Omitted Proofs

Observational Learning in Large Anonymous Games

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Contents

1. Proof of Lemma 6

Proof. Take a limit point $x = (x_0, x_1)$ with $v_0(x_0) > 0$ and $v_1(x_1) < 0$. In the limit, agents want their action to go against the state of the world. Now the simple strategy $\widetilde{\sigma}^T$ is as follows:

$$\widetilde{\sigma}^T\left(\widetilde{\xi}, s\right) = \begin{cases} 1 & \text{if } \widetilde{\xi} = 1 \text{ and } l(s) \leq \underline{k}^T \equiv \frac{v_0(E_{\sigma^T}[X_0])}{-v_1(E_{\sigma^T}[X_1])} \frac{\mathbf{P}_{\sigma^T}\left(\widetilde{\xi}=1|\theta=0\right)}{\mathbf{P}_{\sigma^T}\left(\widetilde{\xi}=1|\theta=1\right)} \\ 1 & \text{if } \widetilde{\xi} = 0 \text{ and } l(s) \leq \bar{k}^T \equiv \frac{v_0(E_{\sigma^T}[X_0])}{-v_1(E_{\sigma^T}[X_1])} \frac{\mathbf{P}_{\sigma^T}\left(\widetilde{\xi}=0|\theta=0\right)}{\mathbf{P}_{\sigma^T}\left(\widetilde{\xi}=0|\theta=1\right)} \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

Given this simple strategy, the approximate improvement is given by:

$$\begin{aligned}
\Delta^T &= \frac{1}{2} \sum_{\theta \in \{0,1\}} \left[\mathbf{P}_{\widetilde{\sigma}^T}\left(a_i = 1 \mid \theta\right) - E_{\sigma^T}\left[X_\theta\right]\right] \cdot v_\theta\left(E_{\sigma^T}\left[X_\theta\right]\right) \\
&= \frac{1}{2} \sum_{\theta \in \{0,1\}} \left[\varepsilon + (1-2\varepsilon)\left[\pi_\theta^T G_\theta(\underline{k}^T) + (1-\pi_\theta^T) G_\theta(\bar{k}^T)\right] - E_{\sigma^T}\left[X_\theta\right]\right] \cdot v_\theta\left(E_{\sigma^T}\left[X_\theta\right]\right) \\
&= \frac{1}{2} \sum_{\theta \in \{0,1\}} v_\theta\left(E_{\sigma^T}\left[X_\theta\right]\right) \left[\varepsilon + (1-2\varepsilon)\left[\pi_\theta^T [G_\theta(\underline{k}^T) - 1] + (1-\pi_\theta^T) G_\theta(\bar{k}^T)\right]\right] \\
&\quad + v_\theta\left(E_{\sigma^T}\left[X_\theta\right]\right) \left[(1-2\varepsilon)\pi_\theta - E_{\sigma^T}\left[X_\theta\right]\right] \\
&= \frac{1}{2} \sum_{\theta \in \{0,1\}} v_\theta\left(E_{\sigma^T}\left[X_\theta\right]\right) \left[(1-2\pi_\theta^T)\varepsilon + (1-2\varepsilon)\left[\pi_\theta^T [G_\theta(\underline{k}^T) - 1] + (1-\pi_\theta^T) G_\theta(\bar{k}^T)\right]\right] \\
&\quad + \frac{1}{2} \sum_{\theta \in \{0,1\}} v_\theta\left(E_{\sigma^T}\left[X_\theta\right]\right) \left[\pi_\theta - E_{\sigma^T}\left[X_\theta\right]\right]
\end{aligned}$$

Thus,

$$\begin{aligned}
\Delta^T &= \frac{1}{2}\left[(1-2\pi_0^T)\varepsilon + (1-2\varepsilon)\left[-\pi_0^T[1 - G_0(\underline{k}^T)] + (1-\pi_0^T) G_0(\bar{k}^T)\right]\right] \cdot v_0\left(E_{\sigma^T}\left[X_0\right]\right) \\
&\quad + \frac{1}{2}\left[(1-2\pi_1^T)\varepsilon + (1-2\varepsilon)\left[-\pi_1^T[1 - G_1(\underline{k}^T)] + (1-\pi_1^T) G_1(\bar{k}^T)\right]\right] \cdot v_1\left(E_{\sigma^T}\left[X_1\right]\right) \\
&\quad + \frac{1}{2} \sum_{\theta \in \{0,1\}} v_\theta\left(E_{\sigma^T}\left[X_\theta\right]\right) \left[\pi_\theta - E_{\sigma^T}\left[X_\theta\right]\right] \\
&= \frac{1}{2}\left[(1-2\varepsilon)(1-\pi_0^T)\left[G_0(\bar{k}^T) - \frac{-v_1\left(E_{\sigma^T}\left[X_1\right]\right)}{v_0\left(E_{\sigma^T}\left[X_0\right]\right)} \frac{(1-\pi_1^T)}{(1-\pi_0^T)} G_1(\bar{k}^T)\right]\right] \cdot v_0\left(E_{\sigma^T}\left[X_0\right]\right)
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{2}\Big[(1-2\varepsilon)\pi_1^T\left[\frac{v_0\left(E_{\sigma^T}\left[X_0\right]\right)}{-v_1\left(E_{\sigma^T}\left[X_1\right]\right)}\frac{\pi_0^T}{\pi_1^T}[1-G_0(\underline{k}^T)]-[1-G_1(\underline{k}^T)]\right]\Big]\cdot v_1\left(E_{\sigma^T}\left[X_1\right]\right)\\
&\quad+\frac{1}{2}(1-2\pi_0^T)\varepsilon\cdot v_0\left(E_{\sigma^T}\left[X_0\right]\right)+\frac{1}{2}(1-2\pi_1^T)\varepsilon\cdot v_1\left(E_{\sigma^T}\left[X_1\right]\right)\\
&\quad+\frac{1}{2}\sum_{\theta\in\{0,1\}}v_\theta\left(E_{\sigma^T}\left[X_\theta\right]\right)\left[\pi_\theta-E_{\sigma^T}\left[X_\theta\right]\right]\\
&=\frac{1}{2}\Big[(1-2\pi_0^T)\varepsilon+(1-2\varepsilon)(1-\pi_0^T)\Big[G_0(\bar{k}^T)-(\bar{k}^T)^{-1}G_1(\bar{k}^T)\Big]\Big]\cdot v_0\left(E_{\sigma^T}\left[X_0\right]\right)\\
&+\frac{1}{2}\Big[(2\pi_1^T-1)\varepsilon+(1-2\varepsilon)\pi_1^T\Big[[1-G_1(\underline{k}^T)]-\underline{k}^T[1-G_0(\underline{k}^T)]\Big]\Big]\cdot\left(-v_1\left(E_{\sigma^T}\left[X_1\right]\right)\right)\\
&\quad+\frac{1}{2}\sum_{\theta\in\{0,1\}}v_\theta\left(E_{\sigma^T}\left[X_\theta\right]\right)\left[\pi_\theta-E_{\sigma^T}\left[X_\theta\right]\right]
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\lim_{T\to\infty}\Delta^T=&\frac{1}{2}\Big[(1-2x_0)\varepsilon+(1-2\varepsilon)(1-x_0)\Big[G_0(\bar{k})-(\bar{k})^{-1}G_1(\bar{k})\Big]\Big]\cdot v_0\left(x_0\right)\\
&+\frac{1}{2}\Big[(2x_1-1)\,\varepsilon+(1-2\varepsilon)x_1\left[[1-G_1(\underline{k})]-\underline{k}[1-G_0(\underline{k})]\right]\Big]\cdot\left(-v_1\left(x_1\right)\right)
\end{aligned}
$$

Again, Corollary 2 leads directly to

$$
\begin{aligned}
&\Big[(1-2\varepsilon)(1-x_0)\Big[G_0(\bar{k})-(\bar{k})^{-1}G_1(\bar{k})\Big]-\varepsilon(2x_0-1)\Big]\cdot v_0\left(x_0\right)\\
&+\Big[(1-2\varepsilon)x_1\left[[1-G_1(\underline{k})]-\underline{k}[1-G_0(\underline{k})]\right]-\varepsilon\left(1-2x_1\right)\Big]\cdot\left(-v_1\left(x_1\right)\right)\leq 0 \blacksquare
\end{aligned}
$$

2. Proof of Lemma 7

Let $\widetilde{NE}_\delta=\left\{x\in[0,1]^2:d\left(x,NE_{(\underline{l},\bar{l})}\right)\leq\delta\right\}$ be the set of all points which are δ–close to elements of $NE_{(\underline{l},\bar{l})}$ and let L^ε denote the set of limit points in a game with mistake probability $\varepsilon>0$. I show first the following Lemma, which is analogous to Lemma 11 in the paper.

LEMMA A.1. LIMIT SET APPROACHES $NE_{(\underline{l},\bar{l})}$. *For any $\delta>0$, $\exists\ \tilde{\varepsilon}>0:L^\varepsilon\subseteq\widetilde{NE}_\delta\ \forall\varepsilon<\tilde{\varepsilon}$.*

Proof. By contradiction. Assume that there exists 1) a sequence of mistake probabilities $\{\varepsilon^n\}_{n=1}^\infty$ with $\lim_{n\to\infty}\varepsilon^n=0$, and 2) an associated sequence $\{x^n\}_{n=1}^\infty$ with $x^n\in L^{\varepsilon^n}$ for all n, but 3) $x^n\notin\widetilde{NE}_\delta$ for all n. Since $x^n\in[0,1]^2$ for all n, this sequence has a convergent subsequence $\{x^{n_m}\}_{m=1}^{\infty}$ with $\lim_{m\to\infty} x^{n_m} = \bar{x} = (\bar{x}_0, \bar{x}_1)$. If $v_0(\bar{x}_0) = v_1(\bar{x}_1) = 0$, then $\bar{x} \in NE$, so for m large enough, $x^{n_m} \in \widetilde{NE}_\delta$. Then, it must be the case that $v_\theta(\bar{x}_\theta) \neq 0$ for some θ.

