Online Appendix for Yuichiro Waki, Richard Dennis, and Ippei Fujiwara, "The Optimal Degree of Monetary-Discretion in a New Keynesian Model with Private Information"

Section E provides microfoundation of the model considered in the paper. Section F compares our approach with that in Amador and Bagwell (2013) and discusses implications of the central bank with "inflation bias."

E Microfoundation

The private sector consists of a continuum of identical households, a continuum of monopolistically competitive intermediate firms that have access to an identical production function, and competitive final good firms. Price stickiness is introduced via Calvo-style price setting, and every period a constant fraction of intermediate firms are chosen randomly and allowed to adjust their prices optimally. The production function for intermediate goods is linear in labor input, and the labor market is neither firm- nor industry-specific. An implication is that all intermediate goods producers face the same marginal cost, which equals the real wage. The final good firms combine intermediate goods using the usual Dixit-Stiglitz aggregater, implying a constant price-elasticity demand function for intermediate goods and a standard price index. For simplicity we assume away real disturbances so that the natural output, or the flexible-price equilibrium output, is constant at its steady state value. Hence, the output gap is defined as the log-deviation of output from its steady state. The government imposes a constant sales tax on the intermediate good firms that corrects monopoly profits in a zero inflation steady state and rebates the tax revenue back to the households in a lump-sum fashion.

At the beginning of each period, the central bank observes its private information and *publicly* sends a message to the mechanism. Markets are complete and households trade claims that are contingent on the history of the central bank's messages.

Let a measurable space $(M, \mathcal{M})$ be a message space. The central bank's (pure) reporting strategy is denoted by $\rho_{CB} := \{\rho_{CB,t}\}_{t=0}^{\infty}$ such that for each t, $\rho_{CB,t}$ maps a relevant history of the central bank into M. We focus on a *public* reporting strategy, which depends on the CB's history only through the history of its past messages, and hence write, for each t, $\rho_{CB,t} : M^t \times \Theta \to M$. A history of messages is said to be an "on-path" history given ρ_{CB} if and only if the central bank's reporting strategy indeed implies the history for some realization of private information. The set of "on-path" message histories at t can be defined recursively as follows: $H_{-1}^{ON,\rho_{CB}} = \emptyset$, and for each $t \geq 0$, $H_t^{ON,\rho_{CB}} = \{(h_{t-1}, m_t) \in M^{t+1} : h_{t-1} \in M_{t-1}^{ON,\rho_{CB}}$ and $\exists \theta \in \Theta, m_t = \rho_{CB,t}(h_{t-1}, \theta)\}$. An interest rate mechanism is given by $\rho_i := \{\rho_{i,t}\}_{t=0}^{\infty}$ such that for each t, $\rho_{i,t} : M^t \to \mathbb{R}$. All private agents take as given the central bank's strategy ρ_{CB} and the stochastic process for messages it generates, and form rational expectations for future messages.

A rational expectation equilibrium (REE) given the interest rate mechanism ρ_i and the central bank's reporting strategy ρ_{CB} is $(p_{-1}, \{(p_t^*, mc_t, c_t, x_t, \pi_t, p_t)\}_{t=0}^{\infty})$ such that for any $t \geq 0$ and any on-path message history $h_t \in H_t^{ON,\rho_{CB}}$,

$$
\begin{aligned}
p_t^*(h_t) &= p_{t-1}(h_{t-1}) + (1-\alpha\beta)\sum_{k=0}^{\infty}(\alpha\beta)^k \mathbb{E}^{h_t}[mc_{t+k}] + \sum_{k=0}^{\infty}(\alpha\beta)^k \mathbb{E}^{h_t}[\pi_{t+k}] \quad (1)\\
c_t(h_t) &= \mathbb{E}^{h_t}[c_{t+1}] - \zeta\{\rho_{i,t}(h_t) - \mathbb{E}^{h_t}[\pi_{t+1}] - r^n\} \quad (2)\\
mc_t(h_t) &= \zeta^{-1}c_t(h_t) + \nu n_t(h_t) \quad (3)\\
p_t(h_t) &= \alpha p_{t-1}(h_{t-1}) + (1-\alpha)p_t^*(h_t) \quad (4)\\
\pi_t(h_t) &= (1-\alpha)(p_t^*(h_t) - p_{t-1}(h_{t-1})) \quad (5)\\
x_t(h_t) &= c_t(h_t) = n_t(h_t). \quad (6)
\end{aligned}
$$

