S.1. Proof of Lemma 6

Take a limit point \( x = (x_0, x_1) \) with \( v_0(x_0) > 0 \) and \( v_1(x_1) < 0 \). In the limit, agents want their action to go against the state of the world. Now the simple strategy \( \tilde{\sigma}^T \) is

\[
\tilde{\sigma}^T(\tilde{\xi}, s) = \begin{cases} 
1 & \text{if } \tilde{\xi} = 1 \text{ and } l(s) \leq k^T = v_0(E_{\sigma^T}[X_0]) P_{\sigma^T}(\tilde{\xi} = 1 | \theta = 0) \\
1 & \text{if } \tilde{\xi} = 0 \text{ and } l(s) \leq k^T = v_0(E_{\sigma^T}[X_0]) P_{\sigma^T}(\tilde{\xi} = 0 | \theta = 0) \\
0 & \text{otherwise.}
\end{cases}
\]

Given this simple strategy, the approximate improvement is given by

\[
\Delta^T = \frac{1}{2} \sum_{\theta \in \{0, 1\}} [P_{\sigma^T}\{a_i = 1 | \theta\} - E_{\sigma^T}[X_\theta]] 
\cdot v_\theta(E_{\sigma^T}[X_\theta])
\]

\[
= \frac{1}{2} \sum_{\theta \in \{0, 1\}} [\epsilon + (1 - 2\epsilon)\pi_\theta \{ G_\theta(k^T) + (1 - \pi_\theta) G_\theta(k^T) \}] - E_{\sigma^T}[X_\theta] 
\cdot v_\theta(E_{\sigma^T}[X_\theta])
\]

\[
= \frac{1}{2} \sum_{\theta \in \{0, 1\}} v_\theta(E_{\sigma^T}[X_\theta])\left[\epsilon + (1 - 2\epsilon)\{ \pi_\theta \{ G_\theta(k^T) - 1 \} + (1 - \pi_\theta) G_\theta(k^T) \}\right]
\]

\[
+ v_\theta(E_{\sigma^T}[X_\theta])\left[1 - 2\epsilon\pi_\theta - E_{\sigma^T}[X_\theta] \right]
\]

\[
= \frac{1}{2} \sum_{\theta \in \{0, 1\}} v_\theta(E_{\sigma^T}[X_\theta])\left[(1 - 2\pi_\theta)\epsilon + (1 - 2\epsilon)\{ \pi_\theta \{ G_\theta(k^T) - 1 \} + (1 - \pi_\theta) G_\theta(k^T) \}\right]
\]

\[
+ \frac{1}{2} \sum_{\theta \in \{0, 1\}} v_\theta(E_{\sigma^T}[X_\theta])\left[ \pi_\theta - E_{\sigma^T}[X_\theta] \right].
\]
Thus,

\[ \Delta^T = \frac{1}{2} \left[ (1 - 2\pi_0^T)e + (1 - 2\varepsilon)[-\pi_0^T - \pi_0^T] + (1 - \pi_0^T)G_0(\tilde{k}^T) \right] \cdot v_0(E_{\sigma^T}[X_0]) \\
+ \frac{1}{2} \left[ (1 - 2\pi_1^T)e + (1 - 2\varepsilon)[-\pi_1^T - \pi_1^T] + (1 - \pi_1^T)G_1(\tilde{k}^T) \right] \cdot v_1(E_{\sigma^T}[X_1]) \\
+ \frac{1}{2} \sum_{\theta \in \{0, 1\}} v_\theta(E_{\sigma^T}[X_\theta]) [\pi_\theta - E_{\sigma^T}[X_\theta]] \\
= \frac{1}{2} \left[ (1 - 2\varepsilon)(1 - \pi_0^T) \left[ G_0(\tilde{k}^T) - \frac{\pi_0^T}{v_0(E_{\sigma^T}[X_0])} \pi_0^T \left[ 1 - G_0(\tilde{k}^T) \right] \right] \cdot v_0(E_{\sigma^T}[X_0]) \\
+ \frac{1}{2} \left[ (1 - 2\varepsilon)\pi_1^T \left[ \frac{v_0(E_{\sigma^T}[X_0])}{-v_1(E_{\sigma^T}[X_1])} \pi_1^T \left[ 1 - G_0(\tilde{k}^T) \right] \right] \cdot v_1(E_{\sigma^T}[X_1]) \\
+ \frac{1}{2} (1 - 2\pi_0^T)e \cdot v_\theta(E_{\sigma^T}[X_\theta]) + \frac{1}{2} (1 - 2\pi_1^T)e \cdot v_1(E_{\sigma^T}[X_1]) \\
+ \frac{1}{2} \sum_{\theta \in \{0, 1\}} v_\theta(E_{\sigma^T}[X_\theta]) [\pi_\theta - E_{\sigma^T}[X_\theta]] \\
= \frac{1}{2} \left[ (1 - 2\varepsilon)(1 - \pi_0^T) \left[ G_0(\tilde{k}^T) - \pi_0^T \left[ 1 - G_0(\tilde{k}^T) \right] \right] \cdot v_0(E_{\sigma^T}[X_0]) \\
+ \frac{1}{2} \left[ (2\pi_1^T - 1)e + (1 - 2\varepsilon)\pi_1^T \left[ 1 - G_1(\tilde{k}^T) - \tilde{k}^T \left[ 1 - G_0(\tilde{k}^T) \right] \right] \right] \cdot v_0(E_{\sigma^T}[X_0]) \\
+ \frac{1}{2} \sum_{\theta \in \{0, 1\}} v_\theta(E_{\sigma^T}[X_\theta]) [\pi_\theta - E_{\sigma^T}[X_\theta]]. \]

Thus,

\[ \lim_{T \to \infty} \Delta^T = \frac{1}{2} \left[ (1 - 2x_0)e + (1 - 2\varepsilon)(1 - x_0) \left[ G_0(\tilde{k}) - \tilde{k}^{-1}G_1(\tilde{k}) \right] \right] \cdot v_0(x_0) \\
+ \frac{1}{2} \left[ (2x_1 - 1)e + (1 - 2\varepsilon)x_1 \left[ 1 - G_1(\tilde{k}) - \tilde{k} \left[ 1 - G_0(\tilde{k}) \right] \right] \right] \cdot (v_1(x_1)). \]

Again, Corollary 2 leads directly to

\[ [(1 - 2\varepsilon)(1 - x_0) \left[ G_0(\tilde{k}) - \tilde{k}^{-1}G_1(\tilde{k}) \right] - \varepsilon(2x_0 - 1)] \cdot v_0(x_0) \\
+ [(1 - 2\varepsilon)x_1 \left[ 1 - G_1(\tilde{k}) - \tilde{k} \left[ 1 - G_0(\tilde{k}) \right] \right] - \varepsilon(1 - 2x_1)] \cdot (v_1(x_1)) \leq 0. \]

S.2. Proof of Lemma 7

Let \( \widetilde{\text{NE}}_\delta = \{ x \in [0, 1]^2 : d(x, \text{NE}_{\{l, \bar{l}\}}) \leq \delta \} \) be the set of all points that are \( \delta \)-close to elements of \( \text{NE}_{\{l, \bar{l}\}} \) and let \( \widetilde{L}^\varepsilon \) denote the set of limit points in a game with mistake probability \( \varepsilon > 0 \). Show first the following lemma, which is analogous to Lemma 11 in the main paper.

**Lemma 11’** (Limit set approaches \( \text{NE}_{\{l, \bar{l}\}} \)). For any \( \delta > 0 \), \( \exists \tilde{\varepsilon} > 0 \), \( \tilde{L}^\varepsilon \subseteq \widetilde{\text{NE}}_\delta \) for \( \varepsilon < \tilde{\varepsilon} \).
Proposition 3 guarantees both that \( \{\epsilon^n\}_{n=1}^{\infty} \) with \( \lim_{n \to \infty} \epsilon^n = 0 \) and (ii) an associated sequence \( \{x^n\}_{n=1}^{\infty} \) with \( x^n \in L^0 \) for all \( n \), but (iii) \( x^n \notin \text{NE}_{\delta} \) for all \( n \). Since \( x^n \in [0, 1]^2 \) for all \( n \), this sequence has a convergent subsequence \( \{x^n\}_{n=1}^{\infty} \) with \( \lim_{n \to \infty} x^{nm} = \bar{x} = (\bar{x}_0, \bar{x}_1) \). If \( v_0(\bar{x}_0) = v_1(\bar{x}_1) = 0 \), then \( \bar{x} \in \text{NE} \), so for \( m \) large enough, \( x^{nm} \in \text{NE}_{\delta} \). Then it must be the case that \( v_0(\bar{x}_\theta) \neq 0 \) for some \( \theta \).

