Supplement to "On Competitive Nonlinear Pricing"

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Abstract

This supplement contains the proofs of technical lemmas and auxiliary results not included in the main paper. Unless stated otherwise, all references to results and equations are to the main paper.

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S.1 Technical Lemmas

Proof of Lemma 1. (Quasiconcavity) Consider a type i and a market maker k and let us hereafter omit the indices i and k for the sake of clarity. Let (q,t) and (q',t') be two trades and let Q^- and $Q^{-\prime}$ be the corresponding solutions to (3). For each $\lambda \in [0,1]$, $\lambda Q^- + (1-\lambda)Q^{-\prime}$ is an admissible candidate in (3). Hence $z^-(\lambda q + (1-\lambda)q', \lambda t + (1-\lambda)t')$ is at least

$$U(\lambda q + (1-\lambda)q' + \lambda Q^- + (1-\lambda)Q^{-\prime}, \lambda t + (1-\lambda)t' + T^-(\lambda Q^- + (1-\lambda)Q^{-\prime})).$$

Because T^- is convex and U is decreasing in transfers, this lower bound is itself at least

$$U(\lambda(q + Q^-) + (1-\lambda)(q' + Q^{-\prime}), \lambda[t + T^-(Q^-)] + (1-\lambda)[t' + T^-(Q^{-\prime})]),$$

and because U is quasiconcave this quantity is at least

$$\min\{U(q + Q^-, t + T^-(Q^-)), U(q' + Q^{-\prime}, t' + T^-(Q^{-\prime}))\},$$

which is $\min\{z^-(q,t), z^-(q',t')\}$ by construction. Notice that all these quantities may be equal if the tariff T^- is locally linear; hence this argument only shows that z^- is weakly quasiconcave.

(Property SC-z) Consider a market maker k and let us hereafter omit the index k for the sake of clarity. Fix some $q < q'$, t, and t'. First, let $\mathcal{T}(Q) \equiv t + T^-(Q - q)$, defined for $Q \geq q$. Similarly, let $\mathcal{T}'(Q) \equiv t' + T^-(Q - q')$, defined for $Q \geq q'$. According to (3), for each i, computing $z_i^-(q,t)$ amounts to maximizing $U_i(Q, \mathcal{T}(Q))$ with respect to $Q \geq q$. Let $\mathcal{Q}_i \geq q$ be the solution to this problem; it is unique as U_i is strictly quasiconcave and strictly decreasing in transfers and $\mathcal{T}$ is convex. Similarly, computing $z_i^-(q',t')$ amounts to maximizing $U_i(Q, \mathcal{T}'(Q))$ with respect to $Q \geq q'$. Let $\mathcal{Q}_i' \geq q'$ be the unique solution to this problem. The proof consists of two steps.

Step 1 We first prove (5). Suppose

$$z_i^-(q,t) < z_i^-(q',t')$$

for some i and let $j > i$. Because $\mathcal{Q}_j \geq q$ is an admissible candidate in the problem that defines $z_i^-(q,t)$, we have

$$U_i(\mathcal{Q}_j, \mathcal{T}(\mathcal{Q}_j)) \leq z_i^-(q,t) < z_i^-(q',t') = U_i(\mathcal{Q}_i', \mathcal{T}'(\mathcal{Q}_i')). \tag{S.1}$$

Two cases may arise.

(i) Suppose first $\mathcal{Q}_j < \mathcal{Q}'_i$. Then

$$z_j^-(q,t) = U_j(\mathcal{Q}_j, \mathcal{T}(\mathcal{Q}_j)) < U_j(\mathcal{Q}'_i, \mathcal{T}'(\mathcal{Q}'_i)) \leq z_j^-(q',t'),$$

where the first inequality follows from (S.1), Assumption SC-U, and the assumptions that $i < j$ and $\mathcal{Q}_j < \mathcal{Q}'_i$, and the second inequality follows from the fact that $\mathcal{Q}'_i \geq q'$ is an admissible candidate in the problem that defines $z_j^-(q',t')$. This shows (5).

(ii) Suppose next $\mathcal{Q}_j \geq \mathcal{Q}'_i$. Because $\mathcal{Q}'_i \geq q' > q$ is an admissible candidate in the problem that defines $z_i^-(q,t)$, we have

$$U_i(\mathcal{Q}'_i, \mathcal{T}(\mathcal{Q}'_i)) \leq z_i^-(q,t) < z_i^-(q',t') = U_i(\mathcal{Q}'_i, \mathcal{T}'(\mathcal{Q}'_i)),$$