Here, p is the logarithm of the nominal price level, p^* is the logarithm of the nominal price set by the price changers, and π_t is the net inflation rate from $t-1$ to t. Consumption, c, labor, n, and marginal costs, mc, are all expressed as log deviations from their respective steady state values. The output gap, x, is the log deviation of output from its steady state level. Note that expectations here depend not only on the history, but also on the central bank's report strategy. Regarding parameters, α is the Calvo probability of not being able to reset price, β is the household's preference discount factor, ζ is the elasticity of intertemporal substitution, and ν is the inverse of the Frisch elasticity of labor supply. The household's and the firms' optimization problems can be found in Chapter 3 of Galì (2008).

Equation (1) is the loglinearized first-order condition for the firms that are able to change their prices at time t, where p_t^* is the nominal price (in logarithm) set by these firms, p_{t-1} is the nominal (aggregate) price level at $t-1$, and mc_{t+k} is the real wage that prevails at time $t+k$. Note that a firm's belief is independent of its own past deviations, because it does not affect the aggregate behavior. Hence, regardless of its own past actions, all firms that can change prices at time t solve the same problem. Because the optimization problem is a univariate, unconstrained one with a strictly concave objective function, the first-order condition is also sufficient.

Equation (2) is the loglinearized Euler equation. Because of equation (6) and the boundedness of x, the household's transversality condition is automatically satisfied. The household's optimality condition is therefore satisfied because of the concavity of the problem.

Equation (3) implies that the real wage equals the household's marginal rate of substitution. Equations (4) and (5) are the aggregate consistency conditions for the price level and inflation, respectively: the $1-\alpha$ fraction of firms change their prices by $p_t^* - p_{t-1}$ on average and inflation must equal $(1-\alpha)(p_t^* - p_{t-1})$. Finally, equation (6) says that the output gap equals the deviation of consumption from its steady state, which in turn equals labor through the resource constraint.

It is straightforward to establish the following proposition:

Proposition 1 *Let* $(p_{-1}, \{(p_t^*, mc_t, c_t, x_t, \pi_t, p_t)\}_{t=0}^{\infty})$ *be a REE given* (ρ_i, ρ_{CB}). *Then* $\{(\pi_t, x_t)\}_{t=0}^{\infty}$ *satisfies for any* $t \geq 0$ *and any on-path history* h_t,

$$\pi_t(h_t) = \kappa x_t(h_t) + \beta \mathbb{E}^{h_t}[\pi_{t+1}] \quad (7)$$

$$x_t(h_t) = \mathbb{E}^{h_t}[x_{t+1}] - \zeta\{\rho_{i,t}(h_t) - \mathbb{E}^{h_t}[\pi_{t+1}] - r^n\}, \quad (8)$$

where $\kappa := (1-\alpha)(1-\alpha\beta)(\zeta^{-1}+\nu)/\alpha$. *Conversely, taking* (ρ_i, ρ_{CB}) *as given, suppose that* $\{(\pi_t, x_t)\}_{t=0}^{\infty}$ *satisfies equations* (7) *and* (8) *for any* $t \geq 0$ *and any on-path history* h_t. *Then for any* p_{-1}, *one can find* $\{(p_t^*, mc_t, c_t, x_t, p_t)\}_{t=0}^{\infty}$ *such that* $(p_{-1}, \{(p_t^*, mc_t, c_t, x_t, \pi_t, p_t)\}_{t=0}^{\infty})$ *is a REE given* (ρ_i, ρ_{CB}).

Hereafter, we call $\{(\pi_t, x_t)\}_{t=0}^{\infty}$ a rational expectation equilibrium given (ρ_i, ρ_{CB}) if it satisfies equations (7) and (8) for any t and any on-path history.