Assume that \( v_1(\bar{x}_1) > 0 \). Pick \( \bar{m} \) large enough so that \( v_1(x^{nm}_1) > 0 \) for all \( m > \bar{m} \). For all \( m \) with \( v_0(x^{nm}_0) \geq 0 \), Lemma 4 implies that \( x^{nm} = (1 - \epsilon^{nm}, 1 - \epsilon^{nm}) \). So if \( v_0(x^{nm}_0) \geq 0 \) infinitely often, then \( \bar{x} = (1, 1) \). As a result, \( \bar{x} \in \text{NE} \), so for \( m \) large enough, \( x^{nm} \in \text{NE}_{\delta} \).

Take next all \( m \) with \( v_0(x^{nm}_0) < 0 \). By Lemma 5, (3) must hold:

\[
\frac{-v_0(x^{nm}_0)}{2} \left[ (1 - 2\epsilon^{nm})x^{nm}_0 [G_0(k^{nm}) - (k^{nm})^{-1}G_1(k^{nm})] - \epsilon(1 - 2x_0) \right] \\
+ \frac{v_1(x^{nm}_1)}{2} \left[ (1 - 2\epsilon^{nm}) (1 - x^{nm}_1) [1 - G_1(k^{nm})] - \epsilon^{nm} [1 - G_0(k^{nm})] \right] \\
- \epsilon^{nm} (2x^{nm}_1 - 1) \leq 0. \tag{S.1}
\]

Proposition 3 guarantees both that \( [1 - G_1(k^{nm})] - \epsilon^{nm} [1 - G_0(k^{nm})] \geq 0 \) and that \( G_0(k^{nm}) - (k^{nm})^{-1}G_1(k^{nm}) \geq 0 \). Then, as (S.1) shows, when \( \epsilon^{nm} \to 0 \), only nonnegative terms may remain. Assume that \( \bar{k} = -[v_0(\bar{x}_0)(1 - \bar{x}_0)]/[v_1(\bar{x}_1)(1 - \bar{x}_1)] < \bar{l} \). Then, for \( \epsilon \) small enough, \( \bar{k} \to \bar{l} \). Proposition 3 implies that

\[
\lim_{m \to \infty} [1 - G_1(k^{nm})] - \epsilon^{nm} [1 - G_0(k^{nm})] > 0.
\]

To summarize, whenever \( \bar{k} < \bar{l} \), (S.1) is not satisfied for small enough \( \epsilon^{nm} \). It must be the case then that \( \bar{k} \geq \bar{l} \). Similarly, if \( \bar{k} > \bar{l} \), then

\[
\lim_{m \to \infty} G_0(k^{nm}) - (k^{nm})^{-1}G_1(k^{nm}) > 0
\]

for small enough \( \epsilon^{nm} \). It must be the case then that \( \bar{k} \leq \bar{l} \).

Analogous arguments (using also Lemma 6) lead to the same result for the case with \( v_1(\bar{x}_1) < 0 \). As a result, \( \bar{x} \in \text{NE}_{\bar{l}, \bar{\delta}} \), so for \( m \) large enough, \( x^{nm} \in \text{NE}_{\delta} \).

The rest of the proof is identical to the proof of Proposition 2 in the paper. \( \square \)

S.3. Example 4: Standard observational learning with mistakes

This corresponds to Example 4 in the paper. Utility is given by \( u(1, X, 1) = u(0, X, 0) = 1 \) and \( u(1, X, 0) = u(0, X, 1) = 0 \). Each agent observes his immediate predecessor: \( M = 1 \).

The signal structure is described by \( v_1([0, s]) = s^2 \) and \( v_0([0, s]) = 2s - s^2 \) with \( s \in (0, 1) \).

**Proof of Example 4.** Let \( \pi \equiv \Pr(\xi = 1 | \theta = 1) \). An agent who observes \( \xi = 1 \) chooses action 1 if and only if \( \frac{\pi^2}{1-\pi} \geq 1 \iff s \geq 1 - \pi \). Similarly, an agent who observes \( \xi =
0 chooses action 1 if and only if $\frac{1 - \pi_j}{1 - \pi} \geq 1 \iff s \geq \pi$. As a result, the likelihood that somebody who observes a sample (that is, not agent 1) will choose the right action is given by

$$\Pr(a_t = 1 \mid \theta = 1, Q(i) \neq 1) = \frac{1}{T-1} \sum_{i=2}^{T} \Pr(a_t = 1 \mid \theta = 1)$$

$$= \epsilon + (1 - 2\epsilon)\left[\pi \Pr(s \geq 1 - \pi) + (1 - \pi) \Pr(s \geq \pi)\right]$$

$$= \epsilon + (1 - 2\epsilon)\left[\pi[1 - (1 - \pi)^2] + (1 - \pi)[1 - \pi^2]\right]$$

$$= \epsilon + (1 - 2\epsilon)\left[\pi \pi - \pi \pi^2 + 2\pi^2 + 1 - \pi - \pi^2 + \pi^3\right]$$

$$= \epsilon + (1 - 2\epsilon)(1 - \pi + \pi^2).$$

Reordering yields

$$\Pr(a_1 = 1 \mid \theta = 1) + \sum_{i=2}^{T} \Pr(a_t = 1 \mid \theta = 1) = \sum_{i=1}^{T-1} \Pr(a_i = 1 \mid \theta = 1) + \Pr(a_T = 1 \mid \theta = 1).$$

Then

$$\epsilon + (1 - 2\epsilon)(1 - \pi + \pi^2) - \pi - \frac{\Pr(a_T = 1 \mid \theta = 1) - \Pr(a_1 = 1 \mid \theta = 1)}{T-1} = 0$$

$$\epsilon + (1 - 2\epsilon)(1 - \pi + \pi^2) - \pi - \Delta = 0$$

$$(1 - 2\epsilon\pi^2 - 2(1 - \epsilon)\pi + 1 - \epsilon - \Delta = 0,$$

where I define $\Delta = \frac{\Pr(a_T = 1 \mid \theta = 1) - \Pr(a_1 = 1 \mid \theta = 1)}{T-1}$. Then

$$\pi = \frac{2(1 - \epsilon) \pm \sqrt{4(1 - \epsilon)^2 - 4(1 - 2\epsilon)(1 - \epsilon - \Delta)}}{2(1 - 2\epsilon)}$$

$$= \frac{1 - \epsilon - \sqrt{(1 - \epsilon)^2 - (1 - 2\epsilon)(1 - \epsilon - \Delta)}}{1 - 2\epsilon}.$$

Note that $\lim_{T \to \infty} \Delta = 0$. Then

$$\pi \to \frac{1 - \epsilon - \sqrt{(1 - \epsilon)^2 - (1 - 2\epsilon)(1 - \epsilon)}}{1 - 2\epsilon}$$

$$= \frac{1 - \epsilon}{1 - 2\epsilon} \left(1 - \sqrt{1 - \frac{1 - 2\epsilon}{1 - \epsilon}}\right) = \frac{1 - \epsilon}{1 - 2\epsilon} \left(1 - \sqrt{\frac{\epsilon}{1 - \epsilon}}\right).$$

Also, as $T \to \infty$, $\pi - \Pr(a_i = 1 \mid \theta) \to 0$. Then $x_1 = \lim_{T \to \infty} \Pr(a_i = 1 \mid \theta) = \frac{1 - \epsilon}{1 - 2\epsilon} \left(1 - \sqrt{\frac{\epsilon}{1 - \epsilon}}\right)$. \qed
S.4. Example 8: Multiple equilibria in a coordination game

Proof of Example 8. Consider a sequence of symmetric strategy profiles \( \{\sigma^T(s, \xi)\} \), where \( \sigma^T(s, \xi) = \sigma(s, \xi) \) does not change with \( T \) and is given by

\[
\sigma(s, \xi) = \begin{cases} 
1 & \text{if } s = 1 \\
0 & \text{if } s = 0 \\
\xi & \text{if } s = 1/2.
\end{cases}
\]

Let \( \pi \equiv \Pr(\xi = 1 | \theta = 1) \). Under \( \sigma(s, \xi) \), the likelihood that somebody who observes a sample (that is, not agent 1) chooses action 1 is given by

\[
\Pr(a_i = 1 | \theta = 1, Q(i) \neq 1) = \frac{1}{T - 1} \sum_{t=2}^{T} \Pr(a_t = 1 | \theta = 1) = \epsilon + (1 - 2\epsilon) \left[ \Pr(s = 1) + \Pr(s = 1/2) \pi \right] = \epsilon + (1 - 2\epsilon) \left[ (1 - \gamma)/100 + 99/100\pi \right].
\]

Reordering yields

\[
\Pr(a_1 = 1 | \theta = 1) + \sum_{t=2}^{T} \Pr(a_t = 1 | \theta = 1) = \sum_{t=1}^{T-1} \Pr(a_t = 1 | \theta = 1) + \Pr(a_T = 1 | \theta = 1).
\]

Then

\[
\frac{\sum_{t=2}^{T} \Pr(a_t = 1 | \theta = 1)}{T - 1} - \frac{\sum_{t=1}^{T-1} \Pr(a_t = 1 | \theta = 1)}{T - 1} = \frac{\Pr(a_T = 1 | \theta = 1) - \Pr(a_1 = 1 | \theta = 1)}{T - 1},
\]

so

\[
\Pr(a_i = 1 | \theta = 1, Q(i) \neq 1) - \pi = \frac{\Pr(a_T = 1 | \theta = 1) - \Pr(a_1 = 1 | \theta = 1)}{T - 1}
\]

\[
\epsilon + (1 - 2\epsilon) \left[ (1 - \gamma)/100 + 99/100\pi \right] - \pi = \Delta.
\]

