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Optimal Dynamic Contracting: the First-Order Approach and Beyond

Online Appendix

Abstract

In this appendix we present the sections and proofs omitted in "Optimal Dynamic Contracting: the First-Order Approach and Beyond" by Marco Battaglini and Rohit Lamba.

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In Section 1 we discuss the relationship between continuous and discrete type models. Sections 2, 3 and 4 provide the proofs excluded from the appendix in the main paper. Section 5 formally states and solves the three types two period example introduced in the main text. And, finally section 6 provides a numerical example of the approximate optimality of monotonic contracts.

Equations from the main text are referenced corresponding to the section they appear in the paper: for example, equation (3.2) means equation (2) in Section 3 of the paper. Moreover, the numbering of new Lemmata here starts off from where we left in the appendix in the main paper, so the first new Lemma here is numbered Lemma A10, and so on.

1 From discrete to continuous types

In this section we formalize the statements made in Section 4.3 of the paper, and show that the continuous case can be seen as the limit of the discrete case, so all problems of the FO-approach in the discrete version are inherited by the continuous version and vice-versa. To keep the notation simple, we assume two periods and $u(\theta, q) = \theta q$. Consider a type set $\Theta = [\underline{\theta}, \overline{\theta}] \subset \mathbb{R}^+$, an associated prior distribution $\Gamma(\theta)$ at $t = 1$ and a conditional distribution $F(\theta' | \theta)$ at $t = 2$ defined on Θ. We assume $\Gamma(\theta)$ is differentiable in θ with density $\mu(\theta)$ and $F(\theta' | \theta)$ is differentiable in both θ, with derivative $F_\theta(\theta' | \theta)$, and θ', with density $f(\theta' | \theta)$. By standard methods we can obtain the following envelope formula (3.4):[1]

$$U'(\theta) = q(\theta) - \int_{\theta'} q(\theta' | \theta) \cdot F_\theta(\theta' | \theta)\, d\theta'$$

and then derive the FO-optimal contract:

$$q(\theta' | \theta) = \theta' + \frac{1 - \Gamma(\theta)}{\mu(\theta)} \frac{F_\theta(\theta' | \theta)}{f(\theta' | \theta)} \tag{1}$$

In the rest of this section, we refer to this as the *continuous model*.

We now explore the connection between the continuous model and the discrete model studied in the previous sections. The continuous model can be derived as the limit of the discrete model as follows. Define $\Theta^N = \{\theta_0, ..., \theta_N\}$ with $\theta_0 = \overline{\theta}$, $\theta_N = \underline{\theta}$ and $\theta_i = \theta_{i+1} + \Delta\theta_N$; and let $\Gamma^N(\theta_i) = \Gamma(\theta_i)$ and $F^N(\theta_j | \theta_i) = F(\theta_j | \theta_i)$. Given this, the probability of a type j at $t = 1$ is $\mu_j^N = \Gamma^N(\theta_j) - \Gamma^N(\theta_{j+1})$ and the probability of a type i at $t = 2$ after a type j at $t = 1$ is $f^N(\theta_j | \theta_i) = F^N(\theta_j | \theta_i) - F^N(\theta_{j+1} | \theta_i)$.[2] In the rest of the section, we refer to this as the *discrete model*.

Consider a sequence of supports Θ^N for $N \to \infty$ such that $\Delta\theta_N \to 0$ as $N \to \infty$ and $\Theta^N \subseteq \Theta^{N+1}$, so that along the sequence the finite approximation of Θ becomes increasingly fine.[3]

[1] See Baron and Besanko [1984], Besanko [1985], Laffont and Tirole [1996], Courty and Li [2000], Eso and Szentes [2007], and Pavan, Segal and Toikka [2014].

[2] In both definitions, we are implicitly assuming a dummy "$N + 1$" type with mass 0.

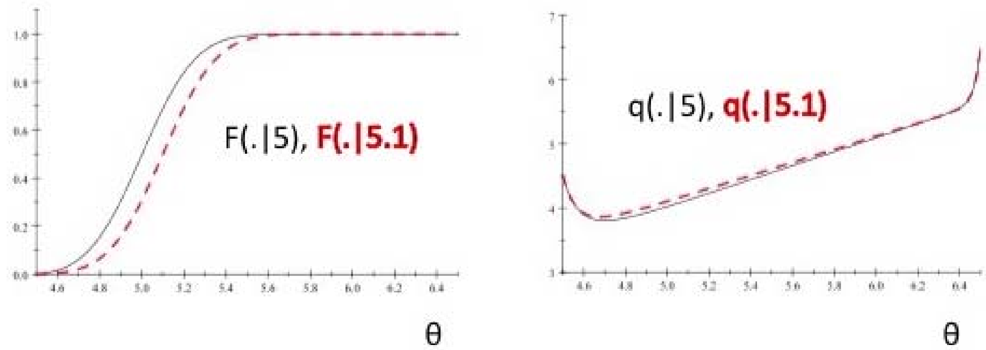

Figure 1: F and q for the Markov process $f_\alpha(\theta'|\theta) = \alpha \cdot e^{-\frac{(\theta'-\theta)^2}{\sigma_\theta(\alpha)}}$.

Using the formula (3.9) derived in the paper, we can write the FO-optimal contract along the sequence as:

$$q_N\left(\theta_j|\theta_i\right) = \theta_j - \tfrac{1-\Gamma^N(\theta_i)}{\mu_i^N} \tfrac{F^N(\theta_j|\theta_i)-F^N(\theta_j|\theta_{i-1})}{f^N(\theta_j|\theta_i)} \Delta\theta_N \tag{2}$$

for any $\theta_j \in \Theta^N$, $\theta_i \in \Theta^N$. Note that μ_i^N can be written as: $\mu_i^N = \frac{\Gamma(\theta_j)-\Gamma(\theta_{j+1})}{\Delta\theta_N} \cdot \Delta\theta_N$. and $f^N\left(\theta_j|\theta_i\right) = \frac{F^N(\theta_j|\theta_i)-F^N(\theta_{j+1}|\theta_i)}{\Delta\theta_N}\Delta\theta_N$. We can therefore rewrite (2) as:

$$q_N\left(\theta_j|\theta_i\right) = \theta_j + \left(1-\Gamma^N\left(\theta_i\right)\right) \frac{\left[F^N\left(\theta_j|\theta_i\right) - F^N\left(\theta_j|\theta_{i-1}\right)\right]/\Delta\theta_N}{\left[\frac{\Gamma(\theta_i)-\Gamma(\theta_{i+1})}{\Delta\theta_N}\right]\left[\frac{F^N(\theta_j|\theta_i)-F^N(\theta_{j+1}|\theta_i)}{\Delta\theta_N}\right]}$$

This condition immediately implies that

$$\lim_{N\to\infty} q_N\left(\theta_j|\theta_i\right) = \theta_j + \frac{1-\Gamma(\theta_i)}{\mu(\theta_i)} \frac{F_\theta\left(\theta_j|\theta_i\right)}{f\left(\theta_j|\theta_i\right)} = q\left(\theta_j|\theta_i\right)$$

since $\mu_i^N/\Delta\theta_N \to \mu(\theta_i)$ and $f^N\left(\theta_j|\theta_i\right)/\Delta\theta_N \to f\left(\theta_j|\theta_i\right)$ as $N \to \infty$. It follows that the limit of the discrete FO-optimal contracts is equal to the continuous FO-optimal contract.[4]

This discussion makes it clear that there is a natural connection between discrete and continuos types dynamic principal-agent models. In the light of this we can present two examples, discretized versions of which are presented in Battaglini and Lamba [2015].

Examples. Consider a two period model. Assume and that types in the first period are distributed uniformly on $[5, 6]$ and consider the transition probabilities: $f_\alpha(\theta'|\theta) = \alpha \cdot e^{-\frac{(\theta'-\theta)^2}{\sigma_\theta(\alpha)}}$ and

[3] For example, consider the sequence $(\theta_0^m, ..., \theta_N^m)$ such that $\theta_0^m = \underline{\theta}$, $\theta_N^m = \overline{\theta}$, $\theta_i^m - \theta_{i-1}^m = (\overline{\theta} - \underline{\theta})/2^m$ and so $N^m = 2^m$.

[4] Since $\Theta^N \subseteq \Theta^{N+1}$, if $\theta_j \in \Theta^N$, $\theta_i \in \Theta^N$, then $\theta_j \in \Theta^M$, $\theta_i \in \Theta^M$ for $M \geq N$, so $\lim_{N\to\infty} q_N^*\left(\theta_i|\theta_j\right)$ is well defined. To extend the contract for points on the real line that do not appear in the sequence of approximations we can consider, for example, the sequence of linear interpolations of the discrete contract. It is immediate to verify that this is a sequence of equicontinuous curves that converges to (1).

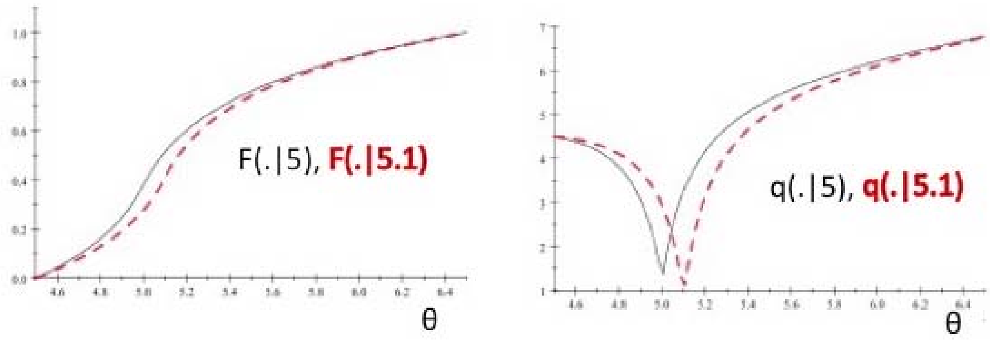

Figure 2: F and q for the Markov process $f_\alpha(\theta'|\theta) = \frac{\alpha}{1+\sigma_\theta(\alpha)|\theta'-\theta|}$.

$f_\alpha(\theta'|\theta) = \frac{\alpha}{1+\sigma_\theta(\alpha)|\theta'-\theta|}$ with $\alpha =$. Note that $\sigma_\theta(\alpha)$ is chosen so that the probabilities sum to one. The larger is α, the higher is the persistence of the types. Figures 1 and 2 show two sample distributions and the associated quantities in period 2. The contract is non-monotonic in two ways: first, for a given history, it is non-monotonic in θ_2. Because of this alone, the FO-optimal contract is not implementable and violates a global constraint. In addition to this, the FO-optimal contract is not monotonic with respect to θ_1; this can be seen from the fact that the contracts with the two different histories cross each other.

2 Proof of Lemma A1

In the proof of Lemma 1 we use the following result:

Lemma A1. *In a FO-relaxed problem:* $IR_N(h^{t-1})$ *can be assumed to hold as equality for all* $h^{t-1} \in H^{t-1}$*;* $IC_{i,i+1}(h^{t-1})$ *can be assumed to hold as an equality for all* $h^{t-1} \in H^{t-1}$ *and* $i = 0, 1, ..., N-1$.

Proof. We proceed in two steps:

Step 1. Suppose that $U(\theta_N|h^{t-1}) = \epsilon > 0$ for some h^{t-1}. If $t = 1$, then decreasing $U(\theta_i|h^0)$ by ϵ for all i does not violate any constraints and increases the monopolist's profit. If $t > 1$, fix h^{t-1} and decrease $U(\theta_i|h^{t-1})$ by ϵ for all θ_i. This does not change any of the constraints and keeps the profit of the monopolist the same.

