$ in a neighborhood of (ω, λ). Hence, an application of Corollary 5 in Milgrom and Segal (2002) implies that $H(\hat{\omega}, \hat{\lambda})$ is differentiable in a neighborhood of (ω, λ) with

$$\begin{aligned}\frac{\partial H}{\partial \hat{\lambda}} &= c(\hat{\lambda}) - h(\hat{\lambda}) + \rho\left[\hat{\omega} - y(\hat{\omega}, \hat{\lambda})\right] - \pi y(\hat{\omega}, \hat{\lambda}) + \Upsilon \\ \frac{\partial H}{\partial \hat{\omega}} &= \alpha U'\left[y(\hat{\omega}, \hat{\lambda})\right] - \hat{\lambda}\pi\end{aligned}$$

Since the maximum correspondence is single valued and upper hemi continuous, it is continuous, which implies that the partial derivatives are continuous as well.

Now the HJB implies that the equation $-(r+\alpha)W(\hat{\omega}) + H(\hat{\omega}, \hat{\lambda}) = 0$ is solved by $\hat{\lambda} = \lambda$ when $\hat{\omega} = \omega$. The above discussion established that the equation is continuously differentiable with respect to $(\hat{\omega}, \hat{\lambda})$ in a neighborhood of (ω, λ). Moreover, $\frac{\partial H}{\partial \lambda} \neq 0$ at (ω, λ) by our assumption that the saving function is non zero. Hence, an application of the Implicit Function Theorem (for example Theorem 13.7 in Apostol 1974) shows that this equation has a unique solution in some neighborhood of ω, and that this function can be written as a continuously differentiable function of $\hat{\omega}$. But $W'(\hat{\omega})$ also solves this equation and, by continuity, must lie in the same neighborhood of ω for $\hat{\omega}$ close enough to ω. Hence, $W'(\hat{\omega})$ must coincide with the continuously differentiable function obtained by the above application of the Implicit Function Theorem. ■

Note that the proposition does not apply at $\omega = 0$ since $H(\omega, \lambda)$ is not differentiable with respect to ω at that point. Indeed, we will show in the next section that $\lim_{\omega \to 0} W''(\omega) = -\infty$.

I.4 Properties of the policy functions

Because the HJB holds with equality, the optimal lumpy consumption problem can be written:

$$\max \alpha\left[U(y) + W(\omega - y)\right] - (\rho + \pi)W'(\omega)y. \tag{14}$$

with respect to $y \in [0, \omega]$.

Proposition I.18 *The optimal lumpy consumption problem (14) is solved by some unique $y(\omega) > 0$, which is continuous, strictly positive and increasing in $\omega > 0$, with $\lim_{\omega \to \infty} y(\omega) = \lim_{\omega \to \infty} \omega - y(\omega) = \infty$. Moreover, there exists some $\bar{\omega} > 0$ such that $y(\omega) = \omega$ if $\omega < \bar{\omega}$, and only if $\omega < \bar{\omega}$ when $\rho + \pi = 0$.*

Proof. Given that $U(y)$ satisfies Inada conditions but $W(\omega)$ does not, it immediately follows that $y(\omega) > 0$. Moreover, the first-order necessary and sufficient conditions for $y(\omega)$ are:

$$\alpha U'(y) - \alpha W'(\omega - y) - (\rho + \pi) W'(\omega) \geq 0, \text{ with `` = '' if } y < \omega.$$

Since this equation is strictly decreasing in y and increasing in ω, it follows that $y(\omega)$ is an increasing function of ω. Moreover, $\lim_{\omega \to \infty} y(\omega) = \infty$ otherwise, given $\lim_{\omega \to \infty} W'(\omega) = 0$, the first-order condition would not be satisfied for ω large. A similar argument shows that $\lim_{\omega \to \infty} \omega - y(\omega) = +\infty$. Finally, evaluating the first order condition at $y = \omega$ we obtain that:

$$\alpha U'(\omega) - \alpha W'(0) - (\rho + \pi) W'(\omega) \geq \alpha U'(\omega) - \alpha W'(0) - (\rho + \pi) W'(0).$$

The right-side is equal to the left-hand side if $\rho + \pi = 0$, and is strictly positive if and only if $\omega < \bar{\omega} = (U')^{-1} \left[\left(1 + \frac{\rho + \pi}{\alpha}\right) W'(0) \right]$. The result follows. ■

Next, we show that the saving function is strictly positive near zero. This provides the basis for establishing that there exists a monetary equilibrium since, according to Proposition I.18, agents only save in cash near $\omega = 0$.

Proposition I.19 *The saving function is strictly positive and decreasing for all $\omega > 0$ and close enough to zero. It is strictly decreasing with SI preferences, and with linear preferences if $\pi > 0$.*

Proof. From Lemma I.12, we know that if the saving function is zero for some $\omega > 0$, then $W'(\omega) = \frac{\alpha U'[y(\omega)]}{r + \alpha + \pi}$. But $W'(\omega) \leq W'(0)$ and $U'(y) \geq U'(\omega)$. Therefore, we obtain that $\omega \geq (U')^{-1} \left[\left(1 + \frac{r + \pi}{\alpha}\right) W'(0) \right]$. Hence, the saving function is non zero for all $\omega > 0$ close enough to zero. Since $W'(\omega)$ is strictly decreasing, we must have $W'(\omega) \neq 1$ for all $\omega > 0$ close enough to zero. Hence, an application of Proposition I.17 shows that the value function is twice continuously differentiable for all $\omega > 0$ and close enough to zero. Moreover,

$$(r + \alpha + \pi) W'(\omega) = \alpha U' [y(\omega)] + W''(\omega) s(\omega)$$

where $s(\omega)$ is the saving function. Since $U'(y) \geq U'(\omega)$ and $W'(\omega) \leq W'(0)$, we obtain that

$$W''(\omega) s(\omega) \leq (r + \alpha + \pi) W'(0) - \alpha U'(\omega) \to -\infty$$

as $\omega \to 0$. Since $W''(\omega) \leq 0$ by concavity, this clearly implies that $s(\omega) > 0$ for all $\omega > 0$ and close enough to zero.

From Proposition I.18 we know that $y = \omega$ for all ω close enough to zero. Therefore, the saving function can be written:

$$s(\omega) = h \left[W'(\omega) \right] - c \left[W'(\omega) \right] - \pi \omega + \Upsilon.$$

Under SI preferences, the result follows because $h(\lambda) - c(\lambda)$ is strictly increasing, while by $W'(\omega)$ is strictly decreasing by Proposition I.15. Under linear preference, the result follows because $c(\lambda) - h(\lambda)$ is weakly increasing. ■

Finally we note that:

Corollary I.20 $\lim_{\omega \to 0} W''(\omega) = -\infty$.

Proof. This follows from the the upper bound derived above, given that thee saving function is strictly positive near $\omega = 0$ ■

II Continuity with respect to the lump sum transfer parameter

To establish equilibrium existence we need to establish that policy functions are continuous with respect to the the lump sum transfer parameter, Υ. As before, we consider $\bar{h}+\Upsilon > 0$, i.e., we restrict Υ to lie in $(-\bar{h}, \infty)$. In the remainder of this section, we depart from our notations and we are explicit about the dependence with respect to (z, Υ) of the various functions under consideration.

We start with a continuity result for the value function:

Lemma II.1 *The value function W is continuous and increasing in $(z, \Upsilon) \in [0, \infty) \times (-\bar{h}, \infty)$.*

Proof. Consider the fixed point problem for the value function, $W = T[W]$ and note the following.

First, it is clear that any policy that is feasible for (z, Υ) is also feasible for any (z', Υ') such that $z' \geq z$ and $\Upsilon' \geq \Upsilon$. Hence, $T[W]$ is increasing in (z, Υ).

Second, let $T_M[W]$ denote the value of the optimization problem (1) but assuming that the horizon is finite and equal to $H > 0$. Then, since utilities are positive, $T_M[W] \leq T[W]$.

Third, consider any $\varepsilon > 0$. Let M_ε be such that $e^{-(r+\alpha)M_\varepsilon} \left[\|u\| + \alpha \left(\|W\| + \|U\|\right)\right] / (r+\alpha) \leq \varepsilon$. The restriction over $[0, H_\varepsilon]$ of any feasible policy over $[0, \infty)$, is a feasible policy over $[0, M_\varepsilon]$. Therefore, $T[W] \leq T_{M_\varepsilon}[W] + \varepsilon$.

Fourth, by inspection of the proof of Lemma I.2, one sees that the Lipchitz property derived for the infinite horizon problem also holds for the finite horizon problem, with the same Lipchitz constant.

Fifth, in the optimization problem (1) with finite horizon M_ε, all the policies that are feasible starting at z with lump sum transfer $\Upsilon' \geq \Upsilon$ are also feasible starting at $z + \Delta(\Upsilon' - \Upsilon)$ with lump sum transfer Υ, where $\Delta(\Upsilon' - \Upsilon) \equiv (\Upsilon' - \Upsilon)/\pi \left(e^{\pi M_\varepsilon} - 1\right)$. Indeed, the extra real balance allow the household to mimic the lump sum transfer over the finite horizon $[0, H_\varepsilon]$: the household can spend $\Upsilon' - \Upsilon$ every period and run out of the extra real balance $\Delta(\Upsilon' - \Upsilon)$ exactly at time M_ε. This shows that $T_{M_\varepsilon}[W](z, \Upsilon') \leq T_{M_\varepsilon}[W](z + \Delta(\Upsilon' - \Upsilon), \Upsilon)$.

With these remarks in mind, consider any (z, Υ) and (z', Υ'). Let $\overline{\Upsilon} = \max\{\Upsilon, \Upsilon'\}$, $\underline{\Upsilon} = \min\{\Upsilon, \Upsilon'\}$, $\overline{z} = \max\{z, z'\}$ and $\underline{z} = \min\{z, z'\}$. We have

$$
\begin{aligned}
|T[W](z, \Upsilon) - T[W](z', \Upsilon')| &\leq T[W](\overline{z}, \overline{\Upsilon}) - T[W](\underline{z}, \underline{\Upsilon}) \\
&= T[W](\overline{z}, \overline{\Upsilon}) - T[W](\overline{z}, \underline{\Upsilon}) + T[W](\overline{z}, \underline{\Upsilon}) - T[W](\underline{z}, \underline{\Upsilon}) \\
&\leq T_{M_\varepsilon}[W](\overline{z}, \overline{\Upsilon}) - T_{M_\varepsilon}[W](\overline{z}, \underline{\Upsilon}) + \varepsilon + T[W](\overline{z}, \underline{\Upsilon}) - T[W](\underline{z}, \underline{\Upsilon}) \\
&\leq T_{M_\varepsilon}[W](\overline{z} + \Delta(\overline{\Upsilon} - \underline{\Upsilon}), \underline{\Upsilon}) - T_{M_\varepsilon}[W](\overline{z}, \underline{\Upsilon}) + \varepsilon + T[W](\overline{z}, \underline{\Upsilon}) - T[W](\underline{z}, \underline{\Upsilon}).
\end{aligned}
$$

where the first inequality follows from monotonicity, and the second one from the above derived inequalities between the value of the finite and the infinite horizon problems. Now, using the Lipchitz properties of $T[W]$ and $T_H[W]$, we obtain:

$$
|T[W](z, \Upsilon) - T[W](z', \Upsilon')| \leq \frac{r+\alpha}{\bar{h}} \left[\Delta(\overline{\Upsilon} - \underline{\Upsilon}) + \overline{z} - \underline{z}\right] + \varepsilon.
$$

We obtain the continuity result by letting $(z', \Upsilon') \to (z, \Upsilon)$, and then $\varepsilon \to 0$. ■

Next, we consider the optimal lumpy consumption problem. Since $W(z,\Upsilon)$ is continuous in (z,Υ) and since the optimal lumpy consumption problem has a unique solution, an application of the Theorem of the Maximum shows that:

Lemma II.2 *The optimal lumpy consumption, $y(z,\Upsilon)$, is continuous in $(z,\Upsilon) \in [0,\infty) \times (-\bar{h},\infty)$.*

Next, we work on the first derivative of the value function, $W'(z,\Upsilon)$. To do so we first need to prove an intermediate result. Let $\bar{z}(\Upsilon) = \sup\{z \geq 0 \,:\, z - y(z,\Upsilon) = 0\}$. We know that, for all $z > \bar{z}(\Upsilon)$, the first-order condition for optimal lumpy consumption holds with equality, i.e. $U'\left[y(z,\Upsilon)\right] = W'\left[z - y(z,\Upsilon), \Upsilon\right]$. Given that, for fixed Υ, this first-order condition is continuous in z, we obtain by letting $z \to \bar{z}(\Upsilon)$ that:

$$U'\left[\bar{z}(\Upsilon)\right] = W'(0,\Upsilon).$$

Hence, $\bar{z}(\Upsilon)$ is continuous in Υ if and only if $W'(0,\Upsilon)$ is continuous in Υ. We establish this in the following Lemma

Lemma II.3 *The function $W'(0,\Upsilon)$ is continuous in $\Upsilon \in (-\bar{h},\infty)$.*

Proof. We note that, for any Υ, $W'(0,\Upsilon)$ solves the HJB: $W(0,\Upsilon) = H(0,\Upsilon,\lambda)$. Moreover, we know from Proposition I.19 that the saving function is strictly positive at zero, hence it follows from Lemma I.10 that the left-derivative of H with respect to λ is strictly positive, i.e., $H_\lambda(0,\Upsilon,\lambda^+) > 0$ when evaluated at $\lambda = W'(0,\Upsilon)$. Together with the fact that $H(0,\Upsilon,\lambda)$ is convex in λ, this implies that the equation $W(0,\Upsilon) = H(0,\Upsilon,\lambda)$ has at most one other solution, $\hat{\lambda}$, and for this solution $H_\lambda(0,\Upsilon,\hat{\lambda}^+) < 0$. Hence, $W'(0,\Upsilon)$ is the unique solution of the HJB equation satisfying $H_\lambda(0,\Upsilon,\lambda^+) \geq 0$.

Now consider a sequence of $\Upsilon_n \to \Upsilon$, the associated sequence $\lambda_n = W'(0,\Upsilon_n)$, and some $(c_n,h_n) \in X(\lambda_n)$. Since saving functions are strictly positive at $z=0$, we have that $h_n - c_n + \Upsilon_n > 0$. Since the sequence of λ_n is bounded, it has at least one accumulation point, $\lambda^\star$. By continuity, this accumulation point satisfies the HJB equation. Corresponding to this accumulation point, there is an accumulation point $(c^\star,h^\star)$ of the sequence (c_n,h_n) which is, by upper hemi continuity, an element of $X(\lambda)$. By continuity, is satisfies $h^\star - c^\star + \Upsilon \geq 0$. Hence, by Lemma I.10, we obtain that $H_\lambda(0,\Upsilon,\lambda^{\star+}) \geq 0$. By the characterization of the previous paragraph, we conclude that $\lambda^\star = W'(0,\Upsilon)$, and the result follows. ∎

Next, we show:

Lemma II.4 *The function $\varphi(x,\Upsilon)$ solving $z - y(z,\Upsilon) = x$ and $z \geq \bar{z}(\Upsilon)$ is continuous in $(x,\Upsilon) \in [0,\infty) \times (-\bar{h},\infty)$.*

Proof. By construction, this function is the unique function of the pair of equations $z - y(z,\Upsilon) = x$ and $z \geq \bar{z}(\Upsilon)$. To show continuity, consider some $\Upsilon_n \to \Upsilon$, $x_n \to x$, and the corresponding sequence $z_n = \varphi(x_n,\Upsilon_n)$. This sequence is bounded since $z_n \leq x_n$ and x_n is convergent, hence it has at least one accumulation point, $z^\star$. By continuity, we find that $z^\star$ solves $z^\star - y(z^\star,\Upsilon) = 0$ and $z^\star \geq \bar{z}(\Upsilon)$, whose unique solution is $\varphi(x,\Upsilon)$. ∎

Finally, we obtain

Lemma II.5 *The first-derivative of the value function with respect to z, $W'(z, \Upsilon)$, is continuous in $(z, \Upsilon) \in [0, \infty) \times (-\bar{h}, \infty)$.*

Proof. The proof follows from the same reasoning as before. We start from the first-order condition of the optimal lumpy consumption problem, at any $z \geq \bar{z}(\Upsilon)$:

$$U'\left[y(z, \Upsilon)\right] = W'\left[z - y(z, \Upsilon), \Upsilon\right].$$

For any $x \geq 0$, evaluate this first order condition at $\varphi(x, \Upsilon)$. We obtain:

$$U'\left[y(\varphi(x, \Upsilon), \Upsilon)\right] = W'(x, \Upsilon).$$

Since $y(z, \Upsilon)$ and $\varphi(x, \Upsilon)$ are both continuous, the result follows. ■

III Further results: SI preferences with $\pi \geq 0$

We now derive some additional properties of the solution under SI preferences. For simplicity we assume here that $\pi \geq 0$. A non-negative inflation rate simplifies some proof because it implies that the saving function is strictly decreasing, and that the target money balance is finite. But one may expect results to extend beyond $\pi \geq 0$.