The following lemma characterizes the conditional expectations in a REE. In particular, it shows that the private sector's belief about the central bank's past private information is irrelevant for the conditional expectations in equations (7) and (8).

Lemma 1 *Let* $\{(\pi_t, x_t)\}_{t=0}^{\infty}$ *be a REE given* (ρ_i, ρ_{CB}). *Then for any* t *and any on-path history* h_t, *both* $\mathbb{E}^{h_t}[\pi_{t+1}]$ *and* $\mathbb{E}^{h_t}[x_{t+1}]$ *are independent of the private sector's belief and given by*

$$\mathbb{E}^{h_t}[\pi_{t+1}] = \int_{\Theta} \pi_{t+1}(h_t, \rho_{CB,t+1}(h_t, \theta_{t+1}))p(\theta_{t+1})d\theta_{t+1},$$

$$\mathbb{E}^{h_t}[x_{t+1}] = \int_{\Theta} x_{t+1}(h_t, \rho_{CB,t+1}(h_t, \theta_{t+1}))p(\theta_{t+1})d\theta_{t+1}.$$

The proof is as follows. Notice first that, because θ is IID over time, the private sector's observation up to time t, h_t, is uninformative about θ_{t+1}. Second, because the central bank uses a public reporting strategy, the way it reports at $t + 1$ depends only on h_t and θ_{t+1}, and not on the true history of private information up to time t. Hence, the private sector's belief about the past realizations of private information is irrelevant.

E.1 Revelation principle

Now we turn to the revelation principle.

Note that the REE given (ρ_i, ρ_{CB}) is not defined for message histories that never occur under the central bank's reporting strategy ρ_{CB}. To define a (public) perfect Bayesian equilibrium (PBE) formally, we need to extend the notion of REE to off-path histories. However, for the revelation principle, it suffices to show that its on-path outcome can be achieved by a direct mechanism for which truth-telling is a PBE.

Take ρ_{CB} as given. We consider a set of reporting strategies such that the central bank's deviation from ρ_{CB} to these strategies is never detectable, and call them *on-path deviation strategies*.[1] For any time t, any on-path message history h_{t-1}, and any history of private information θ^t, an on-path deviation strategy from (h_{t-1}, θ^t) is defined as a report strategy, $\tilde{\rho}_{CB}$, such that: (1) given θ^{t-1}, $\tilde{\rho}_{CB}$ generates the message history h_{t-1}; (2) for any realization of private information from time t on, $\tilde{\theta}_t^\infty := (\tilde{\theta}_t, \tilde{\theta}_{t+1}, ...) \in \Theta^\infty$, the sequence of messages $\tilde{\rho}_{CB}$ generates from time t on given $(\theta^{t-1}, \tilde{\theta}_t^\infty)$ is identical to the message sequence ρ_{CB} generates given $(\theta^{t-1}, \theta_t^\infty)$ for some θ_t^∞.

Lemma 2 (Revelation principle) *Let $(\rho_i, \rho_{CB}, \pi, x)$ be such that (i) (π, x) is a REE given (ρ_i, ρ_{CB}); (ii) for any t, any on-path history of ρ_{CB}, h_{t-1}, and any realization of private information, θ^{t-1}, the central bank does not benefit from deviating from ρ_{CB} to any on-path deviation strategies; and (iii) for any t and any on-path history h_t, the private sector's belief is such that the central bank has been following ρ_{CB}. Then there is a direct mechanism that achieves the same outcome in a PBE in which the central bank reports truthfully.*

First we construct a direct mechanism that achieves the same outcome as the PBE when the central bank reports truthfully. For $t = 0$ and for all θ_0, define

$$\begin{aligned} i_0^D(\theta_0) &= i_0(h_{-1}, \rho_{CB,0}(h_{-1}, \theta_0)), \\ \pi_0^D(\theta_0) &= \pi_0(h_{-1}, \rho_{CB,0}(h_{-1}, \theta_0)), \\ x_0^D(\theta_0) &= x_0(h_{-1}, \rho_{CB,0}(h_{-1}, \theta_0)), \\ h_0(\theta_0) &= (h_{-1}, \rho_{CB,0}(h_{-1}, \theta_0)), \end{aligned}$$