Step 2. Suppose that $IC_{i,i+1}(h^{t-1})$ does not hold as an equality for some $h^{t-1} \in H^{t-1}$ and $i = 0, 1, ..., N-1$. Then, decrease $U(\theta_k|h^{t-1})$ by ϵ for each $k \le i$. If $t = 1$, all the constraints are still satisfied and the monopolist's profit is strictly higher, giving a contradiction. If $t > 1$, this change does not affect any constraint except $IC_{j-1,j}(h^{t-2})$, where θ_j is such that $h^{t-1} = \left(h^{t-2}, \theta_j\right)$. The right hand side of $IC_{j-1,j}(h^{t-2})$ is *reduced* by $\delta \sum_{k \le i}(\alpha_{(j-1)k} - \alpha_{jk})\epsilon = \delta \Delta F\left(\theta_{i+1}|\theta_j\right)\epsilon \ge 0$, where the last inequality follows from first order stochastic dominance. Now, repeat the same procedure, decreasing $U(\theta_k|h^{t-2})$ by $\delta\Delta F(\theta_{i+1}|\theta_j)\epsilon$ for each $k \leq j-1$. We can keep reducing utility vectors backward till the first period, unless h^{t-1} contains θ_0, in which case the backward iteration ends there, to deduce a strictly greater increase in the monopolist's profit. Thus, the changes do not violate any of the constraints and keep the profit of the monopolist larger than or equal to before the change. ■

3 Proof of Lemmata A2-A3

We now prove the lemmata used in the proof of Proposition 2. Recall that $\Delta U(\theta_k \left| h^{t-1}, \theta_i \right.) = U(\theta_k \left| h^{t-1}, \theta_i \right.) - U(\theta_k \left| h^{t-1}, \theta_{i+1} \right.)$. For simplicity of exposition, we will write the proofs for the special case where $u(\theta, q) = \theta q$, and hence $u_\theta(\theta, q) = q$; the arguments are easily generalizable. We have:

Lemma A2. *If $q(\theta_i|h^{t-1})$ and $\Delta U(\theta_k \left| h^{t-1} \right.)$ are non increasing in, respectively, i and k for any h^{t-1}, then (3.4) implies that local upward incentive compatibility constraints are satisfied.*

Proof. Since $IC_{i,i+1}(h^{t-1})$ holds as an equality, we have for any i and h^{t-1}:

$$U(\theta_i|h^{t-1}) = U(\theta_{i+1}|h^{t-1}) + \Delta\theta q(\theta_{i+1}|h^{t-1}) + \delta \sum_{k=0}^{N} \left(\alpha_{ik} - \alpha_{(i+1)k}\right) U(\theta_k|h^{t-1}, \theta_{i+1}).$$

Thus,

$$\begin{aligned}
U(\theta_{i+1}|h^{t-1}) - U(\theta_i|h^{t-1}) &= -\Delta\theta q(\theta_{i+1}|h^{t-1}) - \delta \sum_{k=0}^{N} \left(\alpha_{ik} - \alpha_{(i+1)k}\right) U(\theta_k|h^{t-1}, \theta_{i+1})) \\
&= -\Delta\theta q(\theta_i|h^{t-1}) + \delta \sum_{k=0}^{N} \left(\alpha_{(i+1)k} - \alpha_{ik}\right) U(\theta_k|h^{t-1}, \theta_i) \\
&+ \Delta\theta \left(q(\theta_i|h^{t-1}) - q(\theta_{i+1}|h^{t-1})\right) + \delta \sum_{k=0}^{N} \left(\alpha_{ik} - \alpha_{(i+1)k}\right) \Delta U(\theta_k|h^{t-1}, \theta_i) \\
&\geq -\Delta\theta q(\theta_i|h^{t-1}) + \delta \sum_{k=0}^{N} \left(\alpha_{(i+1)k} - \alpha_{ik}\right) U(\theta_k|h^{t-1}, \theta_i),
\end{aligned}$$

where the last inequality follows from the fact that $q(\theta_i|h^{t-1})$ is non increasing in i and $\sum_{k=0}^{N} (\alpha_{ik} - \alpha_{(i+1)k})\Delta U(\theta_k \left| h^{t-1}, \theta_i \right.) \geq 0$. The second observation follows from the fact that $\Delta U(\theta_k \left| h^{t-1}, \theta_i \right.)$ is non increasing in k, and that $\boldsymbol{\alpha}_i$ first-order stochastically dominates $\boldsymbol{\alpha}_{i+1}$. Thus, $IC_{i+1,i}(h^{t-1})$ holds. ■

Lemma A3. *If $q(\theta_i|h^{t-1})$ and $\Delta U(\theta_k \left| h^{t-1} \right.)$ are non increasing in, respectively, i and k for any h^{t-1} and (3.4) holds, then the local incentive compatibility constraints imply the global incentive compatibility constraints.*

Proof. We show that $IC_{i,i+2}(h^{t-1})$ holds. Since $IC_{i,i+1}(h^{t-1})$ and $IC_{i+1,i+2}(h^{t-1})$ hold as equalities, we have:

$$\begin{aligned}
&U(\theta_i|h^{t-1}) - U(\theta_{i+2}|h^{t-1}) \\
&\quad= \left[U(\theta_i|h^{t-1}) - U(\theta_{i+1}|h^{t-1})\right] + \left[U(\theta_{i+1}|h^{t-1}) - U(\theta_{i+2}|h^{t-1})\right] \\
&\quad= \Delta\theta q(\theta_{i+1}|h^{t-1}) + \delta \sum_{k=0}^{N} \left(\alpha_{ik} - \alpha_{(i+1)k}\right) U(\theta_k|h^{t-1}, \theta_{i+1}) \\
&\quad+ \Delta\theta q(\theta_{i+2}|h^{t-1}) + \delta \sum_{k=0}^{N} \left(\alpha_{(i+1)k} - \alpha_{(i+2)k}\right) U(\theta_k|h^{t-1}, \theta_{i+2}).
\end{aligned}$$

It follows that:

$$\begin{aligned}
&U(\theta_i|h^{t-1}) - U(\theta_{i+2}|h^{t-1}) \\
= \;& 2\Delta\theta q(\theta_{i+2}|h^{t-1}) + \delta \sum_{k=0}^{N} \left(\alpha_{ik} - \alpha_{(i+2)k}\right) U(\theta_k|h^{t-1}, \theta_{i+2}) \\
& +\Delta\theta \left(q(\theta_{i+1}|h^{t-1}) - q(\theta_{i+2}|h^{t-1})\right) + \delta \sum_{k=0}^{N} \left(\alpha_{ik} - \alpha_{(i+1)k}\right) \Delta U(\theta_k|h^{t-1}, \theta_{i+1}) \\
\geq \;& 2\Delta\theta q(\theta_{i+2}|h^{t-1}) + \delta \sum_{k=0}^{N} \left(\alpha_{(i+1)k} - \alpha_{(i+2)k}\right) U(\theta_k|h^{t-1}, \theta_{i+2}),
\end{aligned}$$

where the last inequality follows from the fact that $q(\theta_i|h^{t-1})$ is non increasing in i and $\sum_{k=0}^{N}(\alpha_{ik} - \alpha_{(i+1)k})\Delta U(\theta_k \left|h^{t-1}, \theta_i\right.) \geq 0$. As in the previous lemma, the second observation follows from the fact that $\Delta U(\theta_k \left|h^{t-1}, \theta_i\right.)$ is non increasing in k, and that $\boldsymbol{\alpha}_i$ first-order stochastically dominates $\boldsymbol{\alpha}_{i+1}$. Thus, $IC_{i,i+2}(h^{t-1})$ holds. Similarly we can show that $IC_{i,i+l}(h^{t-1})$ holds for all $l \leq N-i$. Therefore, all global downward incentive constraints are satisfied. In an analogous fashion, we can show that all upward global incentive constraints are satisfied. ■

4 Proof of Lemma A9

Using f_{ij}^{τ} as a shorthand for the ijth element of f_τ, we can write:

$$f_{ij}^{\tau} = e^{-\lambda\tau} \sum\nolimits_{n=0}^{\infty} \left[\widehat{P}_{ij}\right]^n \frac{(\lambda\tau)^n}{k!} = e^{-\lambda\tau}\left(1_{i=j} + \widehat{P}_{ij}\lambda\tau\right) + e^{-\lambda\tau} \sum\nolimits_{n=2}^{\infty} \left[\widehat{P}_{ij}\right]^n \frac{(\lambda\tau)^n}{n!} \tag{3}$$

where $1_{i=j} = 1$ if $i = j$. We first show that the second term in (3) is an $o(\lambda)$. Note that $\sum_{n=2}^{\infty} \left[\widehat{P}_{ij}\right]^n \frac{(\lambda\tau)^n}{n!} \geq 0$ (that is the elements of this matrix are all nonegative) and

$$\begin{aligned}
\sum\nolimits_{n=2}^{\infty} \left[\widehat{P}_{ij}\right]^n \frac{(\lambda\tau)^n}{n!} &= \frac{(\lambda\tau)^2}{n(n-1)} \sum\nolimits_{n=1}^{\infty} \left[\widehat{P}_{ij}\right]^n \frac{(\lambda\tau)^{n-2}}{(n-2)!} \\
&\leq \frac{(\lambda\tau)^2}{n(n-1)} \sum\nolimits_{n=0}^{\infty} \frac{(\lambda\tau)^n}{n!} = \frac{(\lambda\tau)^2}{n(n-1)}
\end{aligned}$$

It follows that $\left[\sum_{n=2}^{\infty}[P_{ij}]^n \frac{(\lambda\tau)^n}{n!}\right]/\lambda \to 0$ as $\lambda \to 0$.

We can therefore write:

$$f_{ij}^{\tau} = e^{-\lambda\tau}\left(1_{i=j} + \widehat{P}_{ij}\lambda\tau\right) + o(\lambda)$$

That is:

$$f_{ii}^{\tau} = e^{-\lambda\tau}\left(1 + (\lambda - \lambda_i)\tau\right) + o(\lambda) \tag{4}$$

$$f_{ij}^{\tau} = e^{-\lambda\tau}\left(\lambda_i P_{i,j}\tau\right) + o(\lambda) \tag{5}$$

Note that $\frac{\lambda_i}{\lambda} \in [0,1]$ so there is a $\eta_i \in [0,1]$ such that $\frac{\lambda_i}{\lambda} \to \eta_i$ as $\lambda \to 0$. From the second equation (equation (5)), setting $\tau = 1$, we have:

$$\frac{f_{ij}}{\lambda} \to \eta_i P_{i,j}$$

as $\lambda \to 0$. From the first equation (equation (4)), using a Taylor expansion, applied to the first term with respect to λ and λ_i evaluated at $(0,0)$, we have:

$$\begin{aligned} f_{ii}(\lambda,\lambda_i) &= f_{ii}^{\tau}(0,0,1) + \left.\frac{\partial f^{ii}(\lambda,\lambda_i,\tau)}{\partial\lambda}\right|_{\lambda_i,\lambda_j=0} \cdot \lambda + \left.\frac{\partial f_{ii}(\lambda,\lambda_i,\tau)}{\partial\lambda_j}\right|_{\lambda_i,\lambda_j=0} \cdot \lambda_j + o(\lambda) \\ &= 1 + \left(-e^{-\lambda}\left(1 + (\lambda - \lambda_i)\right) + e^{-\lambda\tau}\tau\right)_{\lambda,\lambda_j=0} \cdot \lambda - \left[e^{-\lambda_j\tau}\tau\right]_{\lambda,\lambda_j=0} \cdot \lambda_i + o(\lambda) \end{aligned}$$

where note that in the last term we put all factors that converges to zero faster than λ (so also $o(\lambda_i)$). We have therefore:

$$\frac{1 - f_{ii}}{\lambda} \to \eta_i$$

as $\lambda \to 0$. ■

5 The solved example of Section 5

To characterize the optimal contract we first guess which constraints are relevant and then we show that we can ignore the remaining constraints without loss of generality. We focus on a *weakly relaxed program* (henceforth *WR*-program) that constitutes problem (3.3) with $|\Theta| = 3$ and $T = 2$, with the following subset of constraints:

$$\begin{gathered} IR_L, IC_{HM}, IC_{ML}, IC_{HL}, \\ IC_{HM}(M), IC_{ML}(M), IC_{LM}(M), IC_{HM}(L), IC_{ML}(L), IC_{LM}(L) \end{gathered} \tag{6}$$

where IR_L is the individual rationality constraint of type L at $t = 0$, $IC_{i,j}$ is the incentive constraint requiring that type i doesn't want to misreport being type j in period 1, and $IC_{i,j}(k)$ is the incentive constraint requiring that type i doesn't want to misreport being type j in period 2, after the agent reports being type k in period 1. See Figure 3 for an illustration of the constraints.

