Dynamic Games with (Almost) Perfect Information: Appendix B

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In Section B.1, we shall present the model of measurable dynamic games with partially perfect information and show the existence of subgame-perfect equilibria in Proposition B.1. It covers the results in Theorem 3 (Theorem 4) for dynamic games with almost perfect information (perfect information), and in discounted stochastic games.

In Section B.2, we present Lemmas B.1-B.6 as the mathematical preparations for proving Theorem 3. We present in Section B.3 a new equilibrium existence result for discontinuous games with stochastic endogenous sharing rules. The proof of Theorem 3 is given in Section B.4. The proof of Proposition B.1 is provided in Section B.5, which covers Theorem 4 as a special case. One can skip Sections B.2 and B.3 first, and refer to the technical results in these two sections whenever necessary.

B.1 Measurable dynamic games with partially perfect information

In this section, we will generalize the model of measurable dynamic games in three directions. The ARM condition is partially relaxed such that (1) perfect information may be allowed in some stages, (2) the state transitions could have a weakly continuous component in all other stages, and (3) the state transition in any period can depend on the action profile in the current stage as well as on the previous history. The first change allows us to combine the models of dynamic games with perfect and almost perfect information. The second generalization implies that the state transitions need not be

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norm continuous on the Banach space of finite measures. The last modification covers the model of stochastic games as a special case.

The changes are described below.

1. The state space is a product space of two Polish spaces; that is, $S_t = \hat{S}_t \times \tilde{S}_t$ for each $t \geq 1$.

2. For each $i \in I$, the action correspondence A_{ti} from H_{t-1} to X_{ti} is measurable, nonempty and compact valued, and sectionally continuous on $X^{t-1} \times \hat{S}^{t-1}$. The additional component of Nature is given by a measurable, nonempty and closed valued correspondence $\hat{A}_{t0}$ from $\text{Gr}(A_t)$ to $\hat{S}^t$, which is sectionally continuous on $X^t \times \hat{S}^{t-1}$. Then $H_t = \text{Gr}(\hat{A}_{t0}) \times \tilde{S}_t$, and H_∞ is the subset of $X^\infty \times S^\infty$ such that $(x, s) \in H_\infty$ if $(x^t, s^t) \in H_t$ for any $t \geq 0$.

3. The choice of Nature depends not only on the history h_{t-1}, but also on the action profile x_t in the current stage. The state transition $f_{t0}(h_{t-1}, x_t) = \hat{f}_{t0}(h_{t-1}, x_t) \diamond \tilde{f}_{t0}(h_{t-1}, x_t)$, where $\hat{f}_{t0}$ is a transition probability from $\text{Gr}(A_t)$ to $\mathcal{M}(\hat{S}_t)$ such that $\hat{f}_{t0}(\hat{A}_{t0}(h_{t-1}, x_t)|h_{t-1}, x_t) = 1$ for all $(h_{t-1}, x_t) \in \text{Gr}(A_t)$, and $\tilde{f}_{t0}$ is a transition probability from $\text{Gr}(\hat{A}_{t0})$ to $\mathcal{M}(\tilde{S}_t)$.

4. For each $i \in I$, the payoff function u_i is a Borel measurable mapping from H_∞ to $\mathbb{R}_{++}$, which is sectionally continuous on $X^\infty \times \hat{S}^\infty$.

As in Subsection 3.3, we allow the possibility for the players to have perfect information in some stages. For $t \geq 1$, let

$$N_t = \begin{cases} 1, & \text{if } f_{t0}(h_{t-1}, x_t) \equiv \delta_{s_t} \text{ for some } s_t \text{ and} \\ & |\{i \in I \colon A_{ti} \text{ is not point valued}\}| = 1; \\ 0, & \text{otherwise.} \end{cases}$$

Thus, if $N_t = 1$ for some stage t, then the player who is active in the period t is the only active player and has perfect information.

We will drop the ARM condition in those periods with only one active player, and weaken the ARM condition in other periods.