Then

\[
\epsilon - 2\epsilon \left[ (1 - \gamma)/100 + 99/100\pi \right] + (1 - \gamma)/100 - 1/100\pi = \Delta
\]

\[
\epsilon - 2\epsilon (1 - \gamma)/100 - \epsilon 198/100\pi + (1 - \gamma)/100 - 1/100\pi = \Delta
\]

\[
+(1 - \gamma)/100 + \left[ 1 - (1 - \gamma)/50 \right] \epsilon - (1/100 + 198/100\epsilon) \pi = \Delta
\]

\[
+(1 - \gamma) + \left[ 100 - 2(1 - \gamma) \right] \epsilon - (1 + 198\epsilon) \pi = 100\Delta.
\]
Then
\[ \pi = \frac{(1 - \gamma) + [100 - 2(1 - \gamma)]\varepsilon - 100\Delta}{1 + 198\varepsilon}. \]

Proposition 1 guarantees that as the number of agents grows large, the average action is close to its expectation. For low enough \( \varepsilon \) and large enough \( T \), approximately \( X_0|\sigma \xrightarrow{P} \gamma \) and \( X_1|\sigma \xrightarrow{P} 1 - \gamma \). Then
\[ \frac{\Pr(\theta = 1 | \xi = 1)}{\Pr(\theta = 0 | \xi = 1)} \approx \frac{1 - \gamma}{\gamma}. \]

So the sample is informative about the state of the world. To sum up, there is \( \varepsilon \) small and \( T \) large such that \( \sigma \) is indeed an equilibrium.

\[ \square \]

S.5. Proving Lemma 12

I illustrate first the effect of different values of \( \gamma > 1 \) on sampling probabilities. Figure S.1 presents an agent in position 21. The black line shows the probability of observing a predecessor in position \( \tau < 21 \) when \( \gamma = 8 \). With probability higher than 0.998, the agent observes one of his three immediate predecessors. The distribution becomes flatter as \( \gamma \) decreases. The red line shows the distribution when \( \gamma = 1.05 \). In this case, the agent in position 21 observes his immediate predecessor twice as often as he observes the first agent in the sequence. As \( \gamma \to 1 \), sampling approaches uniform random sampling. Instead, as \( \gamma \to \infty \), sampling approaches observing the immediate predecessor.

Next, I present the proof of Lemma 12.

Proof of Lemma 12. A strategy \( \sigma_i \) induces \( \rho_{\theta}(\xi) = P_{\sigma_i}(a_i | \theta, \xi) \). For the rest of this section, I fix the state of the world \( \theta \) and drop its index. Then a strategy \( \sigma_i \) induces a vector \( (\rho(\emptyset), \rho(0), \rho(1)) \). Because of mistakes, \( \varepsilon < \rho(\xi) < 1 - \varepsilon \) for all \( \xi \in \{0, 1, \emptyset\} \).

\[ \text{Figure S.1. Probabilities of different predecessors being observed: geometric sampling.} \]
Assume first that \( \gamma > 1 \). The first agent in the sequence chooses action 1 with probability \( \rho(\emptyset) \). For \( t \geq 2 \),

\[
P_\rho(a_t = 1) = \Pr(\xi_1 = 0) \Pr(a_t = 1 \mid \xi_1 = 0) + \Pr(\xi_1 = 1) \Pr(a_t = 1 \mid \xi_1 = 1)
\]

\[
= \Pr(\xi_1 = 0)\rho(0) + \Pr(\xi_1 = 1)\rho(1)
\]

\[
= [1 - \Pr(\xi_1 = 1)]\rho(0) + \Pr(\xi_1 = 1)\rho(1)
\]

\[
= \rho(0) + [\rho(1) - \rho(0)]\Pr(\xi_1 = 1)
\]

\[
= \rho(0) + [\rho(1) - \rho(0)]\sum_{\tau < t} \Pr(O_t = \tau) \mathbb{1}(a_\tau = 1)
\]

\[
= \rho(0) + [\rho(1) - \rho(0)]\sum_{t=1}^{t-1} \frac{\gamma - 1}{\gamma} \frac{\gamma^t}{\gamma^t - 1} a_\tau.
\]

Define the weighted sum of the past history by \( p_t \equiv \sum_{\tau=1}^{t-1} \frac{\gamma - 1}{\gamma} \frac{\gamma^t}{\gamma^t - 1} a_\tau \) for \( t \geq 2 \). This concept plays a key role in the model:

\[
P_\rho(a_t = 1) = \rho(0) + [\rho(1) - \rho(0)] p_t.
\]

This weighted sum has a recursive nature:

\[
p_{t+1} = \sum_{\tau=1}^{t} \frac{\gamma - 1}{\gamma} \frac{\gamma^\tau}{\gamma^\tau - 1} a_\tau = \frac{\gamma^t - 1}{\gamma^t - 1} \left[ \sum_{\tau=1}^{t-1} \frac{\gamma - 1}{\gamma} \frac{\gamma^\tau}{\gamma^\tau - 1} a_\tau \right] + \frac{\gamma - 1}{\gamma} \frac{\gamma^t}{\gamma^t - 1} a_t
\]

\[
= \frac{\gamma^t - 1}{\gamma^t - 1} p_t + \frac{\gamma - \gamma^t}{\gamma^t - 1} a_t.
\]

In expectation,

\[
E[p_{t+1} \mid I_t] = \frac{\gamma^t - 1}{\gamma^t - 1} E[p_t \mid I] + \frac{\gamma - \gamma^t}{\gamma^t - 1} E[a_t \mid I]
\]

\[
= \frac{\gamma^t - 1}{\gamma^t - 1} E[p_t \mid I] + \frac{\gamma - \gamma^t}{\gamma^t - 1} \left[ \rho(0) + [\rho(1) - \rho(0)] E[p_t \mid I] \right]
\]

\[
= \frac{\gamma^t - 1}{\gamma^t - 1} E[p_t \mid I] + \frac{\gamma - \gamma^t}{\gamma^t - 1} \left[ \rho(0) + [\rho(1) - \rho(0)] E[p_t \mid I] \right]
\]

\[
= E[p_t \mid I] + \frac{\gamma^t - \gamma^t}{\gamma^t - 1} E[p_t \mid I] + \frac{\gamma^t - \gamma^t}{\gamma^t - 1} \left[ [\rho(0) - 1 + \rho(0) - \rho(1)] E[p_t \mid I] \right]
\]

\[
= E[p_t \mid I] + \frac{\gamma^t - \gamma^t}{\gamma^t - 1} \left[ [\rho(0) - 1 + \rho(0) - \rho(1)] E[p_t \mid I] \right]
\]
Let $\rho^* \equiv \frac{\rho(0)}{1 + \rho(0) - \rho(1)}$. Then

$$E[p_{t+1} \mid I] - \rho^* = E[p_t \mid I] - \rho^* \cdot \frac{\gamma^t - \gamma^{t-1}}{\gamma^t - 1} [1 + \rho(0) - \rho(1)] [E[p_t \mid I] - \rho^*]$$

$$= \left[1 - \frac{\gamma^t - \gamma^{t-1}}{\gamma^t - 1} [1 + \rho(0) - \rho(1)] \right] [E[p_t \mid I] - \rho^*]$$

$$= \left[1 - \frac{\gamma - 1}{\gamma} \frac{\gamma^t - \gamma^{t-1}}{\gamma^t - 1} [1 + \rho(0) - \rho(1)] \right] [E[p_t \mid I] - \rho^*]. \quad (S.2)$$

I next provide bounds for the terms (*) and (**) in (S.2):

$$2\varepsilon \leq 1 + \rho(0) - \rho(1) \leq 2 - 2\varepsilon$$

$$\frac{\gamma - 1}{\gamma} \leq \frac{\gamma - 1}{\gamma} \frac{\gamma^t - \gamma^{t-1}}{\gamma^t - 1} \leq 1.$$  

With these bounds, I can also bound the whole term in brackets in (S.2):

$$\frac{\gamma - 1}{\gamma} 2\varepsilon \leq \frac{\gamma - 1}{\gamma} \frac{\gamma^t - \gamma^{t-1}}{\gamma^t - 1} [1 + \rho(0) - \rho(1)] \leq 2 - 2\varepsilon$$

$$\frac{\gamma - 1}{\gamma} 2\varepsilon - 1 \leq \frac{\gamma - 1}{\gamma} \frac{\gamma^t - \gamma^{t-1}}{\gamma^t - 1} [1 + \rho(0) - \rho(1)] - 1 \leq 1 - 2\varepsilon$$

$$\left|1 - \frac{\gamma - 1}{\gamma} \frac{\gamma^t - \gamma^{t-1}}{\gamma^t - 1} [1 + \rho(0) - \rho(1)]\right| \leq 1 - \frac{\gamma - 1}{\gamma} 2\varepsilon.$$