The intuition for modifying the FO-approach to focus on the WR-program is as follows. It is natural to ignore incentive constraints after history $h^1 = \theta_H$, since the contract is typically efficient after this history even in the FO-approach (see (3.7)). Similarly, it is natural to drop

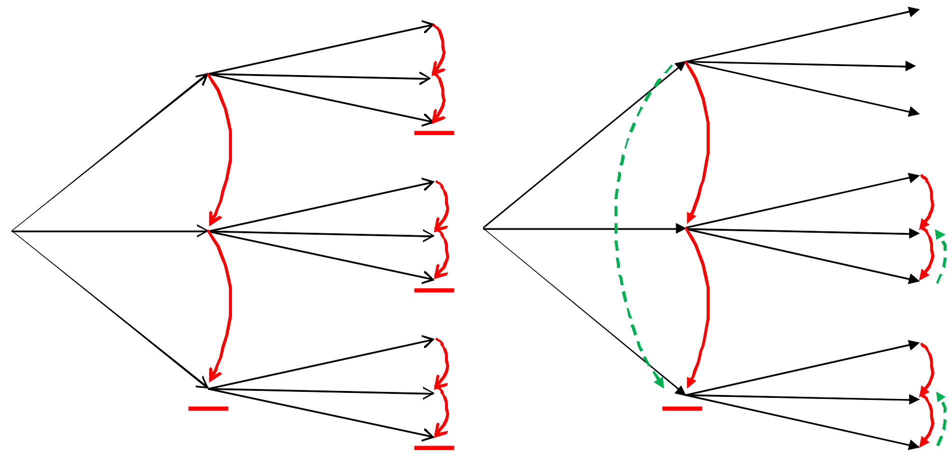

Figure 3: The dashed arrows are the constraints in the *WR-program* that are ignored in the first-order approach.

the individual rationality constraints at $t = 2$, since they are typically not binding even in the FO-approach (any rent left to the lowest type at $t = 2$ can be extracted at $t = 1$, so there is no reason to force these rents to be non-negative). There are, however, two reasons why we need additional constraints. First, we must include IC_{HL} since we know from the previous analysis that it may be violated if ignored. Second, since the second period is terminal, within history monotonicity is a necessary condition; that is, $q(\theta_j|\theta_i)$ is weakly increasing in θ_j. Thus to allow for pooling in period 2 we include $IC_{LM}(h^1)$ for $h^1 = M, L$.

In what follows we prove that there is no loss of generality in restricting attention to the WR-program so we can focus on (6) to solve for the optimal contract. For a given μ_L and δ, the environment is fully described by two parameters, μ_M and α, and therefore it can be represented in the two dimensional box $(\mu_M, \alpha) \in E(\mu_L) = (0, 1 - \mu_L) \times (1/3, 1)$.[5] In the rest of the analysis we will fix μ_L and δ and study how the optimal contract changes as we change μ_M and α.

The following proposition provides a full characterization of the optimal contract. Table 2 details the exact formulas case by case; Figure 4 illustrates the possible cases in the (μ_M, α) space.

Proposition A1. *There exist thresholds $\mu^*(\alpha)$ and $\mu^{**}(\alpha)$, $\mu^*(\alpha) > \mu^{**}(\alpha)$, such that:*

- **Case A:** *For all $\mu_M \geq \mu^*(\alpha)$, IC_{HL} does not bind and there exists a threshold $\alpha_0(\mu_M)$ such that:*
  - **Case A1:** *if $\alpha < \alpha_0(\mu_M)$, the optimal contract is fully separating and FO-optimal.*
  - **Case A2:** *If $\alpha \geq \alpha_0(\mu_M)$, the optimal contract is fully separating after all histories except M; after this history types M and L are pooled: $q(\theta_M|\theta_M) = q(\theta_L|\theta_M)$.*

[5] The thresholds defined below do not depend on the types θ.

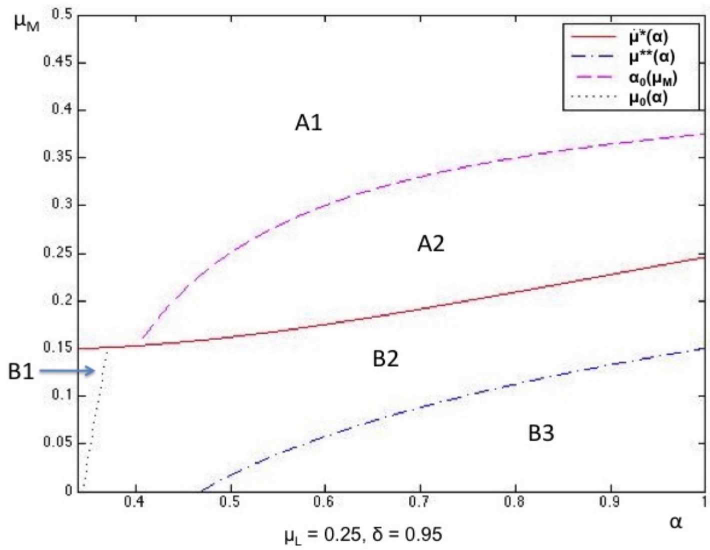

Figure 4: Fully characterized contract when $\mu_L = 0.25$ and $\delta = 0.95$.

- **Case B:** *For all* $\mu_M < \mu^*(\alpha)$*,* IC_{HL} *binds and there exists a threshold* $\mu_0(\alpha)$ *such that:*
  - **Case B1:** *if* $\mu_M \in [\mu^{**}(\alpha), \mu^*(\alpha)) \cap (\mu_0(\alpha), 1)$*, then the optimal contract is fully separating.*
  - **Case B2:** *if* $\mu_M \in [\mu^{**}(\alpha), \mu^*(\alpha)) \cap (0, \mu_0(\alpha)]$*, then the optimal contract is fully separating after all histories except* M*; after this history types* M *and* L *are pooled:* $q(\theta_M|\theta_M) = q(\theta_L|\theta_M)$*.*
  - **Case B3:** *if* $\mu_M < \mu^{**}(\alpha)$ *the optimal contract pools types* M *and* L *in the first period:* $q_M = q_L$*. In the second period, after history* H *the contract is separating and efficient; after histories* M *and* L*, types* M *and* L *are pooled across both histories:* $q(\theta_M|\theta_i) = q(\theta_L|\theta_i)$ *and* $q(\theta_j|\theta_M) = q(\theta_j|\theta_L)$ *for* $i, j = M, L$

While the example solved in Proposition A1 is very special, it presents interesting features that are reminiscent of the features of optimal contracts in multidimensional screening problems. Multiple IC constraints can bind simultaneously to determine the optimal quantities, a fact that is ruled out by assumption in FO-optimal contracts. For example, in regions $A2$ and $B2$ both IC_{HL} and IC_{HM} are binding. Multiple binding IC constraints have been observed in a multidimensional screening problem by, for example, Armstrong and Rochet [1999]. The optimal contract also features a strategic use of bunching in order to minimize the expected rent of the buyer. In regions $A2$ and $B2$, we observe separation in period 1 followed by history-dependent pooling in period 2, which we term *dynamic pooling.* In region $B3$, types are pooled in period 2 across the pooled histories in period 1—it is as if we were in a two-type model following the pooled histories.

An analogous use of bunching to screen types in multidimensional problems, even with a very simple distribution of types, has been documented by Rochet and Chone' [1997]. The similarities between contracts in dynamic and multidimensional environments are not surprising. In a dynamic environment, the expected utility of a type at t is not only given by the time t realization θ_t, but also by the conditional distribution of types $f(\theta_{t+1} | \theta_t)$, a multidimensional object. At the same time, the optimal contract as stated in Proposition A1 features some distinctive characteristics that depend on the dynamic structure of the problem, the most interesting perhaps being the fact that pooling is state dependent and thus dynamic.

5.1 Proof of Proposition A1

To solve the example, we use a simplified notation. Let U_i be the expected utility of type i in the first period and $u_i(h)$ be the expected utility of type i after history h in the second period. Note that since the second period is the terminal period, the expected utility and stage utility are the same. Similarly, we define q_i and $q_i(h)$ to be the first and second period allocations respectively.

In Section 5.1.1 we prove two preliminary results. In Section 5.1.2 we characterize the WR-problem. In Section 5.1.3, we prove the the solution of the WR-problem is optimal.

5.1.1 Preliminary results

The Lemmas here are numbered The following lemma allows to simplify the constraint set (6):

Lemma A10. *In the WR-program, constraints IR_L, IC_{HM}, IC_{ML} bind at the optimum.*

Proof. First, we prove a useful lemma.

Lemma A10.1. *The optimal solution satisfies: $q_L \leq \theta_L$, $q_L(L) \leq \theta_L$ and $q_M(L) \leq \theta_M$.*

Proof. Suppose $q_L > \theta_L$. Then, decrease q_L by ε. Since it only appears on the RHS of incentive constraints and has positive coefficients, this does not violate any of the constraints. Moreover, the change in the monopolist's profit is proportional to

$$\left(\theta_L (q_L - \varepsilon) - \frac{1}{2}(q_L - \varepsilon)^2\right) - \left(\theta_L q_L - \frac{1}{2}q_L^2\right) = (q_L - \theta_L)\,\varepsilon - \frac{1}{2}\varepsilon^2.$$

We can choose ε small enough so that the above expression is positive, giving us a contradiction. We can similarly show that $q_L(L) \leq \theta_L$.

Next, suppose $q_M(L) > \theta_M$. Note that the second period incentive constraints after history L give

$$\Delta\theta q_L(L) \leq u_M(L) - u_L(L) \leq \Delta\theta q_M(L).$$

Without loss of generality, $IC_{ML}(L)$ can be assumed to hold as an equality. Suppose $u_M(L) - u_L(L) > \Delta\theta q_L(L)$. Then, decrease $u_M(L)$ so that $IC_{ML}(L)$ holds as an equality. This does not violate any constraints and keeps the profit of the monopolist the same.