which implies $\mathcal{T}'(\mathcal{Q}'_i) < \mathcal{T}(\mathcal{Q}'_i)$. Moreover, as $q < q'$ and T^- is convex, $\mathcal{T}'(Q) - \mathcal{T}(Q)$ is nonincreasing in $Q \geq q'$. Because $\mathcal{Q}_j \geq \mathcal{Q}'_i \geq q'$ and $\mathcal{T}'(\mathcal{Q}'_i) < \mathcal{T}(\mathcal{Q}'_i)$, it follows that $\mathcal{T}'(\mathcal{Q}_j) < \mathcal{T}(\mathcal{Q}_j)$. Now, as $\mathcal{Q}_j \geq q'$, $\mathcal{Q}_j$ is an admissible candidate in the problem that defines $z_j^-(q',t')$ and thus

$$U_j(\mathcal{Q}_j, \mathcal{T}'(\mathcal{Q}_j)) \leq z_j^-(q',t').$$

Hence, from $\mathcal{T}'(\mathcal{Q}_j) < \mathcal{T}(\mathcal{Q}_j)$, we directly obtain

$$z_j^-(q,t) = U_j(\mathcal{Q}_j, \mathcal{T}(\mathcal{Q}_j)) < U_j(\mathcal{Q}_j, \mathcal{T}'(\mathcal{Q}_j)) \leq z_j^-(q',t').$$

This shows (5).

Step 2 The proof of (4) follows from (5) by continuity. Suppose $z_i^-(q,t) = z_i^-(q',t')$ for some i and let $j > i$. Then, because z_i^- is strictly decreasing in transfers, for any strictly positive ε, we have $z_i^-(q,t+\varepsilon) < z_i^-(q',t')$ and thus $z_j^-(q,t+\varepsilon) < z_j^-(q',t')$ from (5). As z_j^- is continuous, we can take limits as ε goes to zero to obtain (4). Notice that we may have $z_j^-(q,t) = z_j^-(q',t')$ if the tariff T^- is locally linear; hence this argument only shows that the family of functions z_i^- satisfies a weak single-crossing property. The result follows. ■

Proof of Lemma 2. Consider a market maker k and let us hereafter omit the index k for the sake of clarity. Let $\mu^* \equiv \{(q_i^*, t_i^*) : i = 0, \ldots, I\}$ be a menu with nondecreasing quantities such that (7) holds. The proof consists of two steps.

Step 1 We first show that there exists a menu $\mu \equiv \{(q_i, t_i) : i = 0, \ldots, I\}$ that has nondecreasing quantities and satisfies the following conditions:

(a) $\sum_i m_i v_i(q_i, t_i) \geq \sum_i m_i v_i(q_i^*, t_i^*)$.

(b) For each $i \geq 1$, $z_i^-(q_i, t_i) \geq z_i^-(q_{i-1}, t_{i-1})$.

(c) For each $i > 1$, if $q_i > q_{i-1}$, then $z_{i-1}^-(q_{i-1}, t_{i-1}) > z_{i-1}^-(q_i, t_i)$.

Notice that (b) is identical to (7), whereas (c) is a strict version of the upward local incentive-compatibility constraints. Suppose, by way of contradiction, that there is no menu that satisfies conditions (a), (b), and (c). Nevertheless, the set of menus with nondecreasing quantities such that (a) and (b) hold is nonempty, as it contains μ^*. Therefore, we can select in this set a menu μ that maximizes the index $j > 1$ of the first violation of (c). By construction, for this index j, we must have $q_j > q_{j-1}$.

We can even require that (b) holds as an equality at $i = j$ for μ. Indeed, if (b) holds as a strict inequality at $i = j$, we can increase t_j until reaching an equality: this is feasible because z_j^- is weakly quasiconcave and strictly decreasing in transfers. This change in t_j defines a new menu that satisfies conditions (a), (b) for all i, with an equality at $i = j$, and (c) for all $i < j$; but, according to our definition of μ, (c) must still be violated at $i = j$. With a slight abuse of notation, denote this new menu again by μ.

Now, because (b) holds as an equality at $i = j$ and $q_j > q_{j-1}$, the contraposition of (5) in Property SC-z yields $z_{j-1}^-(q_{j-1}, t_{j-1}) \geq z_{j-1}^-(q_j, t_j)$. Recall, however, that (c) is violated at $i = j$. Therefore, the only remaining possibility is that this inequality is in fact an equality. As a result, (b) and (c) hold as equalities at $i = j$ and we face a cycle of binding incentive-compatibility constraints that we now eliminate by pooling types $j - 1$ and j on the same trade. Two cases may arise.