III.1 Target real balances

We first study the target real balances.

Lemma III.1 *Under SI preferences, if $\pi \geq 0$, the saving function $s(z, \Upsilon) \equiv h[W'(z, \Upsilon)] - c[W'(z, \Upsilon)] - \pi z + \Upsilon$ is continuous in (z, Υ).*

Proof. We know that $W'(z, \Upsilon)$ is continuous and strictly decreasing. That the saving function is continuous follows from $X(\lambda)$ being singled valued, and hence continuous. That the saving function is strictly decreasing follows $h'(\lambda) < 0$ and $c'(\lambda) > 0$ and $\pi \geq 0$. ■

Now we define the target real balance:

Lemma III.2 *Let the target real balance be*

$$z^\star(\Upsilon) = \inf\{z > 0 \,|\, s(z, \Upsilon) \leq 0\}.$$

Under SI preferences, if $\pi \geq 0$, $z^\star(\Upsilon)$ is strictly positive, finite, and continuous in Υ.

Proof. By Proposition I.19 we have $z^\star > 0$. Moreover, $z^\star$ is finite since $\lim_{z\to\infty} W'(z) = 0$ and $\lim_{\lambda\to 0} c(\lambda) = +\infty$. To prove continuity, consider some Υ and some $\bar{z}$ such that $s(\bar{z}, \Upsilon) < 0$. Because the saving function is continuous, it is negative at $\bar{z}$ for all Υ' close enough to Υ. Since the saving function is decreasing in z, we conclude that $z^\star(\Upsilon) < \bar{z}$ for all Υ' close enough to Υ. Now consider a sequence $\Upsilon_n \to \Upsilon$, and the corresponding sequence $z_n = z^\star(\Upsilon_n)$. By construction $s(z_n, \Upsilon_n) = 0$ and so by continuity $s(z, \Upsilon) = 0$ for any accumulation point of the sequence z_n. We conclude that all accumulation point of z_n must equal $z^\star(\Upsilon)$, since it is the unique solution of $s(z, \Upsilon)$. Because z_n is bounded by $\bar{z}$ for n large enough, it follows that z_n converges to $z^\star(\Upsilon)$. ■

III.2 Twice continuous differentiability

Next we study the second derivative of the value function over $[0, z^\star]$. By Proposition I.17 we obtain that $W(z)$ is twice continuously differentiable over $(0, z^\star)$, and has an infinite second derivative at $z = 0$. The difficulty lies in establishing that it is also twice continuously differentiable at the target real balance, $z^\star$. Twice continuous differentiability at $z^\star$ is a useful regularity property: it implies that the system of ODE satisfied by optimal real balance is continuously differentiable, and so is well behaved everywhere. We proceed in steps:

Lemma III.3 *Under SI preferences, if $\pi \geq 0$, $V(z)$ is twice continuously differentiable over $[0, \hat{z})$, where $\hat{z} > z^\star$ solves $\hat{z} - y(\hat{z}) = z^\star$.*

Proof. Recall that $y(z)$ solves: $U'[y(z)] \geq W'[z - y(z)]$, with an equality if $z \geq \bar{z}$. If $z < \bar{z}$, then $y(z) = z$ and so is clearly continuously differentiable with $y'(z) = 1$. Since $V'(z) = U[y(z)]$, we obtain likewise that $V(z)$ is twice continuously differentiable with $V''(z) = U''(z)$.

If $z > \bar{z}$, then the first-order condition holds with equality. Moreover, since we assume that $z < \hat{z}$, we have that $z - y(z) < z^\star$. Since $W(z)$ is twice continuously differentiable over $[0, z^\star)$, this implies that the first-order condition defines a continuously differentiable and strictly decreasing implicit function for $y(z)$. Hence, we can apply the implicit function theorem and assert that $y(z)$ is continuously differentiable, with derivative:

$$y'(z) = \frac{W''[z - y(z)]}{W''[z - y(z)] + U''[y(z)]}.$$

and, by implication, $V''(z) = U''[y(z)]\, y'(z)$.

Finally, consider $z = \bar{z}$ in case $\bar{z} < \hat{z}$. Since $\lim_{z \to 0} W''(z) = -\infty$ (Proposition I.20), and since $\lim_{z \to \bar{z}} y(z) = \bar{z}$ it follows that $\lim_{z \to \bar{z}^+} y'(z) = 1$. Since we already know that $y'(z) = 1$ for all $z < \bar{z}$, an application of the mean value theorem implies that $y(z)$ is continuously differentiable at $\bar{z}$ with $y'(\bar{z}) = 1$. By implication, $V(z)$ is twice continuously differentiable at $\bar{z}$ with $V''(\bar{z}) = U''(\bar{z})$.
■

Proposition III.4 *Assume SI preferences and $\pi \geq 0$. Then $W(z)$ is twice continuously differentiable over $(0, \infty)$ except perhaps at $z^\star$ where $W''(z)$ has left- and right-limits at $z^\star$ which are negative solutions of the quadratic equations:*

$$\left\{h'\left[W'(z^\star)+\right] - c'\left[W'(z^\star)+\right]\right\} x^2 - (r + \alpha + 2\pi)\, x + \alpha V''(z^\star) = 0$$
$$\left\{h'\left[W'(z^\star)-\right] - c'\left[W'(z^\star)-\right]\right\} x^2 - (r + \alpha + 2\pi)\, x + \alpha V''(z^\star) = 0.$$

In particular, if $W'(z^\star) \neq \bar{\lambda}$, where $\bar{\lambda}$ is the threshold below which $h(\lambda) = 0$, then $W(z)$ is twice continuously differentiable at $z^\star$.

Proof. For $z \neq z^\star$, the result follows from Proposition I.17. The only potential difficulty arises at $z = z^\star$. To address it, consider the initial value problem:

$$\dot{z}_t = h(\lambda_t) - c(\lambda_t) - \pi z_t + \Upsilon$$
$$\dot{\lambda}_t = (r + \alpha + \pi)\lambda_t - \alpha V'(z_t),$$

starting with initial condition $z_0 \neq z^\star$ close enough to $z^\star$, and $\lambda_0 = W'(z_0)$. From Lemma I.11, we know that $h(\lambda) - c(\lambda)$ is continuously differentiable for $\lambda \neq \bar{\lambda}$ and admits left- and right-limit at $\bar{\lambda}$. From Lemma III.3, we know that $V(z)$ is twice continuously differentiable near $z^\star$. Hence the system satisfies Lipchitz conditions so that standard existence and uniqueness theorems for ODEs apply. By Proposition I.16, it is clear that the unique solution of this problem is obtained by solving the initial value problem $\dot{z}_t = h[W'(z_t)] - c[W'(z_t)] - \pi z_t + \Upsilon$, with initial condition z_0, and letting $\lambda(z_t) = W'(z_t)$.

Note that we must have $z_t \neq z^\star$ at all times. Indeed, suppose towards a contradiction that there is some finite time T at which $z_T = z^\star$. Then $\lambda_T = W'(z^\star)$. But note that, by Lemma I.12,

$[z^\star, W'(z^\star)]$ is a stationary point of the above system of differential equations. Since the above system of ODE satisfies standard Lipchitz conditions, it admits a unique solution for any set of initial condition, and so it follows that $z_t = z^\star$ at all times, which is a contradiction. Also, since the saving function is strictly positive for all $z < z^\star$ and strictly negative for all $z > z^\star$, it follows that (z_t, λ_t) converges towards $[z^\star, W'(z^\star)]$ as time goes to infinity.

Next, we study the asymptotic behavior of the above system of ODE near the stationary point. Assume for now that $W'(z^\star) \neq \bar{\lambda}$ so that the system is continuously differentiable at $[z^\star, W'(z^\star)]$. The Jacobian evaluated at $[z^\star, W'(z^\star)]$ is:

$$J = \begin{pmatrix} -\pi & h'[W'(z^\star)] - c'[W'(z^\star)] \\ -\alpha V''(z^\star) & r + \alpha + \pi. \end{pmatrix}$$

Clearly, the determinant of J is strictly negative, implying that J has two non-zero eigenvalues of opposite sign. Therefore, the stationary point $[z^\star, W'(z^\star)]$ is a saddle. By the stable manifold theorem (see Perko, 2001, chapter 2.7), there is a unique trajectory solving the ODE converging to $[z^\star, W^\star(z^\star)]$, the "saddle path". Moreover, this trajectory is tangent to the subspace associated to the negative eigenvalue of J. Formally, let C_1 and C_2 denote eigenvectors associated with the negative and positive eigenvalues of J. Let (y_1, y_2) denote the coordinates of any point x on the basis formed by (C_1, C_2). These coordinates solve $x = Cy$, where $C \equiv [C_1, C_2]$. Then, any solution of the ODE converging to $[z^\star, W(z^\star)]$ must satisfy $y_{2t} = \psi(y_{1t})$, in a neighborhood of the stationary point, for some continuously differentiable function. The tangency condition is that $\psi'(y_1^\star) = 0$, where $(y_1^\star, y_2^\star)$ denote the coordinate of the stationary point. Hence, z_t and λ_t must satisfy, $C_{21}^{-1} z_t + C_{22}^{-1}\lambda_t = \psi\left(C_{11}^{-1} z_t + C_{12}^{-1}\lambda_t\right)$, where C_{ij}^{-1} denote the elements of the matrix C^{-1}. Taking derivatives and rearranging we obtain that:

$$\frac{\dot{\lambda}_t}{\dot{z}_t}\left[C_{22}^{-1} - C_{12}^{-1}\psi'\left(C_{11}^{-1} z_t + C_{12}^{-1}\lambda_t\right)\right] = -C_{21}^{-1} + C_{11}^{-1}\psi'\left(C_{11}^{-1} z_t + C_{12}^{-1}\lambda_t\right),$$

where we used the fact that $\dot{z}_t \neq 0$. Since $C_{22}^{-1} = \det(C)^{-1} C_{11} \neq 0$, and since $\psi'(y_1^\star) = 0$, we obtain that:

$$\lim_{t\to\infty} \frac{\dot{\lambda}_t}{\dot{z}_t} = \lim_{t\to\infty} W''(z_t) = -\frac{C_{21}^{-1}}{C_{22}^{-1}} = \frac{C_{21}}{C_{11}},$$

where we used the fact that $\dot{\lambda}_t = W''(z_t)\dot{z}_t$ and that $C_{21}^{-1} = -\det(C)^{-1}C_{21}$. Clearly, the same result obtains starting from an initial condition $z_0 > z^\star$. Taken together, this gives us that:

$$\lim_{z\to z^\star} W''(z) = \frac{C_{21}}{C_{11}},$$

and so an application of the mean value theorem shows that $W''(z^\star) = C_{21}/C_{22}$. Finally, a straightforward eigenvector calculation leads to the formula of the proposition.

Finally, if $W'(z^\star) = \bar{\lambda}$ the system of ODE is not continuously differentiable at $z^\star$, however the Jacobian has left- and right- limits as $z \to z^\star$. This allows us to solve for the saddle path separately to left and the right of the stationary point, $[z^\star, W'(z^\star)]$. For example, to obtain the left- derivative, we extend $h(\lambda)-c(\lambda)-\pi z+\Upsilon$ to the right of $\bar{\lambda}$ so that it is continuously differentiable at $\bar{\lambda}$. We can then apply the stable manifold theorem just as before, obtain a saddle path that converges to $z^\star$ from the left, which is clearly also a saddle path for the original ODE. The eigenvector calculation provided the left-limit of $W''(z)$ at $z^\star$. To obtain the right-derivative, we proceed similarly to the right of $\bar{\lambda}$. ■

III.3 The time path of real balances starting at $z = 0$

Next, consider the initial value problem of finding a differentiable function $z(t, \Upsilon)$ such that:

$$\begin{aligned} \dot{z}_t(\Upsilon) &= s\left[z_t(\Upsilon), \Upsilon\right] \\ z_0(\Upsilon) &= 0, \end{aligned} \tag{15}$$

where $s(z, \Upsilon) \equiv h\left[W'(z, \Upsilon)\right] - c\left[W'(z, \Upsilon)\right] - \pi z + \Upsilon$ is continuous in $z \geq 0$ and continuously differentiable in $z > 0$. As a result, to construct the solution starting at $z = 0$, we cannot directly use standard existence theorem because $W''(0, \Upsilon) = \infty$ and so $s(z, \Upsilon)$ fails to be Lipchitz with respect to z at $z = 0$. However, we can construct a solution by running the ODE forward and backward starting at some $\hat{z} \in (0, z^\star)$.

The forward solution. The forward solution is defined as the solution of the initial value problem $\dot{z}_{Ft}(\Upsilon) = s\left[z_{Ft}(\Upsilon), \Upsilon\right]$, with some arbitrary initial condition $z_{F0} = \hat{z} \in (0, z^\star)$. From the proof of Proposition III.4, we already know that the forward solution remains less than $z^\star(\Upsilon)$ at all times, and converges towards $z^\star(\Upsilon)$ as time goes to infinity. Moreover, standard results about continuity with respect to parameters (see, for example, Theorem 2.10 in Tikhonov et al., 1980) show that $z_{Ft}(\Upsilon)$ is continuous, since we have shown that $W'(z, \Upsilon)$ is continuous in (z, Υ).

The backward solution. The backward solution is defined as the solution of the initial value problem $\dot{z}_{Bt}(\Upsilon) = -s\left[z_{Bt}(\Upsilon), \Upsilon\right]$ starting at $z_{B0} = \hat{z} \in (0, z^\star)$, where we extend the saving function to be $s(z, \Upsilon) = s(0, \Upsilon)$ for all $z \leq 0$. Since the saving function is positive over $[0, \hat{z}]$ and continuous, it is bounded away from zero by some $\underline{s} > 0$, so that the backwards solution, $z_{Bt}(\Upsilon)$ must reach zero at some finite time $T_B(\Upsilon) \leq \hat{z}/\underline{s}$. Clearly, standard results about continuity with respect to parameters apply for all $t < T_B(\Upsilon)$. We first establish:

Lemma III.5 *The function $T_B(\Upsilon) \equiv \inf\{t \geq 0 \,:\, z_{Bt}(\Upsilon) = 0\}$ is continuous in $\Upsilon \in (-\bar{h}, \infty)$.*

Proof. Fix some $\varepsilon > 0$ and some $\Upsilon \in (-\bar{h}, \infty)$. Since the function $z_{Bt}(\Upsilon)$ is continuous at $t = T_B(\Upsilon)$ by construction, there exists η_1 such that $|t - T_B(\Upsilon)| < \eta_1$ implies that $|z_{Bt}(\Upsilon)| < \varepsilon$. Now set any $t \neq T_B(\Upsilon)$ such that $|t - T_B(\Upsilon)| < \min\{\varepsilon, \eta_1\}$. Since $t \neq T_B(\Upsilon)$, the ODE satisfies the regularity conditions required for the solution to be continuous with respect to the parameter Υ. Namely, there is some η_2 such that $|\Upsilon' - \Upsilon| < \eta_2$ implies that $|z_{Bt}(\Upsilon) - z_{Bt}(\Upsilon')| < \varepsilon$ and, as a result,

$$|z_{Bt}(\Upsilon')| \leq |z_{Bt}(\Upsilon') - z_{Bt}(\Upsilon)| + |z_{Bt}(\Upsilon)| \leq 2\varepsilon.$$

Now since the saving function is bounded below by $\underline{s}$ we have $\underline{s}|t' - t| \leq |z_{Bt'}(\Upsilon') - z_{Bt}(\Upsilon')|$ for any t'. In particular, when $t' = T_B(\Upsilon')$, $z_{Bt'}(\Upsilon') = 0$ so that this inequality becomes:

$$|T_B(\Upsilon') - t| \leq \frac{|z_{Bt}(\Upsilon')|}{\underline{s}} \leq \frac{2\varepsilon}{\underline{s}}.$$

Taken together we obtain that

$$|T_B(\Upsilon) - T_B(\Upsilon')| \leq |T_B(\Upsilon) - t| + |t - T_B(\Upsilon')| \leq \varepsilon \left(1 + 2/\underline{s}\right),$$

for all $|\Upsilon' - \Upsilon| < \eta_2$, since we chose t such that $|t - T_B(\Upsilon)| \leq \varepsilon$. The result follows. ■

Putting the backward and the forward solution together. We now let

$$z_t(\Upsilon) \equiv \begin{cases} z_{BT_B(\Upsilon)-t}(\Upsilon) & t \leq T_B(\Upsilon) \\ z_{Ft-T_B(\Upsilon)}(\Upsilon) & t \geq T_B(\Upsilon). \end{cases}$$

Our main result is:

Proposition III.6 *The function $z_t(\Upsilon)$ is the unique solution of the initial value problem (15), is strictly increasing in time, converges to $z^\star(\Upsilon)$ as $t \to \infty$, and is continuous in (t, Υ).*

Proof. By construction, $z(t, \Upsilon)$ solves the initial value problem (15), is strictly increasing in t and converges to $z^\star(\Upsilon)$ as $t \to \infty$. To establish uniqueness, we note that any solution of (15) must be strictly increasing, and so must be strictly positive for all $t > 0$. Since the ODEs we consider are continuously differentiable for all $z > 0$, their solution is unique given any initial condition, which implies that any solution of the (15) must coincide with $z(t, \Upsilon)$ for any $t > 0$. That it also coincides with $z(t, \Upsilon)$ for $t = 0$ follows by continuity.