Assume that $v_1(\bar{x}_1) > 0$. Pick $\tilde{m}$ large enough so that $v_1(x_1^{n_m}) > 0$ for all $m > \tilde{m}$. For all m with $v_0(x_0^{n_m}) \geq 0$, Lemma 4 implies that $x^{n_m} = (1-\varepsilon^{n_m}, 1-\varepsilon^{n_m})$. So if $v_0(x_0^{n_m}) \geq 0$ infinitely often, then $\bar{x} = (1,1)$. As a result, $\bar{x} \in NE$, so for m large enough, $x^{n_m} \in \widetilde{NE}_\delta$.

Take next all m with $v_0(x_0^{n_m}) < 0$. By Lemma 5 equation (4) must hold:

$$\begin{aligned}
&\frac{-v_0(x_0^{n_m})}{2}\Bigg[\overbrace{(1-2\varepsilon^{n_m})}^{\to 1}\overbrace{x_0^{n_m}\left[G_0\left(\underline{k}^{n_m}\right) - \left(\underline{k}^{n_m}\right)^{-1} G_1\left(\underline{k}^{n_m}\right)\right]}^{\geq 0} - \overbrace{\varepsilon\left(1-2x_0\right)}^{\to 0}\Bigg] \\
&\quad + \frac{v_1(x_1^{n_m})}{2}\Bigg[\underbrace{(1-2\varepsilon^{n_m})}_{\to 1}\underbrace{\left(1-x_1^{n_m}\right)\left[\left[1-G_1\left(\overline{k}^{n_m}\right)\right] - \overline{k}^{n_m}\left[1-G_0\left(\overline{k}^{n_m}\right)\right]\right]}_{\geq 0} \\
&\quad - \underbrace{\varepsilon^{n_m}\left(2x_1^{n_m}-1\right)}_{\to 0}\Bigg] \leq 0
\end{aligned} \tag{2}$$

Proposition 3 guarantees both that $\left[\left[1-G_1\left(\overline{k}^{n_m}\right)\right] - \overline{k}^{n_m}\left[1-G_0\left(\overline{k}^{n_m}\right)\right]\right] \geq 0$ and that $\left[G_0\left(\underline{k}^{n_m}\right) - \left(\underline{k}^{n_m}\right)^{-1} G_1\left(\underline{k}^{n_m}\right)\right] \geq 0$. Then, as equation (2) shows, when $\varepsilon^{n_m} \to 0$ only non-negative terms may remain. Assume that $\overline{k} = -[v_0(\bar{x}_0)(1-\bar{x}_0)]/[v_1(\bar{x}_1)(1-\bar{x}_1)] < \bar{l}$. Then, for ε small enough, $\overline{k}^{n_m} < \bar{l}$. Proposition 3 implies that

$$\lim_{m\to\infty}\left[\left[1-G_1\left(\overline{k}^{n_m}\right)\right] - \overline{k}^{n_m}\left[1-G_0\left(\overline{k}^{n_m}\right)\right]\right] > 0.$$

To summarize, whenever $\overline{k} < \bar{l}$, equation (2) is not satisfied for small enough ε^{n_m}. It must be the case then that $\overline{k} \geq \bar{l}$. Similarly, if $\underline{k} > \underline{l}$ then

$$\lim_{m\to\infty}\left[G_0\left(\underline{k}^{n_m}\right) - \left(\underline{k}^{n_m}\right)^{-1} G_1\left(\underline{k}^{n_m}\right)\right] > 0$$

for small enough ε^{n_m}. It must be the case then that $\underline{k} \leq \underline{l}$.

Analogous arguments (using also Lemma 6) lead to the same result for the case with $v_1(\bar{x}_1) < 0$. As a result, $\bar{x} \in NE_{(\underline{l},\bar{l})}$, so for m large enough, $x^{n_m} \in \widetilde{NE}_\delta$.

The rest of the proof is identical to the proof of Proposition 2 in the paper. ■

3. Example 4. Standard Observational Learning with Mistakes

This corresponds to Example 4 in the paper. Utility is given by $u(1,X,1) = u(0,X,0) = 1$ and $u(1,X,0) = u(0,X,1) = 0$. Each agent observes his immediate predecessor: $M = 1$. The signal structure is described by $\nu_1[(0,s)] = s^2$ and $\nu_0[(0,s)] = 2s - s^2$ with $s \in (0,1)$.

Proof.

Let $\pi \equiv \Pr(\xi = 1 \mid \theta = 1)$. An agent who observes $\xi = 1$ chooses action one if and only if $\frac{\pi}{1-\pi}\frac{s}{1-s} \geq 1 \Leftrightarrow s \geq 1-\pi$. Similarly, an agent who observes $\xi = 0$ chooses action one if and only if $\frac{1-\pi}{\pi}\frac{s}{1-s} \geq 1 \Leftrightarrow s \geq \pi$. As a result, the likelihood that somebody who observes a sample (that is, not agent one) will choose the right action is given by:

$$\begin{aligned}
\Pr(a_i = 1 \mid \theta = 1, Q(i) \neq 1) &= \frac{1}{T-1}\sum_{t=2}^{T} \Pr(a_t = 1 \mid \theta = 1) \\
&= \varepsilon + (1-2\varepsilon)\left[\pi \Pr(s \geq 1-\pi) + (1-\pi)\Pr(s \geq \pi)\right] \\
&= \varepsilon + (1-2\varepsilon)\left[\pi[1-(1-\pi)^2] + (1-\pi)[1-\pi^2]\right] \\
&= \varepsilon + (1-2\varepsilon)\left[\pi - \pi(1+\pi^2-2\pi) + 1 - \pi - \pi^2 + \pi^3\right] \\
&= \varepsilon + (1-2\varepsilon)\left[\pi - \pi - \pi^3 + 2\pi^2 + 1 - \pi - \pi^2 + \pi^3\right] \\
&= \varepsilon + (1-2\varepsilon)\left(1 - \pi + \pi^2\right)
\end{aligned}$$

Reordering,

$$\Pr(a_1 = 1 \mid \theta = 1) + \sum_{t=2}^{T} \Pr(a_t = 1 \mid \theta = 1) = \sum_{t=1}^{T-1} \Pr(a_t = 1 \mid \theta = 1) + \Pr(a_T = 1 \mid \theta = 1)$$

Then,

$$\begin{aligned}
\varepsilon + (1-2\varepsilon)\left(1-\pi+\pi^2\right) - \pi - \frac{\Pr(a_T = 1 \mid \theta = 1) - \Pr(a_1 = 1 \mid \theta = 1)}{T-1} &= 0 \\
\varepsilon + (1-2\varepsilon)\left(1-\pi+\pi^2\right) - \pi - \Delta &= 0 \\
(1-2\varepsilon)\pi^2 - 2(1-\varepsilon)\pi + 1 - \varepsilon - \Delta &= 0
\end{aligned}$$

So

$$\pi = \frac{2(1-\varepsilon) \pm \sqrt{4(1-\varepsilon)^2 - 4(1-2\varepsilon)(1-\varepsilon-\Delta)}}{2(1-2\varepsilon)}$$
$$= \frac{1-\varepsilon-\sqrt{(1-\varepsilon)^2-(1-2\varepsilon)(1-\varepsilon-\Delta)}}{1-2\varepsilon}$$

As $T \to \infty$, $\Delta \to 0$, then

$$\pi \to \frac{1-\varepsilon-\sqrt{(1-\varepsilon)^2-(1-2\varepsilon)(1-\varepsilon)}}{1-2\varepsilon}$$
$$= \frac{1-\varepsilon}{1-2\varepsilon}\left(1-\sqrt{1-\frac{1-2\varepsilon}{1-\varepsilon}}\right) = \frac{1-\varepsilon}{1-2\varepsilon}\left(1-\sqrt{\frac{\varepsilon}{1-\varepsilon}}\right)$$

Also, as $T \to \infty$, $\pi - \Pr(a_i = 1 \mid \theta) \to 0$. Then, $x_1 = \lim_{T\to\infty} \Pr(a_i = 1 \mid \theta) = \frac{1-\varepsilon}{1-2\varepsilon}\left(1-\sqrt{\frac{\varepsilon}{1-\varepsilon}}\right)$. ■

4. Example 8. Multiple Equilibria in a Coordination Game

Proof. Consider a sequence of symmetric strategy profiles $\{\sigma^T(s,\xi)\}$ where $\sigma^T(s,\xi) = \sigma(s,\xi)$ does not change with T and is given by:

$$\sigma(s,\xi) = \begin{cases} 1 & \text{if } s = 1 \\ 0 & \text{if } s = 0 \\ \xi & \text{if } s = 1/2 \end{cases}$$

Let $\pi \equiv \Pr(\xi = 1 \mid \theta = 1)$. Under $\sigma(s,\xi)$, the likelihood that somebody who observes a sample (that is, not agent one) chooses action 1 is given by:

$$\Pr(a_i = 1 \mid \theta = 1, Q(i) \neq 1) = \frac{1}{T-1}\sum_{t=2}^{T} \Pr(a_t = 1 \mid \theta = 1)$$
$$= \varepsilon + (1-2\varepsilon)\left[\Pr(s=1) + \Pr(s=1/2)\pi\right]$$
$$= \varepsilon + (1-2\varepsilon)\left[(1-\gamma)/100 + 99/100\pi\right]$$

Reordering,

$$\Pr(a_1 = 1 \mid \theta = 1) + \sum_{t=2}^{T} \Pr(a_t = 1 \mid \theta = 1) = \sum_{t=1}^{T-1} \Pr(a_t = 1 \mid \theta = 1) + \Pr(a_T = 1 \mid \theta = 1)$$

Then,

$$\frac{\sum_{t=2}^{T} \Pr(a_t = 1 \mid \theta = 1)}{T-1} - \frac{\sum_{t=1}^{T-1} \Pr(a_t = 1 \mid \theta = 1)}{T-1} = \frac{\Pr(a_T = 1 \mid \theta = 1) - \Pr(a_1 = 1 \mid \theta = 1)}{T-1}$$

So,

$$\Pr(a_i = 1 \mid \theta = 1, Q(i) \neq 1) - \pi = \frac{\Pr(a_T = 1 \mid \theta = 1) - \Pr(a_1 = 1 \mid \theta = 1)}{T-1}$$

$$\varepsilon + (1 - 2\varepsilon)\left[(1-\gamma)/100 + 99/100\pi\right] - \pi = \Delta$$

Then,

$$\begin{aligned}
\varepsilon - 2\varepsilon\left[(1-\gamma)/100 + 99/100\pi\right] + (1-\gamma)/100 - 1/100\pi &= \Delta \\
\varepsilon - 2\varepsilon(1-\gamma)/100 - \varepsilon 198/100\pi + (1-\gamma)/100 - 1/100\pi &= \Delta \\
+(1-\gamma)/100 + [1 - (1-\gamma)/50]\varepsilon - (1/100 + 198/100\varepsilon)\pi &= \Delta \\
+(1-\gamma) + [100 - 2(1-\gamma)]\varepsilon - (1 + 198\varepsilon)\pi &= 100\Delta
\end{aligned}$$

Then,

$$\pi = \frac{(1-\gamma) + [100 - 2(1-\gamma)]\varepsilon - 100\Delta}{1 + 198\varepsilon}$$

Proposition 1 guarantees that as the number of agents grows large, the average action is close to its expectation. For low enough ε and large enough T, approximately $X_0|\sigma \xrightarrow{p} \gamma$ and $X_1|\sigma \xrightarrow{p} 1-\gamma$. Then,

$$\frac{\Pr(\theta = 1 \mid \xi = 1)}{\Pr(\theta = 0 \mid \xi = 1)} \approx \frac{1-\gamma}{\gamma}$$

So the sample is informative about the state of the world. To sum up, there is ε small and

T large such that σ is indeed an equilibrium.

