Online Appendix to "Strategy-proof tie-breaking in matching with priorities"

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Section 1 contains the omitted proofs of Lemma 5, Lemma 6 and Lemma 7. Subsection 1.1 shows that neither (A) nor (B) of Assumption 1 can be dispensed of in Theorem 1. Subsection 1.2 shows that all priority structures having at most a two-way tie at the top (and being otherwise strict) is solvable.

1 Omitted proofs

Lemma 5. *Fix a weak priority structure $\succeq$.*

(1) Let $i, j, k \in I$ be three distinct agents and $o, p \in O$ be two distinct objects such that $i \sim_o j \sim_o k$ and $i \succ_p k \succ_p j$. Let R be a preference profile such that

$$\begin{array}{ccc} R_i & R_j & R_k \\ \hline o & o & p \end{array},$$

and such that for all $z \in I \setminus \{i, j, k\}$ and all $\tilde{o} \in \{o, p\}$ for which $z \succeq_{\tilde{o}} \tilde{i}$ for some $\tilde{i} \in \{i, j, k\}$, $zP_z\tilde{o}$. If f is constrained efficient and strategy-proof, then $f_i(R) = o$.

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(2) Let $i, j, k, l \in I$ be four distinct agents and $o, p, q \in O$ be three distinct objects such that $i \sim_o j \sim_o k \sim_o l$, $\{i, j\} \succ_p k \succ_p l$, and $i \succeq_q l \succ_q j$. Let R be a preference profile such that

$$\begin{array}{cccc} R_i & R_j & R_k & R_l \\ \hline o & o & p & q \end{array},$$

and such that for all $z \in I \setminus \{i, j, k, l\}$ and all $\tilde{o} \in \{o, p, q\}$ for which $z \succeq_{\tilde{o}} \tilde{i}$ for some $\tilde{i} \in \{i, j, k, l\}$, $zP_z\tilde{o}$. If f is constrained efficient and strategy-proof, then $f_i(R) = o$.

(3) Let $i, j, k, l \in I$ be four distinct agents and $o, p, q \in O$ be three distinct objects such that $i \sim_o j \sim_o k$, $i \succ_p l \succ_p k$, and $k \succ_q l \succ_q j$. Let R be a preference profile such that

$$\begin{array}{cccc} R_i & R_j & R_k & R_l \\ \hline o & o & p & q \end{array},$$

and such that for all $z \in I \setminus \{i, j, k, l\}$ and all $\tilde{o} \in \{o, p, q\}$ for which $z \succeq_{\tilde{o}} \tilde{i}$ for some $\tilde{i} \in \{i, j, k, l\}$, $zP_z\tilde{o}$. If f is constrained efficient and strategy-proof, then $f_i(R) = o$.

Proof. (1) By definition of R and stability, object o is assigned to agent i or agent j at R. Assume to the contrary that $f_i(R) \neq o$ and $f_j(R) = o$. The following diagram shows how we derive a contradiction:

$$\begin{array}{ccc}
\begin{array}{c|ccc} R & R_i & R_j & R_k \\ \hline & o & \boxed{o} & p \\ & & & \end{array} & \rightarrow & \begin{array}{c|ccc} R^1 & R_i^1 & R_j & R_k \\ \hline & o & \boxed{o} & p \\ & p & & \end{array} \\
 & & \downarrow \\
\begin{array}{c|ccc} R^3 & R_i^1 & R_j & R_k^2 \\ \hline & o & \boxed{o} & o \\ & p & & p \end{array} & \leftarrow & \begin{array}{c|ccc} R^2 & R_i^1 & R_j & R_k^1 \\ \hline & o & \boxed{o} & p \\ & p & & o \end{array}
\end{array}$$

In moving from R to R^1, we have used strategy-proofness for agent i to infer $f_i(R^1) \neq o$. Since $i \succ_p k$, stability then implies $f_k(R^1) = k$. In moving from R^1 to R^2, we have used strategy-proofness for agent k to infer $f_k(R^2) \neq p$. This is compatible with stability only when $f_i(R^2) = p$. But then constrained efficiency requires that $f_k(R^2) \neq o$, since i and j would otherwise form a stable improvement cycle of $f(R^2)$ at R^2. Finally, in moving from R^2 to R^3 we have used strategy-proofness for k one more time to infer $f_k(R^3) = k$. This is compatible with constrained efficiency only when $f_i(R^3) = p$ and $f_j(R^3) = o$. However, this assignment at R^3 contradicts Lemma 4 in Ehlers and Westkamp (2017) given that $\{i, k\} \succ_p j$

so that $i \to_p j$ is an $(i,j;o,p)$-path which is compatible with the $(k,j;o,p)$-path $k \to_p j$ and $i \sim_o j \sim_o k$. Hence, $f_j(R) = o$ is impossible for any constrained efficient and strategy-proof mechanism f.

(2) By definition of R and stability, object o is assigned at R to agent i or agent j. Assume to the contrary that $f_i(R) \neq o$ and $f_j(R) = o$.

If $i \succ_q l \succ_q j$, we obtain an immediate contradiction to the first part of Lemma 5. Hence, we must have $i \sim_q l \succ_q j$.

Now consider the profile R^1 that is obtained from R by letting i add q as her second most preferred object:

$$\begin{array}{c|cccc} R^1 & R_i^1 & R_j & R_k & R_l \\ \hline & o & \boxed{o} & p & q \\ & q & & & \end{array}.$$

By strategy-proofness, $f_i(R^1) \neq o$. By individual rationality, we are left to consider two possible cases.

Case 1: $f_i(R^1) = q$.

The following diagram shows how to derive a contradiction to the assumed properties of f:

$$\begin{array}{c|cccc} R^1 & R_i^1 & R_j & R_k & R_l \\ \hline & o & \boxed{o} & p & q \\ & \boxed{q} & & & \end{array} \quad \rightarrow \quad \begin{array}{c|cccc} R^2 & R_i^1 & R_j & R_k & R_l^1 \\ \hline & o & \boxed{o} & p & q \\ & \boxed{q} & & & o \end{array} \quad \rightarrow \quad \begin{array}{c|cccc} R^3 & R_i^1 & R_j & R_k & R_l^2 \\ \hline & o & \boxed{o} & p & o \\ & \boxed{q} & & & q \end{array}$$

In moving from R^1 to R^2, we have used strategy-proofness for agent l to infer $f_l(R^2) \neq q$. By definition of R and R^2, this is compatible with stability only when $f_i(R^2) = q$. But then, we must have $f_l(R^2) \neq o$, as otherwise i and l would form a stable improvement cycle of $f(R^2)$ at R^2. In moving from R^2 to R^3, we have again used strategy-proofness for agent l to infer $f_l(R^3) = l$. This is compatible with constrained efficiency only when $f_i(R^3) = q$ and $f_j(R^3) = o$. But the latter is a contradiction to Lemma 4 in Ehlers and Westkamp (2017) given that $\{i, l\} \succ_q j$ so that $i \to_q j$ is a $(i,j;o,q)$-path which is compatible with the $(l,j;o,q)$-path $l \to_q j$. Hence, $f_i(R^1) = q$ is impossible for any constrained efficient and strategy-proof mechanism f.

Case 2: $f_i(R^1) = i$.

By the definition of R^1 and stability, $f_i(R^1) = i$ implies $f_l(R^1) = q$. The following diagram shows how to derive a contradiction:

$$\begin{array}{c|cccc} R^1 & R_i^1 & R_j & R_k & R_l \\ \hline & o & \boxed{o} & p & \boxed{q} \\ & q & & & \end{array} \;\to\; \begin{array}{c|cccc} R^4 & R_i^2 & R_j & R_k & R_l \\ \hline & q & \boxed{o} & p & \boxed{q} \\ & & & & \end{array} \;\to\; \begin{array}{c|cccc} R^5 & R_i^2 & R_j & R_k & R_l^2 \\ \hline & q & o & p & \boxed{o} \\ & & & & q \end{array}$$

In moving from R^1 to R^4, we have used strategy-proofness for agent i to infer $f_i(R^4) = i$. This is compatible with stability only when $f_l(R^4) = q$.

