Stochastic Games with Hidden States (Online Appendix)

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S.1 Proof of Lemma B6

Pick an arbitrary belief μ. If

$$\frac{(1-\delta^{2^{|\Omega|}})2\overline{g}}{\delta^{2^{|\Omega|}}\overline{\pi}^{4^{|\Omega|}}} \geq \overline{g},$$

then the result obviously holds because we have $|\lambda \cdot v^{\omega}(\delta, s^{\omega}) - \lambda \cdot v^{\mu}(\delta, \tilde{s}^{\mu})| \leq \overline{g}$. So in what follows, we assume that

$$\frac{(1-\delta^{2^{|\Omega|}})2\overline{g}}{\delta^{2^{|\Omega|}}\overline{\pi}^{4^{|\Omega|}}} < \overline{g}.$$

Suppose that the initial prior is μ and players play the strategy profile $\tilde{s}^{\mu}$. Let $\Pr(h^t|\mu, \tilde{s}^{\mu})$ be the probability of h^t given the initial prior μ and the strategy profile $\tilde{s}^{\mu}$, and let $\mu^{t+1}(h^t|\mu, \tilde{s}^{\mu})$ denote the posterior belief in period $t+1$ given this history h^t. Let H^{*t} be the set of histories h^t such that $t+1$ is the first period at which the support of the posterior belief μ^{t+1} is in the set Ω^*. Intuitively, H^{*t} is the set of histories h^t such that players will switch their play to $s^{\mu^{t+1}}$ from period $t+1$ on, according to $\tilde{s}^{\mu}$.

Note that the payoff $v^{\mu}(\delta, \tilde{s}^{\mu})$ by the strategy profile $\tilde{s}^{\mu}$ can be represented as the sum of the two terms: The expected payoffs before the switch to s^{μ^t} occurs,

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and the payoffs after the switch. That is, we have

$$\lambda \cdot v^{\mu}(\delta, \tilde{s}^{\mu}) = \sum_{t=1}^{\infty} \left(1 - \sum_{\tilde{t}=0}^{t-1} \sum_{h^{\tilde{t}} \in H^{*\tilde{t}}} \Pr(h^{\tilde{t}}|\mu, \tilde{s}^{\mu})\right)(1-\delta)\delta^{t-1} E\left[\lambda \cdot g^{\omega^t}(a^t)|\mu, \tilde{s}^{\mu}\right]$$
$$+ \sum_{t=0}^{\infty} \sum_{h^t \in H^{*t}} \Pr(h^t|\mu, \tilde{s}^{\mu})\delta^t \lambda \cdot v^{\mu^{t+1}(h^t|\mu,\tilde{s}^{\mu})}(\delta, s^{\mu^{t+1}(h^t|\mu,\tilde{s}^{\mu})})$$

where the expectation operator is taken conditional on that the switch has not happened yet. Note that the term $1 - \sum_{\tilde{t}=0}^{t-1} \sum_{h^{\tilde{t}} \in H^{*\tilde{t}}} \Pr(h^{\tilde{t}}|\mu, \tilde{s}^{\mu})$ is the probability that players still randomize all actions in period t because the switch has not happened by then. To simplify the notation, let ρ^t denote this probability. From Lemma B5, we know that

$$\lambda \cdot v^{\mu^{t+1}(h^t|\mu,\tilde{s}^{\mu})}(\delta, s^{\mu^{t+1}(h^t|\mu,\tilde{s}^{\mu})}) \geq v^*$$

for each $h^t \in H^{*t}$, where

$$v^* = \lambda \cdot v^{\omega}(\delta, s^{\omega}) - \frac{(1-\delta^{2^{|\Omega|}})2\overline{g}}{\delta^{2^{|\Omega|}}\overline{\pi}^{4^{|\Omega|}}}.$$

Applying this and $\lambda \cdot g^{\omega^t}(a^t) \geq -2\overline{g}$ to the above equation, we obtain

$$\lambda \cdot v^{\mu}(\delta, \tilde{s}^{\mu}) \geq \sum_{t=1}^{\infty} \rho^t (1-\delta)\delta^{t-1}(-2\overline{g}) + \sum_{t=0}^{\infty} \sum_{h^t \in H^{*t}} \Pr(h^t|\mu, \tilde{s}^{\mu})\delta^t v^*.$$

Using $\sum_{t=0}^{\infty} \sum_{h^t \in H^{*t}} \Pr(h^t|\mu, \tilde{s}^{\mu})\delta^t = \sum_{t=1}^{\infty}(1-\delta)\delta^{t-1} \sum_{\tilde{t}=0}^{t-1} \sum_{h^{\tilde{t}} \in H^{*\tilde{t}}} \Pr(h^{\tilde{t}}|\mu, \tilde{s}^{\mu}) = \sum_{t=1}^{\infty}(1-\delta)\delta^{t-1}(1-\rho^t)$, we obtain

$$\lambda \cdot v^{\mu}(\delta, \tilde{s}^{\mu}) \geq (1-\delta) \sum_{t=1}^{\infty} \delta^{t-1} \left\{\rho^t(-2\overline{g}) + (1-\rho^t)v^*\right\}. \tag{1}$$

