Supplement to "Dynamic project selection"

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Appendix: Auxiliary technical lemmas

Lemma A.1. *(i) When $c < \underline{c}$, $\Phi^C(\cdot, c)$ on $[\underline{x}, \bar{x}]$ is positive at first, then intersects zero at a point, then is negative, then intersects zero at a point, and then is positive again.*

(ii) The $\Phi^C(\cdot, c)$ is nonnegative on $[\underline{x}, \bar{x}]$ if and only if $c \geq \underline{c}$.

(iii) The $\Phi^A(\cdot, \cdot, c)$ is nonnegative on $[\underline{x}, \bar{x}]^2$ if and only if $c \geq \underline{c}$.

(iv) When $c < \bar{c}$, $\Phi^A(x_1, \cdot, c)$ is quasi-convex on $[x_1, b^{-1}(x_1)]$.

Proof. The proof proceeds in steps.

Step 1. *Claim*. We have $1 - c - a > 0$ and $1 - c - a^2 > 0$ for any $a \in [\underline{x}, \bar{x}]$.

Proof. We have

$$1 - c - a > 1 - c - \bar{x} = \frac{1 - 2c - \sqrt{1-4c}}{2} = \frac{4c^2}{2(1 - 2c + \sqrt{1-4c})} > 0,$$

which, coupled with $a^2 < a$, also implies that $1 - c - a^2 > 0$. ◁

Step 2. *Claim*. Define $\hat{a} = 1 - \sqrt{c}$. Then $\hat{a} \in (\underline{x}, \bar{x})$, and $a \in [\underline{x}, \hat{a}) \cup (\hat{a}, \bar{x}]$ implies that $(a - \hat{a})(c - (1-a)^2) > 0$.

Proof. The inequality follows by the definition of $\hat{a}$ and by inspection. It remains to verify that $\hat{a} \in (\underline{x}, \bar{x})$. Indeed,

$$\hat{a} - \underline{x} = \frac{1 + \sqrt{1-4c} - 2\sqrt{c}}{2} > 0,$$
$$\bar{x} - \hat{a} = \frac{2\sqrt{c} + \sqrt{1-4c} - 1}{2} > 0,$$

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where $c < \bar{c}$. ◁

Step 3. *Claim.* If $\Phi^C(a,c) < 0$ for some $a \in [\underline{x}, \bar{x}]$, then $\Phi^C(\cdot, c)$ is at first positive, then intersects zero at a single point to the left of $\hat{a}$, then is negative, then intersects zero at a single point to the right of $\hat{a}$, and then is again positive.

PROOF. Because $\Phi^C(\underline{x}, c) = 1/(1-\underline{x}) > 0$ and $\Phi^C(\bar{x}, c) = 1/(1-\bar{x}) > 0$, then $\Phi(a,c) \leq 0 \implies a \in (\underline{x}, \bar{x})$. Differentiating yields

$$\begin{aligned}\Phi_1^C(a,c) &= \frac{c(1-c-a^2)}{(1-a)(1-c-a)^2} + \ln\frac{c(1-a)}{a(1-c-a)} \\ &= \frac{\Phi^C(a,c)}{a} + \frac{\left(c-(1-a)^2\right)\left(1-c-a^2\right)}{a(1-a)(1-a-c)^2}.\end{aligned}$$

If $\Phi^C(a,c) = 0$ for some $a \in (\underline{x}, \bar{x})$, then

$$\Phi_1^C(a,c) = \frac{\left(c-(1-a)^2\right)\left(1-c-a^2\right)}{a(1-a)(1-c-a)^2}.$$

By Step 1, the sign of $\Phi_1^C(a,c)$ is the sign of $c-(1-a)^2$, which, by Step 2, switches the sign from negative to positive at $\hat{a} \in (\underline{x}, \bar{x})$. Hence, if a with $\Phi^C(a,c) < 0$ exists, then $\Phi^C(\cdot, c)$ intersects zero twice: once from above and to the left of $\hat{a}$, and once from below and to the right of $\hat{a}$. ◁

Step 4. *Claim.* If $c < \underline{c}$, then $\Phi^C(\hat{a}, c) < 0$; if $c \geq \underline{c}$, then $\Phi^C(\cdot, c)$ is nonnegative on $[\underline{x}, \bar{x}]$.