By strategy-proofness for agent l, $f_l(R^4) = q$ implies $f_l(R^5) \in \{o, q\}$. Suppose first that $f_l(R^5) = q$. In this case, strategy-proofness for agent i would imply that, for $\tilde{R}$ defined by

$$\begin{array}{c|cccc} \tilde{R} & \tilde{R}_i & R_j & R_k & R_l^2 \\ \hline & q & o & p & o \\ & o & & & q \end{array},$$

we must have $f_i(\tilde{R}) \neq q$. Furthermore, if $f_i(\tilde{R}) = o$, stability would require that $f_l(\tilde{R}) = q$. But then i and l would form a stable improvement cycle of $f(\tilde{R})$ at $\tilde{R}$, contradicting constrained efficiency of f. Since $f_i(\tilde{R}) \notin \{q, o\}$, we must have $f_i(\tilde{R}) = i$. But then strategy-proofness for agent i implies that, for $\hat{R}$ defined by

$$\begin{array}{c|cccc} \hat{R} & R_i^1 & R_j & R_k & R_l^2 \\ \hline & o & o & p & o \\ & q & & & q \end{array},$$

we must have $f_i(\hat{R}) = i$. This is compatible with constrained efficiency only if $f_l(\hat{R}) = q$ and $f_j(\hat{R}) = o$. But $f_j(\hat{R}) = o$ is a contradiction to Lemma 4 in Ehlers and Westkamp (2017) given that $\{i, l\} \succ_q j$ so that $i \to_q j$ is an $(i, j; o, q)$-path which is compatible with the $(l, j; o, q)$-path $l \succ_q j$ (and $i \sim_o j \sim_o l$). Since $f_l(R^5) = q$ necessarily leads to a contradiction, we must have $f_l(R^5) = o$.

Strategy-proofness for agent l then requires that, for R^6 defined by

$$\begin{array}{c|cccc} R^6 & R_i^2 & R_j & R_k & R_l^3 \\ \hline & q & o & p & \boxed{o} \end{array},$$

we must have $f_l(R^6) = o$. But since $j \sim_o k \sim_o l$ and $j \succ_p k \succ_p l$, $f_l(R^6) = o$ is a contradiction to the first part of Lemma 5. This completes the proof.

(3) By the definition of R and non-wastefulness, object o is assigned to agent i or agent j at the preference profile R. Assume that, contrary to what we want to show, $f_i(R) \neq o$ and $f_j(R) = o$. Given that $i \succ_p l \succ_p k$ and $k \succ_q l$, it is easy to see that strategy-proofness and constrained efficiency imply

$$
\begin{array}{c|cccc}
R^1 & R_i^1 & R_j & R_k & R_l \\
\hline
 & o & \boxed{o} & p & q \\
 & \boxed{p} & & &
\end{array}
\rightarrow
\begin{array}{c|cccc}
R^2 & R_i^1 & R_j & R_k^1 & R_l \\
\hline
 & o & \boxed{o} & p & q \\
 & \boxed{p} & & o & \\
 & & & \boxed{q} &
\end{array}
\rightarrow
\begin{array}{c|cccc}
R^3 & R_i^1 & R_j & R_k^1 & R_l^1 \\
\hline
 & o & \boxed{o} & p & q \\
 & \boxed{p} & & o & p \\
 & & & \boxed{q} &
\end{array}.
$$

Next, consider

$$
\begin{array}{c|cccc}
R^4 & R_i^1 & R_j & R_k^1 & R_l^2 \\
\hline
 & o & o & p & p \\
 & p & & o & q \\
 & & & q &
\end{array}.
$$

Strategy-proofness for agent l implies $f_l(R^4) = l$. This is compatible with constrained efficiency only if $f_i(R^4) = p$, $f_j(R^4) = o$, and $f_k(R^4) = q$. Strategy-proofness for k implies that

$$
\begin{array}{c|cccc}
R^5 & R_i^1 & R_j & R_k^2 & R_l^2 \\
\hline
 & o & o & o & p \\
 & p & & p & q \\
 & & & \boxed{q} &
\end{array}.
$$

By constrained efficiency, $f_i(R^5) = o$ or $f_j(R^5) = o$.

We argue first that $f_j(R^5) = o$ is impossible. Suppose the contrary. Note that $i \rightarrow_p l \rightarrow_q j$ is an $(i, j; o, q)$-path which is compatible with the $(k, j; o, q)$-path $k \rightarrow_q j$. Furthermore, note that the fact that k ranks p as her second most preferred object is irrelevant since l ranks p first and has strictly higher priority for it than k. Given these observations, it is straightforward to modify the arguments in the proof of Lemma 4 in Ehlers and Westkamp (2017) to show that $f_j(R^5) = o$ implies that f cannot be strategy-proof and constrained efficient. Hence, $f_j(R^5) \neq o$ as was claimed above.

Since $f_j(R^5) \neq o$, we must have $f_i(R^5) = o$. By constrained efficiency, this implies $f_l(R^5) = p$. Then

$$\begin{array}{c|cccc} R^5 & R_i^1 & R_j & R_k^2 & R_l^2 \\ \hline & \boxed{o} & o & o & \boxed{p} \\ & p & & p & q \\ & & & \boxed{q} & \end{array} \rightarrow \begin{array}{c|cccc} R^6 & R_i^1 & R_j & R_k^2 & R_l^1 \\ \hline & o & o & o & \boxed{q} \\ & p & & p & p \\ & & & q & \end{array} \rightarrow \begin{array}{c|cccc} R^7 & R_i^1 & R_j & R_k^2 & R_l \\ \hline & o & o & o & \boxed{q} \\ & p & & p & \\ & & & q & \end{array}.$$

In moving from R^5 to R^6 we have used strategy-proofness for l to infer $f_l(R^6) \neq l$. If $f_l(R^6) = p$, then stability implies $f_i(R^6) = o$ and $f_k(R^6) = q$. But then l and k would form a stable improvement cycle, thus contradicting constrained efficiency. Hence, we must have $f_l(R^6) = q$. By strategy-proofness, $f_l(R^7) = q$. Since $k \succ_q l$, $f_l(R^7) = q$ is compatible with stability only if $f_k(R^7) \in \{o, p\}$. But given that $f_k(R^2) = q$ and k can unilaterally deviate from R^2 to obtain R^7, $f_k(R^7) \in \{o, p\}$ implies that f cannot be strategy-proof. Hence, $\succeq$ is unsolvable and this completes the proof. □

Lemma 6. *Fix a weak priority structure $\succeq$.*

(1) Let $i, j \in I$ be two distinct agents and $o \in O$ be an object such that $i \sim_o j$. If there is an $(i, j; o)$-path $i \rightarrow_{p^1} i^1 \cdots \rightarrow_{p^M} i^M \rightarrow_o j$ which is compatible with a $(j, i; o)$-path $j \rightarrow_{q^1} j^1 \cdots \rightarrow_{q^N} j^N \rightarrow_o i$, then $\succeq$ is unsolvable.

(2) Let $i, j, k, l \in I$ be four distinct agents and $o \in O$ be an object such that $i \sim_o j \sim_o k \sim_o l$. If there exist two objects $p, q \in O$ such that $i \succ_p k \succ_p j$ and $j \succ_q l \succ_q i$, then $\succeq$ is unsolvable.

Proof. (1) Suppose to the contrary that there exists a constrained efficient and strategy-proof mechanism f. Consider the following preference profile:

$$\begin{array}{c|cccc} R & R_i & R_j & R_{i^m} & R_{j^n} \\ \hline & o & o & p^m & q^n \\ & p^1 & q^1 & p^{m+1} & q^{n+1} \end{array}$$

Lemma 3 in Ehlers and Westkamp (2017) implies $f_i(R) = p^1$ and $f_j(R) = q^1$.

Now assume that i deviates to $R_i' : o$. By strategy-proofness, we must have $f_i(R_i', R_{-i}) = i$. We claim $f_j(R_i', R_{-i}) = o$. Otherwise, the construction of R would imply that, for all

$n \in \{0, \ldots, N\}$, $f_{j^n}(R'_i, R_{-i}) = q^{n+1}$. But then $j = j^0, j^1, \ldots, j^N$ would form a stable improvement cycle of $f(R'_i, R_{-i})$ at (R'_i, R_{-i}). Hence, we must have $f_j(R'_i, R_{-i}) = o$.

Next, assume that, starting from R, j deviates to $R'_j : o$. A completely symmetric argument to the one used to establish that $f_j(R'_i, R_{-i}) = o$ shows that we must have $f_i(R'_j, R_{-j}) = o$.

Finally, consider $R'' \equiv (R'_i, R'_j, R_{-i,j})$. Coming from the profile (R'_j, R_{-j}), strategy-proofness for i implies $f_i(R'') = o$. Coming from (R'_i, R_{-i}), strategy-proofness for j implies $f_j(R'') = o$. Since o cannot be allocated to more than one agent and $i \neq j$, we obtain a contradiction. Hence, there cannot be a constrained efficient and strategy-proof mechanism.