(i) Suppose first $v_j(q_j, t_j) \leq v_j(q_{j-1}, t_{j-1})$. We can then build a new menu μ' that only differs from μ in allocating (q_{j-1}, t_{j-1}) to type j. (a) is relaxed by construction. (b) and (c) are unaffected for all $i < j$ and trivially hold at $i = j$ as types $j - 1$ and j are pooled on the same trade. Finally, (b) still holds for all $i > j$ because, by Property SC-z, the downward incentive-compatibility constraints are satisfied as soon as the downward local incentive-compatibility constraints are satisfied. But then μ' satisfies conditions (a) and (b), and any violation of (c) for μ' must take place for a type strictly higher than j, contradicting our definition of μ.

(ii) Suppose next $v_j(q_j, t_j) > v_j(q_{j-1}, t_{j-1})$. We can then build a new menu μ' that only differs from μ in allocating (q_j, t_j) to type $j - 1$. (a) is relaxed because, from $q_j > q_{j-1}$, the contraposition of Property SC-v yields $v_{j-1}(q_j, t_j) > v_{j-1}(q_{j-1}, t_{j-1})$. (b) and (c) are unaffected for all $i < j - 1$ and trivially hold at $i = j$ as types $j - 1$ and j are pooled on the same trade. (b) is unaffected for all $i > j$. At $i = j - 1$, because (c) holds as an equality at $i = j$ for μ, the change from μ to μ' does not affect type $j-1$'s utility and so (b) still holds at $i = j-1$. There remains to check that (c) still holds at $i = j-1$, in case $j > 2$. Because (c) holds as an equality at $i = j$ for μ, the contraposition of (5) in Property SC-z yields

$$z^{-}_{j-2}(q_{j-1}, t_{j-1}) \geq z^{-}_{j-2}(q_j, t_j).$$

We also know that (c) holds at $i = j-1$ for μ, so that

$$z^{-}_{j-2}(q_{j-2}, t_{j-2}) > z^{-}_{j-2}(q_{j-1}, t_{j-1}).$$

These inequalities imply that (c) still holds at $i = j-1$. Once more, μ' satisfies conditions (a) and (b), and any violation of (c) for μ' has to take place for a type strictly higher than j, contradicting our definition of μ.

Step 2 In Step 1, we have shown that, for any menu μ^* with nondecreasing quantities such that (7) holds, there exists a menu μ with nondecreasing quantities that yields an expected profit at least as high as μ^* and satisfies conditions (b) and (c). By continuity of the functions z^{-}_{i}, we can then slightly decrease each transfer in the menu μ to obtain a menu μ' in which both (b) and (c) now hold as strict inequalities. Hence the local incentive-compatibility and type 1's individual-rationality constraint for μ' are slack. Property SC-z together with the fact that quantities in the menu μ' are nondecreasing then ensure that the constraints (6) hold as strict inequalities and thus that the insider has a unique best response to μ'. As the decrease in transfers in μ' relative to μ is arbitrarily small, we can approximate as closely as we want the expected profit from μ and, a fortiori, from μ^*. The result follows. ■

Proof of Lemma 3. We begin with some preliminary remarks on the insider's best response to an arbitrary profile of convex tariffs.

Step 0 Recall that, given a profile $(t^1, \ldots, t^K)$ of convex tariffs, the aggregate demand Q_i of type i is uniquely determined and nondecreasing in i. Given Q_i, type i's utility-maximization problem (1) reduces to minimizing her total payment for Q_i, $T(Q_i)$, as defined by (2). This is a convex problem, so that, by the Kuhn–Tucker theorem (Rockafellar (1970, Corollary 28.3.1)), we can associate to any of its solutions $(q^1_i, \ldots, q^K_i)$ a Lagrange multiplier p_i such that $p_i \in \partial t^k(q^k_i)$ for all k. If there were two different solutions $(q^1_i, \ldots, q^K_i)$ and $(q'^1_i, \ldots, q'^K_i)$ to (2) with different multipliers $p_i < p'_i$, then, because each tariff is convex, we would have $q^k_i \leq q'^k_i$ for all k; but then, as both solutions must sum to Q_i, they would be identical, a contradiction. This shows that all the solutions to (2) must share the same p_i.

Hence we can associate to each type i a marginal price p_i such that, whatever the solution $(q_i^1, \ldots, q_i^K)$ to (2), we have $p_i \in \partial t^k(q_i^k)$ for all k. Finally, we can with no loss of generality adopt the convention that p_i is nondecreasing in i. Indeed, if $p_{i-1} > p_i$ for some $i > 1$, then, because $p_{i-1} \in \partial t^k(q_{i-1}^k)$ and $p_i \in \partial t^k(q_i^k)$ for all k, we have $q_{i-1}^k \geq q_i^k$ for all k. As these quantities sum to Q_{i-1} and Q_i, respectively, and as $Q_{i-1} \leq Q_i$, it follows that $q_{i-1}^k = q_i^k$ for all k. Hence $p_{i-1} \in \partial t^k(q_i^k)$ for all k and we can replace p_i by p_{i-1}. Given this convention, the lower and upper bounds $\underline{s}^k(p_i)$ and $\overline{s}^k(p_i)$ of the supply $s^k(p_i)$ of market maker k at marginal price p_i, as defined by (9), are both nondecreasing in i for all k.