Assumption B.1 (ARM′). *1. For any $t \geq 1$ with $N_t = 1$, S_t is a singleton set $\{\acute{s}_t\}$ and $\lambda_t = \delta_{\acute{s}_t}$.*

2. For each $t \geq 1$ with $N_t = 0$, $\hat{f}_{t0}$ is sectionally continuous on $X^t \times \hat{S}^{t-1}$, where the range space $\mathcal{M}(\hat{S}_t)$ is endowed with topology of weak convergence of measures on

$\hat{S}_t$. *The probability measure* $\tilde{f}_{t0}(\cdot|h_{t-1}, x_t, \hat{s}_t)$ *is absolutely continuous with respect to an atomless Borel probability measure* λ_t *on* $\tilde{S}_t$ *for all* $(h_{t-1}, x_t, \hat{s}_t) \in Gr(\hat{A}_{t0})$, *and* $\varphi_{t0}(h_{t-1}, x_t, \hat{s}_t, \tilde{s}_t)$ *is the corresponding density.*[1]

3. *The mapping* φ_{t0} *is Borel measurable and sectionally continuous on* $X^t \times \hat{S}^t$, *and integrably bounded in the sense that there is a* λ_t*-integrable function* $\phi_t \colon \tilde{S}_t \to \mathbb{R}_+$ *such that* $\varphi_{t0}(h_{t-1}, x_t, \hat{s}_t, \tilde{s}_t,) \leq \phi_t(\tilde{s}_t)$ *for any* $(h_{t-1}, x_t, \hat{s}_t)$.

The following result shows that the existence result is still true in this more general setting.

Proposition B.1. *If an infinite-horizon dynamic game as described above satisfies the ARM′ condition and is continuous at infinity, then it possesses a subgame-perfect equilibrium* f. *In particular, for* $j \in I$ *and* $t \geq 1$ *such that* $N_t = 1$ *and player* j *is the only active player in this period,* f_{tj} *can be deterministic. Furthermore, the equilibrium payoff correspondence* E_t *is nonempty and compact valued, and essentially sectionally upper hemicontinuous on* $X^{t-1} \times \hat{S}^{t-1}$.

Remark B.1. *The result above also implies a new existence result of subgame-perfect equilibria for stochastic games. In the existence result of [6], the state transitions are assumed to be norm continuous with respect to the actions in the previous stage. They did not assume the ARM condition. On the contrary, our Proposition B.1 allows the state transitions to have a weakly continuous component.*

B.2 Technical preparations

The following lemma shows that the space of nonempty compact subsets of a Polish space is still Polish under the Hausdorff metric topology.

Lemma B.1. *Suppose that* X *is a Polish space and* $\mathcal{K}$ *is the set of all nonempty compact subsets of* X *endowed with the Hausdorff metric topology. Then* $\mathcal{K}$ *is a Polish space.*

Proof. By Theorem 3.88 (2) of [1], $\mathcal{K}$ is complete. In addition, Corollary 3.90 and Theorem 3.91 of [1] imply that $\mathcal{K}$ is separable. Thus, $\mathcal{K}$ is a Polish space. □

The following result presents a variant of Lemma 5 in terms of transition correspondences.

[1] In this section, a property is said to hold for λ^t-almost all $h_t \in H_t$ if it is satisfied for λ^t-almost all $\tilde{s}^t \in \tilde{S}^t$ and all $(x^t, \hat{s}^t) \in H_t(\tilde{s}^t)$.

Lemma B.2. *Let X and Y be Polish spaces, and Z a compact subset of $\mathbb{R}^l_+$. Let G be a measurable, nonempty and compact valued correspondence from X to $\mathcal{M}(Y)$. Suppose that F is a measurable, nonempty, convex and compact valued correspondence from $X \times Y$ to Z. Define a correspondence Π from X to Z as follows:*

$$\Pi(x) = \{ \int_Y f(x,y) g(\mathrm{d}y|x) \colon g \text{ is a Borel measurable selection of } G, \\ f \text{ is a Borel measurable selection of } F \}.$$

If F is sectionally continuous on Y, then

1. *the correspondence $\tilde{F} \colon X \times \mathcal{M}(Y) \to Z$ as $\tilde{F}(x,\nu) = \int_Y F(x,y)\nu(\mathrm{d}y)$ is sectionally continuous on $\mathcal{M}(Y)$; and*

2. *Π is a measurable, nonempty and compact valued correspondence.*

3. *If F and G are both continuous, then Π is continuous.*

Proof. (1) For any fixed $x \in X$, the upper hemicontinuity of $\tilde{F}(x,\cdot)$ follows from Lemma 7.