5. Proof of Lemma 12

I illustrate first the effect of different values of $\gamma > 1$ on sampling probabilities. Figure 1 presents an agent in position 21. The black line shows the probability of observing a predecessor in position $\tau < 21$ when $\gamma = 8$. With probability higher than 0.998, the agent observes one of his three immediate predecessors. The distribution becomes flatter as γ decreases. The red line shows the distribution when $\gamma = 1.05$. In this case, the agent in position 21 observes his immediate predecessor twice as often as he observes the first agent in the sequence. As $\gamma \to 1$, sampling approaches uniform random sampling. Instead, as $\gamma \to \infty$ sampling approaches observing the immediate predecessor.

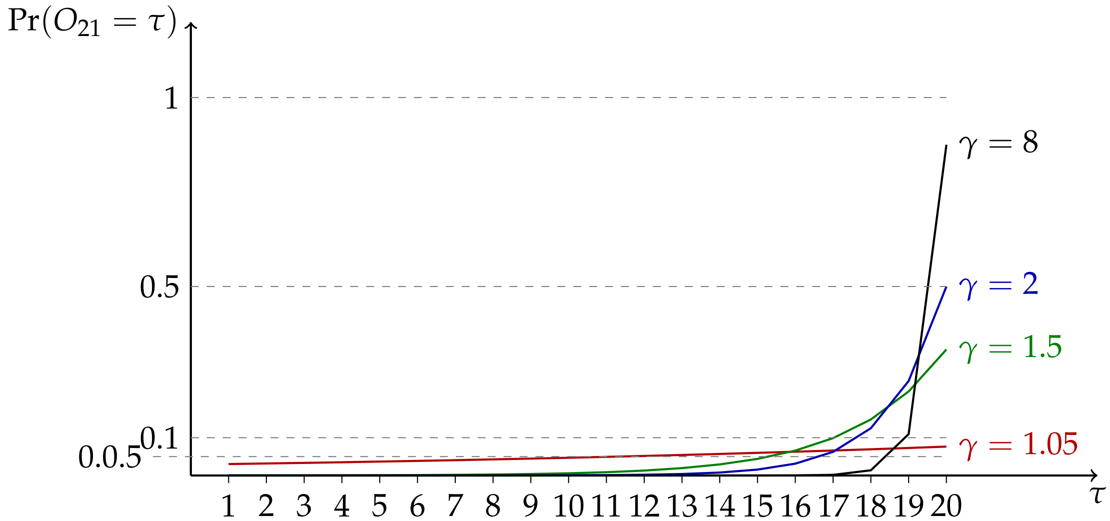

Figure 1: Probabilities of Different Predecessors Being Observed. Geometric Sampling

Next, I present the proof of Lemma 12.

Proof. A strategy σ_i induces $\rho_\theta(\xi) = \mathbf{P}_{\sigma_i}(a_i \mid \theta, \xi)$. For the rest of this section, I fix the state of the world θ and drop its index. Then, a strategy σ_i induces a vector $(\rho(\varnothing), \rho(0), \rho(1))$. Because of mistakes, $\varepsilon < \rho(\xi) < 1 - \varepsilon$ for all $\xi \in \{0, 1, \varnothing\}$.

Assume first that $\gamma > 1$. The first agent in the sequence chooses action 1 with probability $\rho(\varnothing)$. For $t \geq 2$,

$$\begin{aligned}
\mathbf{P}_\sigma(a_t = 1) &= \Pr(\xi_t = 0)\Pr(a_t = 1 \mid \xi_t = 0) + \Pr(\xi_t = 1)\Pr(a_t = 1 \mid \xi_t = 1) \\
&= \Pr(\xi_t = 0)\rho(0) + \Pr(\xi_t = 1)\rho(1) \\
&= [1 - \Pr(\xi_t = 1)]\,\rho(0) + \Pr(\xi_t = 1)\rho(1) \\
&= \rho(0) + [\rho(1) - \rho(0)]\Pr(\xi_t = 1) \\
&= \rho(0) + [\rho(1) - \rho(0)]\sum_{\tau<t}\Pr(O_t = \tau)\mathbb{1}\{a_\tau = 1\} \\
&= \rho(0) + [\rho(1) - \rho(0)]\sum_{\tau=1}^{t-1}\frac{\gamma-1}{\gamma}\frac{\gamma^\tau}{\gamma^{t-1}-1}a_\tau
\end{aligned}$$

Define the weighted sum of the past history by $p_t \equiv \sum_{\tau=1}^{t-1}\frac{\gamma-1}{\gamma}\frac{\gamma^\tau}{\gamma^{t-1}-1}a_\tau$ for $t \geq 2$. This concept plays a key role in the model:

$$\mathbf{P}_\sigma(a_t = 1) = \rho(0) + [\rho(1) - \rho(0)]\,p_t$$

This weighted sum has a recursive nature:

$$\begin{aligned}
p_{t+1} &= \sum_{\tau=1}^{t}\frac{\gamma-1}{\gamma}\frac{\gamma^\tau}{\gamma^t-1}a_\tau = \frac{\gamma^{t-1}-1}{\gamma^t-1}\left[\sum_{\tau=1}^{t-1}\frac{\gamma-1}{\gamma}\frac{\gamma^\tau}{\gamma^{t-1}-1}a_\tau\right] + \frac{\gamma-1}{\gamma}\frac{\gamma^t}{\gamma^t-1}a_t \\
&= \frac{\gamma^{t-1}-1}{\gamma^t-1}p_t + \frac{\gamma^t-\gamma^{t-1}}{\gamma^t-1}a_t
\end{aligned}$$

In expectation,

$$\begin{aligned}
E[p_{t+1} \mid I_t] &= \frac{\gamma^{t-1}-1}{\gamma^t-1}E[p_t \mid I] + \frac{\gamma^t-\gamma^{t-1}}{\gamma^t-1}E[a_t \mid I] \\
&= \frac{\gamma^{t-1}-1}{\gamma^t-1}E[p_t \mid I] + \frac{\gamma^t-\gamma^{t-1}}{\gamma^t-1}[\rho(0) + [\rho(1)-\rho(0)]E[p_t \mid I]] \\
&= \frac{\gamma^t-1+\gamma^{t-1}-\gamma^t}{\gamma^t-1}E[p_t \mid I] + \frac{\gamma^t-\gamma^{t-1}}{\gamma^t-1}[\rho(0) + [\rho(1)-\rho(0)]E[p_t \mid I]] \\
&= E[p_t \mid I] + \frac{\gamma^{t-1}-\gamma^t}{\gamma^t-1}E[p_t \mid I] + \frac{\gamma^t-\gamma^{t-1}}{\gamma^t-1}[\rho(0) + [\rho(1)-\rho(0)]E[p_t \mid I]] \\
&= E[p_t \mid I] + \frac{\gamma^t-\gamma^{t-1}}{\gamma^t-1}[\rho(0) - [1+\rho(0)-\rho(1)]E[p_t \mid I]]
\end{aligned}$$

$$= E\left[p_t \mid I\right] + \frac{\gamma^t - \gamma^{t-1}}{\gamma^t - 1}\left[1 + \rho(0) - \rho(1)\right]\left[\rho^* - E\left[p_t \mid I\right]\right]$$

Let $\rho^* \equiv \frac{\rho(0)}{1+\rho(0)-\rho(1)}$.[1] Then,

$$\begin{aligned}
E\left[p_{t+1} \mid I\right] - \rho^* &= E\left[p_t \mid I\right] - \rho^* - \frac{\gamma^t - \gamma^{t-1}}{\gamma^t - 1}\left[1 + \rho(0) - \rho(1)\right]\left[E\left[p_t \mid I\right] - \rho^*\right] \\
&= \left[1 - \frac{\gamma^t - \gamma^{t-1}}{\gamma^t - 1}\left[1 + \rho(0) - \rho(1)\right]\right]\left[E\left[p_t \mid I\right] - \rho^*\right] \\
&= \left[1 - \underbrace{\frac{\gamma - 1}{\gamma}\frac{\gamma^t}{\gamma^t - 1}}_{(*)}\underbrace{\left[1 + \rho(0) - \rho(1)\right]}_{(**)}\right]\left[E\left[p_t \mid I\right] - \rho^*\right] \qquad (3)
\end{aligned}$$

I next provide bounds for the terms $(*)$ and $(**)$ in equation (3):

$$\begin{gathered}
2\varepsilon \leq 1 + \rho(0) - \rho(1) \leq 2 - 2\varepsilon \\
\frac{\gamma - 1}{\gamma} \leq \frac{\gamma - 1}{\gamma}\frac{\gamma^t}{\gamma^t - 1} \leq 1
\end{gathered}$$

With this bounds, I can also bound the whole term in brackets in equation (3):

$$\begin{aligned}
\frac{\gamma - 1}{\gamma}2\varepsilon \leq \frac{\gamma - 1}{\gamma}\frac{\gamma^t}{\gamma^t - 1}\left[1 + \rho(0) - \rho(1)\right] \leq 2 - 2\varepsilon \\
\frac{\gamma - 1}{\gamma}2\varepsilon - 1 \leq \frac{\gamma - 1}{\gamma}\frac{\gamma^t}{\gamma^t - 1}\left[1 + \rho(0) - \rho(1)\right] - 1 \leq 1 - 2\varepsilon \\
\left|1 - \frac{\gamma - 1}{\gamma}\frac{\gamma^t}{\gamma^t - 1}\left[1 + \rho(0) - \rho(1)\right]\right| \leq 1 - \frac{\gamma - 1}{\gamma}2\varepsilon
\end{aligned}$$

This leads to a simple bound over time:

$$\begin{aligned}
\left|E\left[p_{t+n} \mid I_t\right] - \rho^*\right| &= \prod_{\tau=t}^{t+n-1}\left|1 - \frac{\gamma - 1}{\gamma}\frac{\gamma^t}{\gamma^t - 1}\left[1 + \rho(0) - \rho(1)\right]\right|\left|E\left[p_t \mid I_t\right] - \rho^*\right| \\
&\leq \left(1 - \frac{\gamma - 1}{\gamma}2\varepsilon\right)^{n-1}
\end{aligned}$$

[1]Note that $\rho(0) > \varepsilon$ and $\rho(1) < 1 - \varepsilon$, so $1 + \rho(0) - \rho(1) \geq 1 + \varepsilon - (1 - \varepsilon) = 2\varepsilon$. So $1 + \rho(0) - \rho(1) \neq 0$.