Now, suppose that $(t^1, \ldots, t^K)$ are equilibrium tariffs and that market maker k deviates to some convex tariff t. Consider a nondecreasing family of quantities q_i such that (10) holds for all i; we know from Property SC-z that such a family exists. Denoting by $p_i \in \partial t(q_i)$ a Lagrange multiplier for type i's problem of minimizing her total payment, we can, according to Step 0, require that p_i be nondecreasing in i. In fact, under Assumption QL-U, each type i must purchase $D_i(p_i) = (u_i')^{-1}(p_i)$ in the aggregate, which uniquely pins down the value of p_i given the equilibrium tariffs t^{-k} of the market makers other than k. The proof consists of four steps.

Step 1 Letting $\boldsymbol{p} \equiv (p_1, \ldots, p_I)$ and $\boldsymbol{q} \equiv (q_1, \ldots, q_I)$, consider the piecewise-linear tariff $t_{\boldsymbol{p},\boldsymbol{q}}$ recursively defined by $t_{\boldsymbol{p},\boldsymbol{q}}(0) \equiv 0$ and

$$t_{\boldsymbol{p},\boldsymbol{q}}(q) \equiv t_{\boldsymbol{p},\boldsymbol{q}}(q_{i-1}) + p_i(q - q_{i-1}), \quad i = 1, \ldots, I, \quad q \in (q_{i-1}, q_i],$$

with $q_0 \equiv 0$ by convention. Because the families of marginal prices and quantities p_i and q_i are nondecreasing, the tariff $t_{\boldsymbol{p},\boldsymbol{q}}$ is convex. It is straightforward to check that $t_{\boldsymbol{p},\boldsymbol{q}}(q_i) \geq t(q_i)$ for all i. Moreover, as $p_i = \partial^- t_{\boldsymbol{p},\boldsymbol{q}}(q_i)$, it remains a best response for each type i to purchase q_i from market maker k if the tariffs $(t_{\boldsymbol{p},\boldsymbol{q}}, t^{-k})$ are posted. In fact, under Assumption QL-U, $t_{\boldsymbol{p},\boldsymbol{q}}$ is the highest convex tariff with the property that purchasing q_i from market maker k is a best response for each type i given the equilibrium tariffs t^{-k} of the market makers other than k (see Figure 2).

Step 2 According to Step 1, we can hereafter suppose that market maker k deviates to the tariff $t_{\boldsymbol{p},\boldsymbol{q}}$. As in (9), let $s_{\boldsymbol{p},\boldsymbol{q}}^k(p_i) \equiv \{q : p_i \in \partial t_{\boldsymbol{p},\boldsymbol{q}}(q)\}$ be the supply of market maker k at marginal price p_i when he posts the tariff $t_{\boldsymbol{p},\boldsymbol{q}}$, with lower and upper bounds $\underline{s}_{\boldsymbol{p},\boldsymbol{q}}^k(p_i)$ and $\overline{s}_{\boldsymbol{p},\boldsymbol{q}}^k(p_i)$, respectively. Define a nondecreasing family of quantities $\overline{q}_i$ as follows:

(i) If $\underline{s}_{\boldsymbol{p},\boldsymbol{q}}^k(p_i) < \overline{s}_{\boldsymbol{p},\boldsymbol{q}}^k(p_i)$ and if $I_i^+ \equiv \{j : p_j = p_i > c_j\} \neq \emptyset$, let $\overline{q}_i \equiv \max\{q_j : j \in I_i^+\}$.

(ii) Otherwise, let $\overline{q}_i \equiv \underline{s}_{\boldsymbol{p},\boldsymbol{q}}^k(p_i)$.

Intuitively, there is a single value of $\overline{q}$ for each value of p in $\{p_1, \ldots, p_I\}$: below $\overline{q}$, we find all the types such that $c_i < p$ who trade at marginal price p and to whom market maker k would like to sell higher quantities. Above $\overline{q}$, we find all the types such that $p \leq c_i$ who trade at marginal price p and to whom market maker k would like to sell lower quantities.

