Supplementary Material

Supplement to "Alternating-offer bargaining with the global games information structure"

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Anton Tsoy
Department of Economics, Einaudi Institute for Economics and Finance

B.6 *Interim versions of Theorems 1 and 3*

I first introduce interim counterparts of sets E, $\mathcal{E}$, and IR. Define $U^S_\eta(s|\tau,\rho) = \mathbb{E}_\eta[e^{-r\tau(s,b)}(\rho(s,b)-c(s))|s]$ and $U^B_\eta(b|\tau,\rho) = \mathbb{E}_\eta[e^{-r\tau(s,b)}(v(b)-\rho(s,b))|b]$ players' expected payoffs from the bargaining outcome (τ,ρ) at the interim stage, i.e., after types s and b are realized. For any $x \in [0,1]$, let

$$E(x) = \Big\{ \lim_{\eta\to 0}\big(U^S_\eta(x|\tau_\eta,\rho_\eta), U^B_\eta(x|\tau_\eta,\rho_\eta)\big) : (\tau_\eta,\rho_\eta) \underset{\eta\to 0}{\to} (\tau,\rho) \in DL \Big\}$$

be the set of all interim expected payoff profiles of types on the diagonal $s = x$ and $b = x$ generated by double limits. Let $\mathcal{E}(x)$ be the convex hull of the closure of $E(x)$. Note that when players have access to the public randomization device in the beginning of the game, $\mathcal{E}(x)$ is the set of interim expected payoff profiles of types $s = x$ and $b = x$ that can be approximated by double limits of my model.

I put some preliminary restrictions on $\mathcal{E}(x)$. Clearly, the *feasibility constraint* holds:

$$U^S_\eta(x|\tau_\eta,\rho_\eta) + U^B_\eta(x|\tau_\eta,\rho_\eta) \le \Pi(x).$$

Moreover, Lemma 1 implies that the seller's utility is at least $\underline{U}^S(s) \equiv \max\{y^*(0) - c(s), 0\}$ in any frequent-offer PBE limit, and, symmetrically, the buyer's utility is at least $\underline{U}^B(b) \equiv \max\{v(b) - y^*(1), 0\}$. Hence, for any η, any outcome of the frequent-offer PBE limit (τ_η,ρ_η) satisfies the *interim individual rationality constraints*

$$U^S_\eta(s|\tau_\eta,\rho_\eta) \ge \underline{U}^S(s) \quad \text{and} \quad U^B_\eta(b|\tau_\eta,\rho_\eta) \ge \underline{U}^B(b).$$

Denote by

$$IR(x) = \big\{\big(U^S, U^B\big) : U^S + U^B \le \Pi(x), U^S \ge \underline{U}^S(x), \text{and } U^B \ge \underline{U}^B(x)\big\}$$

the set of feasible, interim individually rational payoffs of types $s = x$ and $b = x$, and denote by

$$PF(x) = \big\{\big(U^S, U^B\big) : U^S + U^B = \Pi(x), U^S \ge \underline{U}^S(x), \text{and } U^B \ge \underline{U}^B(x)\big\}$$

its Pareto frontier. Then I have $\mathcal{E}(x) \subseteq IR(x)$.

Anton Tsoy: tsoianton.ru@gmail.com

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The following theorem is the interim counterpart of Theorem 1. The proof follows directly from the argument in the proof of Theorem 1.

THEOREM 4. *If* $(U^S, U^B) \in PF(x)$ *for some* $x \in [0,1]$, *then* $(U^S, U^B) \in \mathcal{E}(x)$.

I now turn to the interim version of the folk theorem. Suppose that the buyer's value is given by $v_0(b) + \xi$ and the seller's cost is $c(s)$, where v_0, c, and ξ are introduced in Section 3.4. I am interested in the limit of the set $\mathcal{E}(x)$ as $\xi \to 0$, which I denote by $\mathcal{E}_0(x)$. Denote by $IR_0(x)$, $x \in [0,1]$ the limit of $IR(x)$ as $\xi \to 0$. It is easy to see that

$$IR_0(x) = \{(U^S, U^B) : U^S + U^B \le \Pi_0(x),\, U^S \ge 0, \text{and } U^B \ge 0\} \quad \text{for } x \in [0,1],$$

where $\Pi_0(x) \equiv v_0(x) - c(x)$. The following theorem is the interim counterpart of Theorem 3.