In particular,

$$|E[p_{t+n} \mid a_t = 1] - \rho^*| \leq \left(1 - \frac{\gamma - 1}{\gamma} 2\varepsilon\right)^{n-1}$$
$$|E[p_{t+n}] - \rho^*| \leq \left(1 - \frac{\gamma - 1}{\gamma} 2\varepsilon\right)^{t+n-1}$$

So finally,

$$\begin{aligned}
|E[p_{t+n} \mid I_t] - E[p_{t+n}]| &\leq |E[p_{t+n} \mid a_t = 1] - \rho^*| + |E[p_{t+n}] - \rho^*| \\
&\leq \left(1 - \frac{\gamma - 1}{\gamma} 2\varepsilon\right)^{n-1} + \left(1 - \frac{\gamma - 1}{\gamma} 2\varepsilon\right)^{t+n-1} \\
&\leq 2\left(1 - \frac{\gamma - 1}{\gamma} 2\varepsilon\right)^{n-1}
\end{aligned}$$

And turning this into probabilities,

$$\begin{aligned}
|\mathbf{P}_\sigma(a_{t+n} = 1 \mid a_t = 1) - \mathbf{P}_\sigma(a_{t+n} = 1)| &= \Big|\rho(0) + [\rho(1) - \rho(0)] E[p_{t+n} \mid a_t = 1] \\
&\quad - [\rho(0) + [\rho(1) - \rho(0)] E[p_{t+n}]]\Big| \\
&= \Big|[\rho(1) - \rho(0)]\, [E[p_{t+n} \mid a_t = 1] - E[p_{t+n}]]\Big| \\
&\leq 2\Big|[E[p_{t+n} \mid a_t = 1] - E[p_{t+n}]]\Big| \\
&\leq 4\left(1 - \frac{\gamma - 1}{\gamma} 2\varepsilon\right)^{n-1} \\
&\leq \frac{4}{1 - \frac{\gamma-1}{\gamma} 2\varepsilon} \left(1 - \frac{\gamma - 1}{\gamma} 2\varepsilon\right)^{n}
\end{aligned}$$

Next, assume that $\gamma = 1$. Then,

$$\mathbf{P}_\sigma(a_t = 1) = \rho(0) + [\rho(1) - \rho(0)] \frac{1}{t-1} \sum_{\tau=1}^{t-1} a_\tau$$

Define now $p_t \equiv \frac{1}{t-1} \sum_{\tau=1}^{t-1} a_\tau$ for $t \geq 2$, which leads to:

$$p_{t+1} = \frac{1}{t} \sum_{\tau=1}^{t} a_\tau = \frac{t-1}{t} \sum_{\tau=1}^{t-1} a_\tau + \frac{1}{t} a_t = \frac{t-1}{t} p_t + \frac{1}{t} a_t$$

In expectation,

$$\begin{aligned}
E\left[p_{t+1} \mid I_t\right] &= \frac{t-1}{t} E\left[p_t \mid I\right] + \frac{1}{t} E\left[a_t \mid I\right] \\
&= \frac{t-1}{t} E\left[p_t \mid I\right] + \frac{1}{t} E\left[\rho(0) + \left[\rho(1) - \rho(0)\right] p_t \mid I\right] \\
&= \frac{1}{t}\left[t - 1 + \rho(1) - \rho(0)\right] E\left[p_t \mid I\right] + \frac{1}{t}\rho(0)
\end{aligned}$$

So in this case:

$$\begin{aligned}
E\left[p_{t+1} \mid I_t\right] - \rho^* &= \frac{1}{t}\left[t - 1 + \rho(1) - \rho(0)\right] E\left[p_t \mid I\right] + \frac{1}{t}\rho(0) - \rho^* \\
&= \frac{1}{t}\left[\rho(0) - \left[1 + \rho(0) - \rho(1)\right] E\left[p_t \mid I\right]\right] + E\left[p_t \mid I\right] - \rho^* \\
&= \frac{1}{t}\left[1 + \rho(0) - \rho(1)\right]\left[\rho^* - E\left[p_t \mid I\right]\right] + E\left[p_t \mid I\right] - \rho^* \\
&= \left[1 - \frac{1}{t}\left[1 + \rho(0) - \rho(1)\right]\right]\left[E\left[p_t \mid I\right] - \rho^*\right]
\end{aligned}$$

Then,

$$E\left[p_{t+n} \mid I_t\right] - \rho^* = \left[E\left[p_t \mid I\right] - \rho^*\right] \prod_{\tau=0}^{n} \left[1 - \frac{1}{t+\tau}\left[1 + \rho(0) - \rho(1)\right]\right]$$

I present without proof the following remark:

Remark 1. *Let* $0 < a_n < 1$ *for all* n*. Then,* $\prod_{\tau=0}^{\infty} a_n > 0 \Leftrightarrow \sum_{\tau=0}^{\infty}(1 - a_n) < \infty$.

Then, it suffices to show that:

$$\sum_{\tau=0}^{n} \frac{1}{t+\tau}\left[1 + \rho(0) - \rho(1)\right] = \left[1 + \rho(0) - \rho(1)\right] \sum_{\tau=0}^{n} \frac{1}{t+\tau} = \infty$$

and follow the same steps as in the case with $\gamma > 1$. ■

6. Proof of Lemma 13

Proof. I show Proposition 1 by proving that $X|\sigma^T - E[X|\sigma^T]$ converges to zero in L^2 norm. The variance $V(\sigma^\tau)$ as defined by equation (6) is bounded above by

$$V(\sigma^\tau) \leq \frac{1}{T}\left(1 + 4\left(1-2\varepsilon^M\right)^{-1}\frac{\left(1-2\varepsilon^M\right)^{\frac{1}{M}}}{1-\left(1-2\varepsilon^M\right)^{\frac{1}{M}}}\right).$$

Note that $\lim_{T\to\infty} 4\left(1-2\varepsilon^{M(T)}\right)^{-1} = 4$ and $\lim_{T\to\infty}\left(1-2\varepsilon^{M(T)}\right)^{\frac{1}{M(T)}} = 1$. Then, the bound converges to zero whenever $\lim_{T\to\infty} T\left[1-\left(1-2\varepsilon^{M(T)}\right)^{\frac{1}{M(T)}}\right] = \infty$. I need to show that for any $K < \infty$, there exists a $\widetilde{T} < \infty$ such that: $T\left[1-\left(1-2\varepsilon^{M(T)}\right)^{\frac{1}{M(T)}}\right] \geq K$ for all $T \geq \widetilde{T}$. This simplifies to

$$\left(1-\frac{K}{T}\right)^{M(T)} \geq 1-2\varepsilon^{M(T)} \qquad \forall\, T \geq \widetilde{T}.$$

Since $\left(1-\frac{K}{T}\right)^{M(T)} \geq 1-\frac{KM}{T}$, it suffices to show that:

$$1-\frac{KM}{T} \geq 1-2\varepsilon^{M(T)} \quad \Leftrightarrow \quad \frac{\varepsilon^{M(T)}}{M} \geq \frac{K}{2}\frac{1}{T}.$$

$M(T)$ is $o\left(\log(T)\right)$. Then, for any constant $c \geq 0$ there is T large enough such that $M(T) \leq c\log(T)$. Pick $c = (-2\log(\varepsilon))^{-1}$. Note next that the function ε^x/x is decreasing. Then, for T large, $\frac{\varepsilon^{M(T)}}{M(T)} \geq \frac{\varepsilon^{(-2\log(\varepsilon))^{-1}\log(T)}}{(-2\log(\varepsilon))^{-1}\log(T)}$. As a result, it suffices to show that for T large enough:

$$\begin{aligned}
\frac{\varepsilon^{\left[(-2\log(\varepsilon))^{-1}\log(T)\right]}}{(-2\log(\varepsilon))^{-1}\log(T)} &\geq \frac{K}{2}\frac{1}{T}\\
\varepsilon^{(-2\log(\varepsilon))^{-1}\log(T)} &\geq \frac{K}{2}\frac{1}{T}(-2\log(\varepsilon))^{-1}\log(T)\\
T^{(-2\log(\varepsilon))^{-1}\log(\varepsilon)} &\geq \frac{1}{-4\log(\varepsilon)}K\frac{\log(T)}{T}\\
T^{-\frac{1}{2}} &\geq \frac{1}{-4\log(\varepsilon)}K\frac{\log(T)}{T}
\end{aligned}$$

$$\frac{T^{\frac{1}{2}}}{\log(T)} \geq \frac{1}{-4\log(\varepsilon)}K$$

The left hand side goes to the infinity, and the right hand side is constant. Then, there always exists a T such that this holds. This shows the first part of Proposition 1.

Next, I focus on the second part of Proposition 1. Equation (7) in the paper now becomes:

$$\Pr\left(\left|X|\sigma^T - X|\widetilde{\sigma}^T\right| \geq \frac{n}{T}\right) \leq \left[\left(1-2\varepsilon^{M(T)}\right)^{\frac{1}{M(T)}}\right]^n,$$

which holds for all n.