(2) This follows immediately from the first part of Lemma 5, since it implies that at the preference profile

$$\begin{array}{c|cccc} R & R_i & R_j & R_k & R_l \\ \hline & o & o & p & q, \end{array}$$

a constrained efficient and strategy-proof mechanism would have to (A) assign o to i given that $i \sim_o j \sim_o k$ and $i \succ_p k \succ_p j$, and (B) assign o to j given that $i \sim_o j \sim_o l$ and $j \succ_q l \succ_q i$. Since there is only one copy of o and $i \neq j$, (A) and (B) imply that there is no constrained efficient and strategy-proof mechanism. $\square$

Lemma 7. *Let i_1, i_2, i_3, i_4, i_5 be five distinct agents and o_1, o_2, o_3, o_4, o_5 be five distinct objects. Each of the following priority structures is unsolvable:*

$$\boxed{\begin{array}{ccccccc} i_1 & \sim_{o_1} & i_2 & \sim_{o_1} & i_3 & \sim_{o_1} & i_4 \\ \{i_1, i_2\} & \succ_{o_2} & i_3 & \succ_{o_2} & i_4 & & \\ \{i_1, i_2\} & \succ_{o_3} & i_3 & & & & \\ i_2 & \succeq_{o_4} & i_4 & \succ_{o_4} & i_1 & & \\ i_1 & \succeq_{o_5} & i_4 & \succ_{o_5} & i_2 & & \end{array}} \tag{1}$$

$$\boxed{\begin{array}{ccccccc} i_1 & \sim_{o_1} & i_2 & \sim_{o_1} & i_3 & \sim_{o_1} & i_4 \\ \{i_1, i_2\} & \succ_{o_2} & i_3 & \succ_{o_2} & i_4 & & \\ i_4 & \succ_{o_3} & \{i_2, i_5\} & & & & \\ i_2 & \succ_{o_4} & i_5 & \succ_{o_4} & i_1 & & \end{array}} \tag{2}$$

$$\boxed{\begin{array}{lllllllll} i_1 & \sim_{o_1} & i_2 & \sim_{o_1} & i_3 \\ i_1 & \sim_{o_2} & i_4 & \sim_{o_2} & i_5 & \succ_{o_2} & i_2 & \succ_{o_2} & i_3 \\ i_4 & \succ_{o_3} & i_5 & \succ_{o_3} & i_1 \\ \{i_2, i_3\} & \succ_{o_4} & i_4 \end{array}} \tag{3}$$

$$\boxed{\begin{array}{lllll} i_1 & \sim_{o_1} & i_2 & \sim_{o_1} & i_3 \\ \{i_2, i_3\} & \succ_{o_2} & i_4 \\ i_4 & \succ_{o_3} & i_1 & \succ_{o_3} & \{i_2, i_3\} \end{array}} \tag{4}$$

$$\boxed{\begin{array}{lllll} i_1 & \sim_{o_1} & i_2 & \sim_{o_1} & i_3 \\ i_1 & \succ_{o_2} & i_4 \\ i_2 & \succ_{o_3} & i_5 \\ \{i_2, i_4\} & \succ_{o_4} & i_3 & \succ_{o_4} & i_1 \\ \{i_1, i_5\} & \succ_{o_5} & i_3 & \succ_{o_5} & i_2 \end{array}} \tag{5}$$

$$\boxed{\begin{array}{lllll} i_1 & \sim_{o_1} & i_2 & \sim_{o_1} & i_3 \\ \{i_1, i_2\} & \succ_{o_2} & i_5 & \succ_{o_2} & i_3 \\ i_2 & \succ_{o_3} & i_4 \\ i_3 & \succ_{o_4} & i_5 & \succ_{o_4} & i_1 \\ \{i_3, i_4\} & \succ_{o_5} & i_5 & \succ_{o_5} & i_2 \end{array}} \tag{6}$$

Proof. For each of the six priority structures defined in Lemma 7, we will use a proof by contradiction. Throughout the proof, let f be an arbitrary constrained efficient and strategy-proof mechanism.

1. The priority structure in Eq. (1) is unsolvable.

   The proof revolves around the following preference profile:

$$\begin{array}{c|cccc} R & R_{i_1} & R_{i_2} & R_{i_3} & R_{i_4} \\ \hline & o_1 & o_1 & o_3 & o_2 \end{array}.$$

   Claim 1: $f_{i_2}(R) = o_1$.

   We start by considering the following preference profile

$\tilde{R}$	R_{i_1}	R_{i_2}	$\tilde{R}_{i_3}$	$\tilde{R}_{i_4}$
	o_1	o_1	o_2	o_4

Since $\{i_1, i_2\} \succ_{o_2} i_3 \succ_{o_2} i_4$ and $i_2 \succeq_{o_4} i_4 \succ_{o_4} i_1$, the second part of Lemma 5 implies that we must have $f_{i_2}(\tilde{R}) = o_1$.

Next, note that strategy-proofness for agent i_2 implies that i_2 must still obtain object o_1 at the following profile:[1]

R^1	R_{i_1}	$R^1_{i_2}$	$\tilde{R}_{i_3}$	$\tilde{R}_{i_4}$
	o_1	$\boxed{o_1}$	$\boxed{o_2}$	$\boxed{o_4}$
		o_4		

Since i_2 obtains o_1 at R^1, constrained efficiency requires $f_{i_4}(R^1) = o_4$. Next, consider the preference profile

R^2	R_{i_1}	$R^1_{i_2}$	$\tilde{R}_{i_3}$	$R^1_{i_4}$
	o_1	o_1	$\boxed{o_2}$	o_2
		o_4		$\boxed{o_4}$

By strategy-proofness for agent i_4, we must have $f_{i_4}(R^2) \in \{o_2, o_4\}$. Since i_3 ranks o_2 first and $i_3 \succ_{o_2} i_4$, stability implies $f_{i_4}(R^2) \neq o_2$. Hence, we must have $f_{i_4}(R^2) = o_4$. We will now show that $f_{i_4}(R^2) = o_4$ is only possible when $f_{i_2}(R^2) = o_1$. If $i_2 \succ_{o_4} i_4$, then $f_{i_2}(R^2) = o_1$ follows immediately from stability of $f(R^2)$. So suppose that $i_2 \sim_{o_4} i_4$ and that, contrary to what we want to show, $f_{i_1}(R^2) = o_1$. Since $f_{i_4}(R^2) = o_4$ and $f_{i_1}(R^2) = o_1$, we must have $f_{i_2}(R^2) = i_2$. We derive a contradiction to the assumed properties of f using the following sequence of preference profiles:

[1] Recall that boxes indicate assigned objects.

$$
\begin{array}{c|cccc}
R^{2,1} & R_{i_1} & R^2_{i_2} & \tilde{R}_{i_3} & R^1_{i_4} \\ \hline
 & \boxed{o_1} & o_4 & \boxed{o_2} & o_2 \\
 & & & & \boxed{o_4}
\end{array}
\quad\rightarrow\quad
\begin{array}{c|cccc}
R^{2,2} & R_{i_1} & R^2_{i_2} & \tilde{R}_{i_3} & R^2_{i_4} \\ \hline
 & \boxed{o_1} & o_4 & \boxed{o_2} & o_1 \\
 & & & & \boxed{o_4}
\end{array}
$$

$$
\downarrow
$$

$$
\begin{array}{c|cccc}
R^{2,4} & R_{i_1} & R^1_{i_2} & \tilde{R}_{i_3} & R^2_{i_4} \\ \hline
 & \boxed{o_1} & o_1 & \boxed{o_2} & o_1 \\
 & & o_4 & & \boxed{o_4}
\end{array}
\quad\leftarrow\quad
\begin{array}{c|cccc}
R^{2,3} & R_{i_1} & R^3_{i_2} & \tilde{R}_{i_3} & R^2_{i_4} \\ \hline
 & \boxed{o_1} & o_4 & \boxed{o_2} & o_1 \\
 & & o_1 & & \boxed{o_4}
\end{array}
$$