THEOREM 5. *For all* $x \in [0,1]$, $\mathcal{E}_0(x) = IR_0(x)$.

PROOF. The argument for the Pareto frontier of $\mathcal{E}_0(x)$ is analogous to that for $\mathcal{E}$ in the proof of Theorem 3. Note that for $x = 0$ or $x = 1$, $IR_0(x) = \{(0,0)\}$. Then in the double limit that I constructed in Theorem 1, types $s = 0$ and $b = 0$ trade at a price close to $y^*(0)$. Since $v(0) - c(0) \underset{\xi\to 0}{\to} 0$, the expected utilities of those types converge to 0 as $\xi \to 0$. Similarly, the expected utilities of types $s = 1$ and $b = 1$ converge to 0 as $\xi \to 0$, which proves that $\mathcal{E}_0(x) = IR_0(x)$, $x \in \{0,1\}$. Now consider $x \in (0,1)$. There exists ξ small enough such that $y^*(0) < c(x) < v(x) < y^*(1)$. In the double limit constructed in the proof of Theorem 3, the utility of types $s = x$ and $b = x$ converges to 0 as $\xi \to 0$, which proves that $\mathcal{E}_0(x) = IR_0(x)$. □

B.7 *Proof of Lemma 12*

Let $V^B \equiv v(b_\infty) - q^S$, $V^S \equiv q^B - c(s_\infty)$, and $\Delta P \equiv q^S - q^B$. Denote $\phi \equiv \frac{1-\delta^2}{\delta^2 \Delta P}$, $\alpha_B \equiv \alpha^B(0) = \phi V^S$, and $\alpha_S \equiv \alpha^S(0) = \phi V^B$. By (24), $\alpha_B > 0$ and $\alpha_S > 0$. Note that $\phi \underset{\delta\to 1}{\to} 0$. I choose δ sufficiently large so that α_S and α_B are less than 1.

System (25) has steady states $(z, -z)$, $z \in \mathbb{R}$. I am interested in the positive trajectory that approaches the steady state $(0,0)$. Around this steady state, the linearized system can be written in the matrix form

$$\begin{pmatrix} x_{k+1} \\ y_{k+1} \end{pmatrix} = \begin{pmatrix} 1 - \alpha_B + \alpha_S\alpha_B & -\alpha_B(1-\alpha_S) \\ -\alpha_S & 1-\alpha_S \end{pmatrix} \begin{pmatrix} x_k \\ y_k \end{pmatrix}.$$

The matrix has eigenvalues 1 and $\lambda \equiv (1-\alpha_B)(1-\alpha_S) \in (0,1)$. Since one of the eigenvalues is equal to 1, the steady state is unstable, and I cannot conclude that in the neighborhood of the steady state, the nonlinear system will converge to the steady state or that the trajectory will stay positive. Therefore, I construct a particular trajectory that satisfies the desired properties. The proof proceeds in three steps.

Step 1: Conjectured solution. In the first step, I conjecture the form of solution and use the method of indeterminate coefficients to derive it. The following preliminary claim gives the Taylor expansion of $\alpha^B(y)$ and $\alpha^S(x)$.

CLAIM 1. *For any* $x \in (0,1)$ *and* $y \in (0,1)$,

$$\alpha^B(y) \equiv \alpha_B + \phi \sum_{l=1}^{\infty} \gamma_l^B y^l,$$

$$\alpha^S(x) \equiv \alpha_S + \phi \sum_{l=1}^{\infty} \gamma_l^B x^l,$$

where $\gamma_l^B \equiv \frac{d^l c(s_\infty)/dx^l}{l!}$ *and* $\gamma_l^S \equiv \frac{d^l v(b_\infty)/dx^l}{l!}$.

Note that by the regularity conditions on v *and* c, $\gamma_l^B < D$ *and* $\gamma_l^S < D$ *for all* l.