Let $n = \lceil (-2\log(\varepsilon))^{-1}\log(T)T^{\frac{3}{4}} \rceil$. As $\left(1-2\varepsilon^M\right)^{\frac{1}{M}} \leq 1$, then:

$$\begin{aligned}\Pr\left(\left|X|\sigma^T - X|\widetilde{\sigma}^T\right| \geq \frac{n}{T}\right) &\leq \left[\left(1-2\varepsilon^{M(T)}\right)^{\frac{1}{M(T)}}\right]^n \\ &\leq \left[\left(1-2\varepsilon^{M(T)}\right)^{\frac{1}{M(T)}}\right]^{(-2\log(\varepsilon))^{-1}\log(T)T^{\frac{3}{4}}} \\ &\leq \left(1-2\varepsilon^{(-2\log(\varepsilon))^{-1}\log(T)}\right)^{\frac{(-2\log(\varepsilon))^{-1}\log(T)T^{\frac{3}{4}}}{(-2\log(\varepsilon))^{-1}\log(T)}} \\ &= \left(1-2T^{-\frac{1}{2}}\right)^{T^{\frac{3}{4}}}\end{aligned}$$

where I have used the fact that $M(T)$ is $o(\log(T))$, so $M(T) \leq (-2\log(\varepsilon))^{-1}\log(T)$ for T large enough. Moreover, I also used the fact that $\left(1-2\varepsilon^M\right)^{\frac{1}{M}}$ is increasing in M.

I need to show that for all $b > 0$, there exists $\widetilde{T}$, such that $\Pr\left(\left|X|\sigma^T - X|\widetilde{\sigma}^T\right| \geq b\right) < b$ for all $T > \widetilde{T}$. Then, it suffices to show that $\lim_{T\to\infty}\frac{n}{T} = 0$ and $\lim_{T\to\infty}\left(1-2T^{-\frac{1}{2}}\right)^{T^{\frac{3}{4}}} = 0$.

So first, note that:

$$\frac{n}{T} \leq \frac{(-2\log(\varepsilon))^{-1}\log(T)T^{\frac{3}{4}}+1}{T} = \frac{1}{(-2\log(\varepsilon))}\frac{\log(T)}{T^{\frac{1}{4}}} + \frac{1}{T} \to 0,$$

so $\lim_{T\to\infty}\frac{n}{T} = 0$.

Second, note that $\lim_{T\to\infty}\left(1-2T^{-\frac{1}{2}}\right)^{T^{\frac{3}{4}}} = 0 \Leftrightarrow \lim_{T\to\infty} T^{\frac{3}{4}}\log\left(1-2T^{-\frac{1}{2}}\right) = -\infty$.

So using L'Hôpital's rule:

$$\lim_{T\to\infty}\frac{\log\left(1-2T^{-\frac{1}{2}}\right)}{T^{-\frac{3}{4}}}=\lim_{T\to\infty}\frac{\frac{1}{1-2T^{-\frac{1}{2}}}(-2)\left(-\frac{1}{2}\right)T^{-\frac{3}{2}}}{-\frac{3}{4}T^{-\frac{7}{4}}}=\lim_{T\to\infty}-\frac{4}{3}\frac{T^{\frac{1}{4}}}{1-2T^{-\frac{1}{2}}}=-\infty$$

This finishes the proof of the second part of Proposition 1.

Lemma 10 also needs some adjustment to allow for M to grow with T. Equation (9) from the paper becomes:

$$\begin{aligned}\pi_\theta^T - E_{\sigma^T}\left[X_\theta\right] &= \frac{1}{T}\left[\sum_{\tau=1}^{M(T)-1}\overbrace{\mathbf{P}_{\sigma^T}\left(a_\tau=1\right)}^{\leq 1}\left(\sum_{t=\tau}^{\tau+M(T)-1}\overbrace{t^{-1}-1}^{\leq 1}\right)\right.\\ &\left.\qquad-\sum_{\tau=T-M(T)+1}^{T}\underbrace{\mathbf{P}_{\sigma^T}\left(a_\tau=1\right)}_{\leq 1}\underbrace{\left(1-\frac{T-\tau}{M(T)}\right)}_{\leq 1}\right]\\ &\leq \frac{2M(T)}{T}\end{aligned}$$

Since $M(T)$ is $o(\log(T))$, then, $\pi_\theta^T - E_{\sigma^T}\to 0$. This adapts Lemma 10 to the case with growing M. The rest of Proposition 2 does not change. ■

7. Many States of the World and Many Actions

7.1 The Model

States and Actions

There are N_θ equally likely states of the world $\theta\in\Theta=\{1,2,\ldots,N_\theta\}$. Agents must choose between N_a possible actions $a\in\mathcal{A}=\{1,2,\ldots,N_a\}$. Let $X^a\equiv\frac{1}{T}\sum_{j\in\mathcal{I}}\mathbb{1}\left\{a_j=a\right\}$ denote the proportion of agents who choose action a, with realizations $x^a\in[0,1]$. The vector $X=\left(X^1,X^2,\ldots,X^{N_a}\right)$ denotes the proportion of agents choosing each action. Agent i obtains utility $u\left(a_i,X,\theta\right):\mathcal{A}\times[0,1]\times\Theta\to\mathbb{R}$, where $u\left(a_i,X,\theta\right)$ is a continuous function in X.

Private Signals

Conditional on the true state of the world, signals are i.i.d. across individuals and distributed according to F_θ. I assume that F_θ and $F_{\tilde{\theta}}$ are mutually absolutely continuous for any two $\theta, \tilde{\theta} \in \Theta$. Then, no perfectly-revealing signals occur with positive probability, and the following likelihood ratio (Radon-Nikodym derivative) exists $l_{\tilde{\theta},\theta}(s) \equiv \frac{dF_{\tilde{\theta}}}{dF_\theta}(s)$. I also define a likelihood ratio that indicates how likely one state is, relative to all other states:

$$l_\theta(s) = \left(\sum_{\tilde{\theta} \neq \theta} l_{\tilde{\theta},\theta}(s) \right)^{-1}$$

Let $G_\theta(l) \equiv \Pr\left(l_\theta(S) \leq l \mid \theta\right)$. I modify the assumption of signals being of unbounded strength as follows:

DEFINITION. SIGNAL STRENGTH. *Signal strength is* ***unbounded*** *if* $0 < G_\theta(l) < 1$ *for all likelihood ratios* $l \in (0, \infty)$, *and for all states* $\theta \in \Theta$.

Sampling, Strategies and Mistakes

The sampling rule does not change. A strategy is now a function $\sigma_i : \mathcal{S} \times \Xi \rightarrow [\varepsilon, 1 - (N_a - 1)\varepsilon]^{N_a}$ that specifies a probability vector $\sigma_i(s, \xi)$ for choosing each action given the information available. For example, $\sigma_i^a(s, \xi)$ indicates the probability of choosing action $a \in \mathcal{A}$, after receiving signal s and sample ξ.

Definition of Social Learning

I modify the definition of NE to allow for many states and actions. I say that x_θ corresponds to a Nash Equilibrium of the stage game (and denote it by $x_\theta \in NE^\theta$) whenever $u(a, x_\theta, \theta) > u(a^*, x_\theta, \theta)$ for some $a, a^* \in \mathcal{A} \Rightarrow x_\theta^{a^*} = 0$. Then, $x \in NE$ whenever $x_\theta \in NE^\theta$ for all $\theta \in \Theta$.

7.2 Results

Existence and Convergence of Average Action

The proofs of Lemma 1 and Proposition 1 extend directly to a context with many actions and many states. I need to adapt the notation. The random variable $X|\sigma$ is now a matrix. Each element $X_\theta^a|\sigma$ is a random variable that denotes the proportion of agents choosing action a in state θ. So the random variable $X|\sigma = (X_1|\sigma, X_2|\sigma, \ldots, X_{N_\theta}|\sigma)$ has realizations $x = (x_1, x_2, \ldots, x_{N_\theta})$, where each x_θ is itself a vector: $x_\theta = (x_\theta^1, x_\theta^2, \ldots, x_\theta^{N_a})$.

Utility Convergence

In what follows, I provide modified expressions for the expected utility, the utility of the expected average action, and the approximate utility of a deviation. These expressions apply to contexts with many actions and many states.

Agents' expected utility under symmetric profile σ^T is simply

$$u(\sigma^T) \equiv E_{\sigma^T}\left[u\left(a_i, X, \theta\right)\right] = \frac{1}{N_\theta}\sum_{\theta\in\Theta} E_{\sigma^T}\left[\sum_{a\in\mathcal{A}} X_\theta^a \cdot u\left(a, X_\theta, \theta\right)\right].$$

Define the *utility of the expected average action* $\bar{u}^T$ by

$$\bar{u}^T \equiv \frac{1}{N_\theta}\sum_{\theta\in\Theta}\sum_{a\in\mathcal{A}} E_{\sigma^T}\left[X_\theta^a\right]\cdot u\left(a, E_{\sigma^T}\left[X_\theta\right], \theta\right).$$

Define the *approximate utility of the deviation* $\widetilde{u}^T$ by

$$\widetilde{u}^T \equiv \frac{1}{N_\theta}\sum_{\theta\in\Theta}\sum_{a\in\mathcal{A}} \mathbf{P}_{\widetilde{\sigma}^T}\left(a_i = a \mid \theta\right)\cdot u\left(a, E_{\sigma^T}\left[X_\theta\right], \theta\right).$$

The proofs of Lemmas 2 and 3, as well as Corollary 1, extend directly to a context with many actions and many states.

Corollary 2: The Approximate Improvement

Let the *approximate improvement* Δ^T be given now by

$$\Delta^T \equiv \widetilde{u}^T - \bar{u}^T = \frac{1}{N_\theta}\sum_{\theta\in\Theta}\sum_{a\in\mathcal{A}}\left[\mathbf{P}_{\widetilde{\sigma}^T}\left(a_i = a \mid \theta\right) - E_{\sigma^T}\left[X_\theta^a\right]\right]\cdot u\left(a, E_{\sigma^T}\left[X_\theta\right], \theta\right)$$

The proof of Corollary 2 extends directly to a context with many actions and many states.