PROOF. Note that, at $a \in (\underline{x}, \bar{x})$,

$$\Phi_2^C(a,c) = \frac{a}{c}\left(\frac{1-a}{1-c-a}\right)^2 > 0.$$

Furthermore,

$$\Phi^C(\hat{a}, \underline{c}) = 2 - (1-\sqrt{\underline{c}})\ln\frac{(1-\sqrt{\underline{c}})^2}{\underline{c}} = 0,$$

where the first equality is by $\hat{a} = 1-\sqrt{\underline{c}}$ and the second equality is by (8). Combining the two displays above delivers $\Phi^C(\hat{a}, c) < 0$ for any $c < \underline{c}$ and $\Phi^C(\hat{a}, c) \geq 0$ for any $c \geq \underline{c}$.

For $c < \underline{c}$, $\Phi^C(\hat{a}, c) < 0$ and Step 3 imply part (i).

For $c \geq \underline{c}$, $\Phi^C(\hat{a}, c) \geq 0$ and Step 3 imply part (ii). ◁

Step 5. *Claim.* Subject to $x_2 \geq x_1$, $\Phi^A(\cdot, \cdot, c)$ is minimized at $x_2 = x_1$.

PROOF. The claim follows from

$$\Phi_2^A(x_1, x_2, c) = \frac{(1-x_1)}{(1-x_2)^2}\left(\ln\frac{c(1-x_1)}{x_1(1-c-x_2)} + \frac{(1-x_2)\left(1-c-x_2^2\right)}{(1-c-x_2)^2}\right) > 0,$$

where the inequality follows because $1-c-x_2^2>0$ and $1-c-x_2>0$ by Step 1.

Because $\Phi^A(z,z,c)=\Phi^C(z,c)$, Step 5 implies that $\Phi^A(\cdot,\cdot,c)$ has the same minimized value as $\Phi^C(\cdot,c)$ does. Hence, part (iii) is implied by part (ii). $\triangleleft$

Step 6. *Claim.* Define $\kappa(x_1,x_2,c)\equiv(1-x_2)(2c-1+x_2+x_2^2-x_1x_2^2)-c^2$. Then, for some $y^A\in(\underline{x},\bar{x})$ and for any $x_2\in[\underline{x},y^*)\cup(y^*,\bar{x}]$, $(x_2-y^A)\kappa(x_1,x_2,c)>0$.

Proof. First, we show that $\kappa(x_1,\bar{x},c)>0$. Indeed,

$$\kappa(x_1,\bar{x},c)\geq\kappa(\bar{x},\bar{x},c)=c-(1-\sqrt{1-4c})\left(\frac{1}{2}-c\right)>0,$$

where the first inequality is by $\partial\kappa/\partial x_1<0$, and the last inequality follows because $\kappa(\bar{x},\bar{x},\cdot)$ is zero at $c\in\{0,\bar{c}\}$ and is positive at the only critical point ($c=2/9$) in $(0,\bar{c})$.

Next, we show that $\kappa(x_1,\underline{x},c)<0$. Indeed,

$$\kappa(x_1,\underline{x},c)\leq\kappa(\underline{x},\underline{x},c)=c-(1+\sqrt{1-4c})\left(\frac{1}{2}-c\right)<0,$$

where the first inequality is by $\partial\kappa/\partial x_1<0$, and the last inequality follows because $\kappa(\underline{x},\underline{x},\bar{c})=0$ and because $\partial\kappa(\underline{x},\underline{x},c)/\partial c>0$.

Finally, $\partial^2\kappa(x_1,x_2,c)/x_2^2=-6(1-x_1)x_2-2x_1<0$. Hence, $\kappa(x_1,\underline{x},c)<0$ and $\kappa(x_1,\bar{x},c)>0$ imply that, on $(\underline{x},\bar{x})$, $\kappa(x_1,\cdot,c)$ crosses zero and—by $\partial^2\kappa(x_1,x_2,c)/x_2^2<0$—just once, from below, at some $y^A\in(\underline{x},\bar{x})$. $\triangleleft$

Step 7. *Claim.* The $\Phi^A(x_1,\cdot,c)$ can be negative on and only on an interval.