I conjecture that there exists $(\mu_i^x, \mu_i^y)_{i=1}^{\infty}$ such that the solution (25) takes the form[17]

$$\begin{pmatrix} x_k \\ y_k \end{pmatrix} = \sum_{i=1}^{\infty} \lambda^{ik} \begin{pmatrix} \lambda^{i/2}\mu_i^x \\ \mu_i^y \end{pmatrix} \quad \text{for } k = 1, 2, \dots, \tag{74}$$

and in addition, satisfies for all $i = 1, 2, \dots$,

$$\left|\mu_i^x\right| \le u_\delta M^i \quad \text{and} \quad \left|\mu_i^y\right| \le u_\delta M^i \tag{75}$$

for some positive M and u_δ such that

$$M < 1 < \frac{1}{\lambda(1+u_\delta)}. \tag{76}$$

Given this conjecture, I next derive expressions for coefficients μ_i^x and μ_i^y, and in the next step, I will verify that for δ sufficiently close to 1, upper bounds (75) indeed hold.

Series (74) defining (x_k, y_k) are absolutely convergent, as they are dominated by the absolutely convergent series $u_\delta \sum_{i=1}^{\infty} \lambda^{ik} M^i$. Plugging the conjectured solution (74) into system (25), I get

$$\begin{cases} \sum_{i=1}^{\infty} \lambda^{i(k+\frac{1}{2})}\left(\mu_i^x - \mu_i^x\lambda^i - \alpha_B\left(\mu_i^x + \mu_i^y\lambda^{\frac{i}{2}}\right)\right) \\ \quad = \phi\left(\sum_{l=1}^{\infty}\gamma_l^B\left(\sum_{i=1}^{\infty}\mu_i^y\lambda^{i(k+1)}\right)^l\right)\left(\sum_{i=1}^{\infty}\lambda^{i(k+\frac{1}{2})}\left(\mu_i^x + \mu_i^y\lambda^{\frac{i}{2}}\right)\right), \\ \sum_{i=1}^{\infty} \lambda^{ik}\left(\mu_i^y - \mu_i^y\lambda^i - \alpha_S\left(\mu_i^x\lambda^{\frac{i}{2}} + \mu_i^y\right)\right) \\ \quad = \phi\left(\sum_{l=1}^{\infty}\gamma_l^S\left(\sum_{i=1}^{\infty}\mu_i^x\lambda^{ik}\right)^l\right)\left(\sum_{i=1}^{\infty}\lambda^{ik}\left(\mu_i^x\lambda^{\frac{i}{2}} + \mu_i^y\right)\right). \end{cases} \tag{77}$$

[17] This is a natural guess given the eigenvalues of the linearized system.

Consider the first equation in system (77). Since $\lambda \in (0,1)$ and $|\mu_i^y| \le u_\delta M^i$, by Mertens' theorem,

$$\sum_{l=1}^{\infty} \gamma_l^B \left(\sum_{i=1}^{\infty} \mu_i^y \lambda^{i(k+1)} \right)^l = \sum_{l=1}^{\infty} \gamma_l^B \left(\sum_{i=l}^{\infty} \sum_{i_1+\cdots+i_l=i} \mu_{i_1}^y \cdot \ldots \cdot \mu_{i_l}^y \lambda^{i(k+1)} \right). \tag{78}$$

The series in (78) is absolutely convergent by

$$\begin{aligned}
&\sum_{l=1}^{\infty} \sum_{i=l}^{\infty} \left| \lambda^{i(k+1)} \gamma_l^B \sum_{i_1+\cdots+i_l=i} \mu_{i_1}^y \cdot \ldots \cdot \mu_{i_l}^y \right| \\
&\quad \le D \sum_{l=1}^{\infty} \sum_{i=l}^{\infty} \lambda^{i(k+1)} \sum_{i_1+\cdots+i_l=i} \left| \mu_{i_1}^y \cdot \ldots \cdot \mu_{i_l}^y \right| \\
&\quad \le D \sum_{l=1}^{\infty} \sum_{i=l}^{\infty} \lambda^{i(k+1)} \sum_{i_1+\cdots+i_l=i} u_\delta^l M^i \\
&\quad = D \sum_{l=1}^{\infty} u_\delta^l \sum_{i=l}^{\infty} \lambda^{i(k+1)} M^i \binom{i-1}{l-1} \\
&\quad = D \sum_{l=1}^{\infty} u_\delta^l \left(\frac{\lambda^{k+1} M}{1-\lambda^{k+1} M} \right)^l \\
&\quad \le D \sum_{l=1}^{\infty} u_\delta^l \left(\frac{\lambda M}{1-\lambda M} \right)^l,
\end{aligned}$$