Given that $f_{i_2}(R^2) = i_2$, the indicated assignment at $R^{2,1}$ follows immediately from strategy-proofness and stability. Strategy-proofness implies that $f_{i_4}(R^{2,2}) \in \{o_1, o_4\}$ since $f_{i_4}(R^{2,1}) = o_4$. If $f_{i_4}(R^{2,2}) = o_1$, then i_4 would still have to obtain o_1 when she unilaterally deviates to $R^3_{i_4} = o_1$. However, since $i_1 \succ_{o_2} i_3 \succ_{o_2} i_4$ and $i_1 \sim_{o_1} i_3 \sim_{o_1} i_4$, $f_{i_4}(R^3_{i_4}, R^{2,2}_{-i_4}) = o_1$ is incompatible with the tie-breaking rule in the first part of Lemma 5. Hence, we must have $f_{i_4}(R^{2,2}) = o_4$ and $f_{i_1}(R^{2,2}) = o_1$. Strategy-proofness and non-wastefulness imply $f_{i_2}(R^{2,3}) \neq o_4$ and $f_{i_4}(R^{2,3}) = o_4$. If $f_{i_2}(R^{2,3}) = o_1$, then i_2 and i_4 would form a stable improvement cycle of $f(R^{2,3})$ at $R^{2,3}$, a contradiction. Hence, we must have $f_{i_2}(R^{2,3}) = i_2$. By strategy-proofness, we must also have $f_{i_2}(R^{2,4}) = i_2$. Non-wastefulness then implies that $f_{i_4}(R^{2,4}) = o_4$ and $f_{i_1}(R^{2,4}) = o_1$. Given that $\{i_2, i_4\} \succ_{o_4} i_1$, $i_2 \rightarrow_{o_4} i_1$ is an $(i_2, i_1; o_1, o_4)$-path which is compatible with the $(i_4, i_1; o_1, o_4)$-path $i_4 \rightarrow_{o_4} i_1$. Since $i_2 \sim_{o_1} i_4 \sim_{o_1} i_1$, $f_{i_1}(R^{2,4}) = o_1$ contradicts the tie-breaking rule in Lemma 4 in Ehlers and Westkamp (2017). Hence, $f_{i_1}(R^2) = o_1$ also leads to a contradiction when $i_2 \sim_{o_4} i_4$ and we must have $f_{i_2}(R^2) = o_1$.

The following diagram summarizes the remainder of the proof of Claim 1:

$$
\begin{array}{c|cccc}
R^{2} & R_{i_1} & R^1_{i_2} & \tilde{R}_{i_3} & R^1_{i_4} \\ \hline
 & o_1 & \boxed{o_1} & \boxed{o_2} & o_2 \\
 & & o_4 & & \boxed{o_4}
\end{array}
\;\rightarrow\;
\begin{array}{c|cccc}
R^{3} & R^1_{i_1} & R^1_{i_2} & \tilde{R}_{i_3} & R^1_{i_4} \\ \hline
 & o_1 & \boxed{o_1} & o_2 & o_2 \\
 & \boxed{o_2} & o_4 & & \boxed{o_4}
\end{array}
\;\rightarrow\;
\begin{array}{c|cccc}
R^{4} & R^1_{i_1} & R^1_{i_2} & R^1_{i_3} & R^1_{i_4} \\ \hline
 & o_1 & \boxed{o_1} & o_2 & o_2 \\
 & \boxed{o_2} & o_4 & \boxed{o_3} & \boxed{o_4}
\end{array}
$$

$$
\downarrow
$$

$$
\begin{array}{c|cccc}
R^{7} & R^1_{i_1} & R^4_{i_2} & R_{i_3} & R_{i_4} \\ \hline
 & o_1 & \boxed{o_1} & \boxed{o_3} & o_2 \\
 & \boxed{o_2} & o_3 & &
\end{array}
\;\leftarrow\;
\begin{array}{c|cccc}
R^{6} & R^1_{i_1} & R^4_{i_2} & R_{i_3} & R^1_{i_4} \\ \hline
 & o_1 & \boxed{o_1} & \boxed{o_3} & o_2 \\
 & \boxed{o_2} & o_3 & & \boxed{o_4}
\end{array}
\;\leftarrow\;
\begin{array}{c|cccc}
R^{5} & R^1_{i_1} & R^4_{i_2} & R^1_{i_3} & R^1_{i_4} \\ \hline
 & o_1 & \boxed{o_1} & o_2 & o_2 \\
 & \boxed{o_2} & o_3 & \boxed{o_3} & \boxed{o_4}
\end{array}
$$

By strategy-proofness, we must have $f_{i_1}(R^3) \neq o_1$ given that $f_{i_1}(R^2) = i_1$. Since

$i_1 \succ_{o_2} \{i_3, i_4\}$, stability requires $f_{i_1}(R^3) = o_2$ and $f_{i_3}(R^3) = i_3$. Strategy-proofness and stability then imply $f_{i_3}(R^4) = o_3$ and $f_{i_1}(R^4) = o_2$. Hence, $f_{i_2}(R^4) = o_1$ and another application of strategy-proofness yields $f_{i_2}(R^5) = o_1$. Since $f_{i_3}(R^5) = o_3$ if $f_{i_2}(R^5) = o_1$, strategy-proofness requires that $f_{i_3}(R^6) = o_3$ as well. Given that $i_2 \succ_{o_3} i_3$, $f_{i_3}(R^6) = o_3$ implies $f_{i_2}(R^6) = o_1$. Finally, $f_{i_2}(R^6) = o_1$ and $i_1 \succ_{o_2} i_4$ imply $f_{i_1}(R^6) = o_2$ and $f_{i_4}(R^6) \neq o_2$. Strategy-proofness allows us to infer $f_{i_4}(R^7) = i_4$, which is compatible with constrained efficiency only if $f_{i_1}(R^7) = o_2$ and $f_{i_2}(R^7) = o_1$. It is now straightforward to see that two more applications of strategy-proofness from R^7 yield the desired statement that $f_{i_2}(R) = o_1$. This completes the proof of Claim 1. □

Claim 2: $f_{i_1}(R) = o_1$.

Consider the following preference profile

$\hat{R}$	R_{i_1}	R_{i_2}	R_{i_3}	R'_{i_4}
	o_1	o_1	o_2	o_5

Since $\{i_1, i_2\} \succ_{o_2} i_3 \succ_{o_2} i_4$ and $i_1 \succeq_{o_5} i_4 \succ_{o_5} i_2$, the second part of Lemma 5 implies that we must have $f_{i_1}(\hat{R}) = o_1$. A completely analogous argument to that used in the proof of Claim 1 shows that $f_{i_1}(\hat{R}) = o_1$ implies $f_{i_1}(R) = o_1$.[2] □

Combining Claims 1 and 2, we find that a constrained efficient and strategy-proof mechanism for $\succeq$ has to satisfy $f_{i_1}(R) = f_{i_2}(R) = o_1$. Since there is only one copy of o_1, this is a contradiction. Hence, there exists no constrained efficient and strategy-proof mechanism for priority structure in Eq. (1). □

2. The priority structure in Eq. (2) is unsolvable.

   As usual, arrows indicate how we move between profiles and boxes indicate object assignments.

[2] In the arguments so far, we have used that $i_2 \succeq_{o_4} i_4 \succ_{o_4} i_1$, $i_1 \succ_{o_2} i_3 \succ_{o_2} i_4$, and $i_2 \succ_{o_3} i_3$. Since $i_1 \succeq_{o_5} i_4 \succ_{o_5} i_2$, $i_2 \succ_{o_2} i_3 \succ_{o_2} i_4$, and $i_1 \succ_{o_3} i_3$, one just has to switch the roles of i_1 and i_2 and the roles of o_4 and o_5 in the proof of Claim 1 to establish that $f_{i_1}(\hat{R}) = o_1$ implies $f_{i_1}(R) = o_1$. We omit the details.

$$
\begin{array}{ccc}
\begin{array}{c|ccccc}
R & R_{i_1} & R_{i_2} & R_{i_3} & R_{i_4} & R_{i_5} \\
\hline
 & o_4 & \boxed{o_1} & o_2 & o_1 & o_3 \\
 & & & & &
\end{array}
& \rightarrow &
\begin{array}{c|ccccc}
R^1 & R_{i_1} & R_{i_2} & R_{i_3} & R^1_{i_4} & R_{i_5} \\
\hline
 & o_4 & \boxed{o_1} & o_2 & o_1 & o_3 \\
 & & & & \boxed{o_3} &
\end{array} \\
 & & \downarrow \\
\begin{array}{c|ccccc}
R^3 & R^1_{i_1} & R_{i_2} & R_{i_3} & R^1_{i_4} & R^1_{i_5} \\
\hline
 & o_4 & \boxed{o_1} & o_2 & o_1 & o_3 \\
 & o_1 & & & \boxed{o_3} & \boxed{o_4}
\end{array}
& \leftarrow &
\begin{array}{c|ccccc}
R^2 & R_{i_1} & R_{i_2} & R_{i_3} & R^1_{i_4} & R^1_{i_5} \\
\hline
 & o_4 & \boxed{o_1} & o_2 & o_1 & o_3 \\
 & & & & \boxed{o_3} & \boxed{o_4}
\end{array} \\
\downarrow & & \\
\begin{array}{c|ccccc}
R^4 & R^1_{i_1} & R^1_{i_2} & R_{i_3} & R^1_{i_4} & R^1_{i_5} \\
\hline
 & o_4 & \boxed{o_1} & o_2 & o_1 & o_3 \\
 & o_1 & o_4 & & \boxed{o_3} & \boxed{o_4}
\end{array}
& \rightarrow &
\begin{array}{c|ccccc}
R^5 & R^2_{i_1} & R^1_{i_2} & R_{i_3} & R^1_{i_4} & R^1_{i_5} \\
\hline
 & o_1 & \boxed{o_1} & o_2 & o_1 & o_3 \\
 & o_4 & o_4 & & \boxed{o_3} & \boxed{o_4}
\end{array}
\end{array}
$$