Proof. At any (x_1,x_2,c) with $\Phi^A(x_1,x_2,c)=0$, by differentiation and substitution,

$$\Phi_2^A(x_1,x_2,x)=\frac{\kappa(x_1,x_2,c)}{(1-x_2)x_2(1-c-x_2)^2}.$$

The sign of $\Phi_2^A(x_1,x_2,c)$ is the sign of $\kappa(x_1,x_2,c)$, which, by Step 6, switches from negative to positive at $y^A\in(\underline{x},\bar{x})$ as x_2 rises; $\Phi^A(x_1,x_2,c)$ is quasi-convex. Part (iv) follows.

$\square$

Lemma A.2. *The function M^C is uniquely maximized on $[\underline{x},\bar{x}]$ at $\bar{x}$.*

Proof. Recall from the proof of Lemma 3 that M^C has two local maxima: at $\underline{a}$ and at $\bar{x}$. It remains to verify that $M^C(\bar{x})>M^C(\underline{a})$.

Then

$$M^C(\bar{x})=2c\sigma(\bar{x})-\frac{1-V(\bar{x},\bar{x})}{(1-\bar{x})^2}=2c\sigma(\bar{x})-\frac{1}{1-\bar{x}},$$

where the last equality uses $V(\bar{x},\bar{x}) = \bar{x}$, by direct substitution. Furthermore,

$$M^C(\underline{a}) = 2c\sigma(\underline{a}) - \frac{1 - V(\underline{a},\underline{a})}{(1-\underline{a})^2} = 2c\sigma(\underline{a}) - \frac{c}{\underline{a}(1-\underline{a})^2} - \frac{1-\underline{a}}{1-\underline{a}-c},$$

where the last equality follows by substituting $\Phi(\underline{a}, c) = 0$ into the expression for $\frac{1-V(\underline{a},\underline{a})}{(1-\underline{a})^2}$.

As a result,

$$\begin{aligned} M^C(\bar{x}) - M^C(\underline{a}) &= 2c\sigma(\bar{x}) - 2c\sigma(\underline{a}) + \frac{c}{\underline{a}(1-\underline{a})^2} + \frac{1-\underline{a}}{1-\underline{a}-c} - \frac{1}{1-\bar{x}} \\ &= c\left[2\sigma(\bar{x}) - \frac{1}{\bar{x}(1-\bar{x})^2}\right] - c\left[2\sigma(\underline{a}) - \frac{1}{\underline{a}(1-\underline{a})^2}\right] + \frac{1-\underline{a}}{1-\underline{a}-c} > 0, \end{aligned}$$

where the last equality follows from $\bar{x}(1-\bar{x}) = c$ and by rearranging, and the inequality follows because the first bracket exceeds the second bracket, and the fraction $(1-\underline{a})/(1-\underline{a}-c)$ is positive (by $\underline{a} < \bar{x}$). The ordering of the brackets follows from $\bar{x} > \underline{a}$ and the observation

$$\frac{\mathrm{d}}{\mathrm{d}a}\left(2\sigma(a) - \frac{1}{a(1-a)}\right) = \frac{1}{a^2(1-a)^2} > 0, \quad \forall a \in (0,1).$$

To summarize, $M^C(\bar{x}) > M^C(\underline{a})$ and, so, M^C has a unique maximand, $\bar{x}$, on $[\underline{x},\bar{x}]$. □

Lemma A.3. *For M^A defined in (18),* $\arg\max_{a\in[x_1,b^{-1}(x_1)]} M^A(x_1,a) = \{b^{-1}(x_1)\}$*, where b^{-1} is the inverse of b defined in (11). As a result, on $\hat{\mathcal{A}}$, $u(x_1) < b^{-1}(x_1)$ and $\mathcal{F} \subset \mathcal{A}$.*

Proof. By Lemma 4, the only two local maxima of $M^A(x_1,\cdot)$ are $d(x_1)$ and $b^{-1}(x_1)$, so it suffices to show that $M^A(x_1, b^{-1}(x_1)) > M^A(x_1, d(x_1))$. Write

$$M^A\big(x_1, b^{-1}(x_1)\big) = c\eta\big(b^{-1}(x_1)\big) - \frac{1 - V\big(x_1, b^{-1}(x_1)\big)}{1 - b^{-1}(x_1)} = c\eta\big(b^{-1}(x_1)\big) - 1,$$

where the first equality is definitional and the second equality is by $V(x_1, b^{-1}(x_1)) = b^{-1}(x_1)$.