According to Lemma B4, the probability that the support reaches Ω^* within $4^{|\Omega|}$ periods is at least π^*. This implies that the probability that players still randomize all actions in period $4^{|\Omega|}+1$ is at most $1-\pi^*$. Similarly, for each natural number n, the probability that players still randomize all actions in period $n4^{|\Omega|}+1$ is at most $(1-\pi^*)^n$, that is, $\rho^{n4^{|\Omega|}+1} \leq (1-\pi^*)^n$. Then since ρ^t is weakly decreasing in t, we obtain

$$\rho^{n4^{|\Omega|}+k} \leq (1-\pi^*)^n$$

for each $n = 0, 1, \cdots$ and $k \in \{1, \cdots, 4^{|\Omega|}\}$. This inequality, together with $-2\overline{g} \leq v^*$, implies that

$$\rho^{n4^{|\Omega|}+k}(-2\overline{g}) + (1-\rho^{n4^{|\Omega|}+k})v^* \geq (1-\pi^*)^n(-2\overline{g}) + \{1-(1-\pi^*)^n\}v^*$$

for each $n = 0, 1, \cdots$ and $k \in \{1, \cdots, 4^{|\Omega|}\}$. Plugging this inequality into (1), we obtain

$$\lambda \cdot v^{\mu}(\delta, \tilde{s}^{\mu}) \geq (1-\delta)\sum_{n=1}^{\infty}\sum_{k=1}^{4^{|\Omega|}}\delta^{(n-1)4^{|\Omega|}+k-1}\begin{bmatrix} -(1-\pi^*)^{n-1}2\overline{g} \\ +\{1-(1-\pi^*)^{n-1}\}v^* \end{bmatrix}.$$

Since

$$\sum_{k=1}^{4^{|\Omega|}}\delta^{(n-1)4^{|\Omega|}+k-1} = \frac{\delta^{(n-1)4^{|\Omega|}}(1-\delta^{4^{|\Omega|}})}{1-\delta},$$

we have

$$\begin{aligned}\lambda \cdot v^{\mu}(\delta, \tilde{s}^{\mu}) \geq & (1-\delta^{4^{|\Omega|}})\sum_{n=1}^{\infty}\delta^{(n-1)4^{|\Omega|}}\begin{bmatrix} -(1-\pi^*)^{n-1}2\overline{g} \\ +\{1-(1-\pi^*)^{n-1}\}v^* \end{bmatrix} \\ = & -(1-\delta^{4^{|\Omega|}})\sum_{n=1}^{\infty}\{(1-\pi^*)\delta^{4^{|\Omega|}}\}^{n-1}2\overline{g} \\ & + (1-\delta^{4^{|\Omega|}})\sum_{n=1}^{\infty}[(\delta^{4^{|\Omega|}})^{n-1} - \{(1-\pi^*)\delta^{4^{|\Omega|}}\}^{n-1}]v^*.\end{aligned}$$

Plugging $\sum_{n=1}^{\infty}\{(1-\pi^*)\delta^{4^{|\Omega|}}\}^{n-1} = 1/\{1-(1-\pi^*)\delta^{4^{|\Omega|}}\}$ and $\sum_{n=1}^{\infty}(\delta^{4^{|\Omega|}})^{n-1} = 1/(1-\delta^{4^{|\Omega|}})$,

$$\lambda \cdot v^{\mu}(\delta, \tilde{s}^{\mu}) \geq -\frac{(1-\delta^{4^{|\Omega|}})2\overline{g}}{1-(1-\pi^*)\delta^{4^{|\Omega|}}} + \frac{\delta^{4^{|\Omega|}}\pi^*}{1-(1-\pi^*)\delta^{4^{|\Omega|}}}v^*.$$

Subtracting both sides from $\lambda \cdot v^{\omega}(\delta, s^{\omega})$, we have

$$\begin{aligned}&\lambda \cdot v^{\omega}(\delta, s^{\omega}) - \lambda \cdot v^{\mu}(\delta, \tilde{s}^{\mu}) \\ &\leq \frac{(1-\delta^{4^{|\Omega|}})2\overline{g}}{1-(1-\pi^*)\delta^{4^{|\Omega|}}} + \frac{\delta^{4^{|\Omega|}}\pi^*(1-\delta^{2^{|\Omega|}})2\overline{g}}{\{1-(1-\pi^*)\delta^{4^{|\Omega|}}\}\delta^{2^{|\Omega|}}\overline{\pi}^{4^{|\Omega|}}} - \frac{(1-\delta^{4^{|\Omega|}})\lambda \cdot v^{\omega}(\delta, s^{\omega})}{1-(1-\pi^*)\delta^{4^{|\Omega|}}}\end{aligned}$$