For continuity, the only potential difficulty arises at $t = 0$, when the ODE fails to be Lipchitz with respect to z. For this consider some $\varepsilon > 0$ and some neighborhood $[\Upsilon_1, \Upsilon_2]$ of Υ. The uniqueness result shows that the construction of $z(t, \Upsilon)$ does not depend on the particular initial condition $\hat{z}$ chosen for the backward and forward solution. Hence, we are free to pick $\hat{z}$ small enough so that $\hat{z}/\underline{s}|\overline{s} - \underline{s}| < \varepsilon$, where $\underline{s} > 0$ and $\overline{s}$ are, respectively, a lower and an upper bound on the saving function over $(z, \Upsilon) \in [0, \hat{z}] \times [\Upsilon_1, \Upsilon_2]$. Note that this implies that $0 < \hat{z}/\overline{s} \leq T_B(\Upsilon')$. For any $t' \geq \hat{z}/\overline{s}$, we can write:

$$\begin{aligned} |z(t', \Upsilon') - z(0, \Upsilon)| &= |z_B\left[T_B(\Upsilon') - t', \Upsilon'\right] - z_B\left[T_B(\Upsilon), \Upsilon\right]| \\ &\leq |z_B\left[T_B(\Upsilon') - t', \Upsilon'\right] - z_B\left[T_B(\Upsilon') - t', \Upsilon\right]| + |z_B\left[T_B(\Upsilon') - t', \Upsilon\right] - z_B\left[T_B(\Upsilon), \Upsilon\right]| \end{aligned}$$

The first term is less than ε by our choice of $\hat{z}$, given that the two backward solutions $z_B(t, \Upsilon)$ and $z_B(t, \Upsilon')$ can differ by at most $t|\overline{s} - \underline{s}|$ and given that $0 \leq T_B(\Upsilon') - t' \leq T_B(\Upsilon') \leq \hat{z}/\underline{s}$. Turning to the second term, recall that in Lemma III.5 we have shown that $T_B(\Upsilon)$ is continuous. Hence, $T_B(\Upsilon') - t' \to T_B(\Upsilon)$ as $(t', \Upsilon') \to (0, \Upsilon)$. Moreover, for fixed Υ, the backward solution is continuous in time by construction. Hence, the second term is smaller than ε in a neighborhood of $(0, \Upsilon)$. ■

III.4 Time to accumulate balance

Consider some Υ and some $Z > z^\star(\Upsilon)$ such that $Z - y(Z, \Upsilon) < z^\star(\Upsilon)$. Such Z exists since $y(z, \Upsilon) > 0$. By continuity, there is a neighborhood $[\Upsilon_1, \Upsilon_2]$ of Υ such that $Z - y(Z, \Upsilon') < z^\star(\Upsilon')$ for all $\Upsilon' \in [\Upsilon_1, \Upsilon_2]$. Finally for any $x \in [0, Z]$ and $\Upsilon \in [\Upsilon_1, \Upsilon_2]$, consider the time it takes a household to accumulate a x real balance starting from $z_0 = 0$:

$$\mathcal{T}(x, \Upsilon) = \inf\{t \geq 0 \,:\, z_t(\Upsilon) = x\},$$

where $T(x, \Upsilon) = \infty$ if this set is empty. For $x < z^\star(\Upsilon)$, $T(x, \Upsilon)$ is the unique solution of the equation $z_t(\Upsilon) = x$. For $x \geq z^\star(\Upsilon)$, $\mathcal{T}(x, \Upsilon) = \infty$. Since $z_t(\Upsilon)$ is strictly increasing in t and continuous in (t, Υ), we obtain:[3]

Lemma III.7 *The time to accumulate balances, $\mathcal{T}(x, \Upsilon)$, is increasing in x, and continuous in $(x, \Upsilon) \in [0, \infty) \times (-\bar{h}, \infty)$.*

Proof. For (x, Υ) such that $x < z^\star(\Upsilon)$ this follows because the functions $z^\star(\Upsilon)$ and $z_t(\Upsilon)$ are continuous, and because $\mathcal{T}(x, \Upsilon)$ is the unique solution of $z_t(\Upsilon) = x$. For (x, Υ) such that $x > z^\star(\Upsilon)$, this also follows because, for any (x', Υ') close enough to (x, Υ) we have that $x' > z^\star(\Upsilon')$ by continuity, and so $\mathcal{T}(x', \Upsilon') = \infty$. Finally, consider $x = z^\star(\Upsilon)$. We seek to show that, for any sequence (x_n, Υ_n) converging to (x, Υ), $\mathcal{T}(x_n, \Upsilon_n) \to \infty$. Suppose, towards a contradiction, that we can find some sequence such that $\mathcal{T}(x_n, \Upsilon_n)$ is bounded by some M. Then, for all n, $z(M, \Upsilon_n) \geq x_n$. Letting n go to infinity we obtain that $z(M, \Upsilon) \geq x = z^\star(\Upsilon)$, which is a contradiction since $z^\star(\Upsilon)$ is reached in infinite time. ■

Finally, to establish existence and uniqueness of stationary distributions, as well as the existence of equilibrium, we need to establish the following continuity property. For this paragraph, consider some Υ and some $Z > z^\star(\Upsilon)$ such that $Z - y(Z, \Upsilon) < z^\star(\Upsilon)$. Such Z exists since $y(z, \Upsilon) > 0$. By continuity, there is a neighborhood $[\Upsilon_1, \Upsilon_2]$ of Υ such that $Z - y(Z, \Upsilon') < z^\star(\Upsilon')$ for all $\Upsilon' \in [\Upsilon_1, \Upsilon_2]$. Let

$$F(z, z', \Upsilon) = 1 - e^{-\alpha \Delta(z, z', \Upsilon)}, \text{ where } \Delta(z, z', \Upsilon) = \max\left\{\mathcal{T}(z'_+, \Upsilon) - \mathcal{T}[z - y(z, \Upsilon), \Upsilon], 0\right\}.$$

It then follows directly from Lemma III.7 that:

Corollary III.8 *The function $F(z, z', \Upsilon)$ is continuous in $(z, z', \Upsilon) \in [0, Z] \times [0, Z] \times [\Upsilon_1, \Upsilon_2]$.*

III.5 The marginal value of real balance is strictly decreasing in Υ

Our objective in this section is to show:

Proposition III.9 *The marginal value of real balances is strictly decreasing in lump-sum transfer: for any $\Upsilon' > \Upsilon$ and $z \in [0, \infty)$, $W'(z \,|\, \Upsilon') < W'(z \,|\, \Upsilon)$.*

[3] In the Lemma, if $\mathcal{T}(x, \Upsilon) = \infty$, then continuity means that $\lim_{(x', \Upsilon') \to (x, \Upsilon)} \mathcal{T}(x', \Upsilon') = \infty$.

To establish this Proposition, we go back to our basic dynamic programming problem: the one-asset version of the contraction mapping operator we studied in Section I.1.

$$T[f](z \mid \Upsilon) = \sup \mathbb{E}\left[\int_0^\infty e^{-(r+\alpha)t} \left\{u(c_t, \bar{h} - h_t) + \alpha \left[U(y_t) + f(z_t - y_t \mid \Upsilon)\right]\right\} dt\right], \tag{16}$$

with respect to time paths for c_t, y_t, y_t and z_t and subject to:

$$\begin{aligned} \dot{z}_t &= h_t - c_t - \pi z_t + \Upsilon \\ 0 &\leq y_t \leq z_t \\ z_0 &= z, \end{aligned}$$

The argument of our proof goes as follows. We first impose regularity conditions on the continuation value $f(z \mid \Upsilon)$ in the contraction mapping optimization program (16): we assume that $f(z \mid \Upsilon) \in \Phi$, the set of functions $f(z \mid \Upsilon)$ which are concave in $z \in [0, \infty)$, continuously differentiable in $z \in [0, \infty)$, twice continuously differentiable in $z \in (0, \infty)$ except perhaps at one point where the second derivative admits left- and right-limits, such that $\lim_{z \to 0} f''(z \mid \Upsilon) = \infty$, and such that $f'(z \mid \Upsilon)$ is decreasing in Υ. We let $\Phi^\star \subseteq \Phi$ be the set of functions $f(z \mid \Upsilon) \in \Phi$ such that $f'(z \mid \Upsilon)$ is strictly decreasing in Υ.

In Section III.5.1 and Section III.5.2 we use saddle path analysis to establish that $T[\Phi] \subseteq \Phi^\star$. That is: (i) the set Φ of function is stable under the contraction mapping operator T and (ii) the contraction mapping operator maps a continuation value with weakly decreasing marginal value of real balances, into a value with strictly decreasing marginal value of real balances. Since monotonicity properties are preserved by taking limits, value function iteration then shows that the marginal value of real balance, $W'(z \mid \Upsilon)$ is weakly decreasing in Υ. Since we already know from Proposition III.2 that $W(z \mid \Upsilon)$ is continuously differentiable over $[0, \infty)$, and twice continuously differentiable over $(0, \infty)$ except perhaps at one point, this shows that $W \in \Phi$. Applying the contraction mapping once more then shows that $T[W] = W \in \Phi^\star$. Therefore, the marginal value of real balances is strictly decreasing in Υ.

III.5.1 Analysis of the optimization program

We start with the following result:

Lemma III.10 *If $f \in \Phi$, then $T[f]$ is concave in $z \in [0, \infty)$, continuously differentiable in $z \in [0, \infty)$, twice continuously differentiable in $z \in (0, \infty)$ except perhaps at one point where the second derivative admits left- and right-limits, such that $\lim_{z \to 0} f''(z \mid \Upsilon) = \infty$.*

Concavity follows from Lemma I.1 we proved earlier. To obtain the smoothness properties, we study the optimization program (16) using the saddle-path approach described in Section 9, Part II in Kamien and Schwartz (1991).

Optimal lumpy consumption. Let $\hat{z}$ denote the level of real balance at which $W(z)$ is not twice continuously differentiable, if any. We first study the optimal choice of lumpy consumption given the continuation value $W(z)$.

$$\begin{aligned} V(z \mid \Upsilon) &\equiv \max_{0 \le y \le z} U(y) + f(z - y \mid \Upsilon) \\ y(z \mid \Upsilon) &\equiv \arg\max_{0 \le y \le z} U(y) + f(z - y \mid \Upsilon). \end{aligned}$$

The argument of Proposition I.18 imply that $V(y \mid \Upsilon)$ is concave and continuously differentiable over $(0, \infty)$ with $V'(y \mid \Upsilon) = U'[y(z \mid \Upsilon)]$. One also sees from the first-order condition that optimal lumpy consumption is strictly increasing in z. Repeating the the argument in the proof of Lemma III.3 shows that $V(z \mid \Upsilon)$ is continuously differentiable over $(0, \infty)$ except at $\tilde{z}$ such that $\tilde{z} - y(\tilde{z}) = \hat{z}$.

Since $f'(z \mid \Upsilon)$ is decreasing in Υ, one sees that $y(z \mid \Upsilon)$ is weakly increasing in Υ. Indeed if $y(z \mid \Upsilon) = z$, then $U'(z) \ge f'(0 \mid \Upsilon) \ge f'(0 \mid \Upsilon')$, implying that $y(z \mid \Upsilon') = z$. If $y(z \mid \Upsilon) < z$, then $U'[y(z \mid \Upsilon)] \ge f'[z - y(z \mid \Upsilon) \mid \Upsilon] \ge f'[z - y(z \mid \Upsilon) \mid \Upsilon']$, implying that $y(z \mid \Upsilon') \ge y(z \mid \Upsilon)$.

Optimal control via saddle path. The Hamiltonian for the optimization program (16) is $H(z, h, c) = u(c, \bar{h} - h) + \alpha V(z \mid \Upsilon) + \lambda (h - c - \pi z + \Upsilon)$. Following Kamien and Schwartz (1991), the system of ODEs for the state and co-state variables is:

$$\begin{aligned} \dot{z}_t &= h(\lambda_t) - c(\lambda_t) - \pi z_t + \Upsilon \\ (r + \alpha + \pi)\lambda_t &= \alpha V'(z_t \mid \Upsilon) + \dot{\lambda}_t, \end{aligned}$$

where

$$\{c(\lambda), h(\lambda)\} \equiv \arg\max_{c, 0 \le h \le \bar{h}} u(c, \bar{h} - h) + \lambda(h - c).$$

The $\dot{z}_t = 0$ isocline is the function $\lambda \equiv I(z \mid \Upsilon)$ defined implicitly by:

$$h(\lambda) - c(\lambda) - \pi z + \Upsilon = 0.$$

The function $I(z \mid \Upsilon)$, shown in plain green on Figure 1, is strictly increasing in z, goes to infinity as $z \to \frac{\bar{h} + \Upsilon}{\pi}$. The $\dot{\lambda}_t = 0$ isocline is the function $\lambda = J(z \mid \Upsilon)$ defined implicitly by:

$$(r + \alpha + \pi)\lambda = \alpha U'[y(z \mid \Upsilon)].$$

This function $J(z \mid \Upsilon)$ is strictly decreasing in z, goes to infinity as $z \to 0$. One sees easily that the two isoclines have a unique intersection $z^\star > 0$.

Using the arguments in the proof of Proposition III.4 in this Supplementary appendix shows that the system of ODEs defines a unique saddle path, illustrated as the orange curve in Figure 1. This saddle path can be viewed as a strictly decreasing function $z \mapsto \lambda(z \mid \Upsilon)$, which is continuously differentiable over $(0, \infty)$ except perhaps at $z^\star$ where it admits left- and right-limits.[4] Moreover, in Section III.5.4 one easily establishes the following properties:

[4] The lack of differentiability of $\lambda(z \mid \Upsilon)$ arises because the system of ODEs is not continuously differentiable at $(z^\star, \lambda^\star)$ if either $\lambda = \bar{\lambda}$ or $z^\star = \tilde{z}$.

- The saddle path is sandwiched between the isoclines, that is:

$$\begin{aligned} 0 \le z \le z^\star &: \quad I(z \,|\, \Upsilon) \le \lambda(z \,|\, \Upsilon) \le J(z \,|\, \Upsilon) \qquad (17)\\ z > z^\star &: \quad J(z \,|\, \Upsilon) \le \lambda(z \,|\, \Upsilon) \le I(z \,|\, \Upsilon), \qquad (18)\end{aligned}$$

  with strict inequalities for $z \neq z^\star$.

- The saddle path has a finite limit at $z = 0$: $\lim_{z\to 0} \lambda(0 \,|\, \Upsilon) < \infty$.
- The derivative of the saddle path is infinite at zero: $\lim_{z\to 0} \lambda'(z \,|\, \Upsilon) = -\infty$
- The saddle path is the derivative of the maximum attainable utility, that is:

$$\begin{aligned} T[f](z \,|\, \Upsilon) &= T[f](z^\star \,|\, \Upsilon) + \int_{z^\star}^{z} \lambda(x \,|\, \Upsilon)\, dx \\ \text{where } T[f](z^\star \,|\, \Upsilon) &\equiv \frac{u(c^\star, \bar{h} - h^\star) + \alpha V(z^\star)}{r + \alpha}. \end{aligned}$$

III.5.2 The saddle path shifts down with Υ

Next we establish:

Lemma III.11 *Given any continuation value $f \in \Phi$ in the optimization program 16, the marginal value of real balances is strictly decreasing in Υ. That is, for any $\Upsilon' > \Upsilon$ and $z \in [0, \infty)$, $\lambda(z \,|\, \Upsilon') < \lambda(z \,|\, \Upsilon)$.*

Graphically, in the phase diagram show in Figure 1, this means that the saddle path shifts down when Υ increases. Our proof goes in two steps.