This leads to a simple bound over time:

$$|E[p_{t+n} \mid I_t] - \rho^*| = \prod_{\tau=t}^{t+n-1} \left|1 - \frac{\gamma - 1}{\gamma} \frac{\gamma^t - \gamma^{t-1}}{\gamma^t - 1} [1 + \rho(0) - \rho(1)]\right| |E[p_t \mid I] - \rho^*|$$

$$\leq \left(1 - \frac{\gamma - 1}{\gamma} 2\varepsilon\right)^{n-1}.$$  

In particular,

$$|E[p_{t+n} \mid a_t = 1] - \rho^*| \leq \left(1 - \frac{\gamma - 1}{\gamma} 2\varepsilon\right)^{n-1}$$

$$|E[p_{t+n}] - \rho^*| \leq \left(1 - \frac{\gamma - 1}{\gamma} 2\varepsilon\right)^{t+n-1}.$$

---

1Note that $\rho(0) > \varepsilon$ and $\rho(1) < 1 - \varepsilon$, so $1 + \rho(0) - \rho(1) \geq 1 + \varepsilon - (1 - \varepsilon) = 2\varepsilon$ and so $1 + \rho(0) - \rho(1) \neq 0$. 
So finally,

\[
\begin{align*}
|E[p_{t+n} | I_t] - E[p_{t+n}]| &\leq |E[p_{t+n} | a_t = 1] - \rho^*| + |E[p_{t+n}] - \rho^*| \\
&\leq \left(1 - \frac{\gamma - \frac{1}{2} \varepsilon}{\gamma}\right)^{n-1} + \left(1 - \frac{\gamma - \frac{1}{2} \varepsilon}{\gamma}\right)^{t+n-1} \\
&\leq 2 \left(1 - \frac{\gamma - \frac{1}{2} \varepsilon}{\gamma}\right)^{n-1},
\end{align*}
\]

and turning this into probabilities yields

\[
|\mathbf{P}_\sigma(a_{t+n} = 1 | a_t = 1) - \mathbf{P}_\sigma(a_{t+n} = 1)| = |\rho(0) + [\rho(1) - \rho(0)]E[p_{t+n} | a_t = 1] \\
- \left[\rho(0) + [\rho(1) - \rho(0)]E[p_{t+n}]ight] | \\
= \left[\rho(1) - \rho(0)\right][E[p_{t+n} | a_t = 1] - E[p_{t+n}]| \\
\leq 2 [E[p_{t+n} | a_t = 1] - E[p_{t+n}]| \\
\leq 4 \left(1 - \frac{\gamma - \frac{1}{2} \varepsilon}{\gamma}\right)^{n-1} \\
\leq \frac{4}{1 - \frac{\gamma - \frac{1}{2} \varepsilon}{\gamma}} \left(1 - \frac{\gamma - \frac{1}{2} \varepsilon}{\gamma}\right)^{n}.
\]

Next assume that \( \gamma = 1 \). Then

\[
\mathbf{P}_\sigma(a_t = 1) = \rho(0) + [\rho(1) - \rho(0)] \frac{1}{t-1} \sum_{t=1}^{t-1} a_t.
\]

Define now \( p_t \equiv \frac{1}{t-1} \sum_{t=1}^{t-1} a_t \) for \( t \geq 2 \), which leads to

\[
p_{t+1} = \frac{1}{t} \sum_{t=1}^{t} a_t = \frac{t-1}{t} \sum_{t=1}^{t-1} a_t + \frac{1}{t} a_t = \frac{t-1}{t} p_t + \frac{1}{t} a_t.
\]

In expectation,

\[
E[p_{t+1} | I_t] = \frac{t-1}{t} E[p_t | I] + \frac{1}{t} E[a_t | I] \\
= \frac{t-1}{t} E[p_t | I] + \frac{1}{t} E[\rho(0) + [\rho(1) - \rho(0)] p_t | I] \\
= \frac{1}{t} [t-1 + \rho(1) - \rho(0)] E[p_t | I] + \frac{1}{t} \rho(0),
\]
so in this case,
\[
E[p_{t+1} \mid I_t] - \rho^* = \frac{1}{t} \left( t - 1 + \rho(1) - \rho(0) \right) E[p_t \mid I] + \frac{1}{t} \rho(0) - \rho^*
\]
\[
= \frac{1}{t} \left[ \rho(0) - [1 + \rho(0) - \rho(1)] E[p_t \mid I] \right] + E[p_t \mid I] - \rho^*
\]
\[
= \frac{1}{t} \left[ 1 + \rho(0) - \rho(1) \right] [\rho^* - E[p_t \mid I] - \rho^*]
\]
\[
= \left[ 1 - \frac{1}{t} [1 + \rho(0) - \rho(1)] \right] \left[ E[p_t \mid I] - \rho^* \right].
\]

Then
\[
E[p_{t+n} \mid I_t] - \rho^* = \left[ E[p_t \mid I] - \rho^* \right] \prod_{\tau=0}^{n} \left[ 1 - \frac{1}{t + \tau} [1 + \rho(0) - \rho(1)] \right].
\]

I present without proof the following remark.

**Remark 1.** Let \(0 < a_n < 1\) for all \(n\). Then \(\prod_{\tau=0}^{\infty} a_n > 0 \Leftrightarrow \sum_{\tau=0}^{\infty} (1 - a_n) < \infty\).

Then it suffices to show that
\[
\sum_{\tau=0}^{n} \frac{1}{t + \tau} [1 + \rho(0) - \rho(1)] = [1 + \rho(0) - \rho(1)] \sum_{\tau=0}^{n} \frac{1}{t + \tau} = \infty
\]
and follow the same steps as in the case with \(\gamma > 1\).

\[\square\]

**S.6. Proof of Lemma 13**

I show Proposition 1 by proving that \(X[\sigma^T] - E[X|\sigma^T]\) converges to zero in \(L^2\) norm. The variance \(V(\sigma^T)\) as defined by (5) is bounded above by
\[
V(\sigma^T) \leq \frac{1}{T} \left( 1 + 4(1 - 2e^M)^{-1} \left( 1 - 2e^{M(T)} \right)^{\frac{M}{T}} \right).
\]

Note that \(\lim_{T \to \infty} 4(1 - 2e^{M(T)})^{-1} = 4\) and \(\lim_{T \to \infty} (1 - 2e^{M(T)})^{\frac{1}{M(T)}} = 1\). Then the bound converges to zero whenever \(\lim_{T \to \infty} T[1 - (1 - 2e^{M(T)})^{\frac{1}{M(T)}}] = \infty\). I need to show that for any \(K < \infty\), there exists a \(\tilde{T} < \infty\) such that \(T[1 - (1 - 2e^{M(T)})^{\frac{1}{M(T)}}] \geq K\) for all \(T \geq \tilde{T}\). This simplifies to
\[
\left( 1 - \frac{K}{T} \right)^{M(T)} \geq 1 - 2e^{M(T)} \quad \forall T \geq \tilde{T}.
\]

Since \(1 - \frac{K}{T} \geq 1 - \frac{KM}{T}\), it suffices to show that
\[
1 - \frac{KM}{T} \geq 1 - 2e^{M(T)} \iff \frac{e^{M(T)}}{M} \geq \frac{K}{2} \frac{1}{T},
\]
where $M(T)$ is $o(\log(T))$. Then, for any constant $c \geq 0$, there is $T$ large enough such that $M(T) \leq c \log(T)$. Pick $c = (-2 \log(e))^{-1}$. Note next that the function $e^x/x$ is decreasing. Then, for $T$ large, $\frac{M(T)}{T^{\frac{1}{2}} \log(T)} = 0$. As a result, it suffices to show that for $T$ large enough,

$$
\frac{e^{(-2 \log(e))^{-1} \log(T)}}{(-2 \log(e))^{-1} \log(T)} \geq \frac{K}{2T},
$$

$$
\frac{e^{(-2 \log(e))^{-1} \log(T)}}{(-2 \log(e))^{-1} \log(T)} \geq \frac{1}{T} \frac{\log(T)}{K},
$$

$$
T^{\frac{1}{2}} \geq \frac{1}{T} \frac{\log(T)}{T}.
$$

The left hand side goes to infinity and the right hand side is constant. Then there always exists a $T$ such that this holds. This shows the first part of Proposition 1.

Next, I focus on the second part of Proposition 1. Equation (6) in the paper now becomes

$$
\Pr \left( |X|_{\sigma^T} - |X|_{\tilde{\sigma}^T} \geq \frac{n}{T} \right) \leq \left[ (1 - 2e^{M(T)}) \frac{1}{\log(T)} \right]^n,
$$

which holds for all $n$.