Note that $f_{i_2}(R) = o_1$ follows from the first part of Lemma 5 since $i_2 \succ_{o_2} i_3 \succ_{o_2} i_4$ and $i_2 \sim_{o_1} i_3 \sim_{o_1} i_4$. By strategy-proofness for i_4, we must have $f_{i_4}(R^1) \neq o_1$. Since $i_4 \succ_{o_3} i_5$, stability then implies that $f_{i_4}(R^1) = o_3$. By strategy-proofness for i_5, we must have $f_{i_5}(R^2) \neq o_3$. Since $i_5 \succ_{o_4} i_1$, stability then implies that $f_{i_5}(R^2) = o_4$. By strategy-proofness for i_1, we must have $f_{i_1}(R^3) \neq o_4$. If $f_{i_1}(R^3) = o_1$, then non-wastefulness would imply $f_{i_4}(R^3) = o_3$ and $f_{i_5}(R^3) = o_4$. But then i_1, i_4, and i_5 would form a stable improvement cycle at R^3, thus contradicting constrained efficiency of $f(R^3)$. Hence, $f_{i_1}(R^3) = i_1$ (and the indicated assignments at R^3 then follow from constrained efficiency). Strategy-proofness for i_2 implies $f_{i_2}(R^4) = o_1$. The indicated assignments then follow immediately from stability given that $i_4 \succ_{o_3} i_5$ and $i_5 \succ_{o_4} i_1$. Strategy-proofness for i_1 implies that $f_{i_1}(R^5) = i_1$. We will now argue that we must have $f_{i_2}(R^5) = o_1$: Otherwise, strategy-proofness for i_2 would imply that for the preference profile

$$
\begin{array}{c|ccccc}
R^{5,1} & R^2_{i_1} & R_{i_2} & R_{i_3} & R^1_{i_4} & R^2_{i_5} \\
\hline
 & o_1 & o_1 & o_2 & o_1 & o_3 \\
 & o_4 & & & o_3 & o_4
\end{array},
$$

we must have $f_{i_2}(R^{5,1}) = i_2$. Since i_1 can obtain $R^{5,1}$ from R^3 by a unilateral deviation (from $R^1_{i_1}$ to $R^2_{i_1}$) and since $f_{i_1}(R^3) = i_1$, strategy-proofness for i_1 would then imply $f_{i_1}(R^{5,1}) = i_1$. Now note that $f_{i_2}(R^{5,1}) = i_2$ and $f_{i_1}(R^{5,1}) = i_1$ could be compatible with non-wastefulness only if $f_{i_4}(R^{5,1}) = o_1$ and $f_{i_5}(R^{5,1}) = o_4$. But then o_3 would remain unassigned even though $o_3 P_{i_5}(R^{5,1}) o_4$. Hence, $f(R^{5,1})$ cannot be stable if $f_{i_2}(R^5) \neq o_1$.

Next, consider

R^6	$R^2_{i_1}$	$R^1_{i_2}$	R_{i_3}	$R^1_{i_4}$	$R^2_{i_5}$
	o_1	$\boxed{o_1}$	o_2	o_1	$\boxed{o_4}$
	o_4	o_4		$\boxed{o_3}$	

Strategy-proofness and $f_{i_5}(R^5) = o_4$ imply $f_{i_5}(R^6) = o_4$. Since $i_2 \succ_{o_4} i_5$, stability implies $f_{i_2}(R^6) = o_1$. Finally, consider the profile

R^7	$R^2_{i_1}$	$R^2_{i_2}$	R_{i_3}	$R^1_{i_4}$	$R^2_{i_5}$
	o_1	o_3	o_2	o_1	o_4 .
	o_4	o_1		o_3	

By strategy-proofness for i_2, we must have $f_{i_2}(R^7) \in \{o_1, o_3\}$. If $f_{i_2}(R^7) = o_1$, then non-wastefulness would require $f_{i_4}(R^7) = o_3$. But then i_2 and i_4 would form a stable improvement cycle, thus contradicting the constrained efficiency of $f(R^7)$. Hence, we must have $f_{i_2}(R^7) = o_3$. Since $i_4 \succ_{o_3} i_2$, this requires $f_{i_4}(R^7) = o_1$. It is straightforward to show that strategy-proofness implies that i_4 must still obtain o_1 when, starting from R^7, i_2 first deletes o_1 from her preferences (again since $i_4 \succ_{o_3} i_2$), i_1 then deletes o_4 from her preferences and, finally, i_4 deletes o_3 from her preferences. Since $i_1 \succ_{o_2} i_3 \succ_{o_2} i_4$ and $i_1 \sim_{o_1} i_3 \sim_{o_1} i_4$, we obtain a contradiction to the first part of Lemma 5.

3. The priority structure in Eq. (3) is unsolvable.

   Consider the following preference profile

R	R_{i_1}	R_{i_2}	R_{i_3}	R_{i_4}	R_{i_5}
	o_1	o_1	o_1	o_4	o_3 .
	o_2	o_4	o_4	o_2	

Since, for $k \in \{2, 3\}$ and $l \in \{2, 3\} \setminus \{k\}$, $i_1 \to_{o_2} i_k$ and $i_l \to_{o_4} i_4 \to_{o_2} i_k$ are two compatible paths, Lemma 4 in Ehlers and Westkamp (2017) and constrained efficiency immediately imply $f_{i_1}(R) = o_1$. We will now complete the proof by showing that $f_{i_1}(R) = o_1$ is impossible as well. The proof relies on the following sequence of profiles:

R^1	$R^1_{i_1}$	R_{i_2}	R_{i_3}	R_{i_4}	R_{i_5}
	o_2	o_1	o_1	o_4	o_3
	$\boxed{o_1}$	o_4	o_4	o_2	

$\rightarrow$

R^2	$R^1_{i_1}$	R_{i_2}	$R^1_{i_3}$	R_{i_4}	R_{i_5}
	o_2	$\boxed{o_1}$	o_1	o_4	o_3
	o_1	o_4		o_2	

$\downarrow$ $\downarrow$

R^3	$R^1_{i_1}$	$R^1_{i_2}$	R_{i_3}	R_{i_4}	R_{i_5}
	o_2	o_1	$\boxed{o_1}$	o_4	o_3
	o_1		o_4	o_2	

$\rightarrow$

R^4	$R^1_{i_1}$	$R^1_{i_2}$	$R^2_{i_3}$	R_{i_4}	R_{i_5}
	o_2	$\boxed{o_1}$	$\boxed{o_1}$	o_4	o_3
	o_1			o_2	

By strategy-proofness for i_1, we must have $f_{i_1}(R^1) \in \{o_1, o_2\}$. Assume first that $f_{i_1}(R^1) = o_2$. Given that $\{i_2, i_3\} \succ_{o_4} i_4$, stability would then imply $f_{i_4}(R^1) = i_4$. It is easy to see that two applications of strategy-proofness (for i_4 and then for i_1) yield

$\tilde{R}^1$	$R^2_{i_1}$	R_{i_2}	R_{i_3}	$R^2_{i_4}$	R_{i_5}
	$\boxed{o_2}$	o_1	o_1	o_2	o_3 .
		o_4	o_4		

Since $i_4 \succ_{o_3} i_5 \succ_{o_3} i_1$ and $i_1 \sim_{o_2} i_4 \sim_{o_2} i_5$, we obtain a contradiction to the first part of Lemma 5. Hence, we must have $f_{i_1}(R^1) = o_1$. By strategy-proofness for i_3 and constrained efficiency, $f_{i_1}(R^1) = o_1$ implies $f_{i_2}(R^2) = o_1$. Similarly, $f_{i_3}(R^3) = o_1$. But then strategy-proofness for i_2 and i_3 would imply that $f_{i_2}(R^4) = f_{i_3}(R^4) = o_1$, which is impossible. Since every possible allocation of o_1 at R necessarily leads to a contradiction of either strategy-proofness or constrained efficiency, the priority structure in Eq. (3) is unsolvable.