Step 1: the saddle path for Υ is strictly above that for Υ' at $z^\star(\Upsilon)$. Indeed suppose that $z^\star(\Upsilon) < z^\star(\Upsilon')$. Using (17), we obtain:

$$\begin{aligned} \lambda\left[z^\star(\Upsilon) \,|\, \Upsilon'\right] &< J\left[z^\star(\Upsilon) \,|\, \Upsilon'\right] \\ &\le J\left[z^\star(\Upsilon) \,|\, \Upsilon\right] \\ &= \lambda\left[z^\star(\Upsilon) \,|\, \Upsilon\right], \end{aligned}$$

where the inequality on the second line follows because $J(z \,|\, \Upsilon)$ is decreasing in Υ, and the equality on the third line follows by definition of $z^\star(\Upsilon)$. Next suppose that $z^\star(\Upsilon) \ge z^\star(\Upsilon')$. Using (18), we obtain

$$\begin{aligned} \lambda\left[z^\star(\Upsilon) \,|\, \Upsilon'\right] &\le I\left[z^\star(\Upsilon) \,|\, \Upsilon'\right] \\ &< I\left[z^\star(\Upsilon) \,|\, \Upsilon\right] \\ &= \lambda\left[z^\star(\Upsilon) \,|\, \Upsilon'\right], \end{aligned}$$

where the inequality on the second line follows because $I(z \,|\, \Upsilon)$ is strictly decreasing in Υ, and the equality on the third line follows by definition of $z^\star(\Upsilon)$.

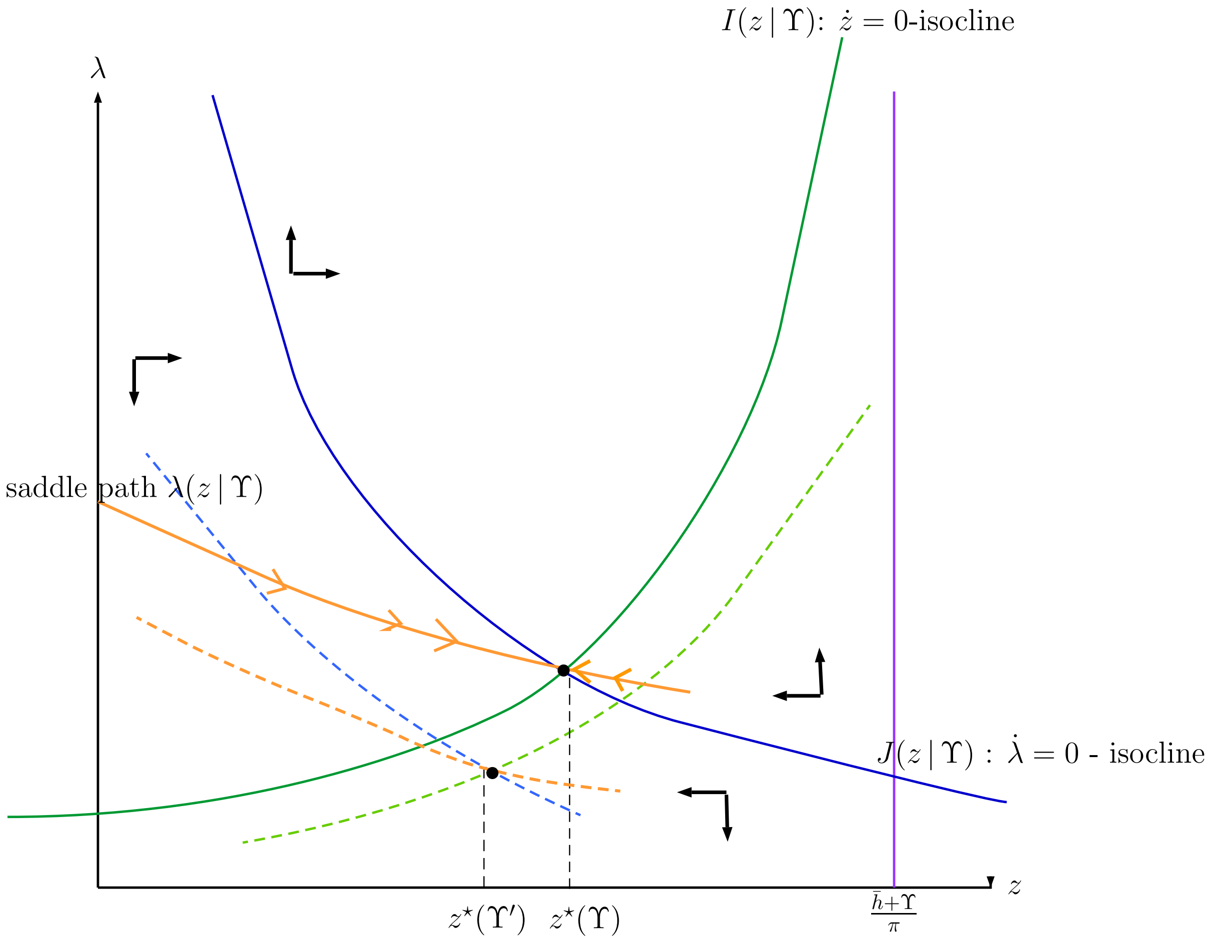

Figure 1: The Phase Diagram

Step 2: the saddle path for Υ is uniformly and strictly above that for Υ'. We already know that this property holds at $z^\star(\Upsilon)$, the saddle path for Υ is above that for Υ'. Now assume, towards a contradiction, that they cross to the left of $z^\star(\Upsilon)$. Then consider largest crossing point less than $z^\star(\Upsilon)$, $\hat{z}$. Clearly, since this is the last point at which the sadde path cross before $z^\star(\Upsilon)$, it has to be the case that the saddle path for Υ' must cross that for Υ from above, that is $\lambda(\hat{z}\,|\,\Upsilon) = \lambda(\hat{z}\,|\,\Upsilon')$ and $\lambda'(\hat{z}\,|\,\Upsilon) \geq \lambda'(\hat{z}\,|\,\Upsilon')$. Letting $\hat{\lambda} \equiv \lambda(\hat{z}\,|\,\Upsilon)$ and using the expression for the derivative of $\lambda(z\,|\,\Upsilon)$, this can be written:

$$\begin{aligned}\frac{(r+\alpha+\pi)\hat{\lambda} - \alpha U'\left[y(\hat{z}\,|\,\Upsilon)\right]}{h(\hat{\lambda}) - c(\hat{\lambda}) - \pi\hat{z} + \Upsilon} &\geq \frac{(r+\alpha+\pi)\hat{\lambda} - \alpha U'\left[y(\hat{z}\,|\,\Upsilon')\right]}{h(\hat{\lambda}) - c(\hat{\lambda}) - \pi\hat{z} + \Upsilon'} \\ &\geq \frac{(r+\alpha+\pi)\hat{\lambda} - \alpha U'\left[y(\hat{z}\,|\,\Upsilon)\right]}{h(\hat{\lambda}) - c(\hat{\lambda}) - \pi\hat{z} + \Upsilon'},\end{aligned}$$

where the inequality on the second line uses that $y(\hat{z}\,|\,\Upsilon') \geq y(\hat{z}\,|\,\Upsilon)$ and that, to the left of $z^\star(\Upsilon)$, we have that $\dot{z}_t > 0$ which can be written as $0 < h(\hat{\lambda}) - c(\hat{\lambda}) - \pi\hat{z} + \Upsilon < h(\hat{\lambda}) - c(\hat{\lambda}) - \pi\hat{z} + \Upsilon'$. Now, to the left of $z^\star(\Upsilon)$, we also have that $\dot{\lambda}_t < 0$, i.e. $(r+\alpha+\pi)\hat{\lambda} - \alpha U'\left[y(\hat{z}\,|\,\Upsilon)\right] < 0$. Therefore, the above inequality implies that $\Upsilon' \leq \Upsilon$, which is a contradiction.

Next, suppose, towards a contradiction, that the two saddle paths cross to the right of $z^\star(\Upsilon)$. Consider the smallest crossing point larger thant $z^\star(\Upsilon)$, $\hat{z}$. Clearly, since this is the first point at which the saddle path cross to the right of $z^\star(\Upsilon)$, it must be that the saddle path for Υ' must cross that for Υ from below. Letting as before $\hat{\lambda} \equiv \lambda(\hat{z}\,|\,\Upsilon)$, this can be written:

$$\begin{aligned}\frac{(r+\alpha+\pi)\hat{\lambda} - \alpha U'\left[y(\hat{z}\,|\,\Upsilon')\right]}{h(\hat{\lambda}) - c(\hat{\lambda}) - \pi\hat{z} + \Upsilon'} &\geq \frac{(r+\alpha+\pi)\hat{\lambda} - \alpha U'\left[y(\hat{z}\,|\,\Upsilon)\right]}{h(\hat{\lambda}) - c(\hat{\lambda}) - \pi\hat{z} + \Upsilon} \\ &\geq \frac{(r+\alpha+\pi)\hat{\lambda} - \alpha U'\left[y(\hat{z}\,|\,\Upsilon')\right]}{h(\hat{\lambda}) - c(\hat{\lambda}) - \pi\hat{z} + \Upsilon},\end{aligned}$$

where the inequality on the second line follows because $y(\hat{z}\,|\,\Upsilon) \leq y(\hat{z}\,|\,\Upsilon')$ and because, to the right of $z^\star(\Upsilon)$, $\dot{z}_t < 0$ so that $h(\hat{\lambda}) - c(\hat{\lambda}) - \pi\hat{z} + \Upsilon < 0$. Now, to the right of $z^\star(\Upsilon)$, we also have that $\dot{\lambda}_t > 0$, which can be written $(r+\alpha+\pi)\hat{\lambda} - \alpha U'\left[y(\hat{z}\,|\,\Upsilon)\right] > 0$. Since $y(\hat{z}\,|\,\Upsilon) \leq y(\hat{z}\,|\,\Upsilon')$, this implies that $(r+\alpha+\pi)\hat{\lambda} - \alpha U'\left[y(\hat{z}\,|\,\Upsilon')\right] > 0$. Combining with the above we find that $\Upsilon' \leq \Upsilon$, another contradiction.

III.5.3 Finishing up the proof

Taking stock of Lemma III.10 and III.11, we obtain the stability result:

Lemma III.12 *If $W(z\,|\,\Upsilon) \in \Phi$, then $T[W](z\,|\,\Upsilon) \in \Phi^\star$.*

Now start the value function iteration with some guess $W^{(0)}(z\,|\,\Upsilon)$ that belongs to Φ and let $W^{(n)}(z\,|\,\Upsilon) \equiv T^n[W^{(0)}](z\,|\,\Upsilon)$ denote the n-th interate. Then, it follows from Lemma III.12 that, $W^{(n)} \in \Phi$ so that for any $z_1 < z_2$ and any $\Upsilon < \Upsilon'$:

$$\frac{W^{(n)}(z_2\,|\,\Upsilon') - W^{(n)}(z_1\,|\,\Upsilon')}{z_2 - z_1} \leq \frac{W^{(n)}(z_2\,|\,\Upsilon) - W^{(n)}(z_1\,|\,\Upsilon)}{z_2 - z_1}. \tag{19}$$

We already know from the analysis of Section I that value function iteration converges to the value of the original optimization problem, i.e. $\lim_{n\to\infty} W^{(n)}(z \mid \Upsilon) = W(z \mid \Upsilon)$. Therefore, going to the limit in the above equation, we obtain:

$$\frac{W(z_2 \mid \Upsilon') - W(z_1 \mid \Upsilon')}{z_2 - z_1} \leq \frac{W(z_2 \mid \Upsilon) - W(z_1 \mid \Upsilon)}{z_2 - z_1}. \tag{20}$$

We also know from the analysis of Section I that the value function is continuously differentiable. This allows us to go to the $z_2 \to z_1$ limit in the above equation, and obtain:

$$W'(z_1 \mid \Upsilon') \leq W'(z_1 \mid \Upsilon).$$

Hence, the marginal value of real balance is decreasing in Υ. But it follows from Proposition III.2 that $W \in \Phi$, so applying Lemma III.12 once again find that $T[W] \in \Phi^\star$. Since $T[W] = W$, this means that the marginal value of real balances is strictly decreasing in Υ.

III.5.4 Omitted proofs

The saddle path is sandwiched between the isoclines. Consider the saddle path to the left of $z^\star$ – the proof for for the saddle path to the right of $z^\star$ is symmetric and is therefore ommitted. Since the saddle path is a strictly decreasing function of z, while the $\dot{z} = 0$-isocline, $I(z \mid \Upsilon)$, is strictly increasing, it is clear that, the the saddle path is always strictly above the isocline for $z < z^\star$: $\lambda^\star(z \mid \Upsilon) > I(z \mid \Upsilon)$. Now turn to the $\dot{\lambda} = 0$ isocline, $J(z \mid \Upsilon)$. From Proposition III.4, we know that the slope of the saddle path near $z^\star$ is given by the negative root of

$$\left\{h'\left[\lambda^\star)+\right] - c'\left[\lambda^\star+\right]\right\} x^2 - (r + \alpha + 2\pi)\, x + \alpha V''(z^\star-) = 0. \tag{21}$$

The slope of the $\dot{\lambda} = 0$-isocline, on the other hand, is:

$$\frac{\alpha}{r + \alpha + \pi} V''(z^\star-) < 0.$$

It is clear that the second order polynomial of (21) is strictly positive when evaluated at $\frac{\alpha}{r+\alpha+\pi} V''(z^\star-)$, which establishes that the slope of the saddle path is strictly larger than the slope of the isocline. Hence, in a neighborhood of $z^\star$, the saddle path is below the $\dot{\lambda} = 0$-osocline. Next assume that the saddle path crosses the $\dot{\lambda} = 0$-isocline at some $z < z^\star$, and consider the last intersection before $z^\star$, $\hat{z}$. Then, it must be that the isocline crosses the saddle path from below, i.e. $J'(z \mid \Upsilon) \geq \lambda^\star(z \mid \Upsilon)$. But this leads to a contradiction because, $J'(\hat{z} \mid \Upsilon) < 0$ and, since $\hat{z}$ belongs to the isocline, $\lambda'(\hat{z} \mid \Upsilon) = 0$.

The saddle path is bounded away from infinity at $z = 0$. Integrating the ODE for λ_t from some $t < 0$ to $t = 0$ gives:

$$\lambda_t = \lambda_0 e^{(r+\alpha+\pi)t} + \alpha \int_t^0 U'\left[y(z_s \mid \Upsilon)\right] e^{-(r+\alpha+\pi)(s-t)}\, ds.$$

The value of the saddle path at $z = 0$ is $\lim_{t\to-\infty} \lambda_t$. The limit as $t \to -\infty$ of the first term is evidently zero. The second term is bounded away from infinity. Indeed, since $\dot{z}_s$ is positive and decreasing over time, we can write

$$\begin{aligned}
0 \le \alpha \int_t^0 U'\left[y(z_s \,|\, \Upsilon)\right] e^{-(r+\alpha+\pi)(s-t)}\, ds \quad &\le \quad \alpha \int_t^0 U'\left[y(z_s \,|\, \Upsilon)\right] \frac{\dot{z}_s}{\dot{z}_0}\, ds \\
&\le \quad \frac{\alpha}{\dot{z}_0} \int_t^0 U'\left[y(z_s \,|\, \Upsilon)\right] \dot{z}_s\, ds \\
&\le \quad \frac{\alpha}{\dot{z}_0} \left\{ U\left[y(z_0 \,|\, \Upsilon)\right] - U\left[y(z_t \,|\, \Upsilon\right]\right\},
\end{aligned}$$

which is bounded away from infinity because $U(y)$ is bounded below.

The derivative of the saddle path is infinity as $z \to 0$. Finally, from the ODE for λ_t, it is clear that $\lim_{t\to-\infty} \dot{\lambda}_t = -\infty$. The ODE for z_t shows that $\lim_{t\to-\infty} \dot{z}_t = h\left[\lambda(0 \,|\, \Upsilon)\right] - c\left[\lambda(0 \,|\, \Upsilon)\right] + \Upsilon \in (0, \infty)$. Therefore $\lim_{z\to 0} \lambda'(z \,|\, \Upsilon) = \lim_{t\to-\infty} \frac{\dot{\lambda}_t}{\dot{z}_t} = -\infty$.

The saddle path is the derivative of the maximum attainable utility. The HJB associated with the control problem is

$$T[f](z) = \max_{c,h,y} \left\{ u(c, \bar{h} - h) + \alpha V(z) + T[f]'(z)\left[h - c - \pi z + \Upsilon\right] \right\},$$

where we omit the Υ argument to simplify notations. Given the saddle path, $\lambda(z)$, we consider the function:

$$(r+\alpha)\mathcal{T}(z) \equiv \mathcal{T}(z^\star) + \int_{z^\star}^{z} \lambda(x)\, dx \text{ where } \mathcal{T}(z^\star) = \frac{u\left[c(\lambda(z^\star)), \bar{h} - h(\lambda(z^\star))\right] + \alpha V(z^\star)}{r+\alpha}.$$

The function $\mathcal{T}(z)$ is continuously differentiable. Using that $\mathcal{T}'(z) = \lambda(z)$, the right-side of the HJB evaluated at $\mathcal{T}'(z)$ is

$$u\left[c(\lambda(z)), \bar{h} - h(\lambda(z))\right] + \alpha V(z) + \lambda(z)\left[h(\lambda(z)) - c(\lambda(z)) - \pi z + \Upsilon\right].$$

It is equal to $(r+\alpha)\mathcal{T}(z^\star)$ at $z = z^\star$. Using the envelope condition, one sees that the derivative of the right-side of the HJB is equal to:

$$\alpha V'(z) - \pi\lambda(z) + \lambda'(z)\left[h(\lambda(z)) - c(\lambda(z)) - \pi z + \Upsilon\right] = (r+\alpha)\lambda(z) = (r+\alpha)\mathcal{T}'(z),$$

where the first equality follows from the ODE for the saddle path. Hence, we obtain that the right-side of the HJB is equal to $(r+\alpha)\mathcal{T}(z)$, establishing that $\mathcal{T}(z)$ satisfies the HJB. A standard optimality verification argument then establishes that $\mathcal{T}(z) = T[f](z)$.