where the first inequality arises via the triangle inequality and $|\gamma_l^B| < D$, the second inequality follows from (75), the first equality arises from the fact that the number of compositions of i into exactly l parts is $\binom{i-1}{l-1}$, the second equality is by summing over i, and the third inequality is by $\lambda^{k+1} < \lambda < 1$. The resulting series is convergent whenever $u_\delta \frac{\lambda M}{1-\lambda M} < 1$, which holds by (76). Therefore, by Fubini's theorem, exchanging the order of summation in (78) results in

$$\begin{aligned}
\sum_{l=1}^{\infty} \gamma_l^B \left(\sum_{i=1}^{\infty} \mu_i^y \lambda^{i(k+1)} \right)^l &= \sum_{l=1}^{\infty} \gamma_l^B \left(\sum_{i=l}^{\infty} \sum_{i_1+\cdots+i_l=i} \mu_{i_1}^y \cdot \ldots \cdot \mu_{i_l}^y \lambda^{i(k+1)} \right) \\
&= \sum_{i=1}^{\infty} \lambda^{i(k+1)} \sum_{l=1}^{i} \sum_{i_1+\cdots+i_l=i} \gamma_l^B \mu_{i_1}^y \cdot \ldots \cdot \mu_{i_l}^y.
\end{aligned}$$

By the absolute convergence of both series on the right-hand side of (77), the product on the right-hand side is equal to the Cauchy product, and so I can rewrite system (77) as

$$\begin{cases} \sum_{i=1}^{\infty} \lambda^{i(k+\frac{1}{2})} \Bigg[\mu_i^x - \mu_i^x \lambda^i - \alpha_B \big(\mu_i^x + \mu_i^y \lambda^{\frac{i}{2}} \big) \\ \quad - \phi \sum_{j=1}^{i-1} \Bigg(\big(\mu_{i-j}^x \lambda^{\frac{j}{2}} + \mu_{i-j}^y \lambda^{\frac{i}{2}} \big) \sum_{l=1}^{j} \gamma_l^B \sum_{j_1+\cdots+j_l=j} \mu_{j_1}^y \cdot \ldots \cdot \mu_{j_l}^y \Bigg) \Bigg] = 0, \\ \sum_{i=1}^{\infty} \lambda^{ik} \Bigg[\mu_i^y - \mu_i^y \lambda^i - \alpha_S \big(\mu_i^x \lambda^{\frac{i}{2}} + \mu_i^y \big) \\ \quad - \phi \sum_{j=1}^{i-1} \Bigg(\big(\mu_{i-j}^x \lambda^{\frac{i-j}{2}} + \mu_{i-j}^y \big) \sum_{l=1}^{j} \gamma_l^S \sum_{j_1+\cdots+j_l=j} \mu_{j_1}^x \cdot \ldots \cdot \mu_{j_l}^x \Bigg) \Bigg] = 0. \end{cases}$$

Setting all coefficients at $\lambda^{i(k+1/2)}$ and λ^{ik} equal to zero results in the system

$$\begin{cases} \mu_i^x - \mu_i^x \lambda^i - \alpha_B \big(\mu_i^x + \mu_i^y \lambda^{\frac{i}{2}} \big) \\ \quad = \phi \sum_{j=1}^{i-1} \Bigg(\big(\mu_{i-j}^x \lambda^{\frac{j}{2}} + \mu_{i-j}^y \lambda^{\frac{i}{2}} \big) \sum_{l=1}^{j} \gamma_l^B \sum_{j_1+\cdots+j_l=j} \mu_{j_1}^y \cdot \ldots \cdot \mu_{j_l}^y \Bigg), \\ \mu_i^y - \mu_i^y \lambda^i - \alpha_S \big(\mu_i^x \lambda^{\frac{i}{2}} + \mu_i^y \big) \\ \quad = \phi \sum_{j=1}^{i-1} \Bigg(\big(\mu_{i-j}^x \lambda^{\frac{i-j}{2}} + \mu_{i-j}^y \big) \sum_{l=1}^{j} \gamma_l^S \sum_{j_1+\cdots+j_l=j} \mu_{j_1}^x \cdot \ldots \cdot \mu_{j_l}^x \Bigg). \end{cases} \tag{79}$$