4. The priority structure in Eq. (4) is unsolvable.

   Consider the following preference profile

R	R_{i_1}	R_{i_2}	R_{i_3}	R_{i_4}
	o_1	o_1	o_1	o_2 .
	o_3	o_2	o_2	o_3

   Note that $i_2 \rightarrow_{o_2} i_4 \rightarrow_{o_3} i_1$ is an $(i_2, i_1; o_1, o_3)$-path which is compatible with the $(i_3, i_1; o_1, o_3)$-path $i_3 \rightarrow_{o_2} i_4 \rightarrow_{o_3} i_1$, and Lemma 4 in Ehlers and Westkamp (2017) implies $f_{i_1}(R) \neq o_1$. Similarly, $i_1 \rightarrow_{o_3} i_2$ and $i_3 \rightarrow_{o_2} i_4 \rightarrow_{o_3} i_2$ are compatible paths, so that Lemma 4 in Ehlers and Westkamp (2017) implies $f_{i_2}(R) \neq o_1$, and $i_1 \to_{o_3} i_3$ and $i_2 \to_{o_2} i_4 \to_{o_3} i_3$ are compatible paths, so that Lemma 4 in Ehlers and Westkamp (2017) implies $f_{i_3}(R) \neq o_1$. Thus, o_1 must remain unallocated at R, contradicting non-wastefulness of $f(R)$.

5. The priority structure in Eq. (5) is unsolvable.

Consider the following preference profile

$$\begin{array}{c|ccccc} R & R_{i_1} & R_{i_2} & R_{i_3} & R_{i_4} & R_{i_5} \\ \hline & o_1 & o_1 & o_4 & o_2 & o_3 \\ & o_2 & o_3 & o_5 & o_4 & o_5 \end{array}.$$

We claim that $f_{i_2}(R) = o_1$. Consider first the preference profile

$$\begin{array}{c|ccccc} R' & R'_{i_1} & R'_{i_2} & R'_{i_3} & R'_{i_4} & R'_{i_5} \\ \hline & o_1 & o_1 & o_4 & o_2 & o_3 \end{array}.$$

Since $i_2 \succ_{o_4} i_3 \succ_{o_4} i_1$, the first part of Lemma 5 implies $f_{i_2}(R') = o_1$. Two applications of strategy-proofness, once for i_1 and once for i_2, imply that $f_{i_2}(R_{i_1}, R_{i_2}, R'_{-\{i_1,i_2\}}) = o_1$. Non-wastefulness then implies that $f_{i_5}(R_{i_1}, R_{i_2}, R'_{-\{i_1,i_2\}}) = o_3$. By strategy-proofness for i_5, we must have $f_{i_5}(R_{i_1}, R_{i_2}, R_{i_5}, R'_{-\{i_1,i_2,i_5\}}) = o_3$. Since $i_2 \succ_{o_3} i_5$ and $i_1 \succ_{o_2} i_4$, stability implies that $f_{i_1}(R_{i_1}, R_{i_2}, R_{i_5}, R'_{-\{i_1,i_2,i_5\}}) = o_2$, $f_{i_2}(R_{i_1}, R_{i_2}, R_{i_5}, R'_{-\{i_1,i_2,i_5\}}) = o_1$, and $f_{i_4}(R_{i_1}, R_{i_2}, R_{i_5}, R'_{-\{i_1,i_2,i_5\}}) = i_4$. By strategy-proofness for i_4, stability, and the assumption that $i_4 \succ_{o_4} i_3$, we must have $f_{i_4}(R_{i_1}, R_{i_2}, R'_{i_3}, R_{i_4}, R_{i_5}) = o_4$ and $f_{i_3}(R_{i_1}, R_{i_2}, R'_{i_3}, R_{i_4}, R_{i_5}) = i_3$. By strategy-proofness for i_3, we must have $f_{i_3}(R_{i_1}, R_{i_2}, R_{i_3}, R_{i_4}, R_{i_5}) \neq o_4$. Stability is easily seen to imply $f_{i_4}(R_{i_1}, R_{i_2}, R_{i_3}, R_{i_4}, R_{i_5}) = o_4$, $f_{i_1}(R_{i_1}, R_{i_2}, R_{i_3}, R_{i_4}, R_{i_5}) = o_2$, and, as we claimed above, $f_{i_2}(R_{i_1}, R_{i_2}, R_{i_3}, R_{i_4}, R_{i_5}) = o_1$.

Next, consider the following preference profile

$$\begin{array}{c|ccccc} \tilde{R} & R_{i_1} & R_{i_2} & \tilde{R}_{i_3} & R_{i_4} & R_{i_5} \\ \hline & o_1 & o_1 & o_5 & o_2 & o_3 \\ & o_2 & o_3 & & o_4 & o_5 \end{array}.$$

We claim that $f_{i_1}(\tilde{R}) = o_1$. The proof of the claim proceeds from the preference profile

$$\begin{array}{c|ccccc} \tilde{R}' & R'_{i_1} & R'_{i_2} & \tilde{R}_{i_3} & R'_{i_4} & R'_{i_5} \\ \hline & o_1 & o_1 & o_5 & o_2 & o_3 \end{array}.$$

and is similar to the proof that $f_{i_2}(R) = o_1$.

Given that $f_{i_1}(\tilde{R}) = o_1$, $i_2 \succ_{o_3} i_5$, and $i_5 \succ_{o_5} i_3$, stability implies $f_{i_3}(\tilde{R}) = i_3$. Furthermore, given that $f_{i_2}(R) = o_1$, $i_1 \succ_{o_2} i_4$, and $i_4 \succ_{o_4} i_3$, stability implies $f_{i_3}(R) = o_5$. But then, since i_3 can obtain R from $\tilde{R}$ by a unilateral deviation from $\tilde{R}_{i_3}$ to R_{i_3}, f cannot be strategy-proof.

6. The priority structure in Eq. (6) is unsolvable.

Consider first the following preference profile

$$\begin{array}{c|ccccc} R^1 & R_{i_1} & R_{i_2} & R_{i_3} & R_{i_4} & R_{i_5} \\ \hline & o_1 & o_1 & o_2 & o_3 & o_5 \end{array}.$$

Because $i_1 \sim_{o_1} i_2 \sim_{o_1} i_3$, $i_1 \succ_{o_2} i_5 \succ_{o_2} i_3$ and $i_3 \succ_{o_5} i_5 \succ_{o_5} i_2$, the third part of Lemma 5 implies $f_{i_1}(R^1) = o_1$. It is straightforward to verify that strategy-proofness and constrained efficiency imply

$$\begin{array}{c|ccccc} \tilde{R}^1 & R^1_{i_1} & R^1_{i_2} & R^1_{i_3} & R^1_{i_4} & R_{i_5} \\ \hline & \boxed{o_1} & o_1 & \boxed{o_2} & o_3 & o_5 \\ & o_2 & \boxed{o_3} & o_4 & \boxed{o_5} & \end{array}.$$

Next, consider first the following preference profile

$$\begin{array}{c|ccccc} R^2 & R_{i_1} & R_{i_2} & R_{i_3} & R_{i_4} & R^1_{i_5} \\ \hline & o_1 & o_1 & o_2 & o_3 & o_4 \end{array}.$$

Because $i_1 \sim_{o_1} i_2 \sim_{o_1} i_3$, $i_2 \succ_{o_2} i_5 \succ_{o_2} i_3$ and $i_3 \succ_{o_4} i_5 \succ_{o_4} i_1$, the third part of Lemma 5 implies $f_{i_2}(R^2) = o_1$. It is straightforward to verify that strategy-proofness and constrained efficiency imply

$\tilde{R}^2$	$R^1_{i_1}$	$R^1_{i_2}$	$R^1_{i_3}$	$R^1_{i_4}$	$R^2_{i_5}$
	o_1	$\boxed{o_1}$	o_2	$\boxed{o_3}$	o_4
	$\boxed{o_2}$	o_3	$\boxed{o_4}$	o_5	$\boxed{o_5}$

.

Since i_5 can obtain $\tilde{R}^2$ from $\tilde{R}^1$ by a unilateral deviation (from R_{i_5} to $R^2_{i_5}$), we obtain that f cannot be strategy-proof. This shows that f cannot be constrained efficient and strategy-proof.