IV Further results: linear preference with $\pi > 0$

IV.1 The target level of real balance

With linear preferences preferences, we define the following three critical levels of real balances:linear

$$
\begin{aligned}
z_1(\Upsilon) &= \inf\left\{z \geq 0 \,:\, W'(z,\Upsilon) \leq 1\right\} \\
z_h(\Upsilon) &= \frac{\bar{h}+\Upsilon}{\pi} \\
z_c(\Upsilon) &= \max\left\{\frac{-\bar{c}+\Upsilon}{\pi}, 0\right\}.
\end{aligned}
$$

Lemma IV.1 *The critical levels $z_1(\Upsilon)$, $z_h(\Upsilon)$ and $z_c(\Upsilon)$ are continuous in $\Upsilon \in (-\bar{h}, \infty)$.*

Proof. This is obvious for $z_h(\Upsilon)$ and $z_c(\Upsilon)$, and the only potential difficulty concerns $z_1(\Upsilon)$. Consider any sequence $\Upsilon_n \to \Upsilon$ and the associated sequence $z_n = z_1(\Upsilon_n)$. Recall that, by concavity $zW'(z,\Upsilon) \leq \|W\|$ implying that z_n is bounded and so it has at least one accumulation point, z. If this accumulation point is strictly positive, then $W'(z_n,\Upsilon_n) = 1$ for all z_n close to the accumulation point, implying by continuity that $W'(z,\Upsilon) = 1$, hence $z = z_1(\Upsilon)$. If the accumulation point is zero, then by continuity we obtain that $W'(0,\Upsilon) \leq 1$, hence $z = z_1(\Upsilon)$. Thus, all accumulation point of z_n are equal to $z_1(\Upsilon)$, implying that $z_n \to z_1(\Upsilon)$. ■

At the first critical level, $z_1(\Upsilon)$, the marginal value of real balance reaches 1. Note that the upper bound of Corollary I.4 implies that $z_1(\Upsilon) = 0$ when Υ is large enough: that is, if the lump transfer is very large, then a household will never find it optimal to supply labor (which, of course, cannot be the basis of an equilibrium otherwise no output would be produced). The second critical level, $z_h(\Upsilon)$, is the stationary point of real balance of a hypothetical household who always works full time. The third critical level, $z_c(\Upsilon)$, is the stationary point of real balance of a hypothetical household who always consume up to its satiation point, $\bar{c}$. If the satiation point is very large, then of course $z_c(\Upsilon) = 0$. Finally, note that $z_h(\Upsilon) > z_c(\Upsilon)$.

With these critical levels in mind, there are two cases to consider. First, if $z_1(\Upsilon) \geq z_c(\Upsilon)$, then a household works full time until its real balance reaches the target $\min\{z_1(\Upsilon), z_h(\Upsilon)\}$. If $z_1(\Upsilon) < z_c(\Upsilon)$, then when the household works full time until its real balance reaches $z_1(\Upsilon)$, and then consume up to its satiation point until its real balances reach the target $z_c(\Upsilon)$. Therefore, in this economy with linear preferences, the target level of real balances is:

$$z^\star(\Upsilon) = \max\left\{z_c(\Upsilon), \min\{z_h(\Upsilon), z_1(\Upsilon)\}\right\} \tag{22}$$

IV.2 An explicit calculation of times to accumulate real balance

We need to consider two cases.

Case 1: if $z_1(\Upsilon) \geq z_c(\Upsilon)$. In this case the target level of real balance is $z^\star(\Upsilon) = \min\{z_1(\Upsilon), z_h(\Upsilon)\} \geq z_c(\Upsilon)$. Then, the ODE of real balance is:

$$\dot{z}_t(\Upsilon) = \begin{cases} \bar{h} - \pi z_t(\Upsilon) + \Upsilon & \text{if } z_t(\Upsilon) < z^\star(\Upsilon) \\ 0 & \text{if } z_t(\Upsilon) = z^\star(\Upsilon). \end{cases}$$

In particular, if $z_h(\Upsilon) \leq z_1(\Upsilon)$, then the saving function is Lipchitz in real balance, and vanishes as real balances approach their target, implying that the target is reached in infinite time. If $z_h(\Upsilon) > z_1(\Upsilon)$, then the saving function remains bounded away from zero as real balance approach their target, and the target is reached in finite time. The time path of real balance has the explicit solution:

$$z_t(\Upsilon) = \min\left\{z^\star(\Upsilon), z_h(\Upsilon)\left(1 - e^{-\pi t}\right)\right\},$$

and the time it takes to reach any x is:

$$\mathcal{T}(x, \Upsilon) = \begin{cases} -\frac{1}{\pi}\log\left(1 - \frac{x}{z_h(\Upsilon)}\right) & \text{if } x \leq z^\star(\Upsilon) \\ +\infty & \text{if } x > z^\star(\Upsilon). \end{cases} \tag{23}$$

Case 2: if $z_1(\Upsilon) < z_c(\Upsilon)$. In this case, when $z = z_1(\Upsilon)$, the saving function is strictly positive even when household consume up to their satiation point, $\bar{c}$. Hence, household continue to accumulate balances until they reach the stationary point $z_c(\Upsilon) = z^\star(\Upsilon)$. In this case, $z^\star(\Upsilon) = z_c(\Upsilon)$ and the ODE of real balance is:

$$\dot{z}_t(\Upsilon) = \begin{cases} \bar{h} - \pi z_t(\Upsilon) + \Upsilon & \text{if } z_t(\Upsilon) < z_1(\Upsilon) \\ -\bar{c} - \pi z_t(\Upsilon) + \Upsilon & \text{if } z_1(\Upsilon) \leq z_t(\Upsilon) \leq z_c(\Upsilon). \end{cases}$$

Note that, in this case, the stationary point, $z^\star(\Upsilon)$, is reached in infinite time. Solving for the time path of real balance as before, we obtain that the time to reach any x is:

$$\mathcal{T}(x, \Upsilon) = \begin{cases} -\frac{1}{\pi}\log\left(1 - \frac{\min\{x, z_1(\Upsilon)\}}{z_h(\Upsilon)}\right) - \frac{1}{\pi}\log\left(1 - \frac{\max\{x - z_1(\Upsilon), 0\}}{z_c(\Upsilon) - z_1(\Upsilon)}\right) & \text{if } x \leq z^\star(\Upsilon) \\ +\infty & \text{if } x > z^\star(\Upsilon). \end{cases} \tag{24}$$

IV.3 A continuity property

To establish existence and uniqueness of stationary distributions, as well as the existence of equilibrium, we need to establish the following continuity property. For this subsection, consider some Υ and some $Z > z^\star(\Upsilon)$ such that $Z - y(Z, \Upsilon) < z^\star(\Upsilon)$. Such Z exists since $y(z, \Upsilon) > 0$. By continuity, there is a neighborhood $[\Upsilon_1, \Upsilon_2]$ of Υ such that $Z - y(Z, \Upsilon') < z^\star(\Upsilon')$ for all $\Upsilon' \in [\Upsilon_1, \Upsilon_2]$. Let

$$F(z, z', \Upsilon) = 1 - e^{-\alpha\Delta(z, z', \Upsilon)}$$

where $\Delta(z, z', \Upsilon) = \max\left\{\mathcal{T}(z'_+, \Upsilon) - \mathcal{T}[z - y(z, \Upsilon), \Upsilon], 0\right\}$. Notice that $\Delta(z, z', \Upsilon) = +\infty$ for all $z' \geq z^\star(\Upsilon)$. We have:

Lemma IV.2 *Consider some* $(z, z', \Upsilon) \in [0, Z] \times [0, Z] \times [\Upsilon_1, \Upsilon_2]$ *such that* z' *is a continuity point of* $z' \mapsto F(z, z', \Upsilon)$*. Then, for any sequence* $(z_n, \Upsilon_n) \to (z, \Upsilon)$*,* $F(z_n, z', \Upsilon_n) \to F(z, z', \Upsilon)$*.*

Proof. One sees easily that the function $z' \mapsto F(z, z', \Upsilon)$ is continuous at z' except at the target when it is reached in finite time, i.e., except at $z' = z^\star(\Upsilon)$ when $\mathcal{T}[z^\star(\Upsilon), \Upsilon] < \infty$. With this in mind we proceed to analyze three cases.

Case 1: if $z' > z^\star(\Upsilon)$. Then $F(z, z', \Upsilon) = 1$. Moreover, by Lemma IV.1, we obtain by continuity that $z' > z^\star(\Upsilon_n)$ for n large enough. Since $z_n - y(z_n, \Upsilon_n) < z^\star(\Upsilon_n)$ by our choice of Z, this implies that $F(z_n, z', \Upsilon_n) = 1$, hence that $F(z_n, z', \Upsilon_n) \to F(z_n, z', \Upsilon)$.

Case 2: if $z' = z^\star(\Upsilon)$ and $\mathcal{T}[z^\star(\Upsilon), \Upsilon] = \infty$. Consider first that $z_1(\Upsilon) \geq z_c(\Upsilon)$, then it must be the case that $z^\star(\Upsilon) = z_h(\Upsilon) > z_c(\Upsilon)$, since otherwise, $z^\star(\Upsilon)$ would be equal to $z_c(\Upsilon)$ and would be reached in finite time. Moreover, from (22), it must also be the case that $z_1(\Upsilon) \geq z_h(\Upsilon)$. Taken together, this implies that $z_1(\Upsilon) > z_c(\Upsilon)$, a strict inequality that must be satisfied for n large enough, by the continuity result of Lemma IV.1. It then follows that, for n large enough, $\Delta(z_n, z', \Upsilon_n)$ is either equal to ∞, or to:

$$-\frac{1}{\pi} \log \left(1 - \frac{z_h(\Upsilon)}{z_h(\Upsilon_n)}\right) + \frac{1}{\pi} \log \left(1 - \frac{z_n - y(z_n, \Upsilon_n)}{z_h(\Upsilon_n)}\right).$$

keeping in mind that $z_n - y(z_n, \Upsilon_n) < Z - y(Z, \Upsilon_n) \to Z - y(Z, \Upsilon) < z_h(\Upsilon)$, one sees that this expression goes to infinity as n goes to infinity. Hence, $F(z_n, z', \Upsilon_n) \to 1 = F(z, z', \Upsilon)$.

Second, consider that $z_1(\Upsilon) < z_c(\Upsilon)$. Then, by (22), $z^\star(\Upsilon) = z_c(\Upsilon)$. In this case the same reasoning as above, but based on the formula (24), shows that $F(z_n, z', \Upsilon_n) \to F(z, z', \Upsilon)$.

Case 3: if $z' < z^\star(\Upsilon)$. If $z_1(\Upsilon) \neq z_c(\Upsilon)$, the result follows by continuity using the explicit formula (23) and (24). If $z_1(\Upsilon) = z_c(\Upsilon)$ then $z^\star(\Upsilon) = z_1(\Upsilon) = z_c(\Upsilon)$, and so our maintained assumption implies that $z' < z^\star(\Upsilon) = z_1(\Upsilon)$. By construction, we also have that $z - y(z, \Upsilon) < z^\star(\Upsilon) = z_1(\Upsilon)$. By the continuity result of Lemma IV.1, these inequalities hold for n large enough. Hence, for n large enough, $\mathcal{T}(x, \Upsilon_n)$ is given by formula (23), both for $x = z'$ and $x = z_n - y(z_n, \Upsilon_n)$. It then follows by continuity that $F(z_n, z', \Upsilon_n) \to F(z, z', \Upsilon)$. ∎

V Equilibrium

V.1 Stationary distribution

Fix some Υ and some $Z > z^\star(\Upsilon)$ such that $Z - y(Z, \Upsilon) < z^\star(\Upsilon)$. Such Z exists since $y(z, \Upsilon) > 0$. By continuity, there is a neighborhood $[\Upsilon_1, \Upsilon_2]$ of Υ such that $Z - y(Z, \Upsilon') < z^\star(\Upsilon')$ for all $\Upsilon' \in [\Upsilon_1, \Upsilon_2]$. Let

$$\Delta(z, z', \Upsilon) = \max\left\{T(z'_+, \Upsilon) - T(z - y(z, \Upsilon), \Upsilon), 0\right\}$$

that we introduced before. In words, the function $\Delta(z, z', \Upsilon)$ gives the time it takes to accumulate strictly more than z' after receiving a lumpy consumption opportunity with real balance z. Equipped with the function $\Delta(z, z', \Upsilon)$, we define the transition probability between money balances prior to the last consumption opportunity and current money balances. Namely, consider a household at time u and let $\tau_u \in (-\infty, u)$ denote its last lumpy consumption opportunity. Then, consider the probability that the current money balance is less than z', conditional on having z money balance at the last last lumpy consumption opportunity:

$$F(z, z', \Upsilon) = \mathbb{P}\left\{z(u) \leq z' \,|\, z(\tau_u) = z\right\} = 1 - e^{-\alpha\Delta(z,z',\Upsilon)}.$$

Notice that $F(z, z', \Upsilon) = 1$ for all $z \in [z^\star(\Upsilon), Z]$. Collecting the results of Corollary III.8 and Lemma IV.2, we obtain:

Proposition V.1 *With SI and linear preferences, the function $F(z, z', \Upsilon)$ has the following properties:*

- *it is continuous in z' except if $z' = z^\star(\Upsilon)$ and $\mathcal{T}[z'(\Upsilon), \Upsilon] < \infty$ when it is right-continuous;*
- *it is decreasing in z and increasing in z';*
- *for any $(z, z', \Upsilon) \in [0, Z] \times [0, Z] \times [\Upsilon_1, \Upsilon_2]$ such that z' is a continuity point of $z' \mapsto F(z, z', \Upsilon)$, and for any sequence $(z_n, \Upsilon_n) \to (z, \Upsilon)$, $F(z_n, z', \Upsilon_n) \to F(z, z', \Upsilon)$.*

For fixed z and Υ, the function $z' \mapsto F(z, z', \Upsilon)$ is increasing, right-continuous, equal to zero at $z' = z - y(z, \Upsilon)$ and equal to one for $z > z' = z^\star$. By Theorem 12.7 in SLP, it thus defines a unique probability measure $Q(z, \cdot, \Upsilon)$ on $[0, Z]$ equipped with the Borel σ-algebra, $\mathcal{B}([0, Z])$. To apply the results of Chapter 12 in SLP, we first show that:

Lemma V.2 *For any $A \in \mathcal{B}([0, Z])$, $z \mapsto Q(z, A, \Upsilon)$ is measurable.*

Proof. Clearly, the property holds for all sets of the form $[0, b)$ and $(a, b]$, for $0 \leq a \leq b \leq Z$, and for the union of any finite and disjoint collection of such sets, the family of which is by Exercise 7.6 in SLP, an algebra generating $\mathcal{B}([0, Z])$. By an application of the monotone convergence theorem (Theorem 7.8) one sees that the collection of sets A such that $Q(z, A, \Upsilon)$ is measurable is a monotone class. Thus, by the monotone class Lemma (Lemma 7.15), it follows that $Q(z, A, \Upsilon)$ is measurable for any $A \in \mathcal{B}([0, Z])$. ■

A stationary distribution of money holding is a solution of the fixed point problem:

$$\lambda = T^{\star}\left[\lambda, \Upsilon\right] \text{ where } T^{\star}\left[\lambda, \Upsilon\right](A) = \int_0^Z Q(z, A, \Upsilon)\lambda(dz), \text{ for all } A \in \mathcal{B}\left([0, Z]\right). \tag{25}$$

The transition probability function $Q(z, \cdot, \Upsilon)$ has one key property: it is monotone, in the sense that a higher z leads to a higher distribution of current money balance, in the sense of first-order stochastic dominance. This follows directly from the observation that $F(z, z', \Upsilon)$ is decreasing in z: a household with higher money balance at its last consumption opportunity will tend to have a higher current money balance. Then a direct application of results in SLP delivers:

Proposition V.3 *The fixed point problem (25) has a unique solution, $\lambda^{\star}(\Upsilon)$, with the following properties:*

- *its support is included in $[0, z^{\star}(\Upsilon)]$;*
- *it does not depend on Z;*
- *it is continuous in Υ in the sense of weak convergence.*