Step 3 A way for market maker k to achieve these objectives consists in decreasing the slope of the tariff $t_{\boldsymbol{p},\boldsymbol{q}}$ between $\underline{s}^k(p_i)$ and $\overline{q}_i$, and in increasing it between $\overline{q}_i$ and $\overline{s}^k(p_i)$. Consider accordingly a small strictly positive ε and let $\hat{t} \equiv t_{\boldsymbol{p}-\varepsilon\mathbf{1}_I,\overline{\boldsymbol{q}}}$, with $\mathbf{1}_I \equiv (1, \ldots, 1) \in \mathbb{R}^I$ and $\overline{\boldsymbol{q}} \equiv (\overline{q}_1, \ldots, \overline{q}_I)$. Notice that, for each i, we have $\partial^- \hat{t}(\overline{q}_i) \leq p_i - \varepsilon < p_i < \partial^+ \hat{t}(\overline{q}_i)$, so that slopes are changed in the right directions (see Figure 3). Let $(\hat{q}_1, \ldots, \hat{q}_I)$ be any best response of the insider to the tariff $\hat{t}$ given the equilibrium tariffs t^{-k} of the market makers other than k. According to the definition of $\overline{q}_i$, two cases may arise.

(i) If $p_i > c_i$, then $\underline{s}^k(p_i) \leq q_i \leq \overline{q}_i$. Then, because for each $q \leq q_i$ the tariff $\hat{t}$ satisfies

$$\partial^- \hat{t}(q) \leq \partial^- \hat{t}(\overline{q}_i) \leq p_i - \varepsilon < p_i$$

and type i has quasilinear utility, we must have $\hat{q}_i \geq q_i$.

(ii) If $p_i \leq c_i$, then $\overline{q}_i \leq q_i \leq \overline{s}^k(p_i)$. Then, because for each $q \geq q_i$ the tariff $\hat{t}$ satisfies

$$\partial^+ \hat{t}(q) \geq \partial^+ \hat{t}(\overline{q}_i) > p_i$$

and type i has quasilinear utility, we must have $\hat{q}_i \leq q_i$.

Step 4 Finally, for any strictly positive ε, we have $\hat{t}(q) = t_{\boldsymbol{p}-\varepsilon\mathbf{1}_I,\overline{\boldsymbol{q}}}(q) \geq t_{\boldsymbol{p},\boldsymbol{q}}(q) - O(\varepsilon)$ for all q (see Figure 3). Thus, for any best response $(\hat{q}_1, \ldots, \hat{q}_I)$ of the insider to the tariff $\hat{t}$ given the equilibrium tariffs t^{-k} of the market makers other than k, we have

$$\begin{aligned}
\sum_i m_i[\hat{t}(\hat{q}_i) - c_i\hat{q}_i] &\geq \sum_i m_i[t_{\boldsymbol{p},\boldsymbol{q}}(\hat{q}_i) - c_i\hat{q}_i] - O(\varepsilon) \\
&\geq \sum_i m_i[t_{\boldsymbol{p},\boldsymbol{q}}(q_i) - c_i q_i] - O(\varepsilon) \\
&\geq \sum_i m_i[t(q_i) - c_i q_i] - O(\varepsilon),
\end{aligned}$$

where the second inequality follows from the fact that $\hat{q}_i \leq q_i$ if $p_i \leq c_i$ and $\hat{q}_i \geq q_i$ if $p_i > c_i$ by Step 3, and the third inequality follows from Step 1. Hence, by posting the tariff $\hat{t}$, market maker k can secure an expected profit within $O(\varepsilon)$ of $\sum_i m_i[t(q_i) - c_i q_i]$, where ε is arbitrarily small. The result follows. ■

Proof of Lemma 4. Consider a market maker k and let us hereafter omit the index k for the sake of clarity. We prove the result for the more general case where the insider's type is distributed over some compact subset $\mathcal{I}$ of $\mathbb{R}$ according to an arbitrary distribution $\boldsymbol{m}$. We assume that the appropriate generalization of SC-v holds, that $\overline{D} \equiv \sup\{D_i(p) : i \in \mathcal{I}\} < \infty$, and that there exists an $\boldsymbol{m}$-integrable function g such that $|\nu_i(q)| \leq g_i$ for all $(i, q) \in \mathcal{I}\times[0, \overline{D}]$, where $\nu_i(q) \equiv v_i(q, pq)$ for all i and q. Now, observe that, if the quantities q_i satisfy the constraints (13), then so do the quantities $\min\{q_i, \overline{q}\}$ for all $\overline{q}$. Hence we can restrict our quest for a solution to problem (12)–(13) to the set of nondecreasing families of quantities q_i such that (13) holds and

$$\int \nu_i(\overline{q}) 1_{\{q_i \geq \overline{q}\}} \, \boldsymbol{m}(\mathrm{d}i) \leq \int \nu_i(q_i) 1_{\{q_i \geq \overline{q}\}} \, \boldsymbol{m}(\mathrm{d}i), \quad \overline{q} \in [0, \|q\|_\infty], \tag{S.2}$$