Since $\lambda \cdot v^{\omega}(\delta, s^{\omega}) \geq -\overline{g}$,

$$
\begin{aligned}
&\lambda \cdot v^{\omega}(\delta, s^{\omega}) - \lambda \cdot v^{\mu}(\delta, \tilde{s}^{\mu}) \\
&\leq \frac{(1-\delta^{4|\Omega|})2\overline{g}}{1-(1-\pi^*)\delta^{4|\Omega|}} + \frac{\delta^{4|\Omega|}\pi^*(1-\delta^{2|\Omega|})2\overline{g}}{\{1-(1-\pi^*)\delta^{4|\Omega|}\}\delta^{2|\Omega|}\overline{\pi}^{4|\Omega|}} + \frac{(1-\delta^{4|\Omega|})\overline{g}}{1-(1-\pi^*)\delta^{4|\Omega|}} \\
&\leq \frac{(1-\delta^{4|\Omega|})3\overline{g}}{1-(1-\pi^*)} + \frac{\pi^*(1-\delta^{2|\Omega|})2\overline{g}}{\{1-(1-\pi^*)\}\delta^{2|\Omega|}\overline{\pi}^{4|\Omega|}} \\
&= \frac{(1-\delta^{4|\Omega|})3\overline{g}}{\pi^*} + \frac{(1-\delta^{2|\Omega|})2\overline{g}}{\delta^{2|\Omega|}\overline{\pi}^{4|\Omega|}}
\end{aligned}
$$

Hence the result follows.

S.2 Proof of Lemma B11

Pick a belief μ whose support is robustly accessible. Suppose that the initial prior is μ^{**}, the opponents play $\tilde{s}^{\mu}_{-i}$, and player i plays a best reply. Let ρ^t denote the probability that players $-i$ still randomize actions in period t. Then as in the proof of Lemma B6, we have

$$v_i^{\mu^{**}}(\tilde{s}^{\mu}_{-i}) \leq \sum_{t=1}^{\infty} \delta^{t-1}\left\{\rho^t \overline{g} + (1-\rho^t)K_i^{\mu}\right\},$$

because the stage-game payoff before the switch to s^{μ}_{-i} is bounded from above by $\overline{g}$, and the continuation payoff after the switch is bounded from above by $K_i^{\mu} = \max_{\tilde{\mu}\in\Delta^{\mu}} v_i^{\tilde{\mu}}(s^{\mu}_{-i})$.

As in the proof of Lemma B6, we have

$$\rho^{n4^{|\Omega|}+k} \leq (1-\pi^*)^n$$

for each $n = 0, 1, \cdots$ and $k \in \{1, \cdots, 4^{|\Omega|}\}$. This inequality, together with $\overline{g} \geq K_i^{\mu}$, implies that

$$\rho^{n4^{|\Omega|}+k}\overline{g} + (1-\rho^{n4^{|\Omega|}+k})v_i^* \leq (1-\pi^*)^n\overline{g} + \{1-(1-\pi^*)^n\}K_i^{\mu}$$

for each $n = 0, 1, \cdots$ and $k \in \{1, \cdots, 4^{|\Omega|}\}$. Plugging this inequality into the first one, we obtain

$$v_i^{\mu^{**}}(\tilde{s}^{\mu}_{-i}) \leq (1-\delta)\sum_{n=1}^{\infty}\sum_{k=1}^{4^{|\Omega|}} \delta^{(n-1)4^{|\Omega|}+k-1}\left[\begin{array}{l} (1-\pi^*)^{n-1}\overline{g} \\ +\{1-(1-\pi^*)^{n-1}\}K_i^{\mu} \end{array}\right].$$

Then as in the proof of Lemma B6, the standard algebra shows

$$v_i^{\mu^{**}}(\tilde{s}_{-i}^{\mu}) \leq \frac{(1-\delta^{4|\Omega|})\overline{g}}{1-(1-\pi^*)\delta^{4|\Omega|}} + \frac{\delta^{4|\Omega|}\pi^* K_i^{\mu}}{1-(1-\pi^*)\delta^{4|\Omega|}}.$$

Since

$$\frac{\delta^{4|\Omega|}\pi^*}{1-(1-\pi^*)\delta^{4|\Omega|}} = 1 - \frac{1-\delta^{4|\Omega|}}{1-(1-\pi^*)\delta^{4|\Omega|}},$$

we have

$$v_i^{\mu^{**}}(\tilde{s}_{-i}^{\mu}) \leq K_i^{\mu} + \frac{(1-\delta^{4|\Omega|})(\overline{g}-K_i^{\mu})}{1-(1-\pi^*)\delta^{4|\Omega|}}.$$

Since $1-(1-\pi^*)\delta^{4|\Omega|} > 1-(1-\pi^*) = \pi^*$ and $K_i^{\mu} \geq -\overline{g}$, the result follows.