Proof. Monotonicity is shown in the paragraph above. The Feller property follows from the third point of Proposition V.1 , together with point b in exercise 12.7, and Theorem 12.8. Next, we verify the mixing condition, Assumption 12.1. For this we let $a = 0$, $b = z^{\star}(\Upsilon)$, and $c = \left[z^{\star}(\Upsilon) + z^{\star}(\Upsilon) - y(z^{\star}, \Upsilon)\right]/2$. Then, $Q\left(z^{\star}, [0, c], \Upsilon\right) = F(z^{\star}, c, \Upsilon) > 0$ since $c > z^{\star}(\Upsilon) - y[z^{\star}(\Upsilon), \Upsilon]$. Moreover, and $Q\left(0, [c, z^{\star}(\Upsilon)], \Upsilon\right) = 1 - F(0, c, \Upsilon) > 0$ since $c < z^{\star}(\Upsilon)$. It thus follows from Theorem 12.12, that there exists a unique stationary distribution, $\lambda(\Upsilon)$. That the support is included in $[0, z^{\star}(\Upsilon)]$ follows because $Q(z, A, \Upsilon) = 0$ for any $A \subseteq (z^{\star}(\Upsilon), Z]$. Given that the support of $\lambda^{\star}$ and $Q(z, \cdot, \Upsilon)$ are all included in $[0, z^{\star}(\Upsilon)]$, it is clear that the stationary distribution does not depend on the particular Z used for its construction: a fixed point for some Z remains a fixed point for $Z' \neq Z$, and so must coincide with the fixed point for Z' by uniqueness. Finally, continuity in Υ follows from the third point of Proposition V.1, together with Theorem 12.13 in SLP. ∎

V.2 Existence of Equilibrium

We proceed to establish that an equilibrium exists. The equilibrium equation can be written as a fixed point problem in the space of real lump-sum transfers, Υ.

$$\Upsilon = \pi \int_0^{\infty} z \, d\lambda^{\star}(z, \Upsilon), \tag{26}$$

where $\lambda^{\star}(\cdot, \Upsilon)$ is the stationary distribution of real balance. We obtain:

Proposition V.4 *Under SI or linear preference, if $\pi > 0$, the equilibrium fixed point equation has at least one solution.*

Proof. First, we note that the stationary distribution cannot be concentrated at $z = 0$, since $Q(z, \{0\}, \Upsilon) = 0$ for all z. Hence, when $\Upsilon = 0$, the left-hand side of (26) is zero and so is less than the right-hand side, which is strictly positive. When $\Upsilon \to \infty$, we have that $W'(z, \Upsilon) \to 0$ for all $z \in [0, \infty)$. This implies that labor supply is zero and consumption is strictly positive for all $z \in [0, z^\star(\Upsilon)]$, hence the saving function is $s(z) < -\pi z + \Upsilon$. Plugging $s\,[z^\star(\Upsilon)] = 0$, it follows that $z < z^\star(\Upsilon) < \Upsilon/\pi$ for all real balance z in the support of the stationary distribution, $\lambda^\star(\Upsilon)$, implying that the right-hand side of (26) is less than the left-hand side. Finally, note that (26) is continuous because, by Proposition V.3, the stationary distribution $\lambda^\star(\Upsilon)$ is continuous in the sense of weak convergence. The result then follows by an application of the intermediate value theorem. ■

V.3 Further results about the equilibrium with linear preferences

We start with the following observation:

Lemma V.5 *In equilibrium, with linear preferences, $z_c < z_1$.*

Proof. Since it takes time to accumulate money balance, we must have $\lambda^\star(\{z^\star\}) < 1$. Together with the market-clearing condition, $\Upsilon = \pi \int z d\lambda^\star(z)$, this implies that $\Upsilon < \pi z^\star$. Since $\Upsilon > \pi z_c$, we obtain that $z_c < z^\star$. Next, note that, since $z^\star$ is a stationary level of money balance, there must be some $(c^\star, h^\star) \in X\,[W'(z^\star)]$ such that $0 = h^\star - c^\star - \pi z^\star + \Upsilon$. Given that $\Upsilon > \pi z^\star$, it follows that $h^\star - c^\star > 0$. With linear preferences, this implies that $W'(z^\star) \geq 1$ and so that $z_1 \geq z^\star$. The result follows. ■

From this observation it follows that:

Lemma V.6 *In equilibrium, with linear preferences, there exists some $\hat{z} > z^\star$ such that the value function $W(z)$ is twice continuously differentiable over $(0, \hat{z}]$, with second derivative*

$$W''(z) = \begin{cases} \frac{(r+\alpha+\pi)W'(z) - \alpha V'(z)}{\bar{h} - \pi z + \Upsilon} & \text{if } z \neq z^\star \\ 0 & \text{if } z = z^\star \text{ and } z^\star = z_1 \\ \frac{\alpha V''(z^\star)}{r+\alpha+2\pi} & \text{if } z = z^\star \text{ and } z^\star = z_h. \end{cases}$$

Proof. There are two cases to consider. First, suppose that $z^\star = z_1 < z_h$. From Proposition **??**, we already know that $W(z)$ is twice continuously differentiable over $(0, z_1)$ and (z_1, z_h) and that its second derivative satisfies:

$$(r + \alpha + \pi)W'(z) = \alpha V'(z) + W''(z) \times \begin{cases} \bar{h} - \pi z + \Upsilon & \text{if } z \in (0, z_1) \\ -\bar{c} - \pi z + \Upsilon & \text{if } z \in (z_1, z_h). \end{cases} \tag{27}$$

The only potential difficulty arises at z_1. From Lemma **??**, we know that $(r + \alpha + \pi)W'(z_1) = \alpha V'(z_1)$. Given that $z_1 \neq \{z_c, z_h\}$, both $\bar{h} - \pi z_1 + \Upsilon \neq 0$ and $-\bar{c} - \pi z_1 + \Upsilon \neq 0$, and so it follows by taking the limit $z \to z_1^-$ and $z \to z_1^+$ in (27) that $W''(z_1^-) = W''(z_1^+) = 0$. An application of the mean value theorem then implies that $W'(z)$ is continuously differentiable at z_1 with $W''(z_1) = 0$.

The second case is when $z^\star = z_h \leq z_1$. As in the previous case, we know from Proposition I.17 that $W(z)$ is twice continuously differentiable over $(0, z^\star)$. Then, one can apply the same saddle-path arguments as in Lemma III.3 and Proposition III.4 and obtain that $W(z)$ is twice continuously differentiable over $(0, \hat{z})$, for some $\hat{z} > z^\star$, with the claimed second derivative. ■

Lemma V.7 *In equilibrium, with linear preferences, $W(z)$ is independent of $\bar{c}$ over $[0, z^\star]$.*

Proof. Consider the value functions $W(z, \bar{c}_1)$ obtained for some $\bar{c}_1$. By direct integration of the HJB, one easily that in the optimization program (1), the maximum is achieved by setting $c_t = 0$ for all t, $h_t = \bar{h}$ until $z_t = z^\star(\bar{c}_1)$, $h_t = \pi z^\star(\bar{c}_1) - \Upsilon$ when $z_t = z^\star$, and $y_t = y\,[z_t]$.

Now consider any $\bar{c}_2 > \bar{c}_1$, and suppose that the optimization program (1) attains a strictly higher value than under $\bar{c}_1$, given $W(z, \bar{c}_1)$. That is, there exists a feasible plan $\{\hat{c}_t, \hat{h}_t, \hat{y}_t, \hat{z}_t\}$ such that:

$$W(z, \bar{c}_1) < \int_0^\infty e^{-(r+\alpha)t} \left\{ \min\{\hat{c}_t, \bar{c}_2\} + \bar{h} - h_t + \alpha \left\{ U\left[\hat{y}_t\right] + W\left[\hat{z}_t - \hat{y}_t, \bar{c}_1\right] \right\} \right\} dt. \tag{28}$$

Note that we can without loss of generality assume that $\hat{c}_t \leq \bar{c}_2$. Indeed, replacing $\hat{c}_t$ by $\min\{\hat{c}_t, \bar{c}_2\}$ and keeping $\hat{h}_t$ and $\hat{y}_t$ the same remains feasible and achieves a higher value. Because the constraint set is linear, any convex combination of the two plans, $x_{\beta t} = \beta x_t + (1-\beta)\hat{x}_t$, $x \in \{c, h, z, y\}$, is feasible. Moreover, since $\hat{c}_t \leq c_2$, $(1-\beta)c_t + \beta \hat{c}_t \leq c_1$ as long as $\beta > 0$ is small enough. Therefore, for small $\beta > 0$:

$$\min\{c_{\beta t}, \bar{c}_1\} = c_{\beta t} = (1-\beta) \min\{c_t, \bar{c}_1\} + \beta \min\{\hat{c}_t, \bar{c}_2\}.$$

Using the concavity of the objective, together with the strict inequality (28), we obtain that:

$$W(z, \bar{c}_1) < \int_0^\infty e^{-(r+\alpha)t} \left\{ \min\{c_{\beta t}, \bar{c}_1\} + \bar{h} - h_{\beta t} + \alpha \left\{ U\left[\hat{y}_{\beta t}\right] + W\left[z_{\beta t} - y_{\beta t}, \bar{c}_1\right] \right\} \right\} dt,$$

which is a contradiction. This shows that

$$W(z, \bar{c}_1) \geq \sup \int_0^\infty e^{-(r+\alpha)t} \left\{ \min\{\hat{c}_t, \bar{c}_2\} + \bar{h} - h_t + \alpha \left\{ U\left[\hat{y}_t\right] + W\left[\hat{z}_t - \hat{y}_t, \bar{c}_1\right] \right\} \right\} dt,$$

over the set of feasible plans. Since the upper bound is clearly achieved for $\{c_t, h_t, y_t, z_t\}$, we obtain $W(z, \bar{c}_1) = W(z, \bar{c}_2)$, as claimed. ■

VI Optimality verification in pure currency economies

In this section, we provide an optimality verification proposition: we show that the value function solving the Hamilton-Jacobi-Bellman equation is a solution of the sequential optimization problem of the a household. The proposition covers all the inter-temporal optimization problems considered in the paper: with linear and SI preferences and lump-sum transfers, with linear preferences and non-linear and possibly discontinuous transfers, and with quadratic preferences. While the general method of proof is standard, it requires some extra work (adapted from Bressan and Hong, 2008) because in the case of discontinuous transfers the value function is not differentiable at the target. See Aguiar, Amador, Farhi and Gopinath (2013) for an earlier application of these methods in a model of sovereign debt crises.

VI.1 The intertemporal household's problem

We start by stating the problem of the household in sequence. Let $\{\mathcal{F}_t,\, t \geq 0\}$ denote the filtration generated by the process for lumpy consumption opportunities. Let the successive arrival times of lumpy consumption opportunities be denoted by $T_1 < T_2 < \ldots$. Consider a household starting with real balance $z_0 \geq 0$. For this household, a *feasible plan* is a collection of stochastic processes, $\{c, h, y, z\}$, with the following properties. First, these processes must satisfy regularity and measurability restrictions: we constrain them to be adapted and left continuous. Second, at each time, these processes must satisfy inequality constraints:

$$c_t \geq 0, \quad 0 \leq h_t \leq \bar{h}, \quad 0 \leq y_t \leq z_t, \quad \text{ and } z_t \geq 0.$$

Third, consumption is assumed to remain bounded over finite horizons, i.e. $\sup_{t\in[0,T]} c_t < \infty$ for all T. Fourth, the real balance process must solve:

$$\begin{aligned} \dot{z}_t &= h_t - c_t - \pi z_t + \Upsilon\left(z_t\right), \text{ almost everywhere over } (T_n, T_{n+1}) \\ z_{T_n^+} &= z_{T_n} - y_{T_n}. \end{aligned}$$

Finally, the real balance process must satisfy the initial condition $z_0 = z$. The *household problem* is, then, to choose a feasible plan in order to maximize the inter-temporal utility:

$$\mathbb{E}\bigg[\int_0^\infty e^{-rt} u\left(c_t, \bar{h} - h_t\right)\, dt + \sum_{n=1}^{\infty} U\left(y_{T_n}\right) e^{-rT_n}\bigg].$$

An *optimal plan* is a feasible plan achieving the maximum attainable utility. To analyze all the cases considered in the paper, we maintain the following assumptions:

Assumption VI.1 *The utility function $u(c, \bar{h} - h)$ is continuous, concave, positive and bounded. The utility function $U(y)$ is continuous, concave, positive and satisifes $U(y) \leq k_U + K_U y$ for some $k_U, K_U > 0$. The transfer is positive, increasing, and satisfies $\Upsilon(z) < k_\Upsilon + (K_\Upsilon + \pi)z$ for some $k_\Upsilon, K_\Upsilon + \pi > 0$ and $K_\Upsilon < r$.*

The condition $K_\Upsilon < r$ implies that real balances grow at a rate smaller than the discount rate, which provides an appropriate "transversality condition" for completing the standard optimality verification argument.

VI.2 The optimality verification argument

We proceed by showing that the value function that is obtained via the Hamilton-Jacobi-Bellman equation is the maximum attainable utility for the household's problem. To cover all the cases analyzed in the paper, we maintain the following assumptions:

Assumption VI.2 *There exists some value function $W(z)$ with the following properties. It is Lipchitz continuous and positive and satisfies $W(z) \leq k_W + K_W z$ for some $k_W, K_W > 0$. It is continuously differentiable over $[0, \infty)$ except perhaps at some $z^\star > 0$. It satisfies, for $z \neq z^\star$:*

$$(r+\alpha)W(z) = \max \left\{ u(c, \bar{h}-h) + \alpha \left[U(y) + W(z-y) \right] + W'(z) \left[h - c - \pi z + \Upsilon(z) \right] \right\} \tag{29}$$

with respect to $c \geq 0$, $h \in \left[0, \bar{h}\right]$ and $y \in [0, z]$. For $z = z^\star$, it satisfies:

$$(r+\alpha)W(z) = \max \left\{ u(c, \bar{h}-h) + \alpha \left[U(y) + W(z-y) \right] \right\} \tag{30}$$

with respect to $c \geq 0$, $h \in [0, \bar{h}]$, $y \in [0, z]$, and subject to $h - c - \pi z + \Upsilon(z) = 0$.

To apply the standard optimality verification argument, we will need the following result, which provides an estimate of the changes in discounted value along any feasible path. The main difficulty in establishing this estimate, and for which we adapt arguments from Bressan and Hong (2008), is that the value function may not be continuously differentiable at $z^\star$.

Lemma VI.1 *For any $T_n < t_1 < t_2 < T_{n+1}$ and any feasible plan,*

$$W(z_{t_1})e^{-rt_1} - W(z_{t_2})e^{-rt_2} \geq \int_{t_1}^{t_2} \left\{ u(c_t, \bar{h} - h_t) + \alpha \left[U(y_t) + W(z_t - y_t) - W(z_t) \right] \right\} e^{-rt} \, dt.$$

Proof. Since consumption flows are bounded over finite horizons, labor flows are bounded, and $\Upsilon(z) \leq k_\Upsilon + (K_\Upsilon + \pi) z$, one easily verifies that z_t and $\dot{z}_t$ remains bounded over $[t_1, t_2]$. Hence, the path for real balance is Lipchitz continuous. Since the value function is Lipchitz continuous as well, it follows that $W(z_t)$ and $W(z_t)e^{-rt}$ are also Lipchitz continuous. Therefore $W(z_t)$ and $W(z_t)e^{-rt}$ are absolutely continuous and thus differentiable almost everywhere (see, for example, Theorem 7.18 in Rudin, 1966). Moreover, their derivative is integrable, that is:

$$\begin{aligned} W(z_{t_1})e^{-rt_1} - W(z_{t_2})e^{-rt_2} &= -\int_{t_1}^{t_2} \frac{d}{dt} \left[W(z_t) e^{-rt} \right] \, dt \\ &= \int_{t_1}^{t_2} \left\{ rW(z_t) - \frac{d}{dt} \left[W(z_t) \right] \right\} e^{-rt} \, dt \end{aligned} \tag{31}$$

Now consider the set $\{t \geq 0 \, : \, z_t \neq z^\star\}$. This set is open, because it is the inverse image of an open set by the continuous function z_t. Therefore, it can be covered by a countable union of disjoint open intervals, i.e. it is equal to $\cup_{i \in I}(a_i, b_i)$. By continuity, it must be the case that $z_t < z^\star$ for all $t \in (a_i, b_i)$, or $z_t > z^\star$ for all $t \in (a_i, b_i)$. Hence for all $t \in (a_i, b_i)$, W is continuously differentiable at z_t. In addition, since z_t is differentiable almost everywhere in (a_i, b_i), with $\dot{z}_t = h_t - c_t - \pi z_t + \Upsilon(z_t)$, we obtain that $\frac{d}{dt}[W(z_t)] = W'(z_t)\dot{z}_t$ almost everywhere in (a_i, b_i). Since $z_t \neq z^\star$, we obtain from the HJB equation (29):

$$rW(z_t) - \frac{d}{dt}[W(z_t)] \geq u(c_t, \bar{h} - h_t) + \alpha [U(y_t) + W(z_t - y_t) - W(z_t)].$$