[1] They are called *on-schedule deviations* in Athey et al. (2004).

where $h_{-1} = \emptyset$. For any $t \geq 1$ and θ^t, recursively define

$$\begin{aligned} i_t^D(\theta^t) &= i_t(h_{t-1}(\theta^{t-1}), \rho_{CB,t}(h_{t-1}(\theta^{t-1}), \theta_t)), \\ \pi_t^D(\theta^t) &= \pi_t(h_{t-1}(\theta^{t-1}), \rho_{CB,t}(h_{t-1}(\theta^{t-1}), \theta_t)), \\ x_t^D(\theta^t) &= x_t(h_{t-1}(\theta^{t-1}), \rho_{CB,t}(h_{t-1}(\theta^{t-1}), \theta_t)), \\ h_t(\theta^t) &= (h_{t-1}(\theta^{t-1}), \rho_{CB,t}(h_{t-1}(\theta^{t-1}), \theta_t)). \end{aligned}$$

Clearly, under truth-telling this direct mechanism achieves the same outcome as does the original non-direct mechanism. I.e. for any t and any realization of private information θ^t, inflation, the output gap, and the nominal interest rate are identical across mechanisms. Condition (i) implies that both the NKPC and the dynamic IS are satisfied for any t and any θ^t:

$$\begin{aligned} \pi_t^D(\theta^t) &= \kappa x_t^D(\theta^t) + \beta \int_{\theta_{t+1}} \pi_{t+1}^D(\theta^t, \theta_{t+1}) p(\theta_{t+1}) d\theta_{t+1}, \\ x_t^D(\theta^t) &= \int_{\theta_{t+1}} x_{t+1}^D(\theta^t, \theta_{t+1}) p(\theta_{t+1}) d\theta_{t+1} - \zeta\{i_t^D(\theta^t) - \int_{\theta_{t+1}} \pi_{t+1}^D(\theta^t, \theta_{t+1}) p(\theta_{t+1}) d\theta_{t+1} - r^n\}. \end{aligned}$$

For each information set of the private sector, which is indexed by report history θ^t, the belief is assigned so that the private sector believes that the central bank has been reporting truthfully.[2]

Truth-telling is a PBE strategy in the direct mechanism for the CB, because deviating to any non-truthful reporting strategy is equivalent to using an on-path deviation strategy in the original non-direct mechanism, which is not profitable by condition (ii).

E.2 Social welfare

In the paper we have assumed that the central bank's private information is about the shock to social welfare, θ, and that it does not enter the structural equations such as the dynamic IS equation or the NKPC.

Our specification of social welfare is a reduced-form one, and it is for tractability. If the household's period utility is given by:

$$u(c_t, n_t) + v(\pi_t, \theta_t),$$

i.e. there is some externality (in the utility sense) of inflation and a taste shock hits it, then under the standard set of assumptions in the New Keynesian literature, its quadratic approximation is consistent with our social welfare specification because

[2] This is the same as the on-path truth-telling in Pavan et al. (2014).

the v function above is additively separable. Although this "inflation in the utility function" specification has an interpretation as a reduced-form representation of inflation tax in monetary models, we have to admit that it is not standard.

If we instead add a shock to some structural parameters in a cashless New Keynesian model, then in general the shock also appears in these structural equations. The problem is that, once the same shock appears in these equations, then the private sector can figure out its true value from observables and information asymmetry may disappear.

F Discussion

F.1 Comparison to Amador and Bagwell (2013)

Amador and Bagwell (2013) establish a necessary and sufficient condition for interval delegation to be optimal in a general static setting. Their results can be used in our framework if we further assume that (a) $\overline{W}$ is a C^2 function and that (b) the function A is given by $A(\pi;\theta) = a(\pi) + \theta\pi$.