Let $n = \left\lfloor (-2 \log(e))^{-1} \log(T) T^{\frac{3}{2}} \right\rfloor$. As $(1 - 2e^M) \frac{1}{T} \leq 1$, then

$$
\Pr \left( |X|_{\sigma^T} - |X|_{\tilde{\sigma}^T} \geq \frac{n}{T} \right) \leq \left[ (1 - 2e^{M(T)}) \frac{1}{\log(T)} \right]^n
$$

$$
\leq \left[ (1 - 2e^{M(T)}) \frac{1}{\log(T)} \right]^{-2 \log(e))^{-1} \log(T) T^{\frac{3}{2}}}
$$

$$
\leq \left( 1 - 2e^{(-2 \log(e))^{-1} \log(T)} \right)^{\frac{1}{(-2 \log(e))^{-1} \log(T) T^{\frac{3}{2}}} - \frac{1}{(-2 \log(e))^{-1} \log(T) T^{\frac{3}{2}}} = \left( 1 - 2T^{-\frac{1}{2}} \right)^{\frac{3}{4}},
$$

where I have used the fact that $M(T)$ is $o(\log(T))$, so $M(T) \leq (-2 \log(e))^{-1} \log(T)$ for $T$ large enough. Moreover, I also used the fact that $(1 - 2e^M) \frac{1}{T}$ is increasing in $M$.

I need to show that for all $b > 0$, there exists $\tilde{T}$, such that $\Pr(|X|_{\sigma^T} - |X|_{\tilde{\sigma}^T} \geq b) < b$ for all $T > \tilde{T}$. Then it suffices to show that $\lim_{T \to \infty} \frac{b}{T} = 0$ and $\lim_{T \to \infty} (1 - 2T^{-\frac{1}{2}})^{\frac{3}{4}} = 0$. 

So first, note that
\[
\frac{n}{T} \leq \frac{(-2 \log(\varepsilon))^{-1} \log(T) T^{\frac{3}{2}} + 1}{T} = \frac{1}{(-2 \log(\varepsilon))} \frac{\log(T)}{T^{\frac{3}{4}}} + \frac{1}{T} \to 0,
\]
so \( \lim_{T \to \infty} \frac{n}{T} = 0 \).

Second, note that \( \lim_{T \to \infty} (1 - 2T^{-\frac{1}{2}})^{\frac{3}{4}} = 0 \Leftrightarrow \lim_{T \to \infty} T^{\frac{1}{4}} \log(1 - 2T^{-\frac{1}{2}}) = -\infty \). So using l'Hôpital's rule,
\[
\lim_{T \to \infty} \frac{T^{\frac{1}{4}}}{(1 - 2T^{-\frac{1}{2}})^{\frac{3}{4}}} = \lim_{T \to \infty} \frac{1}{(-2) T^{-\frac{3}{4}} \left(-\frac{1}{2}\right) T^{-\frac{1}{2}}} = \lim_{T \to \infty} -\frac{4 T^\frac{1}{4}}{3(1 - 2T^{-\frac{1}{2}})} = -\infty.
\]

This finishes the proof of the second part of Proposition 1.

Lemma 10 also needs some adjustment to allow for \( M \) to grow with \( T \). Equation (8) from the paper becomes
\[
\pi^T_\theta - E_{a^T}[X_\theta] = \frac{1}{T} \left[ \sum_{t=1}^{M(T)-1} \left( \sum_{\tau=t}^{\min(M(T),T-1)} P_{a^T}(a_\tau = 1) \right)^{\tau-M(T)-1} \right] - \sum_{\tau=M(T)+1}^{T} \left( 1 - \frac{T - \tau}{M(T)} \right)
\]
\[
\leq \frac{2M(T)}{T}.
\]

Since \( M(T) \) is \( o(\log(T)) \), then, \( \pi^T_\theta - E_{a^T} \to 0 \). This adapts Lemma 10 to the case with growing \( M \). The rest of Proposition 2 does not change.

S.7. Many states of the world and many actions

S.7.1 The model

States and Actions There are \( N_\theta \) equally likely states of the world \( \theta \in \Phi = \{1, 2, \ldots, N_\theta\} \). Agents must choose between \( N_a \) possible actions \( a \in A = \{1, 2, \ldots, N_a\} \). Let \( X^a \equiv \frac{1}{T} \sum_{j \in I} \mathbb{1}[a_j = a] \) denote the proportion of agents who choose action \( a \), with realizations \( x^a \in [0, 1] \). The vector \( X = (X^1, X^2, \ldots, X^{N_\theta}) \) denotes the proportion of agents who choose each action. Agent \( i \) obtains utility \( u(a_i, X, \theta) : A \times [0, 1] \times \Phi \to \mathbb{R} \), where \( u(a_i, X, \theta) \) is a continuous function in \( X \).

Private Signals Conditional on the true state of the world, signals are i.i.d. across individuals and distributed according to \( F_\theta \). I assume that \( F_\theta \) and \( F_\tilde{\theta} \) are mutually absolutely continuous for any two \( \theta, \tilde{\theta} \in \Phi \). Then no perfectly revealing signals occur with positive probability, and the likelihood ratio (Radon–Nikodym derivative) \( \frac{dF_\tilde{\theta}}{dF_\theta}(s) \) exists.
I also define a likelihood ratio that indicates how likely one state is, relative to all other states:

\[ l_{\theta}(s) = \left( \sum_{\theta \neq \theta} l_{\theta, \theta}(s) \right)^{-1}. \]

Let \( G_\theta(l) \equiv \Pr(l_\theta(S) \leq l \mid \theta) \). I modify the assumption of signals being of unbounded strength as follows.

**Definition (Signal strength).** Signal strength is unbounded if \( 0 < G_\theta(l) < 1 \) for all likelihood ratios \( l \in (0, \infty) \) and for all states \( \theta \in \Theta \).

**Sampling Strategies and Mistakes** The sampling rule does not change. A strategy is now a function \( \sigma_i : S \times E \rightarrow [\epsilon, 1 - (N_a - 1)\epsilon]^{N_a} \) that specifies a probability vector \( \sigma_i(s, \xi) \) for choosing each action given the information available. For example, \( \sigma_i^a(s, \xi) \) indicates the probability of choosing action \( a \in A \), after receiving signal \( s \) and sample \( \xi \).

**Definition of Social Learning** I modify the definition of NE to allow for many states and actions. I say that \( x_\theta \) corresponds to a Nash equilibrium of the stage game (and denote it by \( x_\theta \in \text{NE}^\theta \)) whenever \( u(a, x_\theta, \theta) > u(a^*, x_\theta, \theta) \) for some \( a, a^* \in A \Rightarrow x_\theta = 0 \). Then \( x \in \text{NE} \) whenever \( x_\theta \in \text{NE}^\theta \) for all \( \theta \in \Theta \).

**S.7.2 Results**

**Existence and Convergence of Average Action** The proofs of Lemma 1 and Proposition 1 extend directly to a context with many actions and many states. I need to adapt the notation. The random variable \( X|\sigma \) is now a matrix. Each element \( X^a|\sigma \) is a random variable that denotes the proportion of agents choosing action \( a \) in state \( \theta \). So the random variable \( X|\sigma = (X_1|\sigma, X_2|\sigma, \ldots, X_{N_\theta}|\sigma) \) has realizations \( x = (x_1, x_2, \ldots, x_{N_\theta}) \), where each \( x_\theta \) is itself a vector: \( x_\theta = (x^1_\theta, x^2_\theta, \ldots, x^{N_\theta}_\theta) \).

**Utility Convergence** In what follows, I provide modified expressions for the expected utility, the utility of the expected average action, and the approximate utility of a deviation. These expressions apply to contexts with many actions and many states.

Agents’ expected utility under symmetric profile \( \sigma^T \) is simply

\[ u(\sigma^T) \equiv E_{\sigma^T} [u(a, X, \theta)] = \frac{1}{N_\theta} \sum_{\theta \in \Theta} E_{\sigma^T} \left[ \sum_{a \in A} X^a_\theta \cdot u(a, X_\theta, \theta) \right]. \]

Define the utility of the expected average action \( \bar{u}^T \) by

\[ \bar{u}^T = \frac{1}{N_\theta} \sum_{\theta \in \Theta} \sum_{a \in A} E_{\sigma^T} [X^a_\theta] \cdot u(a, E_{\sigma^T}[X_\theta], \theta). \]

Define the approximate utility of the deviation \( \tilde{u}^T \) by

\[ \tilde{u}^T = \frac{1}{N_\theta} \sum_{\theta \in \Theta} \sum_{a \in A} P_{\sigma^T}(a_i = a \mid \theta) \cdot u(a, E_{\sigma^T}[X_\theta], \theta). \]
The proofs of Lemmas 2 and 3, as well as Corollary 1, extend directly to a context with many actions and many states.

**Corollary 2’** (The approximate improvement). *Let the approximate improvement \( \Delta^T \) be given now by*

\[
\Delta^T \equiv \tilde{u}^T - \bar{u}^T = \frac{1}{N_\theta} \sum_{\theta \in \Theta} \sum_{a \in \mathcal{A}} [P_{\sigma_T} (a_i = a \mid \theta) - E_{\sigma_T} [X^a_{\theta}]] \cdot u(a, E_{\sigma_T} [X_{\theta}], \theta).
\]

The proof of Corollary 2’ extends directly to a context with many actions and many states.