After integrating over the closed interval $[a_i, b_i]$ and adding up over $i \in I$:

$$\begin{aligned} & \int_{t \in \cup_{i \in I}[a_i,b_i]} \left\{ rW(z_t) - \frac{d}{dt}[W(z_t)] \right\} dt \\ \geq & \int_{t \in \cup_{i \in I}[a_i,b_i]} \left\{ u(c_t, \bar{h} - h_t) + \alpha [U(y_t) + W(z_t - y_t) - W(z_t)] \right\} dt. \end{aligned} \tag{32}$$

Next consider any $t \in [t_1, t_2] \setminus \cup_{i \in I}[a_i, b_i]$ such that $\dot{z}_t = h_t - c_t - \pi z_t + \Upsilon(z_t)$. By construction, for any $\varepsilon > 0$, $(t, t + \varepsilon)$ cannot be a subset of any (a_i, b_i). Therefore, for any $\varepsilon > 0$, there must exists some $t' \in (t, t + \varepsilon)$ such that $z_{t'} = z^\star$. This implies by continuity that $z_t = z^\star$, and also that $\dot{z}_t = 0$. Given that W is Lipchitz, this also implies that $\frac{d}{dt}[W(z_t)] = 0$ even though W may not be differentiable at $z_t = z^\star$. Using (30) we therefore obtain that:

$$rW(z_t) - \frac{d}{dt}[W(z_t)] = rW(z_t) \geq u(c_t, \bar{h} - h_t) + \alpha [U(y_t) + W(z_t - y_t) - W(z_t)]$$

for any $t \in [t_1, t_2] \setminus \cup_{i \in I}[a_i, b_i]$ such that $\dot{z}_t = h_t - c_t - \pi z_t + \Upsilon(z_t)$. But since $\dot{z}_t = h_t - c_t - \pi z_t + \Upsilon(z_t)$ almost everywhere, we can integrate this inequality and obtain:

$$\begin{aligned} & \int_{t \in [t_1,t_2] \setminus \cup_{i \in I}[a_i,b_i]} \left\{ rW(z_t) - \frac{d}{dt}[W(z_t)] \right\} dt \\ \geq & \int_{t \in [t_1,t_2] \setminus \cup_{i \in I}[a_i,b_i]} \left\{ u(c_t, \bar{h} - h_t) + \alpha [U(y_t) + W(z_t - y_t) - W(z_t)] \right\} dt. \end{aligned} \tag{33}$$

The desired result obtains by combining (32) and (33) and using equality (31). ■

We then obtain the following optimality verification argument.

Proposition VI.2 *Assume that Assumptions VI.1 and VI.2 hold. Then, if the policy functions solving the HJB generate a feasible plan:*

- *The maximum attainable utility of a household starting with z_0 is $W(z_0)$;*
- *The policy functions solving the HJB generate a optimal plan.*

Proof. We adapt arguments from Theorem VII, T1 in Brémaud [1981]. We consider any feasible plan and write:

$$\begin{aligned} e^{-rt} W(z_t) = W(z_0) + & \sum_{0 < T_n \leq t} \left\{ e^{-rT_n} W(z_{T_n+}) - e^{-rT_{n-1}} W(z(T_{n-1+}) \right\} \\ & + e^{-rt} W(z_t) - e^{-r\tau_t} W(z_{\tau_t+}), \end{aligned}$$

where $\tau_t = \sup\{T_n : T_n \leq t\}$. Further:

$$
\begin{aligned}
e^{-rt}W(z_t) = W(z_0) &+ \sum_{0<T_n\leq t} e^{-rT_n}\left\{W(z_{T_n+}) - W(z_{T_n})\right\} \\
&+ \sum_{0<T_n\leq t}\left[e^{-rT_n}W(z_{T_n}) - e^{-rT_{n-1}}W\left(z_{T_{n-1}+}\right)\right] \\
&+ e^{-rt}W(z_t) - e^{-r\tau_t}W(z_{\tau_t+}).
\end{aligned}
$$

The second term on the first line collects jump in the value function when lumpy consumption opportunities arrive, so $z_{T_n+} = z_{T_n} - y_{T_n}$. The terms on the second and the third lines collects changes in the value function in between lumpy consumption opportunities. Using Lemma VI.1 we have

$$
e^{-rT_{n-1}}W\left(z_{T_{n-1}+}\right) - e^{-rT_n}W(z_{T_n}) \geq \int_{T_n}^{T_{n+1}} \left\{u(c_s, \bar{h} - h_s) + \alpha\left[U(y_s) + W(z_s - y_s) - W(z_s)\right]\right\} ds,
$$

with an equality if the feasible plan under consideration is generated by the candidate optimal policy functions. Taken together, we obtain:

$$
\begin{aligned}
W(z_t)\,e^{-rt} &\leq W(z_0) + \int_0^t e^{-rs}\big[W(z_s - y_s) - W(z_s)\big]\,dN_s \\
&\quad - \int_0^t \left\{u(c_s, \bar{h} - h_s) + \alpha\left[U(y_s) + W(z_s - y_s) - W(z_s)\right]\right\} ds \\
&\leq \int_0^t e^{-rs}\big[W(z_s - y_s) + U(y_s) - W(z_s)\big]\,(dN_s - \alpha\,ds) \qquad (34)\\
&\quad - \int_0^t \left[u(c_s, \bar{h} - h_s)\,ds + \alpha U(y_s)\,dN_s\right] \qquad (35)
\end{aligned}
$$

with an equality if the feasible plan under consideration is generated by the candidate optimal policy functions. Note that we have

$$
\begin{aligned}
e^{-rs}\big|W(z_s - y_s) + U(y_s) - W(z_s)\big| &\leq e^{-rs}\big|W(z_s) - W(z_s - y_s)\big| + e^{-rs}U(y_s) \\
&\leq e^{-rs}(k_W + K_W z_s) + e^{-rs}\left[k_U + k_U y(s)\right] \\
&\leq e^{-rs}\left[k_W + k_U + (K_W + K_U)z_s\right],
\end{aligned}
$$

where the last inequality follows because $y_s \leq z_s$. Given our maintained assumptions, it is clear that $z_s e^{-rs}$ is bounded over $[0, t]$, and so it follows that $e^{-rs}\big|W(z_s - y_s) + U(y_s) - W(z_s)\big|$ is bounded as well. Since, in addition, it is a predictable process, it follows by Theorem II, T8 in Brémaud [1981] that

$$
\int_0^t e^{-rs}\big[W(z_s - y_s) + U(y_s) - W(z_s)\big]\left[dN_s - \alpha\,ds\right]
$$

is a martingale, and therefore its time-zero expected value is equal to zero. Taking time-zero expected value on both sides of (35), we obtain after rearranging that

$$\mathbb{E}\left\{ \int_0^t e^{-rs} \left[u\left(c_s, \bar{h} - h_s\right) \, ds + U\left(y_s\right) dN_s \right] \right\} + \mathbb{E}\left[e^{-rt} W\left(z_t\right) \right] \le W\left(z_0\right).$$

with an equality if the feasible plan is generated by the candidate optimal policy functions.

The last step is, as usual, to argue that $\mathbb{E}\left[e^{-rt} W\left(z_t\right) \right] \to 0$ as $t \to \infty$. Since $\dot{z}_s \le \bar{h} - \pi z_s + \Upsilon(z_s) \le \bar{h} + k_\Upsilon + K_\Upsilon z$, we obtain by direct calculations that:

$$e^{-rs} z_s \le e^{-(r - K_\Upsilon)s} z_0 + \frac{\bar{h} + k_\Upsilon}{K_\Upsilon} \left(e^{-(r - K_\Upsilon)t} - e^{-rt} \right). \tag{36}$$

Given our maintained assumptions that $K_\Upsilon < r$ and that $W(z) \le k_w + K_W z$, the result follows.

■

VII Numerical methods

VII.1 Construction of a Stationary Equilibrium

In this section we outline a numerical algorithm to construct a stationary equilibrium. We guess and verify a stationary equilibrium with a numerical method, where the target real balances is finite and the value function is strictly concave, increasing and twice differentiable.

As shown in the mathematical appendix, the value function $W(z)$ solves the following HJB equation:

$$(r+\alpha) W(z) = \max_{c \geq 0, h \in [0,\overline{h}]} \left\{ u\left(c, \overline{h} - h\right) + \alpha V(z) + W'(z)\left[h - c - \pi z + \Upsilon\right] \right\} \tag{37}$$

where $V(z) \equiv \max_{y \in [0,z]} \{U(y) + W(z-y)\}$. Denote $\lambda = W'(z)$, and $c(\lambda)$ and $h(\lambda)$ are the maximizers to the above. Let $y(z)$ denote the solution of y to V. Then $V'(z)$ and $y'(z)$ are given by

$$V'(z) = U'[y(z)] \tag{38}$$

$$y'(z) = \begin{cases} 1 \text{ if } U'(z) \geq W'(0) \\ \frac{W''[z-y(z)]}{W''[z-y(z)]+U''[y(z)]} \text{ if } U'[y(z)] = W'[z-y(z)] \end{cases} \tag{39}$$

Under the premise that W is twice differentiable, the equilibrium dynamics of household's state and co-state is given by the following system of differential equations:

$$\dot{z} = h(\lambda) - c(\lambda) - \pi z + \Upsilon, \tag{40}$$

$$\dot{\lambda} = (r + \alpha + \pi)\lambda - \alpha V'(z). \tag{41}$$

The stationary point $(z^\star, \lambda^\star)$, which is given by

$$\begin{aligned} h(\lambda^\star) &= c(\lambda^\star) + \pi z^\star - \Upsilon, \\ \frac{r+\alpha+\pi}{\alpha}\lambda^\star &= V'(z^\star). \end{aligned}$$

Our novel, recursive method to solve this problem involves two key elements: first rewrite the equilibrium as a system of delay differential equations (DDE), and second modify the time-elimination method (Mulligan and Sala-i-Martin 1993) to solve this system of DDE. The time-elimination method allows us to change the state variable from t to $\zeta \equiv -\lambda$, and then the equilibrium dynamics is fully characterized by the "stable arm" function $z(\zeta)$ with initial condition $z(\zeta_0) = 0$ and boundary condition $z(\zeta^\star) = z^\star$. The stable arm, $z(\zeta)$, is well-defined and unique under the premise that the value function is strictly concave, increasing and twice differentiable (see mathematical appendix for details).

VII.1.1 Computing the Stable Arm: DDE

To compute the system with time-elimination method, we first formulate $z(\zeta; \lambda_0, \Upsilon)$ given λ_0 and Υ. By eliminating the time in (40) and (41), the slope of the stable arm is

$$z'(\zeta) = \frac{\dot{z}}{-\dot{\lambda}} = \frac{h(-\zeta) - c(-\zeta) - \pi z + \Upsilon}{(r+\alpha+\pi)\zeta + \alpha\Omega(\zeta)}. \tag{42}$$

where $\Omega(\zeta) \equiv V'[z(\zeta)]$ is differentiable. We suppress the dependence on λ_0, Υ in z and Ω unless confusion could arise. We formulate $\Omega(\zeta)$ in a recursive way by decomposing z into y and $z-y$ as follows:

$$z = \underbrace{(U')^{-1}(\Omega)}_{y} + \underbrace{\lambda^{-1}[\min\{\Omega, \lambda_0\}]}_{z-y}. \tag{43}$$

Notice that

$$\frac{d\lambda^{-1}(\Omega)}{d\Omega} = z'(-\Omega).$$

Differentiate (43) with respect to ζ to obtain $\Omega(\zeta)$ as the solution to the following DDE:

$$\Omega'(\zeta) = z'(\zeta)\left[U''\left[(U')^{-1}[\Omega(\zeta)]\right]^{-1} + \mathbb{I}[\Omega(\zeta) < \lambda_0]\, z'[-\Omega(\zeta)]\right]^{-1}, \tag{44}$$

where $\mathbb{I}(\Omega < \lambda_0)$ is the indicator function that is equal to 1 if $\Omega < \lambda_0$ and 0 otherwise. Equation (44) is a DDE since (44) depends on the "current time" ζ and the "lag time" $-\Omega(\zeta) < \zeta$. Now the equilibrium is characterized by the system of DDE (42) and (44).

To solve the system of $z(\zeta)$ and $\Omega(\zeta)$, given Υ and $\zeta_0 = \lambda_0$ we start integrating (42) and (44) from the boundary condition $z(-\lambda_0) = 0$ and $\Omega(-\lambda_0) = U'(0)$, which is well-defined by assuming that either $U'(0)$ is bounded or we start with some arbitrarily large value. The integration will result in two functions $z(\zeta)$ and $\Omega(\zeta)$ given λ_0 and Υ. For later use, define $\zeta^\star(\lambda_0, \Upsilon)$ and $z^\star(\lambda_0, \Upsilon)$ as the solution to $h(-\zeta^\star) - c(-\zeta^\star) - \pi(z^\star - \Upsilon) = 0$ and $z^\star = z(\zeta^\star)$.

VII.1.2 Computing the Distribution

Having computed $z(\zeta)$ and $\Omega(\zeta)$, we invert the system back to $\lambda(z)$ and $\Omega(z)$ using the definition $\zeta = -\lambda(z)$. It is invertible since $W'(z)$ is monotone. Notice that by definition we have $y(z) = (U')^{-1}[\Omega(z)]$. Define $\varphi(z)$ as the solution to

$$\varphi - y(\varphi) = z. \tag{45}$$

In other words, $\varphi(z)$ is the level of real balances before the preference shock such that the household will deplete the real balances up to z after the shock. Notice that since $y(z) > 0$ for all $z > 0$, we have $\varphi(z) > z$. Differentiating (45) we have for all $z > 0$

$$\varphi'(z) = \frac{W''[z - y(z)] + U''[\varphi(z)]}{U''[y(z)]} \geq 0.$$

so $\varphi(z)$ is strictly increasing for all $z > 0$. Define $z_d \equiv z^\star - y(z^\star)$, then we have $\varphi(z) \leq z^\star$ if and only if $z \leq z_d$. Define $s(z) \equiv h[\lambda(z)] - c[\lambda(z)] - \pi z + \Upsilon$. which is differentiability continuous, bounded and positive for all $z \in [0, z^\star)$. The equilibrium density for the distribution of real balances $f(z)$ solves the Kolmogorov forward equation (hereafter KFE, which is derived later):

$$\partial_z[s(z) f(z)] = \begin{cases} -\alpha f(z), \text{ if } z > z_d, \\ -\alpha f(z) + \alpha \frac{s[\varphi(z)]}{s(z)} f[\varphi(z)], \text{ if } z < z_d, \end{cases} \tag{46}$$

where ∂_z is the differential functional. The equilibrium density $f(z)$ has a jump at $z = z_d$, which captures the extra flow of the influx of the mass of agents with $z = z^\star$ after a preference shock. The jump is given by

$$f(z_-) = f(z) - \frac{s\left(z_-^\star\right)}{s(z)} f\left(z_-^\star\right), \text{ if } z = z_d. \tag{47}$$

Consider two regions of z: $[z_d, z^\star]$ and $[0, z_d)$. In the first region $[z_d, z^\star]$, the KFE (46) is just a standard ODE:

$$f'(z) = -\frac{\alpha + s'(z)}{s(z)} f(z), \text{ for all } z \in (z_d, z^\star). \tag{48}$$

Fix some arbitrary initial value, says $f(z_d) = 1$ (we will normalize the density function later), we can compute $f(z)$ in this region by integrating the ODE (46) from the initial condition $f(z_d) = 1$ up to the boundary $z = z^\star(\lambda_0, \Upsilon)$. If $s\left(z_-^\star\right) = 0$, then there is also a boundary condition for the KFE, which is given by

$$\lim_{z \uparrow z^\star} s(z) f(z) = 0. \tag{49}$$

On the other hand, if $s\left(z_-^\star\right) > 0$, then there is a probability mass $F(z^\star) - F\left(z_-^\star\right)$ at $z = z^\star$, which is pinned down by a boundary condition

$$F(z^\star) - F\left(z_-^\star\right) = \frac{s\left(z_-^\star\right) f\left(z_-^\star\right)}{\alpha}. \tag{50}$$

Now consider the second region $[0, z_d)$. Transform $z = z_d - t$. and define $\phi(t) = f(z_d - t)$. Using (46), $\phi(t)$ also solves the following DDE

$$\phi'(t) = \left[\frac{\alpha - s'(z_d - t)}{s(z_d - t)}\right] \phi(t) - \alpha \frac{s\left[\varphi(z_d - t)\right]}{s(z_d - t)^2} \phi\left[z_d - \varphi(z_d - t)\right], \text{ for all } t \in (0, z_d) \tag{51}$$

We compute $\phi(t)$ by integrating the DDE (51) from $t = 0$ to $t = z_d$ given the initial value from (47)

$$\phi(0) = 1 - \frac{s\left(z_-^\star\right)}{s(z_d)} f\left(z_-^\star\right). \tag{52}$$

The (unnormalized) density function in this region can be obtained by having $f(z) = \phi(z_d - z)$ for all $z \in [0, z_d)$.