7.3 Alternative Strategy 1: Always Follow a Given Action

I present next a version of Lemma 4 that applies to many actions and many states. Let action $a^* \in \mathcal{A}$ be weakly dominant if

$$u(a^*, x_\theta, \theta) \geq u(a, x_\theta, \theta) \quad \text{for all } \ a \in \mathcal{A} \text{ and for all } \ \theta \in \Theta.$$

Let action $a^* \in \mathcal{A}$ be strictly dominant if

$$u(a^*, x_\theta, \theta) > u(a, x_\theta, \theta) \quad \text{for all } \ a \in \mathcal{A} \text{ and for all } \ \theta \in \Theta.$$

LEMMA A4. DOMINANCE. *If action $a^* \in \mathcal{A}$ is strictly dominant, then $x_\theta^{a^*} = 1 - (N_a - 1)\varepsilon$ for all $\theta \in \Theta$. Assume instead that action $a^* \in \mathcal{A}$ is weakly dominant. If there exists state $\theta \in \Theta$ with $u(a^*, x_\theta, \theta) > u(\tilde{a}, x_\theta, \theta)$, then $x_\theta^{\tilde{a}} = \varepsilon$.*

Proof. Consider the alternative strategy of always choosing action a^*. Because of mistakes this means a^* is chosen with probability $1 - (N_a - 1)\varepsilon$. Then the improvement is as follows:

$$\begin{aligned}
\Delta^T &= \frac{1}{N_\theta} \sum_{\theta \in \Theta} \left[\left[1 - (N_a - 1)\varepsilon - x_\theta^{a^*} \right] u\left(a^*, x_\theta, \theta\right) + \sum_{a \neq a^*} \left(\varepsilon - x_\theta^a\right) \cdot u\left(a, x_\theta, \theta\right) \right] \\
&= \frac{1}{N_\theta} \sum_{\theta \in \Theta} \left[\left[1 - (N_a - 1)\varepsilon - x_\theta^{a^*} \right] u\left(a^*, x_\theta, \theta\right) - \sum_{a \neq a^*} \left(x_\theta^a - \varepsilon\right) \cdot u\left(a, x_\theta, \theta\right) \right]
\end{aligned}$$

Note, that $x_\theta^a - \varepsilon \geq 0$ for all a, θ. Then,

$$\begin{aligned}
&\left[1 - (N_a - 1)\varepsilon - x_\theta^{a^*} \right] u\left(a^*, x_\theta, \theta\right) - \sum_{a \neq a^*} \left(x_\theta^a - \varepsilon\right) \cdot u\left(a, x_\theta, \theta\right) \geq \\
&\left[1 - (N_a - 1)\varepsilon - x_\theta^{a^*} \right] u\left(a^*, x_\theta, \theta\right) - \sum_{a \neq a^*} \left(x_\theta^a - \varepsilon\right) \cdot u\left(a^*, x_\theta, \theta\right) = \\
&\left[\left[1 - (N_a - 1)\varepsilon - x_\theta^{a^*} \right] - \sum_{a \neq a^*} \left(x_\theta^a - \varepsilon\right) \right] \cdot u\left(a^*, x_\theta, \theta\right) = \\
&\underbrace{\left[1 - (N_a - 1)\varepsilon - \sum_{a \in \mathcal{A}} x_\theta^a + (N_a - 1)\varepsilon \right]}_{=0} \cdot u\left(a^*, x_\theta, \theta\right) = 0
\end{aligned}$$

Recall that $\Delta^T \leq 0$, by Corollary 2. Moreover, $\Delta^T \geq 0$. Then, $\Delta^T = 0$. Also, as each term in Δ^T is weakly positive, then all terms in Δ^T must be zero:

$$\left[1 - (N_a - 1)\varepsilon - x_\theta^{a^*}\right] u\left(a^*, x_\theta, \theta\right) - \sum_{a \neq a^*} \left(x_\theta^a - \varepsilon\right) \cdot u\left(a, x_\theta, \theta\right) = 0$$

Assume next that for some action $\tilde{a} \in \mathcal{A}$ in some state $\theta \in \Theta$, $u(a^*, x_\theta, \theta) > u(\tilde{a}, x_\theta, \theta)$. Then,

$$0 = \left[1 - (N_a - 1)\varepsilon - x_\theta^{a^*}\right] u\left(a^*, x_\theta, \theta\right) - \sum_{a \neq a^*} \left(x_\theta^a - \varepsilon\right) \cdot u\left(a, x_\theta, \theta\right) \geq$$

$$\left[1 - (N_a - 1)\varepsilon - x_\theta^{a^*} - \sum_{a \neq a^*, a \neq \tilde{a}} \left(x_\theta^a - \varepsilon\right)\right] u\left(a^*, x_\theta, \theta\right) - \left(x_\theta^{\tilde{a}} - \varepsilon\right) u\left(\tilde{a}, x_\theta, \theta\right) =$$

$$\left[1 - \varepsilon - (1 - x_\theta^{\tilde{a}})\right] u\left(a^*, x_\theta, \theta\right) - \left(x_\theta^{\tilde{a}} - \varepsilon\right) u\left(\tilde{a}, x_\theta, \theta\right) =$$

$$\left(x_\theta^{\tilde{a}} - \varepsilon\right) u\left(a^*, x_\theta, \theta\right) - \left(x_\theta^{\tilde{a}} - \varepsilon\right) u\left(\tilde{a}, x_\theta, \theta\right) =$$

$$\left(x_\theta^{\tilde{a}} - \varepsilon\right) \left[u\left(a^*, x_\theta, \theta\right) - u\left(\tilde{a}, x_\theta, \theta\right)\right]$$

To sum up,

$$\left(x_\theta^{\tilde{a}} - \varepsilon\right) \overbrace{\left[u\left(a^*, x_\theta, \theta\right) - u\left(\tilde{a}, x_\theta, \theta\right)\right]}^{>0} \leq 0$$

So $x_\theta^{\tilde{a}} = \varepsilon$. Similarly, if $u(a^*, x_\theta, \theta) > u(a, x_\theta, \theta)$ for all $a \in \mathcal{A}$ and for all $\theta \in \Theta$, then $x_\theta^{a^*} = 1 - (N_a - 1)\varepsilon$. ■

7.4 Alternative Strategy 2: Improve Upon a Sampled Agent

Consider a possible limit point $x = \left(x_1, x_2, \ldots, x_{N_\theta}\right)$. Assume that action $\tilde{a}$ is not optimal in state θ^*: $u(a^*, x_{\theta^*}, \theta^*) > u(\tilde{a}, x_{\theta^*}, \theta^*)$, but it is still played in the limit: $x_{\theta^*}^{\tilde{a}} > \varepsilon$. As in the case with two states, let $\widetilde{\xi}$ denote the action of one individual selected at random from the sample. Consider an alternative simple strategy $\widetilde{\sigma}$, that makes the agent choose the following action:

$$a_i(\widetilde{\xi}, s) = \begin{cases} a^* & \text{if } \widetilde{\xi} = \tilde{a} \text{ and } l_{\theta^*}(s) \geq k^T \equiv \frac{-\bar{u}}{u\left(a^*, E_{\sigma^T}[X_{\theta^*}], \theta^*\right) - u\left(\tilde{a}, E_{\sigma^T}[X_{\theta^*}], \theta^*\right)} \frac{1}{\mathbf{P}_{\sigma^T}\left(\widetilde{\xi} = \tilde{a} | \theta = \theta^*\right)} \\ \widetilde{\xi} & \text{otherwise} \end{cases}$$

I provide next a version of Lemma 5 in the paper that applies to many actions and many states.

LEMMA A5. IMPROVEMENT PRINCIPLE. *Take any limit point $x \in L$ with $u(a^*, x_{\theta^*}, \theta^*) > u(\tilde{a}, x_{\theta^*}, \theta^*)$. Then,*

$$\widetilde{\Delta}(\varepsilon) + \frac{1 - (N_a - 1)\varepsilon}{N_\theta} \left[x_{\theta^*}^{\tilde{a}} \cdot [u(a^*, x_{\theta^*}, \theta^*) - u(\tilde{a}, x_{\theta^*}, \theta^*)] \right] \\ \times \left[\left[1 - G_{\theta^*}\left(\bar{k}\right) \right] - \bar{k} \left[1 - \widetilde{G}_{\theta^*}\left(\bar{k}\right) \right] \right] \leq 0 \tag{4}$$

with

$$\bar{k} = -\bar{u} \left[\left(u\left(a^*, x_{\theta^*}, \theta^*\right) - u\left(\tilde{a}, x_{\theta^*}, \theta^*\right) \right) x_{\theta^*}^{\tilde{a}} \right]^{-1} \quad \textit{and}$$

$$\widetilde{\Delta}(\varepsilon) = \frac{\varepsilon}{N_\theta} \left[\sum_{\theta \in \Theta} \sum_{a \in \mathcal{A}} \left[1 - (N_a - 1) x_\theta^a \right] u(a, x_\theta, \theta) \right].$$

See section 7.5 for the proof.

The term $\left[\left[1 - G_{\theta^*}\left(\bar{k}\right) \right] - \bar{k} \left[1 - \widetilde{G}_{\theta^*}\left(\bar{k}\right) \right] \right] \geq 0$ in equation (4) decreases in $\bar{k}$ (as shown later in Proposition A3). Moreover, with signals of unbounded strength, this term is strictly positive. Then, whenever $x_\theta^{\tilde{a}} > 0$, there is potential for improvement. The existence of mistakes may present such an improvement. Note however, that $\lim_{\varepsilon \to 0} \widetilde{\Delta}(\varepsilon) = 0$. Then, when mistakes are unlikely the potential for improvement dominates in equation (4).