Using notation

$$A_i \equiv \begin{pmatrix} 1 - \lambda^i - \alpha_B & -\alpha_B \lambda^{i/2} \\ -\alpha_S \lambda^{i/2} & 1 - \lambda^i - \alpha_S \end{pmatrix}, \qquad \mu_i \equiv \begin{pmatrix} \mu_i^x \\ \mu_i^y \end{pmatrix},$$

and

$$\varphi_i = \begin{pmatrix} \varphi_i^x \\ \varphi_i^y \end{pmatrix} \equiv \begin{pmatrix} \phi \sum_{j=1}^{i-1} \Bigg(\big(\mu_{i-j}^x \lambda^{j/2} + \mu_{i-j}^y \lambda^{i/2} \big) \sum_{l=1}^{j} \gamma_l^B \sum_{j_1+\cdots+j_l=j} \mu_{j_1}^y \cdot \ldots \cdot \mu_{j_l}^y \Bigg) \\ \phi \sum_{j=1}^{i-1} \Bigg(\big(\mu_{i-j}^x \lambda^{(i-j)/2} + \mu_{i-j}^y \big) \sum_{l=1}^{j} \gamma_l^S \sum_{j_1+\cdots+j_l=j} \mu_{j_1}^x \cdot \ldots \cdot \mu_{j_l}^x \Bigg) \end{pmatrix}, \tag{80}$$

I can write the system in matrix form as

$$A_i \mu_i = \varphi_i.$$

Since $\det(A_i) = (1 - \lambda^i)(\lambda - \lambda^i) > 0$, for $i \geq 2$, matrix A_i is invertible, and I can solve for all μ_i (with the exception of $i = 1$):

$$\mu_i = A_i^{-1} \varphi_i. \tag{81}$$

Equation (81) expresses μ_i through $\mu_1, \dots, \mu_{i-1}$. For $i = 1$, the equalities in (79) are linearly dependent (as $\det(A_i) = 0$) and the relation between μ_1^x and μ_1^y is given by

$$\mu_1^x = \lambda^{-\frac{1}{2}} \frac{\alpha_B}{\alpha_S}(1-\alpha_S)\mu_1^y. \tag{82}$$

Equations (81) and (82) give the desired expressions for μ_i^x and μ_i^y through the parameters of the model.

Step 2: Verify bounds. In this step, I verify that for μ_i given by (82) and (81), bounds (75) and (76) indeed hold, and so my derivation in Step 1 is justified. Fix any $M < 1$. Let

$$u_\delta = \lambda^{-1/4} - 1.$$

Then $\frac{1}{\lambda(1+u_\delta)} = \lambda^{-3/4} > 1$ and so (76) holds. If $\frac{\alpha_B}{\alpha_S}(1-\alpha_S) \le \lambda^{\frac{1}{2}}$, then I set

$$\begin{aligned} \mu_1^y &= u_\delta M, \\ \mu_1^x &= \lambda^{-\frac{1}{2}} \frac{\alpha_B}{\alpha_S}(1-\alpha_S)\mu_1^y, \end{aligned} \tag{83}$$

and otherwise, set

$$\begin{aligned} \mu_1^x &= u_\delta M, \\ \mu_1^y &= \lambda^{\frac{1}{2}} \frac{\alpha_S}{\alpha_B(1-\alpha_S)}\mu_1^x. \end{aligned} \tag{84}$$

The next claim verifies *(75)*.

Claim 2. *There exists $\hat{\delta} \in (0,1)$ such that for any $\delta \in (\hat{\delta}, 1)$ such that for μ_1^x and μ_1^y defined above in (83) and (84) and μ_i^x and μ_i^y defined in (81), bounds (75) hold.*

Proof. The proof is by induction on i. By (83) and (84), $|\mu_1^x| \le u_\delta M$ and $|\mu_i^y| \le u_\delta M$, which proves the base of induction. Now, I prove the inductive step. Suppose that the statement is true for all $j < i$. I show that $|\mu_i^x| < u_\delta M^i$ and $|\mu_i^y| < u_\delta M^i$. I can find the closed-form solution to system (81),

$$\left|\mu_i^x\right| = \frac{\left|(1-\lambda^i-\alpha_S)\varphi_i^x + \alpha_B \lambda^{i/2}\varphi_i^y\right|}{(1-\lambda^i)(\lambda-\lambda^i)} \le \frac{4\max\{1-\lambda^i, \alpha_S, \alpha_B\} \cdot \max\{|\varphi_i^x|, |\varphi_i^y|\}}{(1-\lambda^i)(\lambda-\lambda^i)},$$

and the same upper bound holds for $|\mu_i^y|$. Thus, it is sufficient to show that

$$\frac{4\max\{(1-\lambda^i), \alpha_S, \alpha_B\} \cdot \max\{|\varphi_i^x|, |\varphi_i^y|\}}{(1-\lambda^i)(\lambda-\lambda^i)} < u_\delta M^i.$$