If $IC_{LM}(L)$ holds as an equality, then we must have $q_M(L) = q_L(L) \leq \theta_L < \theta_M$, giving a contradiction. If $IC_{LM}(L)$ does not hold as an equality, then we can decrease $q_M(L)$ by ε without disturbing any of the constraints. Moreover, the change in the monopolist's profit is proportional to the following expression:

$$\left(\theta_M \left(q_M(L) - \varepsilon\right) - \frac{1}{2}\left(q_M(L) - \varepsilon\right)^2\right) - \left(\theta_M q_M(L) - \frac{1}{2} q_M(L)^2\right) = \left(q_M(L) - \theta_M\right)\varepsilon - \frac{1}{2}\varepsilon^2.$$

We can choose ε so small that the above expression is positive, giving us a contradiction. ■

Now, we show that IR_L binds. Suppose not. Decrease U_H, U_M, U_L by the same small amount. The first period incentive compatibility constraints continue to hold and the second period constraints are unaffected. This increases the profit of the monopolist without disturbing any of the constraints, giving us a contradiction. Thus, $U_L = 0$. Next, we show that IC_{ML} binds. Suppose not. Decrease U_M by ε. Then, all the constraints are satisfied and we increase the monopolist's profit, giving us a contradiction. Using these two binding constraints we can eliminate U_L and U_M from the maximization problem. In particular, IC_{HM} can now be written as

$$U_H \geq \Delta\theta \left(q_M + q_L\right) + \delta \frac{3\alpha - 1}{2} \left[\left(u_H(M) - u_M(M)\right) + \left(u_M(L) - u_L(L)\right)\right]$$

Also, IC_{HL} is given by

$$U_H \geq 2\Delta\theta q_L + \delta \frac{3\alpha - 1}{2} \left[u_H(L) - u_L(L)\right]$$

First, note that at least one of IC_{HM} and IC_{HL} must bind. If not, then we can decrease U_H and increase the monopolist's profit. Suppose IC_{HM} does not bind. Then, IC_{HL} must bind. Thus, we can eliminate U_H from the maximization problem. In particular, IC_{HM} can now be written as

$$\Delta\theta q_L + \delta \frac{3\alpha - 1}{2} \left[u_H(L) - u_M(L)\right] \geq \Delta\theta q_M + \delta \frac{3\alpha - 1}{2} \left[u_H(M) - u_M(M)\right] \tag{7}$$

Second, we claim that if IC_{ML} and IC_{HL} bind and IC_{HM} does not bind, then $IC_{HM}(L)$ binds. Suppose $u_H(L) - u_M(L) > \Delta\theta q_M(L)$. Decrease $u_H(L)$ by ε (and so U_H by $\delta(\alpha_{HH} - \alpha_{LH})\varepsilon$ and U_M by $\delta(\alpha_{MH} - \alpha_{LH})\varepsilon$), thereby, increasing the profit of the monopolist without disturbing any of the remaining constraints, giving us a contradiction. Thus, $IC_{HM}(L)$ must bind.

Using $IC_{HM}(M)$ and the binding $IC_{HM}(L)$ we can rewrite (7) to obtain:

$$\Delta\theta q_L + \delta \frac{3\alpha - 1}{2} \Delta\theta q_M(L) \geq \Delta\theta q_M + \delta \frac{3\alpha - 1}{2} \Delta\theta q_M(M)$$

Since IC_{HM} does not bind, it is easy to see that $q_M = \theta_M$ and $q_i(M) = \theta_i$ for any i. By Lemma A10.1, we have $q_L \leq \theta_L$ (and thus $q_L < \theta_M$) and $q_M(L) \leq \theta_M$. These clearly contradict the above inequality. Thus, we must have that IC_{HM} binds. ■

5.1.2 Characterization of the optimal WR-contract

We can now use the equalities implied by Lemma A10 to reduce the number of free variables in the optimization problem. In particular we can eliminate the period 1 utility vectors. Define $\omega_{HM}(i) = u_H(i) - u_M(i)$ and $\omega_{ML}(i) = u_M(i) - u_L(i)$ for $i = M, L$. The variable $\omega_{kl}(i)$ is the net utility of reporting to be type k rather than a type l after history i. Using this notation, we can rewrite the *WR-program* as a maximization problem in which the control variables are the quantities $\mathbf{q}$ and second period marginal utilities $\boldsymbol{\omega}$:

$$\max_{\langle \boldsymbol{\omega}, \mathbf{q} \rangle} \left\{ \begin{array}{c} \sum_{i=H,M,L} \mu_i \left[\theta_i q_i - \frac{1}{2} q_i^2 + \delta \sum_{k=H,M,L} \alpha_{ik} \left(\theta_k q_k(i) - \frac{1}{2} q_k(i)^2 \right) \right] \\ -\mu_H \left[\Delta\theta q_M + \delta \frac{3\alpha-1}{2} \omega_{HM}(M) \right] \\ -(\mu_H + \mu_M) \left[\Delta\theta q_L + \delta \frac{3\alpha-1}{2} \omega_{ML}(L) \right] \end{array} \right\} \quad (8)$$

subject to:

$$[\lambda]: \quad \Delta\theta q_M + \delta \frac{3\alpha - 1}{2} \omega_{HM}(M) \geq \Delta\theta q_L + \delta \frac{3\alpha - 1}{2} \omega_{HM}(L)$$

$$[\lambda_{HM}(M)]: \quad \omega_{HM}(M) \geq \Delta\theta q_M(M) \quad | \quad [\lambda_{HM}(L)]: \quad \omega_{HM}(L) \geq \Delta\theta q_M(L)$$

$$[\lambda_{ML}(M)]: \quad \omega_{ML}(M) \geq \Delta\theta q_L(M) \quad | \quad [\lambda_{ML}(L)]: \quad \omega_{ML}(L) \geq \Delta\theta q_L(L)$$

$$[\lambda_{LM}(M)]: \quad \omega_{ML}(M) \leq \Delta\theta q_M(M) \quad | \quad [\lambda_{LM}(L)]: \quad \omega_{ML}(L) \leq \Delta\theta q_M(L)$$

where the variables in the square brackets on the left are the Lagrange multipliers associated with the constraints. Program (8) is a standard maximization problem, but it is complicated by a still significantly large number of constraints. The key difference between (8) and the FO-approach is the global constraint IC_{HL} and the presence of the local upward constraints $IC_{LM}(M)$ and $IC_{LM}(L)$. We cannot ignore any of these three constraints. Moreover now we cannot assume without loss of generality that all local downward incentive constraints are binding at $t = 2$, so the envelope formula (4) in Section 3 cannot be directly applied. Hence, we still have utilities in the objective function.

We start the analysis of (8) with the first order conditions. It is easy to see that the H type always gets the efficient quantity. After history H, moreover, quantities are always efficient, implying: $q_H = q_H(M) = q_H(L) = \theta_H$ and $q_H(H) = \theta_H, q_M(H) = \theta_M, q_L(H) = \theta_L$. The remaining first-order conditions are given by:

$$[q_M]: \quad \mu_M (\theta_M - q_M) - \mu_H \Delta\theta + \lambda\Delta\theta = 0$$

$$[q_L]: \quad \mu_L (\theta_L - q_L) - (\mu_H + \mu_M) \Delta\theta - \lambda\Delta\theta = 0$$

$$[q_M(M)]: \quad \mu_M \delta\alpha (\theta_M - q_M(M)) - \lambda_{HM}(M)\Delta\theta + \lambda_{LM}(M)\Delta\theta = 0$$

$$[q_L(M)]: \quad \mu_M \delta \frac{1-\alpha}{2} (\theta_L - q_L(M)) - \lambda_{ML}(M)\Delta\theta = 0$$

$$[q_M(L)]: \quad \mu_L \delta \frac{1-\alpha}{2} (\theta_M - q_M(L)) - \lambda_{HM}(L)\Delta\theta + \lambda_{LM}(L)\Delta\theta = 0$$

$$
\begin{aligned}
[q_L(L)]: &\quad \mu_L \delta\alpha\left(\theta_L - q_L(L)\right) - \lambda_{ML}(L)\Delta\theta = 0 \\
[\omega_{HM}(M)]: &\quad -\mu_H \delta \frac{3\alpha-1}{2} + \lambda\delta\frac{3\alpha-1}{2} + \lambda_{HM}(M) = 0 \\
[\omega_{ML}(M)]: &\quad \lambda_{ML}(M) - \lambda_{LM}(M) = 0 \\
[\omega_{HM}(L)]: &\quad -\lambda\delta\frac{3\alpha-1}{2} + \lambda_{HM}(L) = 0 \\
[\omega_{ML}(L)]: &\quad -\left(\mu_H + \mu_M\right)\delta\frac{3\alpha-1}{2} + \lambda_{ML}(L) - \lambda_{LM}(L) = 0
\end{aligned}
$$

The following result characterizes when we can ignore the IC_{HL} constraint:

Lemma A11. *There exists a threshold $\mu^*(\alpha)$ such that the global incentive constraint IC_{HL} can be ignored if and only if $\mu_M \geq \mu^*(\alpha)$.*

Proof. We first characterize the optimal allocation assuming $\lambda = 0$. We then derive the conditions under which the assumption of $\lambda = 0$ is admissible.

Assuming $\lambda = 0$, we have

$$
q_M = \theta_M - \frac{\mu_H}{\mu_M}\Delta\theta \quad \text{and} \quad q_L = \theta_L - \frac{\mu_H + \mu_M}{\mu_L}\Delta\theta. \tag{9}
$$

Clearly, $\lambda = 0$ implies $\lambda_{HM}(L) = 0$. Also, it is easy to show that $\lambda_{LM}(L) = 0$, else $q_M(L) > \theta_M$, which contradicts lemma A10.1. We therefore have $\lambda_{ML}(L) = \left(\mu_H + \mu_M\right)\delta\frac{3\alpha-1}{2}$, and the solution after history L is given by:

$$
q_M(L) = \theta_M \quad \text{and} \quad q_L(L) = \theta_L - \frac{\mu_H + \mu_M}{\mu_L}\frac{3\alpha-1}{2\alpha}\Delta\theta. \tag{10}
$$

Next, note that we must have $\lambda_{HM}(M) = \mu_H\delta\frac{3\alpha-1}{2}$ and $\lambda_{ML}(M) = \lambda_{LM}(M)$. We have two possible cases:

Case A1. $\lambda_{ML}(M) = \lambda_{LM}(M) = 0$. In this case:

$$
q_M(M) = \theta_M - \frac{\mu_H}{\mu_M}\frac{3\alpha-1}{2\alpha}\Delta\theta \quad \text{and} \quad q_L(M) = \theta_L \tag{11}
$$

For this to be a solution, we must have $\theta_M - \frac{\mu_H}{\mu_M}\frac{3\alpha-1}{2\alpha}\Delta\theta \geq \theta_L$, so $\alpha \leq \alpha_0(\mu_M)$ where

$$
\alpha_0(\mu_M) = \frac{\mu_H}{3\mu_H - 2\mu_M}.
$$

We conclude that for $\alpha \leq \alpha_0(\mu_M)$ the solution is given by $q_H = \theta_H$, $q_H(j) = \theta_H$, $q_j(H) = \theta_j$ for all $j = H, M, L$ in addition to (9)-(11).

Case A2. $\lambda_{ML}(M) = \lambda_{LM}(M) > 0$. Then, $q_M(M)$ and $q_L(M)$ are both equal to a constant q. From the first order condition with respect to $q_M(M)$ and $q_L(M)$ we have:

$$
q_M(M) = q_L(M) = \frac{2\alpha}{1+\alpha}\theta_M + \frac{1-\alpha}{1+\alpha}\theta_L - \frac{\mu_H}{\mu_M}\frac{3\alpha-1}{1+\alpha}\Delta\theta. \tag{12}
$$

We conclude that for $\alpha > \alpha_0(\mu_M)$ the solution is given by $q_H = \theta_H$, $q_H(j) = \theta_H$, $q_j(H) = \theta_j$ for all $j = H, M, L$, (9)-(10) and (12).