**S.7.3 Alternative Strategy 1: Always follow a given action**

I present next a version of Lemma 4 that applies to many actions and many states. Let action \( a^* \in \mathcal{A} \) be weakly dominant if

\[
u(a^*, x_\theta, \theta) \geq \nu(a, x_\theta, \theta) \quad \text{for all } a \in \mathcal{A} \text{ and for all } \theta \in \Theta.
\]

Let action \( a^* \in \mathcal{A} \) be strictly dominant if

\[
u(a^*, x_\theta, \theta) > \nu(a, x_\theta, \theta) \quad \text{for all } a \in \mathcal{A} \text{ and for all } \theta \in \Theta.
\]

**Lemma 4’** (Dominance). *If action \( a^* \in \mathcal{A} \) is strictly dominant, then \( x^a_{\theta} = 1 - (N_a - 1)\varepsilon \) for all \( \theta \in \Theta \). Assume instead that action \( a^* \in \mathcal{A} \) is weakly dominant. If there exists state \( \theta \in \Theta \) with \( \nu(a^*, x_\theta, \theta) > \nu(\tilde{a}, x_\theta, \theta) \), then \( x^a_{\theta} = \varepsilon \).

**Proof.** Consider the alternative strategy of always choosing action \( a^* \). Because of mistakes, this means \( a^* \) is chosen with probability \( 1 - (N_a - 1)\varepsilon \). Then the improvement is

\[
\Delta^T = \frac{1}{N_\theta} \sum_{\theta \in \Theta} \left[ 1 - (N_a - 1)\varepsilon - x^a_{\theta} \right] \nu(a^*, x_\theta, \theta) + \sum_{a \neq a^*} (\varepsilon - x^a_{\theta}) \cdot \nu(a, x_\theta, \theta)
\]

\[
= \frac{1}{N_\theta} \sum_{\theta \in \Theta} \left[ 1 - (N_a - 1)\varepsilon - x^a_{\theta} \right] \nu(a^*, x_\theta, \theta) - \sum_{a \neq a^*} (x^a_{\theta} - \varepsilon) \cdot \nu(a, x_\theta, \theta).
\]

Note that \( x^a_{\theta} - \varepsilon \geq 0 \) for all \( a, \theta \). Then

\[
\left[ 1 - (N_a - 1)\varepsilon - x^a_{\theta} \right] \nu(a^*, x_\theta, \theta) - \sum_{a \neq a^*} (x^a_{\theta} - \varepsilon) \cdot \nu(a, x_\theta, \theta)
\]

\[
\geq \left[ 1 - (N_a - 1)\varepsilon - x^a_{\theta} \right] \nu(a^*, x_\theta, \theta) - \sum_{a \neq a^*} (x^a_{\theta} - \varepsilon) \cdot \nu(a^*, x_\theta, \theta)
\]

\[
= \left[ 1 - (N_a - 1)\varepsilon - x^a_{\theta} \right] - \sum_{a \neq a^*} (x^a_{\theta} - \varepsilon) \cdot \nu(a^*, x_\theta, \theta).
\]
Recall that $\Delta^T \leq 0$ by Corollary 2. Moreover, $\Delta^T \geq 0$. Then $\Delta^T = 0$. Also, as each term in $\Delta^T$ is weakly positive, then all terms in $\Delta^T$ must be zero:

$$\left(1 - (N_a - 1)\varepsilon - x^a_\theta \right) \cdot u(a^*, x_\theta, \theta) - \sum_{a \neq a^*} \left(x^a_\theta - \varepsilon \right) \cdot u(a, x_\theta, \theta) = 0.$$ 

Assume next that for some action $\tilde{a} \in A$ in some state $\theta \in \Theta$, $u(a^*, x_\theta, \theta) > u(\tilde{a}, x_\theta, \theta)$. Then

$$0 = \left(1 - (N_a - 1)\varepsilon - x^a_\theta \right) \cdot u(a^*, x_\theta, \theta) - \sum_{a \neq a^*} \left(x^a_\theta - \varepsilon \right) \cdot u(a, x_\theta, \theta)$$

$$\geq \left(1 - (N_a - 1)\varepsilon - x^a_\theta \right) \cdot u(a^*, x_\theta, \theta) - \left(x^\tilde{a}_\theta - \varepsilon \right) u(\tilde{a}, x_\theta, \theta)$$

$$= \left(1 - \varepsilon - (1 - x^\tilde{a}_\theta) \right) u(a^*, x_\theta, \theta) - \left(x^\tilde{a}_\theta - \varepsilon \right) u(\tilde{a}, x_\theta, \theta)$$

$$= (x^\tilde{a}_\theta - \varepsilon) u(a^*, x_\theta, \theta) - (x^\tilde{a}_\theta - \varepsilon) u(\tilde{a}, x_\theta, \theta)$$

$$= (x^\tilde{a}_\theta - \varepsilon) \left[ u(a^*, x_\theta, \theta) - u(\tilde{a}, x_\theta, \theta) \right].$$

To sum up,

$$\frac{\left(x^\tilde{a}_\theta - \varepsilon \right) \left[ u(a^*, x_\theta, \theta) - u(\tilde{a}, x_\theta, \theta) \right]}{\geq 0} \leq 0.$$

So $x^\tilde{a}_\theta = \varepsilon$. Similarly, if $u(a^*, x_\theta, \theta) > u(a, x_\theta, \theta)$ for all $a \in A$ and for all $\theta \in \Theta$, then $x^{a^*}_\theta = 1 - (N_a - 1)\varepsilon$. \hspace{1cm} $\blacksquare$

### S.7.4 Alternative Strategy 2: Improve upon a sampled agent

Consider a possible limit point $x = (x_1, x_2, \ldots, x_{N_\theta})$. Assume that action $\tilde{a}$ is not optimal in state $\theta^*$: $u(a^*, x_{\theta^*}, \theta^*) > u(\tilde{a}, x_{\theta^*}, \theta^*)$, but it is still played in the limit: $x^\tilde{a}_\theta > \varepsilon$. As in the case with two states, let $\tilde{\xi}$ denote the action of one individual selected at random from the sample. Consider an alternative simple strategy $\tilde{\sigma}$, which makes the agent choose the action

$$a_i(\tilde{\xi}, s) = \begin{cases} 
    a^* & \text{if } \tilde{\xi} = \tilde{a} \text{ and } l_{\theta^*}(s) \geq k^T = \frac{1}{u(a^*, E_{\sigma^T}[X_{\theta^*}], \theta^*) - u(\tilde{a}, E_{\sigma^T}[X_{\theta^*}], \theta^*)} \mathbf{P}_{\sigma^T}(\tilde{\xi} = \tilde{a} | \theta = \theta^*) \\
    \tilde{\xi} & \text{otherwise.}
\end{cases}$$
I provide next a version of Lemma 5 in the paper that applies to many actions and many states.

**Lemma 5’ (Improvement principle).** Take any limit point \( x \in L \) with \( u(a^*, x_{\theta^*}, \theta^*) > u(\tilde{a}, x_{\theta^*}, \theta^*) \). Then

\[
\Delta(x) + \frac{1 - (N_a - 1)\varepsilon}{N_\theta} \left[ x_{\theta^*}^2 \cdot \left[ u(a^*, x_{\theta^*}, \theta^*) - u(\tilde{a}, x_{\theta^*}, \theta^*) \right] \right] \\
\times \left[ [1 - G_{\theta^*}^{\text{r}}(\bar{k})] - \bar{k}[1 - \tilde{G}_{\theta^*}^{\text{r}}(\bar{k})] \right] \leq 0
\]  

(S.3)

with

\[
\bar{k} = -\bar{u} \left[ (u(a^*, x_{\theta^*}, \theta^*) - u(\tilde{a}, x_{\theta^*}, \theta^*)) x_{\theta^*}^2 \right]^{-1} \\
\Delta(x) = \frac{\varepsilon}{N_\theta} \left[ \sum_{\theta \in \Theta} \sum_{a \in A} \left[ 1 - (N_a - 1)x_{\theta}^2 \right] u(a, x, \theta) \right] .
\]

See Section S.7.5 for the proof.

The term \( \left[ [1 - G_{\theta^*}^{\text{r}}(\bar{k})] - \bar{k}[1 - \tilde{G}_{\theta^*}^{\text{r}}(\bar{k})] \right] \geq 0 \) in (S.3) decreases in \( \bar{k} \) (as shown later in Proposition 3). Moreover, with signals of unbounded strength, this term is strictly positive. Then, whenever \( x_{\theta}^2 > 0 \), there is potential for improvement. The existence of mistakes may present such an improvement. Note, however, that \( \lim_{\varepsilon \to 0} \Delta(x) = 0 \). Then when mistakes are unlikely, the potential for improvement dominates in (S.3).