□

1.1 Necessity of Assumption 1 in Ehlers and Westkamp (2017)

In this section, we present two examples: in the first example, the priority structure satisfies Assumption 1 (B) in Ehlers and Westkamp (2017) but violates Assumption 1 (A) in Ehlers and Westkamp (2017); in the second example, the priority structure satisfies Assumption 1 (A) in Ehlers and Westkamp (2017) but violates Assumption 1 (B) in Ehlers and Westkamp (2017). For both examples, we show that a constrained efficient and strategy-proof mechanism exists even though the priority structures are not strict, HET, or TAU. This shows that both parts of Assumption 1 in Ehlers and Westkamp (2017) are necessary for Theorem 1 in Ehlers and Westkamp (2017) to hold.

Example 1. Let $I = \{1, \dots, 6\}$ and $O = \{o, p_1, p_2\}$. Priorities are as follows:

$\succeq$	$\succeq_o$	$\succeq_{p_1}$	$\succeq_{p_2}$
	$1,2,3$	6	6
	4	5	5
	5	4	4
	6	3	1
		2	2
		1	3

This priority structure violates Assumption 1 (A) (because $\succeq$ is not strict and $\succeq$ does not contain any four-way tie), but satisfies in Assumption 1 (B) in Ehlers and Westkamp

(2017). We will show below that $\succeq$ is solvable by constructing a variant of the DA that yields a constrained efficient and strategy-proof mechanism.[3]

Example 2. Let $I = \{1, \ldots, 6\}$ and $O = \{o, p_1, p_2, p_3, p_4\}$. Priorities are as follows:

$\succeq$	$\succ_o$	$\succeq_{p_1}$	$\succeq_{p_2}$	$\succeq_{p_3}$	$\succeq_{p_4}$
	$1,2,3,4,5,6$	6	5	6	6
		5	6	4	5
		4	4	5	2
		3	1	2	4
		2	2	1	1
		1	3	3	3

Note that this priority structure violates Assumption 1 (B) (since, e.g., $6 \succ_{p_1} 1$ and $6 \succeq_q 1$ for all $q \in O$) but satisfies Assumption 1 (A) in Ehlers and Westkamp (2017) (because there is a four-way tie at $\succeq_o$).[4,5]

We will now proceed to construct a *deferred acceptance algorithm with tie-breaking (DAT)* that, as we will show below, is constrained efficient and strategy-proof mechanism for both examples. Let R be an arbitrary preference profile for the six agents in one of the above examples.

Step 1: Exogenous tie-breaking

For Example 1, define the weak priority structure $\succeq'$ by setting $1 \sim'_o 3 \succ'_o 2 \succ'_o 4 \succ'_o 5 \succ'_o 6$ and $\succeq'_p = \succeq_p$ for $p \in \{p_1, p_2\}$.

For Example 2, define the weak priority structure $\succeq'$ by setting $6 \succ'_o 5 \succ'_o 4 \succ'_o 2 \succ'_o 1 \sim'_o 3$ and $\succeq'_p = \succeq_p$ for $p \in \{p_1, p_2, p_3, p_4\}$.

[3]This example can be extended to an arbitrary number of agents $1, \ldots, N$ as follows: Let $1 \sim_o 2 \sim_o 3 \succ_o 4 \succ_o \ldots \succ_o N$, $N \succ_{p_1} \ldots \succ_{p_1} 4 \succ_{p_1} 3 \succ_{p_1} 2 \succ_{p_1} 1$, and $N \succ_{p_2} \ldots \succ_{p_2} 4 \succ_{p_2} 1 \succ_{p_2} 2 \succ_{p_2} 3$. It is easy to see that all arguments below continue to hold for this extended example.

[4]The priority structure does, however, satisfy the weaker requirement that the priority structure is *connected* in the sense that there is no subset of agents $J \subsetneq I$ such that $J \succeq_o I \setminus J$ for all o.

[5]In order to extend this type of example to an arbitrary number of agents $1, \ldots, N$, let there be $N-1$ objects $o, p_1, \ldots, p_{N-2}$ such that $N \sim_o \ldots \sim_o 1$, $N \succ_{p_1} \ldots \succ_{p_1} 1$, $N-1 \succ_{p_2} N \succ_{p_2} N-2 \succ_{p_2} \ldots \succ_{p_2} 4 \succ_{p_2} 1 \succ_{p_2} 2 \succ_{p_2} 3$, $N \succ_{p_3} N-2 \succ_{p_3} N-1 \succ_{p_3} N-3 \succ_{p_3} \ldots \succ_{p_3} 4 \succ_{p_3} 2 \succ_{p_3} 1 \succ_{p_3} 3$, $\ldots$, $N \succ_{p_{N-2}} \ldots \succ_{p_{N-2}} 5 \succ_{p_{N-2}} 2 \succ_{p_{N-2}} 4 \succ_{p_{N-2}} 1 \succ_{p_{N-2}} 3$. As shown in our earlier working paper, Ehlers and Westkamp (2011), the just described construction can be used to characterize all solvable priority structures within the class of priority structures where ties are restricted to occur only at the bottom of priority rankings.

Step 2: DA without tie-breaking

Run a DA in which rejections are determined by $\succeq'$ and let μ^1 be the temporary assignment at the end of this algorithm.[6]

Stop, if μ^1 is a matching and proceed to Step 3 otherwise. Note that, given our construction of $\succeq'$, the only possibility for the procedure to proceed to Step 3 is that $\mu^1(1) = \mu^1(3) = o$.

Step 3: Endogenous tie-breaking

If there is an object p such that $\mu^1(2) = p$ and $3 \succ_p 2 \succ_p 1$, let o reject 1. If there is an object $q \in O$ that is acceptable to agent 1 and most preferred among the ones which have not received any proposals in Step 2, match 1 to q and stop. Otherwise, leave 1 unmatched and stop.

In any other case, let o reject 3. If there is an object $q \in O$ that is acceptable to agent 3 and most preferred among the ones which have not received any proposals in Step 2, match 3 to q and stop. Otherwise, leave 3 unmatched and stop.

Given a preference profile R, let $DAT(R)$ denote the outcome of the above procedure.

Claim 1: For any R, $DAT(R)$ is constrained efficient.

Fix a preference profile R and let $\mu \equiv DAT(R)$.

We argue first that μ is stable. Note that all rejections in Step 2 respect all strict priority rankings for the priority structures in Example 1 and Example 2. Hence, the only possibility for μ to violate the stability condition is that the algorithm reaches Step 3 and there is an object $p \in O \setminus \{o\}$ and an agent $j \neq i$ s.t. $pP_i\mu(i)$, $\mu(j) = p$, and $i \succ_p j$ for some $i \in \{1, 3\}$, say for $i = 1$. But by the rules of Step 3, 1 will only be rejected by o if $\mu(2) = p$ for some p such that $3 \succ_p 2 \succ_p 1$. For both examples, the last observation implies that $j \succeq_{\mu^1(j)} 1$ for all $j \in I\backslash\{1\}$ such that $\mu^1(j) \neq j$. Hence, μ must be stable.

Next, we will show that μ is also constrained efficient. Suppose to the contrary that there is a stable improvement cycle $i_1, \ldots, i_m$. Assume w.l.o.g. that i_1 is among the first (in the

[6]Formally, in each round, let (A) each agent apply to the most preferred object that has not rejected him yet, (B) each object p reject all but the highest priority agents according to $\succeq'_p$, and (C) stop, if there were no new proposals in (A).