where $\|q\|_\infty \equiv \inf\{q : \boldsymbol{m}[\{i \in \mathcal{I} : q_i \leq q\}] = 1\}$. Notice that this set contains the null family and is thus nonempty. We claim that any nondecreasing family of quantities q_i in this set yields an expected profit at most equal to that provided by the quantities $\min\{D_i(p), \|q\|_\infty\}$. This is obvious if $\|q\|_\infty = 0$. If $\|q\|_\infty > 0$, then, for each $\varepsilon \in (0, \|q\|_\infty]$, applying (S.2) to $\overline{q} = \|q\|_\infty - \varepsilon$ implies that there exists j such that $q_j > \|q\|_\infty - \varepsilon$ and

$$\nu_j^k(\|q\|_\infty - \varepsilon) \leq \nu_j^k(q_j).$$

The contraposition of SC-v then yields[1]

$$\nu_i(\|q\|_\infty - \varepsilon) \leq \nu_i(q_j), \quad i \leq j.$$

Because the quantities q_i are nondecreasing, this, in particular, holds for all i such that $q_i < \|q\|_\infty - \varepsilon$. As the functions ν_i are weakly quasiconcave, it follows that, for each i such that $q_i < \|q\|_\infty - \varepsilon$, the function ν_i is nondecreasing over $[0, \|q\|_\infty - \varepsilon]$. Because this is true for all $\varepsilon \in (0, \|q\|_\infty]$, we obtain that, for each i such that $q_i < \|q\|_\infty$, the function ν_i is nondecreasing over $[0, \|q\|_\infty]$. Hence we can choose the quantities $\min\{D_i(p), \|q\|_\infty\}$ instead of the quantities q_i without reducing the expected profit, as claimed. This implies that problem (12)–(13) reduces to

$$\sup\left\{\int \nu_i(\min\{D_i(p), \overline{q}\}) \, \boldsymbol{m}(\mathrm{d}i) : \overline{q} \in [0, \overline{D}]\right\}. \tag{S.3}$$

As the functions ν_i are continuous, Lebesgue's dominated convergence theorem (Aliprantis and Border (2006, Theorem 11.21)) ensures that the objective function in problem (S.3) is

[1] Strictly speaking, the contraposition of SC-v states that $v_j^k(q', t') > v_j^k(q, t)$ implies $v_i^k(q', t') > v_i^k(q, t)$. However, because the profit functions are continuous and strictly decreasing in transfers, we can easily show as in Step 2 of the proof of Property SC-z that $v_j^k(q', t') \geq v_j^k(q, t)$ implies $v_i^k(q', t') \geq v_i^k(q, t)$, which is the implication we use here.

continuous in $\overline{q}$, and, hence, that this problem has a solution. Therefore, problem (12)–(13) has a solution with limit-order quantities at price p. Finally, if the functions ν_i are strictly quasiconcave, the above reasoning shows that they are strictly increasing over the relevant ranges, so that any solution to problem (12)–(13) is of this form. The result follows. ■

Proof of Lemma 5. Recall that, given a profile $(t^1, \ldots, t^K)$ of convex tariffs, the aggregate trade (Q_i, T_i) of type i is uniquely determined, and that we can associate to type i a Lagrange multiplier p_i as in Step 0 of the proof of Lemma 3. To find an efficient allocation, we first solve for each i

$$\max\left\{\sum_k v_i^k(q_i^k, t^k(q_i^k)) : (q_i^1, \ldots, q_i^K) \in A^1 \times \cdots \times A^K\right\},$$

subject to constraint i in (15). Because all market makers have identical quasilinear profit functions, this problem reduces to

$$\min\left\{\sum_k c_i(q_i^k) : (q_i^1, \ldots, q_i^K) \in A^1 \times \cdots \times A^K\right\}, \tag{S.4}$$

subject to

$$\sum_k q_i^k = Q_i \quad \text{and} \quad \underline{s}^k(p_i) \leq q_i^k \leq \overline{s}^k(p_i), \quad k = 1, \ldots, K, \tag{S.5}$$

where the latter constraints ensure that $(q_i^1, \ldots, q_i^K)$ is a best response of type i to the tariffs $(t^1, \ldots, t^K)$. We now show that the family of problems (S.4)–(S.5) indexed by i admits a family of solutions with nondecreasing individual quantities. Notice first that each of these problems has a nonempty compact set of solutions. Hence there exists a family of solutions $(q_1^1, \ldots, q_1^K, \ldots, q_I^1, \ldots, q_I^K)$ to the family of problems (S.4)–(S.5) that minimizes the following criterion for violations of monotonicity:

$$\sum_k \sum_{i>1} \max\{q_{i-1}^k - q_i^k, 0\}. \tag{S.6}$$

Suppose, by way of contradiction, that this minimum is strictly positive. Then, at the minimum, we have

$$q_{i-1}^k > q_i^k \tag{S.7}$$

for some $i > 1$ and k. As $\underline{s}^k(p_i)$ and $\overline{s}^k(p_i)$ are nondecreasing in i, this implies

$$\underline{s}^k(p_{i-1}) \leq \underline{s}^k(p_i) \leq q_i^k < q_{i-1}^k \leq \overline{s}^k(p_{i-1}) \leq \overline{s}^k(p_i). \tag{S.8}$$