To characterize the necessary and sufficient condition for $\lambda = 0$, we need to verify that given the solution defined above, IC_{HL} is satisfied. Plugging in the values of Case A1, we obtain:

$$\theta_M - \frac{\mu_H}{\mu_M}\Delta\theta + \delta\frac{3\alpha-1}{2}\left(\theta_M - \frac{\mu_H}{\mu_M}\frac{3\alpha-1}{2\alpha}\Delta\theta\right) \geq \theta_L - \frac{\mu_H+\mu_M}{\mu_L}\Delta\theta + \delta\frac{3\alpha-1}{2}\theta_M, \tag{13}$$

that is,

$$\mu_M \geq \frac{\mu_L(1-\mu_L)\left(1+\frac{\delta}{\alpha}\left(\frac{3\alpha-1}{2}\right)^2\right)}{1+\mu_L\left(1+\frac{\delta}{\alpha}\left(\frac{3\alpha-1}{2}\right)^2\right)} = \mu_1^*(\alpha) \tag{14}$$

Plugging in the values of Case A2, we obtain:

$$\theta_M - \frac{\mu_H}{\mu_M}\Delta\theta + \delta\frac{3\alpha-1}{2}\left(\frac{2\alpha}{1+\alpha}\theta_M + \frac{1-\alpha}{1+\alpha}\theta_L - \frac{\mu_H}{\mu_M}\frac{3\alpha-1}{1+\alpha}\Delta\theta\right) \geq \theta_L - \frac{\mu_H+\mu_M}{\mu_L}\Delta\theta + \delta\frac{3\alpha-1}{2}\theta_M, \tag{15}$$

that is,

$$\mu_M \geq \frac{\mu_L(1-\mu_L)\left(1+\delta\frac{(3\alpha-1)^2}{2(1+\alpha)}\right)}{1+\mu_L\left(1-\delta\frac{3\alpha-1}{1+\alpha}(1-2\alpha)\right)} = \mu_2^*(\alpha) \tag{16}$$

Let us define $\mu^*(\alpha) = \min\{\mu_1^*(\alpha), \mu_2^*(\alpha)\}$. We have the following result.

Lemma A11.1. *If α, μ_M is such that $\mu_M \geq \mu^*(\alpha)$ and $\alpha \leq \alpha_0(\mu_M)$ then the optimal contract is as described in Case A1 presented above. If $\mu \geq \mu^*(\alpha)$ and $\alpha > \alpha_0(\mu_M)$ then the optimal contract is as described in Case A2 presented above.*

Proof. We first prove that when $\alpha \leq \alpha_0(\mu_M)$, then $\mu_M \geq \mu^*(\alpha)$ implies $\mu_M \geq \mu_1^*(\alpha)$. To this end, we prove the counterpositive: when $\alpha \leq \alpha_0(\mu_M)$, $\mu_M < \mu_1^*(\alpha)$ implies $\mu_M < \mu^*(\alpha)$. Note that: 1. the left hand side of (13) and (15) are the same; 2. the right hand side of (13) is not larger than the right hand side of (15) if and only if $\frac{\mu_M}{\mu_H} \leq \frac{2\alpha}{3\alpha-1}$, that is if $\alpha \leq \alpha_0(\mu_M)$. It follows that if $\mu_M < \mu_1^*(\alpha)$, then neither (13) nor (15) hold, implying $\mu_M < \mu_2^*(\alpha)$ as well: we therefore conclude that $\mu_M < \mu^*(\alpha)$. Given this, the conditions $\mu_M \geq \mu^*(\alpha)$ *and* $\alpha \leq \alpha_0(\mu_M)$ imply the conditions $\mu_M \geq \mu_1^*(\alpha)$ *and* $\alpha \leq \alpha_0(\mu_M)$, so by the discussion presented above, the allocation described in Case A1 is an optimal solution of the *WR-problem.* By a similar argument, we can prove that when $\alpha > \alpha_0(\mu_M)$, then $\mu_M \geq \mu^*(\alpha)$ implies $\mu_M \geq \mu_2^*(\alpha)$. This implies that when we have $\mu_M \geq \mu^*(\alpha)$ *and* $\alpha > \alpha_0(\mu_M)$, then the allocation described in Case A2 is an optimal solution of the *WR-problem.* ■

Finally note that Cases A1 and A2 described above are the only possible allocations consistent with $\lambda = 0$. So, if $\mu_M < \mu^*(\alpha)$, the Largrange multiplier of IC_{HL} must be binding. ■

Cases A1 and A2 follow from Lemma A11.1. For the remaining cases we first prove a useful lemma.

Lemma A12. *The optimal solution satisfies:* $q_L \leq \theta_L - \frac{\mu_H+\mu_M}{\mu_L}\Delta\theta$, $q_L(L) \leq \theta_L - \frac{\mu_H+\mu_M}{\mu_L}\frac{3\alpha-1}{2\alpha}\Delta\theta$ and $q_L(M) \leq \theta_L$.

Proof. We proceed in 3 steps.

Step 1. Suppose $q_L > \theta_L - \frac{\mu_H+\mu_M}{\mu_L}\Delta\theta$. Now, decrease q_L by ϵ. All the constraints are still satisfied. The change in the monopolist's profit is given by

$$\begin{aligned} & \mu_L\left[-\theta_L\epsilon - \frac{1}{2}\left((q_L-\epsilon)^2-(q_L)^2\right)\right] + (\mu_H+\mu_M)\Delta\theta\epsilon \\ = \ & \mu_L\left[\left(q_L - \left(\theta_L - \frac{\mu_H+\mu_M}{\mu_L}\Delta\theta\right)\right)\epsilon - \frac{1}{2}\epsilon^2\right], \end{aligned}$$

which is greater than zero for small enough ϵ, giving us a contradiction.

Step 2. Suppose $q_L(L) > \theta_L - \frac{\mu_H+\mu_M}{\mu_L}\frac{3\alpha-1}{2\alpha}\Delta\theta$. Now, decrease $q_L(L)$ by ϵ and $\omega_{ML}(L)$ by $\Delta\theta\epsilon$. All the constraints are still satisfied. The change in the monopolist's profit is given by

$$\begin{aligned} & \mu_L\delta\alpha\left[-\theta_L\epsilon - \frac{1}{2}\left((q_L(L)-\epsilon)^2-(q_L(L))^2\right)\right] + (\mu_H+\mu_M)\delta\frac{3\alpha-1}{2}\Delta\theta\epsilon \\ = \ & \mu_L\delta\alpha\left[\left(q_L(L) - \left(\theta_L - \frac{\mu_H+\mu_M}{\mu_L}\frac{3\alpha-1}{2\alpha}\Delta\theta\right)\right)\epsilon - \frac{1}{2}\epsilon^2\right], \end{aligned}$$

which is greater than zero for small enough ϵ, giving us a contradiction.

Step 3. Suppose $q_L(M) > \theta_L$. Now, decrease $q_L(M)$ by ϵ and $\omega_{ML}(M)$ by $\Delta\theta\epsilon$. All the constraints are still satisfied. The change in the monopolist's profit is given by

$$\mu_M\delta\frac{1-\alpha}{2}\left[-\theta_L\epsilon - \frac{1}{2}\left((q_L(M)-\epsilon)^2-(q_L(M))^2\right)\right] = \mu_M\delta\frac{1-\alpha}{2}\left[\left[(q_L(M)-\theta_L\right)\epsilon - \frac{1}{2}\epsilon^2\right],$$

which is greater than zero for small enough ϵ, giving us a contradiction. ■

Keep in mind that $\lambda > 0 \Rightarrow \lambda_{HM}(L) > 0$. It follows from the first order condition with respect to $\omega_{HM}(L)$. Next, in order to characterize the quantities after history M, we prove a useful lemma.

Lemma A13. $\lambda > 0 \Rightarrow \lambda_{HM}(M) > 0$.

Proof. Assume by contradiction that $\lambda_{HM}(M) = 0$. Then, we must have $\lambda_{ML}(M) = \lambda_{LM}(M) = 0$. Assuming them strictly positive gives us $q_M(M) = q_L(M)$. Also, from the first order condition for $q_M(M)$, we obtain $q_M(M) > \theta_M$, implying $q_L(M) > \theta_M > \theta_L$, a contradiction to Lemma A12. Thus, $\lambda = \mu_H$ and $q_M = q_M(M) = \theta_M$.

Next, we note that if $\lambda > 0$, then $q_M(L) < \theta_M$. To see this point, consider the first-order condition with respect to $q_M(L)$. Since, $\lambda_{HM}(L) > 0$, if $\lambda_{LM}(L) = 0$ then it follows immediately that $q_M(L) < \theta_M$. If $\lambda_{LM}(L) > 0$, then $q_M(L) = q_L(L) < \theta_L < \theta_M$, where the first inequality follows from Lemma A12.

Using these facts, we can now write:

$$\Delta\theta q_M+\delta\frac{3\alpha-1}{2}\omega_{HM}(M)=\Delta\theta\cdot\theta_M+\delta\frac{3\alpha-1}{2}\omega_{HM}(M)\geq\Delta\theta\cdot\theta_M+\delta\frac{3\alpha-1}{2}\Delta\theta q_M(M)$$

$$=\Delta\theta\cdot\theta_M+\delta\frac{3\alpha-1}{2}\Delta\theta\cdot\theta_M>\Delta\theta q_L+\delta\frac{3\alpha-1}{2}\Delta\theta q_M(L)=\Delta\theta q_L+\delta\frac{3\alpha-1}{2}\omega_{HM}(L). \tag{17}$$

The strict inequality proven in (17) contradicts $\lambda>0$. Thus, we must have $\lambda_{HM}(M)>0$ as requested. This completes the proof of Lemma A13. ■

We divide the reminder of the proof of proposition A1 into two steps. First we assume that $IC_{LM}(L)$ is not binding and we characterize the parameter region in which this assumption is correct. This will allow us to define the regions B1 and B2 described in the statement of the proposition. Then we characterize the optimal contract when $IC_{LM}(L)$ is binding, region B3.

Characterization of Regions B1 and B2 Let us assume $\lambda_{LM}(L)=0$. Since $\mu_M<\mu^*(\alpha)$, we have $\lambda>0$. From the first order conditions, we obtain:

$$q_M = \theta_M-\frac{\mu_H-\lambda}{\mu_M}\Delta\theta,\ \ q_L=\theta_L-\frac{\mu_H+\mu_M+\lambda}{\mu_L}\Delta\theta \tag{18}$$

$$q_M(L) = \theta_M-\frac{\lambda}{\mu_L}\frac{3\alpha-1}{1-\alpha}\Delta\theta,\ \ q_L(L)=\theta_L-\frac{\mu_H+\mu_M}{\mu_L}\frac{3\alpha-1}{2\alpha}\Delta\theta \tag{19}$$

Since $\lambda>0$, we have $\lambda_{HM}(M)>0$ and $\lambda_{HM}(L)>0$. Thus,

$$q_M+\delta\frac{3\alpha-1}{2}q_M(M)=q_L+\delta\frac{3\alpha-1}{2}q_M(L) \tag{20}$$

There are two relevant cases. We use λ_1 to denote λ from Case B1 and λ_2 from Case B2.

Case B1. $\lambda_{ML}(M)=\lambda_{LM}(M)=0$. Then, from the first-order conditions:

$$q_M(M)=\theta_M-\frac{\mu_H-\lambda_1}{\mu_M}\frac{3\alpha-1}{2\alpha}\Delta\theta \quad\text{and}\quad q_L(M)=\theta_L \tag{21}$$

Substituting, the values from (18)-(19) and (21) in equation (20) we obtain:

$$\frac{1+\lambda_1}{\mu_L}+\delta\frac{3\alpha-1}{2}\frac{\lambda_1}{\mu_L}\frac{3\alpha-1}{1-\alpha}=\frac{\mu_H-\lambda_1}{\mu_M}+\delta\frac{3\alpha-1}{2}\frac{\mu_H-\lambda_1}{\mu_M}\frac{3\alpha-1}{2\alpha} \tag{22}$$

which gives:

$$\lambda_1=\lambda_1(\alpha)=\frac{\frac{\mu_H}{\mu_M}\left(1+\delta\frac{3\alpha-1}{2}\frac{3\alpha-1}{2\alpha}\right)-\frac{1}{\mu_L}}{\frac{1}{\mu_M}\left(1+\delta\frac{3\alpha-1}{2}\frac{3\alpha-1}{2\alpha}\right)+\frac{1}{\mu_L}\left(1+\delta\frac{3\alpha-1}{2}\frac{3\alpha-1}{1-\alpha}\right)} \tag{23}$$

Clearly, for this case to be valid, we must justify the assumption that $\lambda_{ML}(M)=\lambda_{LM}(M)=0$. A necessary and sufficient condition for this is $q_M(M)\geq q_L(M)$. Given (21), this condition can be rewritten as: $\frac{\mu_H-\lambda_1}{\mu_M}\frac{3\alpha-1}{2\alpha}\leq 1$, where λ_1 is given by (23). This condition is implied by:

$$\mu_M\geq\frac{1+(1-\mu_L)b_0(\alpha)-\mu_L c_0(\alpha)a_0(\alpha)}{b_0(\alpha)\left(1+c_0(\alpha)\right)}=\mu_0(\alpha),$$

where

$$a_0(\alpha) = 1 + \delta\frac{3\alpha-1}{2}\frac{3\alpha-1}{2\alpha},\ b_0(\alpha) = 1 + \delta\frac{3\alpha-1}{2}\frac{3\alpha-1}{1-\alpha},\ \text{and } c_0(\alpha) = \frac{2\alpha}{3\alpha-1}$$

It follows that (under the assumption that $\lambda_{LM}(L) = 0$) the solution is given by (18)-(19), (21) and (23) when when $\mu_M \geq \mu_0(\alpha)$.