course of above algorithm) agents in the stable improvement cycle to be rejected by the object he is pointing to in the cycle. There has to be an agent j who causes the rejection of i_1, i.e. an agent j who was rejected by all objects he prefers to o_2 when he applied to o_2 in the DA with tie-breaking and for whom $j \succeq_{o_2} i_1$. By our assumption that i_1 was among the first agents to be rejected by the object he is pointing to in the stable improvement cycle, we must have $j \neq i_2$ and hence $\mu(j) \neq o_2$. Furthermore, since the welfare of agents is weakly decreasing during the course of the above algorithm, we must have $o_2 P_j \mu(j)$. These arguments imply that $j \sim_{o_2} i_1$ and, consequently, that $o_2 = o$: otherwise, we would have $i_2 \succ_{o_2} j \succ_{o_2} i_1$ and i_1 could not be among the highest priority agents who desires $o_2 = p$ at μ, thus contradicting the definition of a stable improvement cycle. Next, we will show that $\{i_1, i_2, j\} = \{1, 2, 3\}$. For Example 1, we can immediately infer the just mentioned statement from the original priority ranking $\succeq_o$. For Example 2, it is easy to see that if $\{i_1, i_2, j\} \neq \{1, 2, 3\}$, then the form of $\succeq'_o$ implies $i_1 = 4$, $j = 5$, and $i_2 = 6$. However, by our previous arguments, 6 was temporarily matched to an object $p \in O \setminus \{o\}$ s.t. pR_6o_3 when 4 was rejected by o. Since the only agent who can displace j at o is 6 and the only agent who can displace 6 at p is 5, 6 can't be rejected by o_3 during the DA with tie-breaking. Hence, for both examples, we must have $\{i_1, i_2, j\} = \{1, 2, 3\}$. Now assume first that $i_1 = 2$. This is possible only in Example 1 and we can assume w.l.o.g. that $j = 1$ and $i_2 = 3$. Since each object can appear at most once in a stable improvement cycle and since no no agent in $\{1, 2, 3\}$ can displace an agent in $\{4, 5, 6\}$ at either p_1 or p_2 in Example 1, we must have $\{i_1, \ldots, i_m\} \subseteq \{1, 2, 3\}$. This is easily seen to imply that $m = 2$ and $1 \succ_{o_1} 2 \succ_{o_1} 3$. But then, the rules of Step 3 of the DA with tie-breaking immediately imply that 3 could not have displaced 1 at $o_2 = o$ subsequently to being rejected by o_1. Hence, we must have, w.l.o.g., $i_1 = 1$. In Example 1, the only agent who can displace 1 at o is agent 3. Hence, it would have to be the case that $j = 3$. But then, 3 could not have subsequently been displaced by $i_2 = 2$, thus contradicting $\mu(i_2) = o_2 = o$. We are left to consider the case of $i_1 = 1$ for Example 2. Here, we must have $j = 3$ and $i_2 = 2$ given that $2 \succ'_o 1 \succ'_o 3$. But by the rules of Step 3, 1 cannot displace 2 subsequently to losing a tie-breaking decision against 3. This completes the proof of constrained efficiency.

Claim 2: DAT is strategy-proof.

Suppose that, contrary to what we want to show, there are a preference profile R, an agent i, and a misreport $\hat{R}_i$ such that $DAT_i(\hat{R}_i, R_{-i}) P_i DAT_i(R)$. Let $\mu \equiv DAT(R)$ and

$\hat{\mu} \equiv DAT(\hat{R}_i, R_{-i})$, and let μ^1 and $\hat{\mu}^1$ be the temporary assignments at the end of Step 2 of the DAT under R and $\hat{R} \equiv (\hat{R}_i, R_{-i})$, respectively.

Note first that the DA with tie-breaking has to reach Step 3 for R and $\hat{R}$, and that the tie-breaking decision in Step 3 has to be different for R and $\hat{R}$: if, say, 1 wins the tie-breaking decision at R and $\hat{R}$, the DA with tie-breaking would be equivalent to a DA with strict priorities in which 1 has strictly higher priority for o than 3. Since the DA with strict priorities is strategy-proof, there cannot be a an agent who can profitably manipulate at R.

Next, we will argue that $i \notin \{2, 4, 5, 6\}$. Note first that all agents apart from 1 and 3 receive their final allocation in the second step of DAT. Now consider first Example 1. Here, the temporary assignment at the end of Step 2 of the DAT under R and $\hat{R}$ is equivalent to the final outcome of a DA in which the "priority-ranking" of object o is given by $\{1,3\} \succ''_o \{1\} \succ''_o \{3\} \succ''_o \{2\} \succ''_o \{4\} \succ''_o \{5\} \succ''_o \{6\}$ and the priority ranking for all other objects is as in $\succeq$. Since these "priorities" induce substitutable preferences that satisfy the law of aggregate demand, such a DA is strategy-proof (Hatfield and Milgrom, 2005). This implies that no agent in $\{2, 4, 5, 6\}$ can manipulate DAT for Example 1. Next, consider Example 2. By our construction of $\succeq'_o$ for Example 2, we can infer that no agent in $\{2, 4, 5, 6\}$ could have applied to o under R and $\hat{R}$. But then the temporary assignment at the end of Step 2 of the DAT is equivalent to a DA in which the "priorities" of o are given by $\{1,3\} \succ''_o \{1\} \succ''_o \{3\} \succ''_o \{2\} \succ''_o \{4\} \succ''_o \{5\} \succ''_o \{6\}$ and the priority ranking for all other objects is as in $\succeq$. By the same arguments as above, this implies that no agent in $\{2, 4, 5, 6\}$ can profitably manipulate DAT.

To complete the proof, we will now show that $i = 1$ is impossible (the arguments in the case of $i = 3$ are completely symmetric). Assume first that $\mu(1) = o$ and $\hat{\mu}(1) \neq o$. For both examples, we must have $\hat{\mu}^1(2) = p_1$ by the rules of Step 3. Furthermore, since the tie-breaking decision between 1 and 3 is different at R and $\hat{R}$, we must have $\mu^1(2) \neq \hat{\mu}^1(2)$. It is easy to see that for both examples, we must have $\mu^1(2) = p_2$ since otherwise, 1 would either fail to win the tie-breaking decision against 3 at R (if $\mu^1(2) = p_1$) or would not be able to affect the pre tie-breaking assignment (if $\mu^1(2) \in O \setminus \{o, p_1, p_2\}$). We immediately obtain that oP_1p_2 and $p_2\hat{P}_1o$. But then, in order for the deviation to $\hat{R}_1$ to be profitable for 1, it would have to be the case that, subsequently to displacing 2 at p_2, 1 is rejected by p_2 <u>and</u> 2 ends up temporarily assigned to p_1 at the end of Step 2. It is straightforward to check that the just mentioned configuration is impossible for both types of examples. Next, assume that

$\mu(1) \neq o$ and $\hat{\mu}(1) = o$. By the rules of Step 3, we must have $\mu^1(2) = p_1$ and $\hat{\mu}^1(2) \neq p_1$. It is easy to see that in order for 1 to be able to influence the pre tie-breaking assignment, we must have $\hat{\mu}^1(2) = p_2$. But then, it has to be the case that 1 displaced 2 at p_2 during Step 2 of the DAT for profile R. For Example 1, this immediately implies $\mu(1) = p_2$ and we obtain a contradiction to the assumption that 1 and 3 compete for o in Step 3. For Example 2, 1 must have been displaced at some point of Step 2 of the DAT under profile R. But this is possible only if $\mu^1(2) = p_3$ and $\mu^1(4) = p_2$, thus contradicting our assumption that $\mu^1(2) = p_1$. This completes the proof.

1.2 Two-way ties at the top

We say that $\succeq$ is a *two-way ties at the top (TWT)* priority structure, if, for all $o \in O$, there exist $i(o), j(o) \in I$ such that (A) $i(o) \succeq_o j(o)$, (B) for all $k \in I \setminus \{i(o), j(o)\}$, $j(o) \succ_o k$, and (C) $\succeq_o |_{I\setminus\{i(o)\}}$ is strict. We will now argue that if $\succeq$ is a TWT priority structure, then the following two-step procedure induces a constrained efficient and strategy-proof mechanism:

1. Let $\succeq'$ be a strict priority structure that respects all strict priority rankings in $\succeq$, i.e. assume that $i \succ'_o j$ whenever $i \succ_o j$.

2. For any preference profile R, choose the outcome of the DA-algorithm with respect to R and $\succeq'$.

Note that the mechanism induced by the procedure just described is strategy-proof since the DA mechanism for strict priority structures is strategy-proof and since the same strict priority ranking $\succeq'$ is used for all preference profiles. To see that the outcome of above procedure is always constrained efficient, let R be a preference profile and μ be the matching chosen by the above procedure. If μ is not constrained efficient, then μ contains a SIC, say $i_1, \ldots, i_m$. Note that i_l desires $\mu(i_{l+1})$ and $i_l \in D_{\mu(i_{l+1})}(\mu)$ for all $l \in \{1, \ldots, m\}$ (where $m + 1 := 1$). Choose an agent from $i_1, \ldots, i_m$ who is among the first ones rejected in DA by the object he desires, say i_1. But then i_1 is rejected by $\mu(i_2)$ because some other agent $j \in I\setminus\{i_1, i_2\}$ applied to $\mu(i_2)$. Note that $\mu(j) \neq \mu(i_2)$ and $\mu(i_2)P_j\mu(j)$. If $j \succ_{\mu(i_2)} i_1$, then $\mu(i_2)P_j\mu(j)$ implies $i_1 \notin D_{\mu(i_2)}(\mu)$, a contradiction. Thus, we must have $j \sim_{\mu(i_2)} i_1$ and i_1 and j are tied at the top of $\succeq_{\mu(i_2)}$. But then j is never rejected by $\mu(i_2)$ and we must have $\mu(j) = \mu(i_2)$, again a contradiction.

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