7.5 Proof of Lemma A5

Proof. Let $\rho_\theta^T(a|\tilde{a}) \equiv \mathbf{P}_{\sigma^T}\left(a_i = a|\theta, \tilde{\xi} = \tilde{a}\right)$. In general, the improvement is given by:

$$\begin{aligned}
\Delta^T &= \frac{1}{N_\theta}\sum_{\theta\in\Theta}\sum_{a\in\mathcal{A}}\Bigg[\varepsilon + [1-(N_a-1)\varepsilon]\sum_{a'\in\mathcal{A}}\rho_\theta(a|a')\mathbf{P}_{\sigma^T}\left(\tilde{\xi}=a'|\theta\right) \\
&\quad - E_{\sigma^T}\left[X_\theta^a\right]\Bigg]u(a, E_{\sigma^T}\left[X_\theta\right], \theta) \\
&= \left[\frac{\varepsilon}{N_\theta}\sum_{\theta\in\Theta}\sum_{a\in\mathcal{A}}u(a, E_{\sigma^T}\left[X_\theta\right], \theta)\right] \\
&\quad + \frac{1-(N_a-1)\varepsilon}{N_\theta}\left[\sum_{\theta\in\Theta}\sum_{a\in\mathcal{A}}\sum_{a'\in\mathcal{A}}\rho_\theta(a|a')\mathbf{P}_{\sigma^T}\left(\tilde{\xi}=a'|\theta\right)u(a, E_{\sigma^T}\left[X_\theta\right], \theta)\right] \\
&\quad - \frac{1-(N_a-1)\varepsilon}{N_\theta}\left[\sum_{\theta\in\Theta}\sum_{a\in\mathcal{A}}E_{\sigma^T}\left[X_\theta^a\right]u(a, E_{\sigma^T}\left[X_\theta\right], \theta)\right] \\
&\quad - \frac{(N_a-1)\varepsilon}{N_\theta}\left[\sum_{\theta\in\Theta}\sum_{a\in\mathcal{A}}E_{\sigma^T}\left[X_\theta^a\right]u(a, E_{\sigma^T}\left[X_\theta\right], \theta)\right]
\end{aligned}$$

Let

$$\begin{aligned}
\tilde{\Delta}^T(\varepsilon) &\equiv \frac{\varepsilon}{N_\theta}\left[\sum_{\theta\in\Theta}\sum_{a\in\mathcal{A}}u(a, E_{\sigma^T}\left[X_\theta\right], \theta) - (N_a-1)\left[\sum_{\theta\in\Theta}\sum_{a\in\mathcal{A}}E_{\sigma^T}\left[X_\theta^a\right]u(a, E_{\sigma^T}\left[X_\theta\right], \theta)\right]\right] \\
&= \frac{\varepsilon}{N_\theta}\left[\sum_{\theta\in\Theta}\sum_{a\in\mathcal{A}}\left[1-(N_a-1)E_{\sigma^T}\left[X_\theta^a\right]\right]u(a, E_{\sigma^T}\left[X_\theta\right], \theta)\right]
\end{aligned}$$

and:

$$J(\varepsilon) \equiv \frac{1-(N_a-1)\varepsilon}{N_\theta}$$

Then,

$$\Delta^T = \tilde{\Delta}^T(\varepsilon) + J(\varepsilon)\sum_{\theta\in\Theta}\sum_{a\in\mathcal{A}}\left[\sum_{a'\in\mathcal{A}}\rho_\theta(a|a')\mathbf{P}_{\sigma^T}\left(\tilde{\xi}=a'|\theta\right) - E_{\sigma^T}\left[X_\theta^a\right]\right]u(a, E_{\sigma^T}\left[X_\theta\right], \theta) \quad (5)$$

But

$$= \frac{1}{N_\theta}\sum_{\theta\in\Theta}\sum_{a\in\mathcal{A}}\left[\sum_{a'\in\mathcal{A}}\rho_\theta(a|a')\mathbf{P}_{\sigma^T}\left(\tilde{\xi}=a'|\theta\right) - E_{\sigma^T}\left[X_\theta^a\right]\right]u(a, E_{\sigma^T}\left[X_\theta\right], \theta)$$

$$
\begin{aligned}
&= \frac{1}{N_\theta}\left[\sum_{\theta\in\Theta}\sum_{a\in\mathcal{A}}\sum_{a'\in\mathcal{A}}\rho_\theta(a|a')\mathbf{P}_{\sigma^T}\left(\widetilde{\xi}=a'|\theta\right)u(a,E_{\sigma^T}\left[X_\theta\right],\theta)\right]\\
&\quad -\frac{1}{N_\theta}\left[\sum_{\theta\in\Theta}\sum_{a\in\mathcal{A}}E_{\sigma^T}\left[X_\theta^a\right]u(a,E_{\sigma^T}\left[X_\theta\right],\theta)\right]\\
&= \frac{1}{N_\theta}\left[\sum_{\theta\in\Theta}\sum_{a\in\mathcal{A}}\sum_{a'\in\mathcal{A}}\rho_\theta(a|a')\mathbf{P}_{\sigma^T}\left(\widetilde{\xi}=a'|\theta\right)u(a,E_{\sigma^T}\left[X_\theta\right],\theta)\right]\\
&\quad -\frac{1}{N_\theta}\left[\sum_{\theta\in\Theta}\sum_{a'\in\mathcal{A}}E_{\sigma^T}\left[X_\theta^{a'}\right]u(a',E_{\sigma^T}\left[X_\theta\right],\theta)\right]\\
&= \frac{1}{N_\theta}\sum_{\theta\in\Theta}\sum_{a'\in\mathcal{A}}\left[\sum_{a\in\mathcal{A}}\rho_\theta(a|a')\mathbf{P}_{\sigma^T}\left(\widetilde{\xi}=a'|\theta\right)u(a,E_{\sigma^T}\left[X_\theta\right],\theta)\right.\\
&\quad \left.-E_{\sigma^T}\left[X_\theta^{a'}\right]u(a',E_{\sigma^T}\left[X_\theta\right],\theta)\right]
\end{aligned}
$$

As a result, the improvement in equation (5) can be expressed as:

$$
\begin{aligned}
\Delta^T = \widetilde{\Delta}^T(\varepsilon) + J(\varepsilon)\sum_{\theta\in\Theta}\sum_{a'\in\mathcal{A}}&\left[\sum_{a\in\mathcal{A}}\rho_\theta(a|a')\mathbf{P}_{\sigma^T}\left(\widetilde{\xi}=a'|\theta\right)u(a,E_{\sigma^T}\left[X_\theta\right],\theta)\right.\\
&\left.-E_{\sigma^T}\left[X_\theta^{a'}\right]u(a',E_{\sigma^T}\left[X_\theta\right],\theta)\right]
\end{aligned}
$$

In particular, for the simple strategy $\widetilde{\sigma}$,

$$
\begin{aligned}
\Delta^T &= \widetilde{\Delta}^T(\varepsilon) + J(\varepsilon)\sum_{\theta\in\Theta}\left[\rho_\theta(a^*|\tilde{a})\mathbf{P}_{\sigma^T}\left(\widetilde{\xi}=\tilde{a}|\theta\right)u(a^*,E_{\sigma^T}\left[X_\theta\right],\theta)\right.\\
&\quad\left.+\left[1-\rho_\theta(a^*|\tilde{a})\right]\mathbf{P}_{\sigma^T}\left(\widetilde{\xi}=\tilde{a}|\theta\right)u(\tilde{a},E_{\sigma^T}\left[X_\theta\right],\theta)-E_{\sigma^T}\left[X_\theta^{\tilde{a}}\right]u(\tilde{a},E_{\sigma^T}\left[X_\theta\right],\theta)\right]\\
&= \widetilde{\Delta}^T(\varepsilon) + J(\varepsilon)\sum_{\theta\in\Theta}\left[\rho_\theta(a^*|\tilde{a})\mathbf{P}_{\sigma^T}\left(\widetilde{\xi}=\tilde{a}|\theta\right)\left[u(a^*,E_{\sigma^T}\left[X_\theta\right],\theta)-u(\tilde{a},E_{\sigma^T}\left[X_\theta\right],\theta)\right]\right.\\
&\quad\left.+\left[\mathbf{P}_{\sigma^T}\left(\widetilde{\xi}=\tilde{a}|\theta\right)-E_{\sigma^T}\left[X_\theta^{\tilde{a}}\right]\right]u(\tilde{a},E_{\sigma^T}\left[X_\theta\right],\theta)\right]
\end{aligned}
$$

Let

$$
\widetilde{\widetilde{\Delta}}^T \equiv J(\varepsilon)\sum_{\theta\in\Theta}\left[\mathbf{P}_{\sigma^T}\left(\widetilde{\xi}=\tilde{a}|\theta\right)-E_{\sigma^T}\left[X_\theta^{\tilde{a}}\right]\right]u(\tilde{a},E_{\sigma^T}\left[X_\theta\right],\theta)\Bigg]
$$

Then,

$$
\begin{aligned}
\Delta^T &= \widetilde{\Delta}^T(\varepsilon) + \widetilde{\widetilde{\Delta}}^T \\
&+ J(\varepsilon) \sum_{\theta\in\Theta} \left[\rho_\theta(a^*|\tilde{a}) \mathbf{P}_{\sigma^T}\left(\widetilde{\xi} = \tilde{a}|\theta\right) \left[u(a^*, E_{\sigma^T}\left[X_\theta\right], \theta) - u(\tilde{a}, E_{\sigma^T}\left[X_\theta\right], \theta)\right]\right] \\
&= \widetilde{\Delta}^T(\varepsilon) + \widetilde{\widetilde{\Delta}}^T \\
&+ J(\varepsilon) \Bigg[\sum_{\theta\in\Theta, \theta\neq\theta^*} \left[\rho_\theta(a^*|\tilde{a}) \mathbf{P}_{\sigma^T}\left(\widetilde{\xi} = \tilde{a}|\theta\right) \left[u(a^*, E_{\sigma^T}\left[X_\theta\right], \theta) - u(\tilde{a}, E_{\sigma^T}\left[X_\theta\right], \theta)\right]\right] \\
&\qquad + \rho_{\theta^*}(a^*|\tilde{a}) \mathbf{P}_{\sigma^T}\left(\widetilde{\xi} = \tilde{a}|\theta^*\right) \left[u(a^*, E_{\sigma^T}\left[X_{\theta^*}\right], \theta^*) - u(\tilde{a}, E_{\sigma^T}\left[X_{\theta^*}\right], \theta^*)\right] \Bigg]
\end{aligned}
$$

Now, let

$$
-\bar{u} \equiv \min_{a\in\mathcal{A}, a'\in\mathcal{A}, \theta\in\Theta, x_\theta\in[0,1]^{N_a}} \left[u(a, x_\theta, \theta) - u(a', x_\theta, \theta)\right]
$$

This minimum exists since there is a finite number of states and actions, and the utility functions are continuous in X. Then,

$$
\left[u(a^*, E_{\sigma^T}\left[X_{\theta^*}\right], \theta^*) - u(\tilde{a}, E_{\sigma^T}\left[X_{\theta^*}\right], \theta^*)\right] \geq -\bar{u}
$$