Notice that $\frac{\alpha_S}{1-\lambda^i} < \frac{\alpha_S}{1-\lambda}$ for $i \ge 2$, and by l'Hospital rule, $\lim_{\delta\to 1} \frac{\alpha_S}{1-\lambda} = \lim_{\delta \to 1} \frac{\alpha_S}{\alpha_S+\alpha_B-\alpha_S\alpha_B} = \frac{V^S}{V^S+V^B} < 1$. Hence, for sufficiently large δ and all $i \ge 2$, I have $\frac{\alpha_S}{1-\lambda^i} < 1$, and by an analogous argument, $\frac{\alpha_B}{1-\lambda^i} < 1$. Therefore, $\frac{\max\{1-\lambda^i, \alpha_S, \alpha_B\}}{1-\lambda^i} \le 1$ for sufficiently large δ and it remains to show that

$$\frac{4\max\{|\varphi_i^x|, |\varphi_i^y|\}}{\lambda - \lambda^i} < u_\delta M^i$$

for sufficiently large δ. I will show that $\frac{|\varphi_i^x|}{(\lambda-\lambda^i)u_\delta M^i} < \frac{1}{4}$, and by symmetric argument, $\frac{|\varphi_i^y|}{(\lambda-\lambda^i)u_\delta M^i} < \frac{1}{4}$. Recall from (80) that

$$\begin{aligned}
\varphi_i^x &= \phi \sum_{j=1}^{i-1}\left(\left(\mu_{i-j}^x \lambda^{\frac{j}{2}} + \mu_{i-j}^y \lambda^{\frac{i}{2}}\right) \sum_{l=1}^{j} \gamma_l^B \sum_{j_1+\cdots+j_l=j} \mu_{j_1}^y \cdot \ldots \cdot \mu_{j_l}^y\right) \\
&\leq \phi \sum_{j=1}^{i-1} \lambda^{\frac{j}{2}} \left(\sum_{l=1}^{j} |\gamma_l^B| \sum_{j_1+\cdots+j_l=j} |\mu_{i-j}^x \mu_{j_1}^y \cdot \ldots \cdot \mu_{j_l}^y|\right) \\
&\quad + \phi \lambda^{\frac{i}{2}} \sum_{j=1}^{i-1} \sum_{l=1}^{j} |\gamma_l^B| \sum_{j_1+\cdots+j_l=j} |\mu_{i-j}^y \mu_{j_1}^y \cdot \ldots \cdot \mu_{j_l}^y| \\
&\leq \phi \sum_{j=1}^{i-1} \lambda^{\frac{j}{2}} \sum_{l=1}^{j} |\gamma_l^B| \sum_{j_1+\cdots+j_l=j} u_\delta^{l+1} M^i + \phi \lambda^{\frac{i}{2}} \sum_{j=1}^{i-1} \sum_{l=1}^{j} |\gamma_l^B| \sum_{j_1+\cdots+j_l=j} u_\delta^{l+1} M^i \\
&= \phi \sum_{j=1}^{i-1} \lambda^{\frac{j}{2}} \sum_{l=1}^{j} |\gamma_l^B| u_\delta^{l+1} M^i \binom{j-1}{l-1} + \phi \lambda^{\frac{i}{2}} \sum_{j=1}^{i-1} \sum_{l=1}^{j} |\gamma_l^B| u_\delta^{l+1} M^i \binom{j-1}{l-1} \\
&\leq 2\phi u_\delta M^i D \sum_{j=1}^{i-1} \lambda^{\frac{j}{2}} \sum_{l=1}^{j} u_\delta^l \binom{j-1}{l-1} \\
&\leq 2\phi u_\delta M^i D \sum_{j=1}^{i-1} \lambda^{\frac{j}{2}} u_\delta (1+u_\delta)^{j-1} \\
&= 2\phi u_\delta M^i D \frac{u_\delta \lambda^{\frac{1}{2}} \left(1 - \lambda^{\frac{i-1}{2}} (1+u_\delta)^{i-1}\right)}{1 - \lambda^{\frac{1}{2}}(1+u_\delta)} \\
&= 2\phi u_\delta M^i D \lambda^{\frac{1}{2}} \left(1 - \lambda^{\frac{i-1}{2}}\right),
\end{aligned}$$