The intervals $[\underline{s}^k(p_{i-1}), \overline{s}^k(p_{i-1})]$ and $[\underline{s}^k(p_i), \overline{s}^k(p_i)]$ then have a nontrivial intersection, so it must be that $p_{i-1} = p_i$. Therefore, for each l, $\underline{s}^l(p_{i-1}) = \underline{s}^l(p_i)$ and $\overline{s}^l(p_{i-1}) = \overline{s}^l(p_i)$. Moreover, because $q_{i-1}^k > q_i^k$ and $Q_{i-1} \leq Q_i$, there exists $l \neq k$ such that

$$q_{i-1}^l < q_i^l. \tag{S.9}$$

Summing up, we have

$$\underline{s}^l(p_{i-1}) = \underline{s}^l(p_i) \leq q_{i-1}^l < q_i^l \leq \overline{s}^l(p_{i-1}) = \overline{s}^l(p_i). \tag{S.10}$$

Given (S.8) and (S.10), we can slightly decrease q_{i-1}^k and increase q_{i-1}^l by a strictly positive amount ε, so that all constraints are still satisfied. This modification strictly decreases the criterion (S.6), so that $q_{i-1}^k - \varepsilon$ and $q_{i-1}^l + \varepsilon$ cannot be part of a solution to problem (S.4)–(S.5) for type $i-1$. We thus obtain

$$c_{i-1}(q_{i-1}^k - \varepsilon) + c_{i-1}(q_{i-1}^l + \varepsilon) > c_{i-1}(q_{i-1}^k) + c_{i-1}(q_{i-1}^l).$$

As c_{i-1} is convex, this implies $q_{i-1}^k - \varepsilon < q_{i-1}^l$ and, therefore, $q_{i-1}^k \leq q_{i-1}^l$ as ε is arbitrary. Alternatively, we can slightly increase q_i^k and decrease q_i^l by the same strictly positive amount ε. We similarly obtain

$$c_i(q_i^k + \varepsilon) + c_i(q_i^l - \varepsilon) > c_i(q_i^k) + c_i(q_i^l),$$

which implies $q_i^l \leq q_i^k$. Using (S.7) then yields $q_i^l \leq q_i^k < q_{i-1}^k \leq q_{i-1}^l$, which contradicts (S.9). The result follows. ■

S.2 On the Riemann Approximation (34) of (29)–(30)

In this section, we prove that the Riemann approximation (34) of (29)–(30) is uniform in $\boldsymbol{\chi}$. As a preliminary remark, observe that, when maximising (29)–(30), we can with no loss of generality focus on nondecreasing quantity schedules χ in a uniformly bounded set: the first requirement follows from the fact that the family of functions $\zeta^{*-k}(\cdot, \theta)$ satisfies the strict single-crossing property, and the second requirement follows from the fact that, under Biais, Martimort, and Rochet's (2000) responsiveness assumption $c'(\theta) < 1$, quantities in an optimal schedule are bounded above by

$$\hat{\chi}(\overline{\theta}) \equiv \arg\max \{\zeta^{*-k}(q, \overline{\theta}) - c(\overline{\theta})q : q \geq 0\} = \frac{1}{K} \arg\max \{u(Q, \overline{\theta}) - c(\overline{\theta})Q : Q \geq 0\},$$

that is, a fraction $1/K$ of the efficient quantity for type $\overline{\theta}$. Denote by

$$X \equiv \{\chi : [\underline{\theta}, \overline{\theta}] \to \mathbb{R} : \chi \text{ is nondecreasing and } \chi(\theta) \in [0, \hat{\chi}(\overline{\theta})] \text{ for all } \theta \in [\underline{\theta}, \overline{\theta}]\}$$

the corresponding set of quantity schedules.

Now, each $\chi \in X$, being nondecreasing, has at most countably many discontinuities. Because it is a continuous function of $(\chi(\theta), \theta)$, the same holds for the integrand in (30); it is thus Riemann-integrable (Aliprantis and Border (2006, Theorem 11.30)), so that the Riemann sum in (34) converges to the integral in (30). What we need, however, is a stronger result, namely, that (34) approximates (29)–(30) uniformly in $\chi \in X$. The key observation in that respect is that, if the functions f, u, and c are sufficiently regular, then the indirect utility function $\zeta^{*,-k}$ is twice continuously differentiable. This property is notably satisfied in the uniform-quadratic example studied by Biais, Martimort, and Rochet (2013), and we hereafter assume this to be the case. In particular, the Taylor-Lagrange approximations in (31)–(33) are valid.