Evaluating V in (10) at $(x_1, d(x_1))$ and using $\Phi^A(x_1, d(x_1), c) = 0$ (by (19)), one can write

$$V\big(x_1, d(x_1)\big) = (1-x_1)d(x_1) - \frac{\big(1-d(x_1)\big)c\big(1-c-x_1 d(x_1)\big)}{d(x_1)\big(1-c-d(x_1)\big)} - c + x_1.$$

Then

$$\begin{aligned} M^A\big(x_1, d(x_1)\big) &= c\eta\big(d(x_1)\big) - \frac{1 - V\big(x_1, d(x_1)\big)}{1 - d(x_1)} \\ &= c\left[\eta\big(d(x_1)\big) - \frac{1}{d(x_1)\big(1-d(x_1)\big)}\right] + x_1 - \frac{c(1-x_1)}{1-c-d(x_1)} - 1, \end{aligned}$$

where the first equality is definitional and the second equality follows by substituting $V(x_1, d(x_1))$ and rearranging.

Then, suppressing the argument x_1 in $b^{-1}(x_1)$ and in $d(x_1)$, for compactness,

$$M^A(x_1, b^{-1}) - M^A(x_1, d) = c\left[\eta(b^{-1}) - \frac{1}{b^{-1}(1-b^{-1})} - \left(\eta(d) - \frac{1}{d(1-d)}\right)\right] + \frac{b(d)(b^{-1}-x_1) + x_1(1-b^{-1})}{b^{-1}(1-b(d))} > 0,$$

where the first equality follows by using the definitions of b and b^{-1} and rearranging, and the inequality uses $b^{-1}(x_1) > d(x_1)$ and

$$\frac{\mathrm{d}}{\mathrm{d}y}\left(\eta(y) - \frac{1}{y(1-y)}\right) = \frac{1}{y^2(1-y)} > 0$$

to conclude that the bracket in the first line is positive; and uses $x_1 < b^{-1}(x_1) < 1$ to conclude that the fraction in the second line is positive too. That is, $M^A(x_1, b^{-1}(x_1)) > M^A(x_1, d(x_1))$, as desired.

The conclusion that, on $\hat{\mathcal{A}}$, $u(x_1) < b^{-1}(x_1)$ and $\mathcal{F} \subset \mathcal{A}$ follows by inspection of Lemma's 4 Figure 7 (just validated by showing that $M^A(x_1, b^{-1}(x_1)) > M^A(x_1, d(x_1))$). □

Lemma A.4. *On $\hat{\mathcal{B}}$, $w(x_1) < b^{-1}(x_1)$ and $\mathcal{F} \subset \mathcal{B}$.*

Proof. To conclude that $w(x_1) < b^{-1}(x_1)$, we show that $V(x_1, b^{-1}(x_1)) > B(x_1, b^{-1}(x_1))$.

Note that $V(x_1, b^{-1}(x_1)) = b^{-1}(x_1)$ and, from the definition of B in (22),

$$B(x_1, b^{-1}(x_1)) = 1 - (1 - b^{-1}(x_1))\left(\frac{1 - C(x_1)}{1 - x_1} + c\eta(b^{-1}(x_1)) - c\eta(x_1)\right).$$

Then

$$V(x_1, b^{-1}(x_1)) - B(x_1, b^{-1}(x_1)) = (1 - b^{-1}(x_1))\left(\frac{1 - C(x_1)}{1 - x_1} - 1 + c\eta(b^{-1}(x_1)) - c\eta(x_1)\right),$$

where

$$\begin{aligned}\frac{1 - C(x_1)}{1 - x_1} &= (1 - x_1)\left(\frac{1 - V(\underline{a}, \underline{a})}{(1 - \underline{a})^2} + 2c[\sigma(x_1) - \sigma(\underline{a})]\right) \\ &= (1 - x_1)\left(\frac{1}{1 - b(\underline{a})} + \frac{c}{\underline{a}(1 - \underline{a})^2} + 2c[\sigma(x_1) - \sigma(\underline{a})]\right).\end{aligned}$$

The first equality in the display above uses the definition of C in (13). The second equality uses the definitions of V in (10) and b in (11), and the condition $\Phi^C(\underline{a}, c) = 0$ in (15), which characterizes $\underline{a}$.