It follows that g_x and π_S are C^1 functions, and therefore that S is a C^2 function. Let $b(\pi) := a(\pi) + S(\pi)$. Problem (**P1**) can then be rewritten as:

$$\max_{\pi(.),\delta(.)} \int \left[b(\pi(\theta)) + (\theta - \lambda)\pi(\theta) + \delta(\theta)\right] p(\theta) d\theta$$

subject to

$$b(\pi(\theta)) + \theta\pi(\theta) + \delta(\theta) \geq b(\pi(\theta')) + \theta\pi(\theta') + \delta(\theta'), \quad \forall \theta, \theta',$$
$$\delta(\theta) \leq 0, \quad \forall \theta,$$

where λ denotes the Lagrange multiplier on equation (15), which can be either positive or negative. With $w(\pi,\theta) := b(\pi(\theta)) + (\theta - \lambda)\pi(\theta)$, this formulation is a special case of Amador and Bagwell's, and their theoretical results can be used to prove the optimality of interval delegation. When equation (15) is slack ($\lambda = 0$), there is no conflict of interest between the principal and the agent, and it is optimal to leave the agent's choice unrestricted. When $\lambda > 0$ ($\lambda < 0$), the principal prefers lower (higher) inflation than the agent would choose on average, and the principal finds it optimal to restrict the agent's choice by imposing an upper (lower) bound on inflation. The value of λ changes with π^e_-, suggesting that the optimal upper and lower limits are history-dependent and vary with π^e_-.[3]

[3]We thank an anonymous referee for suggesting us to use this formulation to contrast our approach with Amador and Bagwell's. The discussion based on the sign of λ is also based on

Although this approach nicely relates promised inflation and the agent's bias, the above assumptions (a) and (b) are stronger than the assumptions in our paper and are not guaranteed to hold. In particular, $\overline{W}$ is an endogenously determined object and imposing direct assumptions about its property may not be strongly supported. We do not directly impose any assumptions on the value function, $\overline{W}$, but prove that it is a strictly concave C^1 function by showing that the Bellman operator $\mathbb{T}$ maps the space of weakly concave functions into its proper subspace, the space of strictly concave C^1 functions.

F.2 A central bank with an "inflation bias"

Throughout the paper we have assumed that the central bank is benevolent and that its payoff is identical to social welfare. Our analysis is easily extended to a situation where the central bank has a linear bias in inflation. Suppose that the central bank's payoff function is the same as we have assumed, but that the momentary social welfare function is given by

$$R^{SW}(\pi, x, \theta) = A(\pi; \theta) + \gamma\pi + B(x),$$

where γ is a constant. Clearly, when $\gamma < 0$ the central bank's marginal payoff from inflation is higher than the marginal social welfare, and in this sense it has "inflation bias."

Consider problem (**P1**) in this setting. For each π^e_-, the constraint set is unchanged, while the objective function changes to

$$\begin{aligned} & \int_{\underline{\theta}}^{\overline{\theta}} \left[A(\pi(\theta); \theta) + \gamma\pi(\theta) + S(\pi(\theta); F) + \delta(\theta)\right] p(\theta) d\theta \\ = & \int_{\underline{\theta}}^{\overline{\theta}} \left[\tilde{R}(\pi(\theta), \theta) + \delta(\theta)\right] p(\theta) d\theta + \gamma \int_{\underline{\theta}}^{\overline{\theta}} \pi(\theta) p(\theta) d\theta. \end{aligned}$$

However, equation (15) implies that the last term equals $\gamma\pi^e_-$, which is taken as given in the maximization problem. Therefore the solution to this problem is identical to that of the problem we studied. Denoting the Bellman operator for this problem with inflation bias by $\mathbb{T}^{IB}$, it satisfies

$$\mathbb{T}^{IB} F(\pi^e_-) = \mathbb{T} F(\pi^e_-) + \gamma\pi^e_-$$

for all π^e_-. One implication of this relationship is that, when $\gamma < 0$, $\mathbb{T}^{IB} F$ is not peaked at π^{e*}_- but at a value that is lower than π^{e*}_-, where $\partial \mathbb{T} F / \partial \pi^e_- = -\gamma > 0$.

his or her comment.

References

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