Then,

$$
\begin{aligned}
\Delta^T &\geq \widetilde{\Delta}^T(\varepsilon) + \widetilde{\widetilde{\Delta}}^T + J(\varepsilon) \mathbf{P}_{\sigma^T}\left(\widetilde{\xi} = \tilde{a}|\theta^*\right) \left[u(a^*, E_{\sigma^T}\left[X_{\theta^*}\right], \theta^*) - u(\tilde{a}, E_{\sigma^T}\left[X_{\theta^*}\right], \theta^*)\right] \\
&\quad \times \left[-\frac{\bar{u} \sum_{\theta\in\Theta, \theta\neq\theta^*} \left[\rho_\theta(a^*|\tilde{a}) \mathbf{P}_{\sigma^T}\left(\widetilde{\xi} = \tilde{a}|\theta\right)\right]}{\mathbf{P}_{\sigma^T}\left(\widetilde{\xi} = \tilde{a}|\theta^*\right) \left[u(a^*, E_{\sigma^T}\left[X_{\theta^*}\right], \theta^*) - u(\tilde{a}, E_{\sigma^T}\left[X_{\theta^*}\right], \theta^*)\right]} + \rho_{\theta^*}(a^*|\tilde{a}) \right] \\
&= \widetilde{\Delta}^T(\varepsilon) + \widetilde{\widetilde{\Delta}}^T + J(\varepsilon) \mathbf{P}_{\sigma^T}\left(\widetilde{\xi} = \tilde{a}|\theta^*\right) \left[u(a^*, E_{\sigma^T}\left[X_{\theta^*}\right], \theta^*) - u(\tilde{a}, E_{\sigma^T}\left[X_{\theta^*}\right], \theta^*)\right] \\
&\quad \times \left[\rho_{\theta^*}(a^*|\tilde{a}) - k^T \sum_{\theta\in\Theta, \theta\neq\theta^*} \left[\rho_\theta(a^*|\tilde{a}) \mathbf{P}_{\sigma^T}\left(\widetilde{\xi} = \tilde{a}|\theta\right)\right]\right] \\
&\geq \widetilde{\Delta}^T(\varepsilon) + \widetilde{\widetilde{\Delta}}^T + J(\varepsilon) \mathbf{P}_{\sigma^T}\left(\widetilde{\xi} = \tilde{a}|\theta^*\right) \left[u(a^*, E_{\sigma^T}\left[X_{\theta^*}\right], \theta^*) - u(\tilde{a}, E_{\sigma^T}\left[X_{\theta^*}\right], \theta^*)\right] \\
&\quad \times \left[\rho_{\theta^*}(a^*|\tilde{a}) - k^T \sum_{\theta\in\Theta, \theta\neq\theta^*} \rho_\theta(a^*|\tilde{a}) \right]
\end{aligned}
$$

$$= \Delta_*^T \equiv \widetilde{\Delta}^T(\varepsilon) + \widetilde{\widetilde{\Delta}}^T + J(\varepsilon)\mathbf{P}_{\sigma^T}\left(\widetilde{\xi} = \tilde{a}|\theta^*\right)\left[u(a^*, E_{\sigma^T}\left[X_{\theta^*}\right], \theta^*) - u(\tilde{a}, E_{\sigma^T}\left[X_{\theta^*}\right], \theta^*)\right]$$
$$\times \left[\left[1 - G_{\theta^*}\left(k^T\right)\right] - k^T\left[1 - \widetilde{G}_{\theta^*}\left(k^T\right)\right]\right]$$

Note that $\lim_{T\to\infty} \widetilde{\widetilde{\Delta}}^T = 0$. Let $\widetilde{\Delta}(\varepsilon) \equiv \lim_{T\to\infty} \widetilde{\Delta}^T(\varepsilon)$. Finally, note that, as in proof in the paper, $\lim_{T\to\infty} k^T = \bar{k}$. Then,

$$\lim_{T\to\infty} \Delta_*^T = \widetilde{\Delta}(\varepsilon) + \frac{1-(N_a-1)\varepsilon}{N_\theta}\left[x_{\theta^*}^{\tilde{a}}\left[u(a^*, x_{\theta^*}, \theta^*) - u(\tilde{a}, x_{\theta^*}, \theta^*)\right]\right]$$
$$\times \left[\left[1 - G_{\theta^*}\left(\bar{k}\right)\right] - \bar{k}\left[1 - \widetilde{G}_{\theta^*}\left(\bar{k}\right)\right]\right] \quad \blacksquare$$

7.6 Strategic Learning

Lemmas A4 and A5 are the main building blocks to show how Proposition 2 also applies to a context with many states and many actions. I present this formally.

PROPOSITION A2. STRATEGIC LEARNING. *Assume signals are of unbounded strength. Then there is strategic learning.*

The proof of Proposition A3 requires modifying Proposition 3 and Lemma 11 in the paper. With these results in hand, the proof of Proposition A2 is analogous to the proof of Proposition 2 in the main text. Lemma 11 extends directly to a context with many actions and many states. I present next a version of Proposition 3 in the paper that applies to many states of the world.

PROPOSITION A3. *For all $l \in (\underline{l}, \bar{l})$, $G_\theta(l)$ satisfies:*

$$l > \frac{G_\theta(l)}{\widetilde{G}_\theta(l)} \quad \textit{and} \quad l < \frac{1 - G_1(l)}{1 - G_0(l)} \tag{6}$$

Moreover, if $k' \geq k$ then,

$$\left[1 - G_1(k)\right] - k\left[1 - G_0(k)\right] \geq \left[1 - G_1\left(k'\right)\right] - k'\left[1 - G_0\left(k'\right)\right] \tag{7}$$

Proof. The proof follows that from Proposition 11 in Monzón and Rapp [2014], but here the likelihood ratio G_θ indicates how likely state θ, relative to all other states. Note first that

$$\begin{aligned} l_\theta(s)^{-1} &= \sum_{\tilde{\theta}\neq\theta} l_{\tilde{\theta},\theta}(s) = \sum_{\tilde{\theta}\neq\theta} \frac{dF_{\tilde{\theta}}}{dF_\theta}(s) \\ dF_\theta(s) l_\theta(s)^{-1} &= \sum_{\tilde{\theta}\neq\theta} dF_{\tilde{\theta}}(s) \\ dF_\theta(s) &= l_\theta(s) \sum_{\tilde{\theta}\neq\theta} dF_{\tilde{\theta}}(s) \end{aligned}$$

Recall that $\widetilde{G}_\theta(L) \equiv \sum_{\tilde{\theta}\neq\theta} \Pr(l_\theta(s) \leq L \mid \tilde{\theta})$.

$$\begin{aligned} G_\theta(L) &= \int_{\{S\in\mathcal{S}:l_\theta(s)\leq L\}} dF_\theta = \int_{\{S\in\mathcal{S}:l_\theta(s)\leq L\}} l_\theta(s) \sum_{\tilde{\theta}\neq\theta} dF_{\tilde{\theta}}(s) \\ &< \int_{\{S\in\mathcal{S}:l_\theta(s)\leq L\}} L \sum_{\tilde{\theta}\neq\theta} dF_{\tilde{\theta}}(s) = L \sum_{\tilde{\theta}\neq\theta} \int_{\{S\in\mathcal{S}:l_\theta(s)\leq L\}} dF_{\tilde{\theta}}(s) \\ &= L\widetilde{G}_\theta(L) \end{aligned}$$

Similarly,

$$\begin{aligned} 1 - G_\theta(L) &= \int_{\{S\in\mathcal{S}:l_\theta(s)> L\}} dF_\theta = \int_{\{S\in\mathcal{S}:l_\theta(s)> L\}} l_\theta(s) \sum_{\tilde{\theta}\neq\theta} dF_{\tilde{\theta}}(s) \\ &> \int_{\{S\in\mathcal{S}:l_\theta(s)> L\}} L \sum_{\tilde{\theta}\neq\theta} dF_{\tilde{\theta}}(s) = L \sum_{\tilde{\theta}\neq\theta} \int_{\{S\in\mathcal{S}:l_\theta(s)> L\}} dF_{\tilde{\theta}}(s) \\ &= L\left[1 - \widetilde{G}_\theta(L)\right] \end{aligned}$$

This shows that equation (6) holds. I mover next to the second part. Take $k' > k$.

$$\begin{aligned} \left[1 - G_\theta\left(k\right)\right] - \left[1 - G_\theta\left(k'\right)\right] &= G_\theta\left(k'\right) - G_\theta\left(k\right) = \int_{S\in\mathcal{S}:k\leq l_\theta(S)\leq k'} dF_\theta \\ &= \int_{S\in\mathcal{S}:k\leq l_\theta(S)\leq k'} l_\theta(S) \sum_{\tilde{\theta}\neq\theta} dF_{\tilde{\theta}} \\ &\geq k \int_{S\in\mathcal{S}:k\leq l_\theta(S)\leq k'} \sum_{\tilde{\theta}\neq\theta} dF_{\tilde{\theta}} = k\left[\widetilde{G}_\theta\left(k'\right) - \widetilde{G}_\theta\left(k\right)\right] \\ &= k\left[1 - \widetilde{G}_\theta\left(k\right)\right] - k\left[1 - \widetilde{G}_\theta\left(k'\right)\right] \end{aligned}$$

$$\geq k\left[1-\widetilde{G}_{\theta}\left(k\right)\right]-k'\left[1-\widetilde{G}_{\theta}\left(k'\right)\right]$$

Then,

$$\left[1-G_{\theta}\left(k\right)\right]-\left[1-G_{\theta}\left(k'\right)\right]\geq k\left[1-\widetilde{G}_{\theta}\left(k\right)\right]-k'\left[1-\widetilde{G}_{\theta}\left(k'\right)\right]$$
$$\left[1-G_{\theta}\left(k\right)\right]-k\left[1-\widetilde{G}_{\theta}\left(k\right)\right]\geq\left[1-G_{\theta}\left(k'\right)\right]-k'\left[1-\widetilde{G}_{\theta}\left(k'\right)\right]$$

This shows that equation (7) holds. ■

References

MONZÓN, I. AND M. RAPP (2014): "Observational Learning with Position Uncertainty," *Journal of Economic Theory*, 154, 375–402.