A first implication of this is that the $O(1/I)$ term in the approximation (34) of (29) is uniform in $\chi \in X$. Indeed, the difference between the sums in (29) and (34) can be uniformly bounded as follows:

$$\begin{aligned}
&\Bigg|\sum_{i=1}^{I}\left[m_i - \frac{\overline{\theta}-\underline{\theta}}{I}\, f(\theta_i)\right][\zeta^{*-k}(\chi(\theta_i),\theta_i) - c(\theta_i)\chi(\theta_i)] \\
&\qquad - \sum_{i=1}^{I}[1-F(\theta_i)]\left[\zeta^{*-k}(\chi(\theta_i),\theta_{i+1}) - \zeta^{*-k}(\chi(\theta_i),\theta_i) - \frac{\overline{\theta}-\underline{\theta}}{I}\,\frac{\partial \zeta^{*-k}}{\partial \theta}\,(\chi(\theta_i),\theta_i)\right]\Bigg| \\
&\leq \sum_{i=1}^{I}\left|m_i - \frac{\overline{\theta}-\underline{\theta}}{I}\, f(\theta_i)\right| \max\{|\zeta^{*-k}(q,\theta) - c(\theta)q| : (q,\theta) \in [0,\hat{\chi}(\overline{\theta})] \times [\underline{\theta},\overline{\theta}]\} \\
&\qquad + I \max\Bigg\{\left|\zeta^{*-k}(q,\theta_{i+1}) - \zeta^{*-k}(q,\theta_i) - \frac{\overline{\theta}-\underline{\theta}}{I}\,\frac{\partial \zeta^{*-k}}{\partial \theta}\,(q,\theta_i)\right| \\
&\qquad\qquad : q \in [0,\hat{\chi}(\overline{\theta})] \text{ and } i = 1,\ldots,I\Bigg\} \\
&\leq I\, O\left(\frac{1}{I^2}\right) + \frac{(\overline{\theta}-\underline{\theta})^2}{2I}\left(\max\left\{\left|\frac{\partial^2 \zeta^{*-k}}{\partial \theta^2}\,(q,\theta)\right| : (q,\theta) \in [0,\hat{\chi}(\overline{\theta})] \times [\underline{\theta},\overline{\theta}]\right\} + o(1)\right) \\
&= O\left(\frac{1}{I}\right).
\end{aligned}$$

To conclude the proof, we thus only need to check that the Riemann sum in (34) converges to the integral in (30) at rate $1/I$, uniformly in χ. Define

$$H^*(q,\theta) \equiv \left[\zeta^{*-k}(q,\theta) - c(\theta)q - \frac{1-F(\theta)}{f(\theta)}\,\frac{\partial \zeta^{*-k}}{\partial \theta}\,(q,\theta)\right] f(\theta),$$

which is continuously differentiable in (q, θ) under our regularity assumptions. Therefore, for each $\chi \in X$, $H^*(\chi(\theta), \theta)$ has finite total variation V^*_χ over $[\underline{\theta}, \overline{\theta}]$. In particular, letting

$$\overline{H}^*_q \equiv \max\left\{\left|\frac{\partial H^*}{\partial q}\,(q,\theta)\right| : (q,\theta) \in [0,\hat{\chi}(\overline{\theta})] \times [\underline{\theta},\overline{\theta}]\right\},$$

$$\overline{H}_\theta^* \equiv \max\left\{\left|\frac{\partial H^*}{\partial\theta}(q,\theta)\right| : (q,\theta)\in[0,\hat{\chi}(\overline{\theta})]\times[\underline{\theta},\overline{\theta}]\right\},$$

we obtain a uniform bound for V_χ^*,

$$V_\chi^* \leq \overline{V}^* \equiv \overline{H}_q^*\hat{\chi}(\overline{\theta}) + \overline{H}_\theta^*(\overline{\theta}-\underline{\theta}), \quad \chi \in X.$$

Finally, using a standard inequality (Pólya and Szegö (1978, Part Two, Chapter 1, §2, 9)), we obtain a uniform bound for the difference between the Riemann sum in (34) and the integral in (30),

$$\left|\frac{\overline{\theta}-\underline{\theta}}{I}\sum_{i=1}^{I} H^*(\chi(\theta_i),\theta_i) - \int_{\underline{\theta}}^{\overline{\theta}} H^*(\chi(\theta),\theta)\,\mathrm{d}\theta\right| \leq \frac{(\overline{\theta}-\underline{\theta})V_\chi^*}{I} \leq \frac{(\overline{\theta}-\underline{\theta})\overline{V}^*}{I}.$$

The result follows. ■

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