Then, substituting the display above into its precursor display gives

$$
\begin{aligned}
&\frac{V\big(x_1, b^{-1}(x_1)\big) - B\big(x_1, b^{-1}(x_1)\big)}{1 - b^{-1}(x_1)} \\
&\quad = (1 - x_1)\left(\frac{1}{1 - b(\underline{a})} + \frac{c}{\underline{a}(1 - \underline{a})^2} + 2c\big[\sigma(x_1) - \sigma(\underline{a})\big]\right) \\
&\qquad + c\big[\eta\big(b^{-1}(x_1)\big) - \eta(x_1)\big] - 1 \\
&\quad = \left(\frac{(1 - x_1)}{1 - b(\underline{a})} + \frac{c}{b^{-1}(x_1)\big(1 - b^{-1}(x_1)\big)} - 1\right. \\
&\qquad + (1 - x_1)c\left[2\sigma(x_1) - \frac{1}{x_1(1 - x_1)^2} - \left(2\sigma(\underline{a}) - \frac{1}{\underline{a}(1 - \underline{a})^2}\right)\right] \\
&\qquad \left. + c\left[\eta\big(b^{-1}(x_1)\big) - \frac{1}{b^{-1}(x_1)\big(1 - b^{-1}(x_1)\big)} - \left(\eta(x_1) - \frac{1}{x_1(1 - x_1)}\right)\right]\right).
\end{aligned}
$$

Note that, using the definition of b in (11),

$$
\begin{aligned}
&\frac{(1 - x_1)}{1 - b(\underline{a})} + \frac{c}{b^{-1}(x_1)\big(1 - b^{-1}(x_1)\big)} - 1 \\
&\quad = \frac{(1 - x_1)}{1 - b(\underline{a})} + \frac{x_1}{b^{-1}(x_1)} - 1 \\
&\quad = \frac{x_1\big[1 - b^{-1}(x_1)\big] + b(\underline{a})\big[b^{-1}(x_1) - x_1\big]}{b^{-1}(x_1)\big(1 - b(\underline{a})\big)} > 0,
\end{aligned}
$$

where the inequality follows from $x_1 < b^{-1}(x_1) < 1$. Moreover,

$$
2\sigma(x_1) - \frac{1}{x_1(1 - x_1)^2} - \left(2\sigma(\underline{a}) - \frac{1}{\underline{a}(1 - \underline{a})^2}\right) > 0
$$

by $x_1 > \underline{a}$ and by

$$
\frac{\mathrm{d}}{\mathrm{d}y}\left(2\sigma(y) - \frac{1}{y(1 - y)^2}\right) = \frac{1}{y^2(1 - y)^2} > 0
$$

for any $y \in (0, 1)$. Finally,

$$
\eta\big(b^{-1}(x_1)\big) - \frac{1}{b^{-1}(x_1)\big(1 - b^{-1}(x_1)\big)} - \left(\eta(x_1) - \frac{1}{x_1(1 - x_1)}\right) > 0
$$

by $b^{-1}(x_1) > x_1$ and by

$$
\frac{\mathrm{d}}{\mathrm{d}y}\left(\eta(y) - \frac{1}{y(1 - y)}\right) = \frac{1}{y^2(1 - y)} > 0
$$

for $y \in (0, 1)$. Thus, $V(x_1, b^{-1}(x_1)) - B(x_1, b^{-1}(x_1)) > 0$, as required.

To show that $(\mathcal{F} \cap \hat{\mathcal{B}}) \subset \mathcal{B}$, from (22) and (10), write

$$\frac{B(x) - V(x)}{1 - x_2} = (1 - x_1)\left(\frac{1}{1 - b(x_2)} + \frac{c}{1 - x_2}\big[\eta(x_1) - \eta\big(b(x_2)\big)\big]\right) - \frac{1 - C(x_1)}{1 - x_1} - c\big[\eta(x_2) - \eta(x_1)\big].$$

Differentiating and then simplifying gives

$$\frac{\mathrm{d}}{\mathrm{d}x_2}\left(\frac{B(x) - V(x)}{1 - x_2}\right) = -\frac{c\Phi^A(x, c)}{x_2(1 - x_2)}.$$

As a result, because $\Phi^A(x, c) < 0$ implies that $B(x) > V(x)$, $\mathcal{B}$ covers $\mathcal{F}$, the failure region, on $\hat{\mathcal{B}}$. That is, $\mathcal{F} \cap \hat{\mathcal{B}} \subset \mathcal{B}$. □

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