Case B2. For $\mu_M < \mu_0(\alpha)$ we must have $\lambda_{ML}(M) = \lambda_{LM}(M) > 0$. In this case, we must have:

$$q_M(M) = q_L(M) = \frac{2\alpha}{1+\alpha}\theta_M + \frac{1-\alpha}{1+\alpha}\theta_L - \frac{\mu_H - \lambda_2}{\mu_M}\frac{3\alpha-1}{1+\alpha}\Delta\theta \tag{24}$$

Substituting $q_M(M)$ and $q_M(L)$ equation (20) we obtain:

$$\frac{1+\lambda_2}{\mu_L} + \delta\frac{3\alpha-1}{2}\left(\frac{\lambda_2}{\mu_L}\frac{3\alpha-1}{1-\alpha} - \frac{1-\alpha}{1+\alpha}\right) = \frac{\mu_H-\lambda_2}{\mu_M} + \delta\frac{3\alpha-1}{2}\frac{\mu_H-\lambda_2}{\mu_M}\frac{3\alpha-1}{1+\alpha} \tag{25}$$

which gives

$$\lambda_2 = \frac{\frac{\mu_H}{\mu_M}\left(1+\delta\frac{3\alpha-1}{2}\frac{3\alpha-1}{1+\alpha}\right) - \left(\frac{1}{\mu_L} - \delta\frac{3\alpha-1}{2}\frac{1-\alpha}{1+\alpha}\right)}{\frac{1}{\mu_M}\left(1+\delta\frac{3\alpha-1}{2}\frac{3\alpha-1}{1+\alpha}\right) + \frac{1}{\mu_L}\left(1+\delta\frac{3\alpha-1}{2}\frac{3\alpha-1}{1-\alpha}\right)} \tag{26}$$

It follows that (under the assumption that $\lambda_{LM}(L) = 0$) the solution is given by (18)-(19), (24) and (26) when $\mu_M < \mu_0(\alpha)$.

We now complete the analysis of this section by characterizing the conditions under which we can ignore the $IC_{LM}(L)$ constraint and so $\lambda_{LM}(L) = 0$. It is easy to see that $IC_{LM}(L)$ is satisfied if and only if $q_M(L) \geq q_L(L)$. We have $q_M(L) \geq q_L(L)$ if and only if:

$$\lambda_i \leq \left(\frac{1}{\mu_L}\frac{3\alpha-1}{1-\alpha}\right)^{-1}\left(1 + \frac{1-\mu_L}{\mu_L}\frac{3\alpha-1}{2\alpha}\right) \tag{27}$$

Thus, for Case B1 we have,

$$\frac{\frac{\mu_H}{\mu_M}\left(1+\delta\frac{3\alpha-1}{2}\frac{3\alpha-1}{2\alpha}\right) - \frac{1}{\mu_L}}{\frac{1}{\mu_M}\left(1+\delta\frac{3\alpha-1}{2}\frac{3\alpha-1}{2\alpha}\right) + \frac{1}{\mu_L}\left(1+\delta\frac{3\alpha-1}{2}\frac{3\alpha-1}{1-\alpha}\right)} \leq \left(\frac{1}{\mu_L}\frac{3\alpha-1}{1-\alpha}\right)^{-1}\left(1+\frac{1-\mu_L}{\mu_L}\frac{3\alpha-1}{2\alpha}\right)$$

Define

$$a_1(\alpha,\mu_L) = 1 + \delta\frac{3\alpha-1}{2}\frac{3\alpha-1}{2\alpha},\ b_1(\alpha,\mu_L) = 1 + \delta\frac{3\alpha-1}{2}\frac{3\alpha-1}{1-\alpha},\ \text{and}$$

$$c_1(\alpha,\mu_L) = \left(\frac{1}{\mu_L}\frac{3\alpha-1}{1-\alpha}\right)^{-1}\left(1+\frac{1-\mu_L}{\mu_L}\frac{3\alpha-1}{2\alpha}\right)$$

We can then write the previous inequality as:

$$\mu_M \geq \frac{\mu_L a_1(\alpha,\mu_L)\left[1-\mu_L - c_1(\alpha,\mu_L)\right]}{1 + a_1(\alpha,\mu_L)\mu_L + b_1(\alpha,\mu_L)c_1(\alpha,\mu_L)} = \mu_1^{**}(\alpha)$$

Next, for Case B2, we have $q_M(L) \geq q_L(L)$ iff,

$$\frac{\frac{\mu_H}{\mu_M}\left(1+\delta\frac{3\alpha-1}{2}\frac{3\alpha-1}{1+\alpha}\right) - \left(\frac{1}{\mu_L} - \delta\frac{3\alpha-1}{2}\frac{1-\alpha}{1+\alpha}\right)}{\frac{1}{\mu_M}\left(1+\delta\frac{3\alpha-1}{2}\frac{3\alpha-1}{1+\alpha}\right) + \frac{1}{\mu_L}\left(1+\delta\frac{3\alpha-1}{2}\frac{3\alpha-1}{1-\alpha}\right)} \leq \left(\frac{1}{\mu_L}\frac{3\alpha-1}{1-\alpha}\right)^{-1}\left(1+\frac{1-\mu_L}{\mu_L}\frac{3\alpha-1}{2\alpha}\right)$$

Define

$$a_2(\alpha,\mu_L) = 1+\delta\frac{3\alpha-1}{2}\frac{3\alpha-1}{1+\alpha},\ b_2(\alpha,\mu_L) = \frac{1}{\mu_L}-\delta\frac{3\alpha-1}{2}\frac{1-\alpha}{1+\alpha}$$

$$c_2(\alpha,\mu_L) = \left(\frac{1}{\mu_L}\frac{3\alpha-1}{1-\alpha}\right)^{-1}\left(1+\frac{1-\mu_L}{\mu_L}\frac{3\alpha-1}{2\alpha}\right),\ d_2(\alpha,\mu_L) = 1+\delta\frac{3\alpha-1}{2}\frac{3\alpha-1}{1-\alpha}$$

Rearranging, we obtain:

$$\mu_M \geq \frac{\mu_L a_2(\alpha,\mu_L)\left[1-\mu_L-c_2(\alpha,\mu_L)\right]}{\mu_L\left(a_2(\alpha,\mu_L)+b_2(\alpha,\mu_L)\right)+b_2(\alpha,\mu_L)\,c_2(\alpha,\mu_L)} = \mu_2^{**}(\alpha)$$

Let us define $\mu^{**}(\alpha) = \min\{\mu^*(\alpha), \mu_1^{**}(\alpha), \mu_2^{**}(\alpha)\}$. We have:

Lemma 14. *If $\mu_M \in [\mu^{**}(\alpha), \mu^*(\alpha)]$ and $\mu_M \geq \mu_0(\alpha)$, then the solution of the WR-problem is given by the solution in Case B1 presented above. If $\mu_M \in [\mu^{**}(\alpha), \mu^*(\alpha)]$ and $\mu_M < \mu_0(\alpha)$, then the solution of the WR-problem is given by the solution in Case B2 presented above.*

Proof. We first show that if $\mu_M \in [\mu^{**}(\alpha), \mu^*(\alpha)]$ *and* $\mu_M \geq \mu_0(\alpha)$, then $\mu_M \in [\mu_1^{**}(\alpha), \mu^*(\alpha)]$ and $\mu_M \geq \mu_0(\alpha)$. This implies that the solution is given by Case B1. Assume $\mu_M < \mu_1^{**}(\alpha)$. In this case, (27) does not hold with λ_1. This implies that (27) does not hold with λ_2 as well if $\lambda_2 \geq \lambda_1$. Subtracting equation (25) from equation (22), we get

$$(\lambda_1-\lambda_2)\left[\frac{1}{\mu_L}+\frac{1}{\mu_M}+\delta\frac{3\alpha-1}{2}\left(\frac{1}{\mu_L}\frac{3\alpha-1}{1-\alpha}+\frac{1}{\mu_M}\frac{3\alpha-1}{1+\alpha}\right)\right] = \delta\frac{3\alpha-1}{2}\frac{1-\alpha}{1+\alpha}\left[\frac{\mu_H-\lambda_1}{\mu_M}\frac{3\alpha-1}{2\alpha}-1\right] \tag{28}$$

So, we have that $\lambda_2 \geq \lambda_1$ if:

$$\frac{\mu_H-\lambda_1}{\mu_M}\frac{3\alpha-1}{2\alpha}-1 \leq 0,$$

which is implied by $\mu_M \geq \mu_0(\alpha)$. It follows that if $\mu_M < \mu_1^{**}(\alpha)$, then $\mu_M < \mu^{**}(\alpha)$, a contradiction. We conclude that it must be $\mu_M \geq \mu_1^{**}(\alpha)$.

We now show that if $\mu_M \in [\mu^{**}(\alpha), \mu^*(\alpha)]$ and $\mu_M < \mu_0(\alpha)$, then $\mu_M \in [\mu_2^{**}(\alpha), \mu^*(\alpha)]$ *and* $\mu_M < \mu_0(\alpha)$. This implies that the solution is given by Case B2. Assume $\mu_M < \mu_2^{**}(\alpha)$. In this case, (27) does not hold with λ_2 . This implies that (27) does not hold with λ_1 as well if $\lambda_1 \geq \lambda_2$. From (28) we have that this always true if $\mu_M < \mu_0(\alpha)$. It follows that if $\mu_M < \mu_2^{**}(\alpha)$, then $\mu_M < \mu^{**}(\alpha)$, a contradiction. We conclude that it must be $\mu_M \geq \mu_2^{**}(\alpha)$. ■

Characterization of Region B3 Finally, we characterize the contract when $\mu_M < \mu^{**}(\alpha)$ and so both $\lambda > 0$ and $\lambda_{LM}(L) > 0$. This is region B3. In this case:

$$q_M = \theta_M - \frac{\mu_H-\lambda}{\mu_M}\Delta\theta \quad \text{and} \quad q_L = \theta_L - \frac{\mu_H+\mu_M+\lambda}{\mu_L}\Delta\theta \tag{29}$$

We also have that $\lambda_{LM}(L) > 0$ implies $q_M(L) = q_L(L)$, so:

$$q_M(L) = q_L(L) = \frac{1-\alpha}{1+\alpha}\theta_M + \frac{2\alpha}{1+\alpha}\theta_L - \frac{\mu_H+\mu_M+\lambda}{\mu_L}\frac{3\alpha-1}{1+\alpha}\Delta\theta \tag{30}$$

From Lemma A12, we have $q_L(L) \leq \theta_L - \frac{\mu_H+\mu_M}{\mu_L}\frac{3\alpha-1}{2\alpha}\Delta\theta$. Also, when $\lambda_{LM}(L) > 0$, the above inequality is strict. Thus, substituting the optimal value of $q_L(L)$, we obtain:

$$1 - \frac{\lambda}{\mu_L}\frac{3\alpha-1}{1-\alpha} + \frac{\mu_H+\mu_M}{\mu_L}\frac{3\alpha-1}{2\alpha} < 0 \tag{31}$$

Note that as $\lambda_{LM}(L)$ converges to zero, (31) is the exact violation of $\mu_M \geq \mu^{**}(\alpha)$, that is, inequality (27).