### S.7.5 Proof of Lemma 5’

Let \( \rho_\theta^T(a|\tilde{a}) \equiv \mathbb{P}_{\sigma^T}(a_i = a|\theta, \tilde{\xi} = \tilde{a}) \). In general, the improvement is given by

\[
\Delta^T = \frac{1}{N_\theta} \sum_{\theta \in \Theta} \sum_{a \in A} \left[ \varepsilon + [1 - (N_a - 1)\varepsilon] \sum_{a' \in A} \rho_\theta(a|a') \mathbb{P}_{\sigma^T}(\tilde{\xi} = a'|\theta) \\
- E_{\sigma^T}[X_{\theta}^a] u(a, E_{\sigma^T}[X_{\theta}], \theta) \right] \\
= \left[ \frac{\varepsilon}{N_\theta} \sum_{\theta \in \Theta} \sum_{a \in A} u(a, E_{\sigma^T}[X_{\theta}], \theta) \right] \\
+ \frac{1 - (N_a - 1)\varepsilon}{N_\theta} \left[ \sum_{\theta \in \Theta} \sum_{a \in A} \sum_{a' \in A} \rho_\theta(a|a') \mathbb{P}_{\sigma^T}(\tilde{\xi} = a'|\theta) u(a, E_{\sigma^T}[X_{\theta}], \theta) \right] \\
- \frac{1 - (N_a - 1)\varepsilon}{N_\theta} \left[ \sum_{\theta \in \Theta} \sum_{a \in A} E_{\sigma^T}[X_{\theta}^a] u(a, E_{\sigma^T}[X_{\theta}], \theta) \right] \\
- \frac{(N_a - 1)\varepsilon}{N_\theta} \left[ \sum_{\theta \in \Theta} \sum_{a \in A} E_{\sigma^T}[X_{\theta}^a] u(a, E_{\sigma^T}[X_{\theta}], \theta) \right].
\]
Let
\[ \Delta^T(\varepsilon) \equiv \frac{\varepsilon}{N_\theta} \left[ \sum_{\theta \in \Theta} \sum_{a \in A} u(a, E_{\sigma^T}[X_\theta], \theta) - (N_a - 1) \left[ \sum_{\theta \in \Theta} \sum_{a \in A} E_{\sigma^T}[X^a_\theta] u(a, E_{\sigma^T}[X_\theta], \theta) \right] \right] \]
\[ = \frac{\varepsilon}{N_\theta} \left[ \sum_{\theta \in \Theta} \sum_{a \in A} \left[ 1 - (N_a - 1) E_{\sigma^T}[X^a_\theta] \right] u(a, E_{\sigma^T}[X_\theta], \theta) \right] \]

and
\[ J(\varepsilon) \equiv \frac{1 - (N_a - 1)\varepsilon}{N_\theta}. \]

Then
\[ \Delta^T = \Delta^T(\varepsilon) + J(\varepsilon) \sum_{\theta \in \Theta} \sum_{a \in A} \left[ \sum_{a' \in A} \rho_\theta(a|a') P_{\sigma^T}(\tilde{\xi} = a'|\theta) - E_{\sigma^T}[X^a_\theta] \right] \]
\[ \times u(a, E_{\sigma^T}[X_\theta], \theta). \] (S.4)

However,
\[ = \frac{1}{N_\theta} \sum_{\theta \in \Theta} \sum_{a \in A} \sum_{a' \in A} \rho_\theta(a|a') P_{\sigma^T}(\tilde{\xi} = a'|\theta) u(a, E_{\sigma^T}[X_\theta], \theta) \]
\[ - \frac{1}{N_\theta} \sum_{\theta \in \Theta} \sum_{a \in A} \sum_{a' \in A} E_{\sigma^T}[X^a_\theta] u(a, E_{\sigma^T}[X_\theta], \theta) \]
\[ = \frac{1}{N_\theta} \sum_{\theta \in \Theta} \sum_{a \in A} \sum_{a' \in A} \rho_\theta(a|a') P_{\sigma^T}(\tilde{\xi} = a'|\theta) u(a, E_{\sigma^T}[X_\theta], \theta) \]
\[ - \frac{1}{N_\theta} \sum_{\theta \in \Theta} \sum_{a \in A} E_{\sigma^T}[X^a_\theta] u(a, E_{\sigma^T}[X_\theta], \theta) \]
\[ = \frac{1}{N_\theta} \sum_{\theta \in \Theta} \sum_{a' \in A} \sum_{a \in A} \rho_\theta(a|a') P_{\sigma^T}(\tilde{\xi} = a'|\theta) u(a, E_{\sigma^T}[X_\theta], \theta) \]
\[ - E_{\sigma^T}[X^a_\theta] u(a', E_{\sigma^T}[X_\theta], \theta) \].

As a result, the improvement in (S.4) can be expressed as
\[ \Delta^T = \Delta^T(\varepsilon) + J(\varepsilon) \sum_{\theta \in \Theta} \sum_{a \in A} \left[ \sum_{a' \in A} \rho_\theta(a|a') P_{\sigma^T}(\tilde{\xi} = a'|\theta) u(a, E_{\sigma^T}[X_\theta], \theta) \right. \]
\[ - \left. E_{\sigma^T}[X^a_\theta] u(a', E_{\sigma^T}[X_\theta], \theta) \right]. \]
In particular, for the simple strategy $\tilde{\sigma}$,

$$\Delta^T = \tilde{\Delta}^T(\varepsilon) + J(\varepsilon) \sum_{\theta \in \Theta} [\rho_\theta(a^\ast|\tilde{a})]P_{\sigma^T}(\tilde{\xi} = \tilde{a}|\theta)u(a^\ast, E_{\sigma^T}[X_\theta], \theta)$$

$$+ \left[1 - \rho_\theta(a^\ast|\tilde{a})\right]P_{\sigma^T}(\tilde{\xi} = \tilde{a}|\theta)u(\tilde{a}, E_{\sigma^T}[X_\theta], \theta) - E_{\sigma^T}[X_\theta]\sum_{\theta \in \Theta} \left[\rho_\theta(a^\ast|\tilde{a})\right]u(\tilde{a}, E_{\sigma^T}[X_\theta], \theta)$$

$$= \tilde{\Delta}^T(\varepsilon) + J(\varepsilon) \sum_{\theta \in \Theta} [\rho_\theta(a^\ast|\tilde{a})]P_{\sigma^T}(\tilde{\xi} = \tilde{a}|\theta)[u(a^\ast, E_{\sigma^T}[X_\theta], \theta) - u(\tilde{a}, E_{\sigma^T}[X_\theta], \theta)]$$

$$+ \left[\tilde{P}_{\sigma^T}(\tilde{\xi} = \tilde{a}|\theta) - E_{\sigma^T}[X_\theta]\right]u(\tilde{a}, E_{\sigma^T}[X_\theta], \theta)].$$

Let

$$\tilde{\Delta}^T(\varepsilon) \equiv J(\varepsilon) \sum_{\theta \in \Theta} [\rho_\theta(a^\ast|\tilde{a})]P_{\sigma^T}(\tilde{\xi} = \tilde{a}|\theta) - E_{\sigma^T}[X_\theta]\sum_{\theta \in \Theta} \left[\rho_\theta(a^\ast|\tilde{a})\right]u(\tilde{a}, E_{\sigma^T}[X_\theta], \theta)].$$

Then

$$\Delta^T = \tilde{\Delta}^T(\varepsilon) + \tilde{\Delta}^T(\varepsilon)$$

$$+ J(\varepsilon) \sum_{\theta \in \Theta} [\rho_\theta(a^\ast|\tilde{a})]P_{\sigma^T}(\tilde{\xi} = \tilde{a}|\theta)[u(a^\ast, E_{\sigma^T}[X_\theta], \theta) - u(\tilde{a}, E_{\sigma^T}[X_\theta], \theta)]$$

$$= \tilde{\Delta}^T(\varepsilon) + \tilde{\Delta}^T(\varepsilon)$$

$$+ J(\varepsilon) \sum_{\theta \in \Theta, \theta \neq \theta^*} [\rho_\theta(a^\ast|\tilde{a})]P_{\sigma^T}(\tilde{\xi} = \tilde{a}|\theta)[u(a^\ast, E_{\sigma^T}[X_\theta], \theta) - u(\tilde{a}, E_{\sigma^T}[X_\theta], \theta)]$$

$$+ \rho_\theta^*(a^\ast|\tilde{a})P_{\sigma^T}(\tilde{\xi} = \tilde{a}|\theta^*)[u(a^\ast, E_{\sigma^T}[X_{\theta^*}], \theta^*) - u(\tilde{a}, E_{\sigma^T}[X_{\theta^*}], \theta^*)].$$

Now let

$$-\tilde{u} \equiv \min_{a \in A, a' \in A, \theta \in \Theta, x \in [0,1], \chi_\theta} \left[u(a, x, \theta) - u(a', x, \theta)\right].$$

This minimum exists since there is a finite number of states and actions, and the utility functions are continuous in $X$. Then

$$[u(a^\ast, E_{\sigma^T}[X_{\theta^*}], \theta^*) - u(\tilde{a}, E_{\sigma^T}[X_{\theta^*}], \theta^*)] \geq -\tilde{u}.$$