where the first inequality is due to the triangle inequality, the second inequality arises via the inductive hypothesis, the first equality makes use of the fact that the number of compositions of j into exactly l parts is $\binom{j-1}{l-1}$, the fourth inequality is by $\lambda^j > \lambda^i$ for $j < i$ and $|\gamma_l^B| < D$, the fifth inequality is by summing over l, the second equality is the summation over j, and the last equality is plugging in $u_\delta = \lambda^{-1/4} - 1$. Thus, I need to show that

$$2\phi D \frac{\lambda^{\frac{1}{4}} \left(1 - \lambda^{\frac{i-1}{4}}\right)}{\lambda - \lambda^i} < \frac{1}{4}.$$

This inequality holds for sufficiently large δ, as $\phi \to 0$ as $\delta \to 1$ and

$$\lim_{\delta\to 1}\frac{\lambda^{\frac{1}{4}}(1-\lambda^{\frac{i-1}{4}})}{\lambda-\lambda^{i}}=\lim_{\delta\to 1}\frac{1-\lambda^{\frac{i-1}{4}}}{1-\lambda^{i-1}}=\lim_{\delta\to 1}\frac{-\frac{i-1}{4}\lambda^{\frac{i-5}{4}}}{-i\lambda^{i-2}}=\frac{i-1}{4i}.$$

This completes the proof of the inductive step and the claim. □

Step 3: Check solution. In this step, I verify that the candidate trajectories (x_k, y_k) given by (74) that I have constructed indeed satisfy all conditions of Lemma 12. The convergence to $(0,0)$ follows immediately by taking the limit $k\to\infty$ of (74) and noting that $\lambda<1$. Note that I still have one free parameter left: M that pins down μ_1^x and μ_1^y in (83) or (84). I choose M so that

$$x_1+y_1=\lambda^{3/2}\mu_i^x+\lambda\mu_i^y=2\eta,$$

and so the initial condition in (25) is satisfied. In follows from (74) that x_k and y_k are decreasing in k and so

$$x_k+y_{k+1}\le x_k+y_k\le x_1+y_1=2\eta,$$

verifying inequalities (28) and (29).

Finally, I show that x_k and y_k are positive. Observe that

$$\begin{aligned}x_k&=\sum_{i=1}^{\infty}\lambda^{i(k+1/2)}\mu_i^x\\&=\lambda^{k+1/2}\left(\mu_1^x+\sum_{i=2}^{\infty}\lambda^{i(k+1/2)}\mu_i^x\right)\\&\ge\lambda^{k+1/2}\left(\mu_1^x-\sum_{i=2}^{\infty}\lambda^{i(k+1/2)}u_\delta M^i\right)\\&\ge\lambda^{k+1/2}\left(\mu_1^x-u_\delta\frac{\lambda^{2k+1}M^2}{1-\lambda^{(k+1/2)}M}\right).\end{aligned}$$

Hence, for sufficiently large k, x_k is positive whenever μ_1^x is. Since from (82) the sign of μ_1^y and μ_1^x is the same, y_k is positive for sufficiently large k. Thus, I have shown that x_k and y_k are positive starting from some k_0. By rearranging terms in the first equality of (25), $x_k=\frac{x_{k+1}+\alpha^B(y_{k+1})y_{k+1}}{1-\alpha^B(y_{k+1})}$. Observe that for δ sufficiently close to 1, $\alpha^B(y)\in(0,1)$ for all $y>0$. Hence, x_k is positive whenever x_{k+1} and y_{k+1} are positive. Analogously, it can be shown from the second equality of (25) that for sufficiently large δ, y_k is positive whenever x_{k+1} and y_{k+1} are positive. This proves that x_k and y_k are positive for all $k=1,2,\ldots,$ when δ is sufficiently close to 1.

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