To characterize the quantities after history M, we now show that $\lambda_{ML}(M) = \lambda_{LM}(M) > 0$.

Lemma A15. $\lambda, \lambda_{LM}(L) > 0 \Rightarrow \lambda_{ML}(M) = \lambda_{LM}(L) > 0$.

Proof. Suppose $\lambda_{ML}(M) = \lambda_{LM}(M) = 0$. Then,

$$q_M(M) = \theta_M - \frac{\mu_H - \lambda}{\mu_M}\frac{3\alpha-1}{2\alpha}\Delta\theta \quad \text{and} \quad q_L(M) = \theta_L.$$

From $\theta_M - \frac{\mu_H-\lambda}{\mu_M}\frac{3\alpha-1}{2\alpha}\Delta \geq \theta_L$, we have:

$$\frac{2\alpha}{3\alpha-1} - \frac{\mu_H-\lambda}{\mu_M} \geq 0. \tag{32}$$

Since $\lambda, \lambda_{LM}(L) > 0$, using $q_M(M) \geq q_L(M) = \theta_L > q_L(L) = q_M(L)$, we get $q_L > q_M$. This implies

$$\left(1 - \frac{\mu_H-\lambda}{\mu_M} + \frac{\mu_H+\mu_M+\lambda}{\mu_L}\right) < 0.$$

Using equation (32). we get

$$\frac{\mu_H+\mu_M+\lambda}{\mu_L}\frac{3\alpha-1}{1-\alpha} < 1. \tag{33}$$

Now, inequality (31) can be written as

$$\begin{aligned} 1 &< \frac{\mu_H+\mu_M+\lambda}{\mu_L}\frac{3\alpha-1}{1-\alpha} - \frac{\mu_H+\mu_M}{\mu_L}(3\alpha-1)\left(\frac{1}{1-\alpha} + \frac{1}{2\alpha}\right) \\ &= \frac{\mu_H+\mu_M+\lambda}{\mu_L}\frac{3\alpha-1}{1-\alpha} - \frac{\mu_H+\mu_M}{\mu_L}\frac{3\alpha-1}{2\alpha}\frac{1+\alpha}{2\alpha} \end{aligned}$$

which contradicts condition (33). ■

It follows that

$$q_M(M) = q_L(M) = \frac{2\alpha}{1+\alpha}\theta_M + \frac{1-\alpha}{1+\alpha}\theta_L - \frac{\mu_H-\lambda}{\mu_M}\frac{3\alpha-1}{1+\alpha}\Delta\theta$$

Finally, substituting the optimal values in IC_{HL} as equality, we obtain:

$$\left(1 - \frac{\mu_H-\lambda}{\mu_M} + \frac{\mu_H+\mu_M+\lambda}{\mu_L}\right) = 0 \tag{34}$$

that implies $q_M = q_L$. Note that equation (34) gives the value of λ, which uniquely defines the solution at the optimum. In particular note that type M and L are treated as *one*, that is,

$$q_M = q_L \quad \text{and} \quad q_M(M) = q_L(M) = q_M(L) = q_L(L) \tag{35}$$

We conclude that the solution of the *WR-problem* in region B3 ($\mu_M < \mu^{**}(\alpha)$) is given by: (29),(30), (35) and (34).

This concludes the complete characterization of the optimal allocations in the *WR-problem.* Table 2 summarizes the solution of the describing the optimal allocation for each possible case. ■

5.1.3 The optimal WR-contract is the optimal contract

We prove the lemma as follows. Let $\mathbf{U} = U(h^t)$ be the vector of expected utilities, mapping an history h^t to the corresponding agent's expected utility. First, we construct a vector of utilities $\mathbf{U}$ using the solution of the *WR-problem*, $\langle \boldsymbol{\omega}, \mathbf{q} \rangle$. We then show that the solution $\langle \mathbf{U}, \mathbf{q} \rangle$ satisfies all the constraints of the seller's profit maximization problem and it maximizes profits. We proceed in two steps:

Step 1. We set $u_L(M)$, $u_L(L)$, $u_L(H)$ all equal to zero. We also define:

$$u_M(M) = \omega_{ML}(M), u_M(L) = \omega_{ML}(L), u_M(H) = \Delta\theta q_L(H)$$
$$u_H(M) = \omega_{ML}(M) + \omega_{HM}(M), u_H(L) = \omega_{ML}(L) + \omega_{HM}(L), u_H(H) = \Delta\theta \left(q_L(H) + q_M(H)\right)$$

Since IR_L, IC_{ML} and IC_{HM} hold as an equality, we must have:

$$\begin{aligned} U_L &= 0, \\ U_M &= \Delta\theta q_L + \delta \frac{3\alpha - 1}{2} \omega_{ML}(L), \text{ and} \\ U_H &= U_M + \Delta\theta q_M + \delta \frac{3\alpha - 1}{2} \omega_{HM}(M). \end{aligned}$$

Step 2. We now show that $\langle \mathbf{U}, \mathbf{q} \rangle$ satisfies all the constraints of the profit maximizing problem. By construction it is immediate that $\langle \mathbf{U}, \mathbf{q} \rangle$ satisfies all the constraints in the *WR-problem.* It remains to be shown that it also satisfies the other constraints,

$$\begin{gathered} IR_H, IR_M, IC_{MH}, IC_{LM}, IC_{LH}, \\ IC_{HM}(H), IC_{ML}(H), IR_L(H), IR_L(M), IR_L(L) \\ IC_{MH}(H), IC_{LM}(H), IC_{LH}(H), IC_{HL}(H)), IC_{MH}(M), \\ IC_{LH}(M), IC_{HL}(M), IC_{MH}(L), IC_{LH}(L), IC_{HL}(L). \end{gathered} \tag{36}$$

First, we show that IR_M is satisfied. From IC_{ML} we have

$$\begin{aligned} U_M &= U_L + \Delta\theta q_L + \delta \frac{3\alpha - 1}{2} \left[u_M(L) - u_L(L)\right] \\ &= \Delta\theta q_L + \delta \frac{3\alpha - 1}{2} \left[u_M(L) - u_L(L)\right] \qquad \text{[Using } IR_L\text{]} \\ &\geq \Delta\theta q_L + \delta \frac{3\alpha - 1}{2} \Delta q_L(L) > 0 \qquad \text{[Using } IC_{ML}(L)\text{]} \end{aligned}$$

Similarly, we can show that IR_H is satisfied. To prove the remaining constraints we need the following properties of the solution of the WR-problem.

Lemma A16. *For all parameter configurations, in the solution to the WR-problem we have: 1.* $q_i(H) = \theta_i$ *for* $i = M, L, H$, $q_M(M) < \theta_M, q_L(M) \leq \theta_L$, *and* $q_L(M) \geq q_L(L)$ *2.* $\omega_{HM}(M) = \Delta\theta q_M(M)$ *and, without loss of generality,* $\omega_{ML}(M) = \Delta\theta q_L(M)$, $\omega_{HM}(L) = \Delta\theta q_M(L)$; *3. quantities at* $t = 2$ *are nondecreasing in type after any history; 4.* $q_H \geq q_M \geq q_L$.

Proof. Point 1 follow from the solution characterized in Section 5.1.2 of this appendix (for convenience the quantities are reported in Table 2). The first part of Point 2 ($IC_{HM}(M)$ always binds) follows from the first-order condition for $\omega_{HM}(M)$ (when $\lambda = 0$) and Lemma 13 (when $\lambda > 0$). The second part follows from the fact that $IC_{ML}(M)$ can be assumed to hold as an equality. Suppose $\omega_{ML}(M) > \Delta\theta q_L(M)$. Then can decrease $\omega_{ML}(M)$ so that this holds as an equality. No constraint is violated and the profit of the monopolist is unaffected. Similarly, we show that $IC_{HM}(L)$ can be assumed to hold as an equality, implying $\omega_{HM}(L) = \Delta\theta q_M(L)$. Point 3 follows from incentive compatibility constraints for the second (terminal) period. We now turn to Point 4. From the fact that in the solution to the *WR-problem*, $q_H = \theta_H$ and the fact that (as shown in Section 5.1.2 of this appendix) $q_i \leq \theta_i$ for $i = H, M, L$, we have $q_H \geq q_i$ $i = M, L$. We, therefore, only need to prove that $q_M \geq q_L$. We will show this result case by case for all regions A1, A2, B1, B2 and B3. In cases $A1$ and $A2$, from (9) we have $q_M \geq q_L$ if and only if

$$1 - \frac{\mu_H}{\mu_M} + \frac{\mu_H + \mu_M}{\mu_L} \geq 0,$$

that is, $\frac{1}{\mu_L} \geq \frac{\mu_H}{\mu_M}$. In regions A1 and A2 we have $\mu_M \geq \mu^*(\alpha)$, as defined in Lemma 5.2. This condition can be written as

$$\frac{1}{\mu_L} \geq \frac{\mu_H}{\mu_M} + \delta\frac{3\alpha - 1}{2}\frac{\mu_H}{\mu_M}\frac{3\alpha - 1}{2\alpha} \quad \text{and} \quad \frac{1}{\mu_L} \geq \quad \frac{\mu_H}{\mu_M} + \delta\frac{3\alpha - 1}{2}\left(\frac{1 - \alpha}{1 + \alpha} + \frac{\mu_H}{\mu_M}\frac{3\alpha - 1}{1 + \alpha}\right).$$

clearly implying $\frac{1}{\mu_L} \geq \frac{\mu_H}{\mu_M}$. For case $B3$, we show in Section 4.1.2 of this appendix that $q_M = q_L$. We now show that in regions $B1$ and $B2$ we have $q_M \geq q_L$ as well. In these region we have $\mu \in [\mu^{**}(\alpha), \mu^*(\alpha)]$. We have $q_M \geq q_L$ if and only if $1 - \frac{\mu_H - \lambda}{\mu_M} + \frac{\mu_H + \mu_M + \lambda}{\mu_L} \geq 0$. It is clear from the first-order condition for $\omega_{HM}(L)$ that $\lambda > 0$ implies $\lambda_{HM}(L) > 0$, thus, $\omega_{HM}(L) = \Delta\theta q_M(L)$. Therefore, we have in regions B1 and B2,

$$q_M + \delta\frac{3\alpha - 1}{2}q_M(M) = q_L + \delta\frac{3\alpha - 1}{2}q_M(L).$$

When $\mu_M \geq \mu_0(\alpha)$, substituting optimal values (summarized in Table 2) we have

$$1 - \frac{\mu_H - \lambda_1}{\mu_M} + \frac{\mu_H + \mu_M + \lambda_1}{\mu_L} + \delta\frac{3\alpha - 1}{2}\left[\frac{\lambda_1}{\mu_L}\frac{3\alpha - 1}{1 - \alpha} - \frac{\mu_H - \lambda_1}{\mu_M}\frac{3\alpha - 1}{2\alpha}\right] = 0.$$

That can be re written as:

$$\left(1 - \frac{\mu_H - \lambda_1}{\mu_M} + \frac{\mu_H + \mu_M + \lambda_1}{\mu_L}\right)\left(1 + \delta\frac{(3\alpha - 1)^2}{4\alpha}\right) = \delta\frac{(3\alpha - 1)^2}{4\alpha}\left[1 + \frac{\mu_H + \mu_M}{\mu_L} - \frac{\lambda_1}{\mu_L}\frac{3\alpha - 1}{1 - \alpha}\right].$$

We know from (27) that right hand side of the above equation is non-negative. Thus, $1 - \frac{\mu_H - \lambda_1}{\mu_M} + \frac{\mu_H + \mu_M + \lambda_1}{\mu_L} \geq 0$.