Then

$$\Delta^T \geq \tilde{\Delta}^T(\varepsilon) + \tilde{\Delta}^T(\varepsilon)$$

$$+ J(\varepsilon)P_{\sigma^T}(\tilde{\xi} = \tilde{a}|\theta^*)[u(a^\ast, E_{\sigma^T}[X_{\theta^*}], \theta^*) - u(\tilde{a}, E_{\sigma^T}[X_{\theta^*}], \theta^*)]$$

$$\times \left[-\tilde{u} \sum_{\theta \in \Theta, \theta \neq \theta^*} [\rho_\theta(a^\ast|\tilde{a})]P_{\sigma^T}(\tilde{\xi} = \tilde{a}|\theta)\right]$$

$$+ \rho_\theta^*(a^\ast|\tilde{a})P_{\sigma^T}(\tilde{\xi} = \tilde{a}|\theta^*)[u(a^\ast, E_{\sigma^T}[X_{\theta^*}], \theta^*) - u(\tilde{a}, E_{\sigma^T}[X_{\theta^*}], \theta^*)] + \rho_\theta^*(a^\ast|\tilde{a}).$$
Moreover, if many states of the world, and many states, I present next a version of Proposition 3 in the paper that applies to Proposition 2 in the main text. Lemma 11 extends directly to a context with many actions. I present this formally.

\[
\Delta^T(\epsilon) + J(\epsilon)P_{x^l}(\tilde{\xi} = \tilde{a}|\theta)\left[u(a^*, E_{\theta^l}[X_{\theta^l}], \theta^*) - u(\tilde{a}, E_{\theta^l}[X_{\theta^l}], \theta^*)\right]
\times \left[\rho_\theta(a^*|\tilde{a}) - k^T \sum_{\theta \in \Theta, \theta \neq \theta^l} \rho_\theta(a^*|\tilde{a})\right]
\geq \Delta^T(\epsilon) + J(\epsilon)P_{x^l}(\tilde{\xi} = \tilde{a}|\theta)\left[u(a^*, E_{\theta^l}[X_{\theta^l}], \theta^*) - u(\tilde{a}, E_{\theta^l}[X_{\theta^l}], \theta^*)\right]
\times \left[\rho_\theta(a^*|\tilde{a}) - k^T \sum_{\theta \in \Theta, \theta \neq \theta^l} \rho_\theta(a^*|\tilde{a})\right]
\equiv \tilde{\Delta}^T(\epsilon) + \tilde{\Delta}^T + J(\epsilon)P_{x^l}(\tilde{\xi} = \tilde{a}|\theta)\left[u(a^*, E_{\theta^l}[X_{\theta^l}], \theta^*) - u(\tilde{a}, E_{\theta^l}[X_{\theta^l}], \theta^*)\right]
\times \left[\rho_\theta(a^*|\tilde{a}) - k^T \sum_{\theta \in \Theta, \theta \neq \theta^l} \rho_\theta(a^*|\tilde{a})\right]
\times \left[[1 - G_{\theta^l}(k^T)] - k^T \left[1 - \tilde{G}_{\theta^l}(k^T)\right]\right].
\]

Note that \(\lim_{T \to \infty} \frac{\tilde{\Delta}^T}{N^T} = 0\). Let \(\tilde{\Delta}(\epsilon) \equiv \lim_{T \to \infty} \tilde{\Delta}^T(\epsilon)\). Finally, note that, as in the proof in the paper, \(\lim_{T \to \infty} k^T = \tilde{k}\). Then

\[
\lim_{T \to \infty} \Delta^T(\epsilon) = \tilde{\Delta}(\epsilon) + \frac{1 - (N_a - 1)\epsilon}{N_\theta} \left[x_{\theta^l}^l [u(a^*, x_{\theta^l}, \theta^*) - u(\tilde{a}, x_{\theta^l}, \theta^*)]\right]
\times \left[[1 - G_{\theta^l}(\tilde{k})] - \tilde{k} [1 - \tilde{G}_{\theta^l}(\tilde{k})]\right].
\]

S.7.6 Strategic learning

Lemmas 4' and 5' are the main building blocks to show how Proposition 2 also applies to a context with many states and many actions. I present this formally.

**Proposition 2' (Strategic learning).** Assume signals are of unbounded strength. Then there is strategic learning.

The proof of Proposition 3' requires modifying Proposition 3 and Lemma 11 in the paper. With these results in hand, the proof of Proposition 2' is analogous to the proof of Proposition 2 in the main text. Lemma 11 extends directly to a context with many actions and many states. I present next a version of Proposition 3 in the paper that applies to many states of the world.

**Proposition 3'.** For all \(l \in [\tilde{l}, \tilde{l}]\), \(G_\theta(l)\) satisfies

\[
l \geq \frac{G_\theta(l)}{G_\theta(\tilde{l})} \quad \text{and} \quad l \leq \frac{1 - G_1(l)}{1 - G_0(l)}.
\]  

(S.5)

Moreover, if \(k' \geq k\), then

\[
[1 - G_1(k)] - k [1 - G_0(k)] \geq [1 - G_1(k')] - k' [1 - G_0(k')].
\]  

(S.6)
Proof. The proof follows that from Proposition 11 in Monzón and Rapp (2014), but here the likelihood ratio \( G_\theta \) indicates how likely state \( \theta \) is relative to all other states. Note first that

\[
 l_\theta(s)^{-1} = \sum_{\hat{\theta} \neq \theta} l_{\hat{\theta}, \theta}(s) = \sum_{\hat{\theta} \neq \theta} \frac{dF_{\hat{\theta}}(s)}{dF_\theta(s)}
\]

\[
 dF_\theta(s) l_\theta(s)^{-1} = \sum_{\hat{\theta} \neq \theta} dF_{\hat{\theta}}(s)
\]

\[
 dF_\theta(s) = \sum_{\hat{\theta} \neq \theta} dF_{\hat{\theta}}(s).
\]

Recall that \( \tilde{G}_\theta(L) \equiv \sum_{\hat{\theta} \neq \theta} \Pr(l_{\hat{\theta}, \theta}(s) \leq L \mid \hat{\theta}) \):

\[
 G_\theta(L) = \int_{\{S \in S : l_\theta(s) \leq L\}} dF_\theta = \int_{\{S \in S : l_\theta(s) \leq L\}} l_\theta(s) \sum_{\hat{\theta} \neq \theta} dF_{\hat{\theta}}(s)
\]

\[
 < \int_{\{S \in S : l_\theta(s) \leq L\}} L \sum_{\hat{\theta} \neq \theta} dF_{\hat{\theta}}(s) = L \sum_{\hat{\theta} \neq \theta} \int_{\{S \in S : l_\theta(s) \leq L\}} dF_{\hat{\theta}}(s)
\]

\[
 = L \tilde{G}_\theta(L).
\]

Similarly,

\[
 1 - G_\theta(L) = \int_{\{S \in S : l_\theta(s) > L\}} dF_\theta = \int_{\{S \in S : l_\theta(s) > L\}} l_\theta(s) \sum_{\hat{\theta} \neq \theta} dF_{\hat{\theta}}(s)
\]

\[
 > \int_{\{S \in S : l_\theta(s) > L\}} L \sum_{\hat{\theta} \neq \theta} dF_{\hat{\theta}}(s) = L \sum_{\hat{\theta} \neq \theta} \int_{\{S \in S : l_\theta(s) > L\}} dF_{\hat{\theta}}(s)
\]

\[
 = L[1 - \tilde{G}_\theta(L)].
\]

This shows that (S.5) holds. I move next to the second part. Take \( k' > k \):

\[
 [1 - G_\theta(k)] - [1 - G_\theta(k')] = G_\theta(k') - G_\theta(k) = \int_{S \in S : k \leq l_\theta(S) \leq k'} dF_\theta
\]

\[
 = \int_{S \in S : k \leq l_\theta(S) \leq k'} l_\theta(S) \sum_{\hat{\theta} \neq \theta} dF_{\hat{\theta}}
\]

\[
 \geq k \int_{S \in S : k \leq l_\theta(S) \leq k} \sum_{\hat{\theta} \neq \theta} dF_{\hat{\theta}} = k[\tilde{G}_\theta(k') - \tilde{G}_\theta(k)]
\]

\[
 = k[1 - \tilde{G}_\theta(k)] - k[1 - \tilde{G}_\theta(k')] = k[1 - \tilde{G}_\theta(k)] - k'[1 - \tilde{G}_\theta(k')].
\]
Then

\[ [1 - G_{\theta}(k)] - [1 - G_{\theta}(k')] \geq k [1 - \tilde{G}_{\theta}(k)] - k'[1 - \tilde{G}_{\theta}(k')] \]
\[ [1 - G_{\theta}(k)] - k[1 - \tilde{G}_{\theta}(k)] \geq [1 - G_{\theta}(k')] - k'[1 - \tilde{G}_{\theta}(k')]. \]

This shows that (S.6) holds.

References