VII.1.3 Computing the Stationary Equilibrium and Welfare Cost

Finally, we use the transversality condition and government's balanced budget condition to solve λ_0 and Υ. The transversality condition implies another boundary condition:

$$(r + \alpha + \pi) \zeta^\star(\lambda_0, \Upsilon) + \alpha \Omega\left[\zeta^\star(\lambda_0, \Upsilon)\right] = 0. \tag{53}$$

The lump-sum transfer is defined as $\Upsilon = \pi \mathbb{E}(z)$, which implies

$$\Upsilon = \frac{\pi \int_0^{z^\star(\lambda_0, \Upsilon)} z dF(z)}{\int_0^{z^\star(\lambda_0, \Upsilon)} dF(z)}, \tag{54}$$

where $F(z) \equiv \int_0^z f(z)\,dz$ is the cumulative density function. So we have two equations to solve for the two unknown λ_0 and Υ.

Recall that the welfare under inflation π is given by

$$\mathcal{W}_\pi = \int_0^{z_\pi^\star} \left\{u\left[c_\pi(z), \overline{h} - h_\pi(z)\right] + \alpha U\left[y_\pi(z)\right]\right\} dF_\pi(z).$$

Define the welfare cost of inflation Δ_π as the solution to

$$\mathcal{W}_\pi = \int_0^{z_\pi^\star} \left\{u\left[(1-\Delta_\pi) c_0(z), \overline{h} - h_0(z)\right] + \alpha U\left[(1-\Delta_\pi) y_0(z)\right]\right\} dF_0(z).$$

In other words, the welfare cost of inflation is defined as the percentage of households' consumption that a social planner would be willing to give up in order to have inflation zero instead of π.

VII.1.4 Special Case: Laissez-Faire

The time-elimination method is also convenient to construct the equilibrium under the special case of zero money growth and full depletion, which does not involve any fixed-point problem. The stationary point $(z^\star, \lambda^\star)$ is given by

$$\begin{aligned} h(\lambda^\star) &= c(\lambda^\star) && (55) \\ z^\star &= \left(U'\right)^{-1}\left[\left(\frac{r+\alpha}{\alpha}\right)\lambda^\star\right], && (56) \end{aligned}$$

The Jacobian of (40) and (41) at $z = z^\star$ and $\lambda = \lambda^\star$ is then given by

$$J = \begin{pmatrix} 0 & h'(\lambda^\star) - c'(\lambda^\star) \\ -\alpha U''(z^\star) & r+\alpha \end{pmatrix},$$

where we have used that $V''(z^\star) = U''(z^\star)$. Then the negative eigenvalue of J (corresponding to the stable arm) is given by

$$\xi = -\frac{r+\alpha}{2}\left[\left[1 - \frac{4\alpha U''(z^\star)}{(r+\alpha)^2}\left[h'(\lambda^\star) - c'(\lambda^\star)\right]\right]^{1/2} - 1\right]. \tag{57}$$

Define p as the slope of the stable arm at $z = z^\star$ and $\lambda = \lambda^\star$, which is given by

$$p = \frac{\xi}{h'(\lambda^\star) - c'(\lambda^\star)}. \tag{58}$$

In the model with linear preferences, (55) is replaced with $h(\lambda^\star) = c(\lambda^\star) = 0$, $\lambda^\star = 1$ in (56), and $p = \frac{\alpha}{r+\alpha} U''(z^\star)$. These equations solve $\lambda^\star$ and $z^\star$. Under full depletion and zero money growth, the dynamic system is reduced to the following ODEs of $\lambda(z)$ and $f(z)$:

$$\begin{aligned} \lambda'(z) &= \frac{(r+\alpha)\lambda - \alpha U'(z)}{h(\lambda) - c(\lambda)}, && (59) \\ f'(z) &= -\frac{\alpha + \lambda'(z)\left[h'(\lambda) - c'(\lambda)\right]}{h(\lambda) - c(\lambda)} f(z) && (60) \end{aligned}$$

A convenient way to construct the stable arm is to: first, integrate (59) backward from $z = z^\star$ to $z = 0$ with initial value $\lambda(z^\star) = \lambda^\star$ and $\lambda'(z^\star) = p$[5]. Then we solve for the stable arm $\lambda(z)$. We integrate (60) forward from $z = 0$ to $z = z^\star$ with initial value $f(0) = 1$, so we cannot integrate backward like $\lambda(z)$). If $s\left(z_-^\star\right) > 0$ then we construct the probability mass $1 - F\left(z_-^\star\right)$ by the KFE boundary condition (50). After that we solve $f(z)$ (unnormalized). The initial values of λ_0 and Υ are set to $\lambda_0 = \lambda(0)$ and $\Upsilon = \pi \int_0^{z^\star} z f(z)\, dz / \int_0^{z^\star} f(z)\, dz$.

VII.1.5 Special Case: Linear Preferences

So far we need to solve a system of two DDE and one KFE. The system can be further simplified under linear preferences. Eliminating the time in (40) and (41), then using the fact that $\lambda = W'(z)$ and $V'(z) = U'[y(z)]$, we have

$$W''(z) = \frac{\dot{\lambda}}{\dot{z}} = \frac{(r + \alpha + \pi) W'(z) - \alpha U'[y(z)]}{\overline{h} - \pi z + \Upsilon}.$$

Shifting the state variable to $z - y(z)$, we have for all $z > (U')^{-1}(\lambda_0)$

$$W''[z - y(z)] = \frac{(r + \alpha + \pi) U'[y(z)] - \alpha U'[y(z - y(z))]}{\overline{h} - \pi [z - y(z) - \Upsilon]}, \tag{61}$$

where we have used the fact that $U'[y(z)] = V'(z) = W'[z - y(z)]$. Substituting (61) into (39), we have

$$y' = \begin{cases} 1 \text{ if } z \leq (U')^{-1}(\lambda_0) \\ \left[1 + U''(y) \frac{h - \pi(z - y - \Upsilon)}{(r + \alpha + \pi) U'(y) - \alpha U'[y(z - y)]}\right]^{-1} \text{ if } z > (U')^{-1}(\lambda_0) \end{cases} \tag{62}$$

The equilibrium features full depletion if and only if $y'(z) = 1$ for all $z \in [0, z^\star]$. Notice that (39) is also a DDE but no longer depending on λ. Then now $z^\star$ is simply given by

$$z^\star = \min\left\{\frac{h}{\pi} + \Upsilon, y^{-1} \circ (U')^{-1}\left(1 + \frac{r + \pi}{\alpha}\right)\right\}. \tag{63}$$

The equilibrium features binding labor if $z^\star$ takes the first term on the right hand side of (63), otherwise the equilibrium features slack labor. Under linear preferences we have $s(z) = \overline{h} - \pi z + \Upsilon$, which is again independent to λ. Thus, the KFE (46) now is also independent to λ. In sum, the stationary equilibrium can be reduced to the system of 1 DDE (62) and 1 KFE (46).

The KFE can be further simplified under linear preferences. Notice that given $s(z) = \overline{h} - \pi z + \Upsilon$, the KFE (48) with initial value $f(z_d) = 1$ admits the following closed-form solution (unnormalized):

$$f(z) = \left[\frac{s(z)}{s(z_d)}\right]^{\frac{\alpha}{\pi} - 1}, \text{ for all } z \in (z_d, z^\star). \tag{64}$$

[5] $\lambda'(z^\star)$ involves zero dividing zero so it is pinned down by the eigenvector associated with the negative eigenvalue, which corresponds to the stable arm.

If $s\left(z_{-}^{\star}\right) > 0$, then there is a probability mass at $z = z^{\star}$ given by the KFE boundary condition (50), which is simply

$$F\left(z^{\star}\right) - F\left(z_{-}^{\star}\right) = \frac{s\left(z_d\right)^{1-\frac{\alpha}{\pi}} s\left(z_{-}^{\star}\right)^{\frac{\alpha}{\pi}}}{\alpha}. \tag{65}$$

Finally, using the closed-form (64), the KFE (51) is reduced to an ODE given by

$$\phi'\left(t\right) = \frac{\alpha+\pi}{s\left(z_d - t\right)}\phi\left(t\right) - \alpha s\left(z_d\right)^{1-\frac{\alpha}{\pi}} \frac{s\left[\varphi\left(z_d - t\right)\right]^{\frac{\alpha}{\pi}}}{s\left(z_d - t\right)^2}, \text{ for all } t \in \left[0, z_d\right]. \tag{66}$$

The initial condition from (52) becomes

$$\phi\left(0\right) = 1 - \left[\frac{s\left(z_{-}^{\star}\right)}{s\left(z_d\right)}\right]^{\frac{\alpha}{\pi}}. \tag{67}$$

VII.2 Derivation of Kolmogorov Forward Equation

In this section we derive the KFE (46) used in the previous section. The law of motion of real balances is given by

$$\dot{z} = s\left(z\right) \equiv h\left[\lambda\left(z\right)\right] - c\left[\lambda\left(z\right)\right] - \pi z + \Upsilon, \text{ where } z \in \left[0, z^{\star}\right],$$

and agent's real balances reduces by $y\left(z\right)$ after a preference shock, which arrives at the Poisson rate α. Suppose $s\left(z\right)$ is continuous, bounded and positive for all $z \in \left[0, z^{\star}\right)$. Recall that $z_t = \int_0^t s\left(z_s\right) ds$, and let T denote the solution to $z^{\star} = \int_0^T s\left(z_s\right) ds$.

We use a discrete time, discrete state-space model to obtain the Kolmogorov forward equation and the boundary conditions for the density $f\left(z\right)$. Fix any integer n, then there exist $\Delta_n > 0$ and a sequence $\{z_i\}_{i=0}^n$ such that $z_0 = 0$, $z_n = z^{\star}$ and $z_i = z_{i-1} + s\left(z_{i-1}\right)\Delta_n$ for any $i > 0$. To see it, fix any n and $\Delta_n > 0$ and construct $z_i = z_{i-1} + s\left(z_{i-1}\right)\Delta_n$, since $s\left(z\right)$ is bounded, we have $z_n \to \infty$ if $\Delta_n \to \infty$; $z_n \to 0$ if $\Delta_n \to 0$. So there must exist Δ_n such that $z_n = z^{\star}$. Divide $\left[0, z^{\star}\right]$ into $n+1$ discrete states $\{z_i\}_{i=0}^n$. Let $I\left(z\right)$ be the interval function such that $I\left(z\right) = \left[z_{i-1}, z_i\right]$ and $z \in \left(z_{i-1}, z_i\right]$. Let $D\left(z\right)$ be the correspondence such that $D\left(z\right) \subseteq \{z_i\}_{i=0}^n$ and $D\left(z\right) - y\left[D\left(z\right)\right] \subseteq I\left(z\right)$. As n goes to infinity, Δ_n converges to zero, $\{z_i\}_{i=0}^n$ converges to the continuous time process $z_t = \int_0^t s\left(z_\tau\right) d\tau$, $I\left(z\right)$ converges to z, and $D\left(z\right)$ converges to $\varphi\left(z\right)$.

Now let $f_n\left(z_i, t\right)$ denote the fraction of agents with real balances z_i at time t for fixed n. With a slight abuse of notation, let $f_n\left(z\right)$ be the stationary distribution. We are interested in characterizing the density $f\left(z\right) \equiv \lim_{n\to\infty} \frac{f_n(z)}{s(z)\Delta_n}$. For any $i \neq n$, the dynamics of z implies

$$f_n\left(z_i, t+\Delta_n\right) = \left(1 - \alpha\Delta_n\right) f_n\left(z_{i-1}, t\right) + \alpha\Delta_n \sum_{z \in D(z_i)} f_n\left(z, t\right). \tag{68}$$

In any period of length Δ_n, a fraction $\alpha\Delta_n$ of agents are hit by a preference shock. Thus the fraction of agent with $z = z_i$ at $t + \Delta_n$ are a fraction $1 - \alpha\Delta_n$ of those who were agents with $z = z_{i-1}$ at t but not hit by a preference shock, plus the sum of fraction $\alpha\Delta_n$ of those who were agents with $z \in D(z_i)$ at t and hit by a preference shock. Now impose stationarity of f_n. Dividing both side by Δ_n and rearrange terms, we have:

$$s(z_i)\frac{f_n(z_i)}{s(z_i)\Delta_n} - s(z_{i-1})\frac{f_n(z_{i-1})}{s(z_{i-1})\Delta_n} = -\alpha s(z_{i-1})\Delta_n \frac{f_n(z_{i-1})}{s(z_{i-1})\Delta_n} + \alpha \sum_{z\in D(z_i)} f_n(z).$$

Suppose $\varphi(z)$ has probability mass at some $z>0$. Since $D(z)$ converges to $\varphi(z)$, $\sum_{z'\in D(z)} f_n(z')$ converges to $F[\varphi(z)] - F\left[\varphi(z)_-\right]$. Taking the limit as n goes to infinity and eliminating the term with Δ_n, we have:

$$s(z)[f(z) - f(z_-)] = \alpha\left\{F[\varphi(z)] - F\left[\varphi(z)_-\right]\right\}, \tag{69}$$

which implies (47) by taking $z = z^\star$. Suppose $\varphi(z)$ does not have probability mass (atomless) at some $z>0$. Dividing both side by $s(z_{i-1})\Delta_n$ we have

$$\frac{s(z_i)\frac{f_n(z_i)}{s(z_i)\Delta_n} - s(z_{i-1})\frac{f_n(z_{i-1})}{s(z_{i-1})\Delta_n}}{s(z_{i-1})\Delta_n} = -\alpha\frac{f_n(z_{i-1})}{s(z_{i-1})\Delta_n} + \alpha\sum_{z\in D(z_i)}\frac{s(z)}{s(z_{i-1})}\frac{f_n(z)}{s(z)\Delta_n}.$$

Taking the limit as n converges to infinity, $D(z)$ converges to a function $\varphi(z)$ and $\sum_{z\in D(z_i)}\frac{s(z)}{s(z_{i-1})}\frac{f_n(z)}{s(z)\Delta_n}$ converges to $\frac{s[\varphi(z)]}{s(z)} f[\varphi(z)]$ if $\varphi(z) \le z^\star$ (corresponding to $D(z_i) \neq \emptyset$), and 0 otherwise. Then we obtain Kolmogorov forward equations for all $z \in (0, z^\star)$

$$\partial_z[s(z)f(z)] = \begin{cases} -\alpha f(z) + \alpha\frac{s[\varphi(z)]}{s(z)} f[\varphi(z)], \text{ if } \varphi(z) \le z^\star, \\ -\alpha f(z), \text{ otherwise.}\end{cases}, \tag{70}$$

where ∂_z is the differential functional defined as $\partial_z G(z) = \lim_{\Delta\to 0}\frac{G(z)-G(z-\Delta)}{\Delta}$. It implies (46).

To obtain boundary conditions, the dynamics of $z = z^\star = z_n$ imply

$$f_n(z_n, t+\Delta_n) = (1-\alpha\Delta_n) f_n(z_{n-1}, t) + (1-\alpha\Delta_n) f_n(z_n, t).$$

In any period of length Δ_n, a fraction $\alpha\Delta_n$ of agents are hit by a preference shock. Thus the fraction of agent with $z = z_n$ at $t+\Delta_n$ are a fraction $1-\alpha\Delta_n$ of those who were agents with $z = z_{n-1}$ at t but not hit by a preference shock, plus the sum of fraction $1-\alpha\Delta_n$ of those with $z = z_n$ at t not hit by a preference shock. Now impose stationarity of f_n. Rearranging terms we have

$$\alpha f_n(z_n) = (1-\alpha\Delta_n) s(z_{n-1}) \frac{f_n(z_{n-1})}{s(z_{n-1})\Delta_n}.$$

If $s\left(z^\star_-\right) = 0$, then we have the boundary condition

$$\lim_{z\uparrow z^\star} s(z) f(z) = 0, \tag{71}$$

which implies (49). If $s\left(z^\star_-\right) \neq 0$, then there is probability mass at $z = z^\star$, and $f_n(z_n)$ converges to $F(z^\star) - F\left(z^\star_-\right)$. The boundary condition becomes

$$s\left(z^\star_-\right) f\left(z^\star_-\right) = \alpha\left[F(z^\star) - F\left(z^\star_-\right)\right], \tag{72}$$

which implies (50). Finally, for another boundary $z = z_0 = 0$, the discrete time, discrete state-space KFE is given by

$$f_n(0, t + \Delta_n) = (1 - \alpha\Delta_n) f_n(0, t) + \alpha\Delta_n \sum_{z \in D(0)} f_n(z, t) . \tag{73}$$

Taking Δ_n to zero then we have both sides of (73) equal to $f(0)$, which does not impose any condition on $f(0)$.

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