When $\mu_M < \mu_0(\alpha)$, substituting optimal values again (see Table 1) we have

$$1 - \frac{\mu_H - \lambda_2}{\mu_M} + \frac{\mu_H + \mu_M + \lambda_2}{\mu_L} + \delta\frac{3\alpha - 1}{2}\left[\frac{\lambda_2}{\mu_L}\frac{3\alpha - 1}{1 - \alpha} - \frac{\mu_H - \lambda_2}{\mu_M}\frac{3\alpha - 1}{1 + \alpha} - \frac{1 - \alpha}{1 + \alpha}\right] = 0.$$

That can be rewritten as:

$$\left(1 - \frac{\mu_H - \lambda_2}{\mu_M} + \frac{\mu_H + \mu_M + \lambda_2}{\mu_L}\right)\left(1 + \delta\frac{(3\alpha - 1)^2}{2(1 + \alpha)}\right) = \delta\frac{\alpha(3\alpha - 1)}{1 + \alpha}\left[\begin{array}{c} 1 + \frac{\mu_H + \mu_M}{\mu_L}\frac{3\alpha - 1}{2\alpha} \\ -\frac{\lambda_2}{\mu_L}\frac{3\alpha - 1}{1 - \alpha}\end{array}\right].$$

We know that (27) is always verified in the relevant range. Using this condition we can see that right hand side of the above equation is non-negative. Thus, we we have $1 - \frac{\mu_H - \lambda_2}{\mu_M} + \frac{\mu_H + \mu_M + \lambda_2}{\mu_L} \geq 0$. ■

Consider the first period constraints. To show that IC_{LM} holds it is sufficient to prove:

$$\begin{aligned} 0 &= U_L \geq \theta_L q_M + \delta\left[\alpha u_L(L) + \frac{1 - \alpha}{2}u_M(M) + \frac{1 - \alpha}{2}u_H(M)\right] \quad (37) \\ &= U_M - \Delta\theta q_M - \delta\frac{3\alpha - 1}{2}u_L(M) \\ &= U_M - \Delta\theta q_M - \delta\frac{3\alpha - 1}{2}q_L(M) \end{aligned}$$

Since $U_M = \Delta\theta q_L + \delta\frac{3\alpha - 1}{2}q_L(L)$, (37) can be written as:

$$q_M + \delta\frac{3\alpha - 1}{2}q_L(M) \geq q_L + \delta\frac{3\alpha - 1}{2}q_L(L)$$

The fact that this inequality is satisfied follows from Point 1 and 4 in Lemma A16. (In the following, when we mention a point, we refer to the points of Lemma A16.)

Next, we show that IC_{MH} holds. From IC_{HM} we have:

$$U_H = U_M + \Delta\theta q_M + \delta\frac{3\alpha - 1}{2}\left[u_H(M) - u_M(M)\right]$$

Thus,

$$\begin{aligned} U_M &= U_H - \Delta\theta q_M - \delta\frac{3\alpha - 1}{2}\left[u_H(M) - u_M(M)\right] \\ &= U_H - \Delta\theta q_H - \delta\frac{3\alpha - 1}{2}\left[u_H(H) - u_M(H)\right] \\ &+ \Delta\theta(q_H - q_M) + \delta\frac{3\alpha - 1}{2}\left[(u_H(H) - u_M(H)) - (u_H(M) - u_M(M))\right] \\ &> U_H - \Delta\theta q_H - \delta\frac{3\alpha - 1}{2}\left[u_H(H) - u_M(H)\right]. \end{aligned}$$

The last inequality follows from the observation that:

$$u_H(H) - u_M(H) \geq \Delta\theta q_M(H) = \Delta\theta\theta_M > \Delta\theta q_M(M) = u_H(M) - u_M(M), \quad (38)$$

where the first inequality follows from the definition of $u_i(H)$, the first equality and the second inequality follow from Point 1. From (38) and the fact that $q_H > q_M$ (Point 4), it follows that IC_{MH} holds. We now turn to IC_{LH}. Using IC_{LM} first and then IC_{MH}, we have:

$$\begin{aligned}
U_L &\geq U_M - \Delta\theta q_M - \delta\frac{3\alpha-1}{2}\left[u_M(M) - u_L(M)\right] \\
&\geq U_H - \Delta\theta q_H - \delta\frac{3\alpha-1}{2}\left[u_H(H) - u_M(H)\right] - \Delta\theta q_M - \delta\frac{3\alpha-1}{2}\left[u_M(M) - u_L(M)\right] \\
&= U_H - 2\Delta\theta q_H - \delta\frac{3\alpha-1}{2}\left[u_H(H) - u_L(H)\right] \\
&+ \Delta\theta\left(q_H - q_M\right) + \delta\frac{3\alpha-1}{2}\left[\left(u_M(H) - u_L(H)\right) - \left(u_M(M) - u_L(M)\right)\right] \\
&> U_H - 2\Delta\theta q_H - \delta\frac{3\alpha-1}{2}\left[u_H(H) - u_L(H)\right],
\end{aligned}$$

The last inequality follows from the observation that:

$$u_M(H) - u_L(H) \geq \Delta\theta q_L(H) = \Delta\theta\theta_L \geq \Delta\theta q_L(M) = u_M(M) - u_L(M), \tag{39}$$

where the first inequality follows from the definition of $u_i(H)$, the first equality and the second inequality follow from Point 1. From (39) and $q_H > q_M$ (Point 4), it follows that IC_{LH} holds.

Consider now the second period constraints. The constraints $IR_L(M)$, $IR_L(L)$ $IR_L(H)$, $IC_{ML}(H)$, and $IC_{HM}(H)$) follow immediately by the definition of the utilities at $t = 2$. The proof that $\langle \mathbf{U}, \mathbf{q} \rangle$ solves the seller's problem is therefore completed if we prove that it satisfies the constraints in the last two lines of (36). This result follows from the fact that the local downward incentive constraints are satisfied in period 2 and quantities are weakly monotonic after any history (Point 3). Finally, to see that the contract is optimal, we note that it maximizes expected profits in the less restricted *WR-problem*, so it must be optimal in the seller's problem. Note moreover that since the original problem is concave in q this is in fact the unique solution (in quantities). ■

6 Numerical solution of the example in Section 6

We consider a three-type, three-period model with a uniform prior and the Markov process: $f(\theta\,|\theta) = \alpha$, $f(\theta|\theta') = (1-\alpha)/2$ for $\theta \neq \theta'$, and calculate the loss in expected profit from using (i) the optimal monotonic contract and (ii) the repeated optimal static contract. The loss is expressed as a percentage of the profit in the optimal contract in the Table 1 in Figure 5. As can be seen, the approximation by the optimal monotonic contract is quite good for all cases, with a loss of profit that is never higher than 0.06%.

Table 1

				α			
δ=0.95	**0.38**	**0.48**	**0.58**	**0.68**	**0.78**	**0.88**	**0.98**
μ_H**=0.5**	0.01	0.01	0.02	0.02	0.01	0.01	0.00
μ_M**=0.1**	11.00	9.87	8.49	6.87	4.98	2.86	0.51
μ_H**=0.5**	0.01	0.02	0.04	0.06	0.06	0.04	0.01
μ_M**=0.2**	10.70	9.62	8.32	6.77	4.96	2.87	0.51
μ_H**=0.5**	0.01	0.01	0.02	0.03	0.03	0.02	0.01
μ_M**=0.3**	10.01	9.87	8.51	6.91	5.06	3.93	0.52
μ_H**=0.3**	0.01	0.01	0.01	0.02	0.02	0.01	0.00
μ_M**=0.1**	10.75	9.73	8.45	6.91	5.08	2.95	0.53
μ_H**=0.3**	0.01	0.01	0.01	0.03	0.04	0.03	0.01
μ_M**=0.2**	10.61	9.61	8.37	6.87	5.08	3.98	0.54
μ_H**=0.3**	0.01	0.01	0.01	0.02	0.02	0.02	0.01
μ_M**=0.3**	10.41	9.42	8.20	6.72	4.97	2.92	0.53

Figure 5: Percentage loss of optimal objective (monopolist's profit) by using monotonic contracts (in bold) and repetition of the static optimum.

<table>
<tr><th></th><th>q_H</th><th>q_M</th><th>q_L</th><th>$q_H(\theta)$</th><th>$q_\theta(H)$</th><th>$q_M(M)$</th><th>$q_L(M)$</th><th>$q_M(L)$</th><th>$q_L(L)$</th></tr>
<tr><td>A1</td><td rowspan="5">θ_H</td><td rowspan="2">$\theta_M - \frac{\mu_H}{\mu_M}\Delta\theta$</td><td rowspan="2">$\theta_L - \frac{\mu_H + \mu_M}{\mu_L}\Delta\theta$</td><td rowspan="5">$\theta_H$</td><td rowspan="5">$\theta$</td><td>$\theta_M - \frac{\mu_H}{\mu_M}\frac{3\alpha - 1}{2\alpha}\Delta\theta$</td><td>$\theta_L$</td><td rowspan="2">$\theta_M$</td><td rowspan="4">$\theta_L - \frac{\mu_H + \mu_M}{\mu_L}\frac{3\alpha - 1}{2\alpha}\Delta\theta$</td></tr>
<tr><td>A2</td><td colspan="2">$\frac{2\alpha}{1+\alpha}\theta_M + \frac{1-\alpha}{1+\alpha}\theta_L - \frac{\mu_H}{\mu_M}\frac{3\alpha - 1}{1+\alpha}\Delta\theta$</td></tr>
<tr><td>B1</td><td>$\theta_M - \frac{\mu_H - \lambda_1}{\mu_M}\Delta\theta$</td><td>$\theta_L - \frac{\mu_H + \mu_M + \lambda_1}{\mu_L}\Delta\theta$</td><td>$\theta_M - \frac{\mu_H - \lambda_1}{\mu_M}\frac{3\alpha - 1}{2\alpha}\Delta\theta$</td><td>$\theta_L$</td><td>$\theta_M - \frac{\lambda_1}{\mu_L}\frac{3\alpha - 1}{1-\alpha}\Delta\theta$</td></tr>
<tr><td>B2</td><td>$\theta_M - \frac{\mu_H - \lambda_2}{\mu_M}\Delta\theta$</td><td>$\theta_L - \frac{\mu_H + \mu_M + \lambda_2}{\mu_L}\Delta\theta$</td><td colspan="2">$\frac{2\alpha}{1+\alpha}\theta_M + \frac{1-\alpha}{1+\alpha}\theta_L - \frac{\mu_H - \lambda_2}{\mu_M}\frac{3\alpha - 1}{1+\alpha}\Delta\theta$</td><td>$\theta_M - \frac{\lambda_2}{\mu_L}\frac{3\alpha - 1}{1-\alpha}\Delta\theta$</td></tr>
<tr><td>B3</td><td colspan="2">$\theta_M - \frac{\mu_H - \lambda}{\mu_M}\Delta\theta = \theta_L - \frac{\mu_H + \mu_M + \lambda}{\mu_M}\Delta\theta$</td><td colspan="4">$\frac{2\alpha}{1+\alpha}\theta_M + \frac{1-\alpha}{1+\alpha}\theta_L - \frac{\mu_H - \lambda}{\mu_M}\frac{3\alpha - 1}{2\alpha}\Delta\theta = \frac{1-\alpha}{1+\alpha}\theta_M + \frac{2\alpha}{1+\alpha}\theta_L - \frac{\mu_H + \mu_M + \lambda}{\mu_L}\frac{3\alpha - 1}{1+\alpha}\Delta\theta$</td></tr>
</table>

Table 2: The optimal contract when $N=3$ and $T=2$.