Next, we shall show the lower hemicontinuity. Fix any $x \in X$. Suppose that $\{\nu_j\}_{j \geq 0}$ is a sequence in $\mathcal{M}(Y)$ such that $\nu_j \to \nu_0$ as $j \to \infty$. Pick an arbitrary point $z_0 \in \tilde{F}(x,\nu_0)$. Then there exists a Borel measurable selection f of $F(x,\cdot)$ such that $z_0 = \int_Y f(y)\nu_0(\mathrm{d}y)$.

By Lemma 3 (Lusin's theorem), for each $k \geq 1$, there exists a compact subset $D_k \subseteq Y$ such that f is continuous on D_k and $\nu_0(Y \setminus D_k) < \frac{1}{3kM}$, where $M > 0$ is the bound of Z. Define a correspondence $F_k \colon Y \to Z$ as follows:

$$F_k(y) = \begin{cases} \{f(y)\}, & y \in D_k; \\ F(x,y), & y \in Y \setminus D_k. \end{cases}$$

Then F_k is nonempty, convex and compact valued, and lower hemicontinuous. By Theorem 3.22 in [1], Y is paracompact. Then by Lemma 3 (Michael's selection theorem), F_k has a continuous selection f_k.

For each k, since $\nu_j \to \nu_0$ and f_k is bounded and continuous, $\int_Y f_k(y)\nu_j(\mathrm{d}y) \to \int_Y f_k(y)\nu_0(\mathrm{d}y)$ as $j \to \infty$. Thus, there exists a subsequence $\{\nu_{j_k}\}$ such that $\{j_k\}$ is an increasing sequence, and for each $k \geq 1$,

$$\left\| \int_Y f_k(y)\nu_{j_k}(\mathrm{d}y) - \int_Y f_k(y)\nu_0(\mathrm{d}y) \right\| < \frac{1}{3k},$$

where $\|\cdot\|$ is the Euclidean norm on $\mathbb{R}^l$.

Since f_k coincides with f on D_k, $\nu_0(Y \setminus D_k) < \frac{1}{3kM}$, and Z is bounded by M,

$$\left\| \int_Y f_k(y)\nu_0(\mathrm{d}y) - \int_Y f(y)\nu_0(\mathrm{d}y) \right\| < \frac{2}{3k}.$$

Thus,

$$\left\| \int_Y f_k(y)\nu_{j_k}(\mathrm{d}y) - \int_Y f(y)\nu_0(\mathrm{d}y) \right\| < \frac{1}{k}.$$

Let $z_{j_k} = \int_Y f_k(y)\nu_{j_k}(\mathrm{d}y)$ for each k. Then $z_{j_k} \in \tilde{F}(x, \nu_{j_k})$ and $z_{j_k} \to z_0$ as $k \to \infty$. By Lemma 1, $\tilde{F}(x, \cdot)$ is lower hemicontinuous.

(2) Since G is measurable and compact valued, there exists a sequence of Borel measurable selections $\{g_k\}_{k\geq 1}$ of G such that $G(x) = \overline{\{g_1(x), g_2(x), \ldots\}}$ for any $x \in X$ by Lemma 2 (5). For each $k \geq 1$, define a correspondence Π^k from X to Z by letting $\Pi^k(x) = \tilde{F}(x, g_k(x)) = \int_Y F(x,y) g_k(\mathrm{d}y|x)$. Since F is convex valued, so is Π^k. By Lemma 5, Π^k is also measurable, nonempty and compact valued.

Fix any $x \in X$. It is clear that $\Pi(x) = \tilde{F}(x, G(x))$ is nonempty valued. Since $G(x)$ is compact, and $\tilde{F}(x, \cdot)$ is compact valued and continuous, $\Pi(x)$ is compact by Lemma 2. Thus, $\overline{\bigcup_{k\geq 1} \Pi^k(x)} \subseteq \Pi(x)$.

Fix any $x \in X$ and $z \in \Pi(x)$. There exists a point $\nu \in G(x)$ such that $z \in \tilde{F}(x, \nu)$. Since $\{g_k(x)\}_{k\geq 1}$ is dense in $G(x)$, it has a subsequence $\{g_{k_m}(x)\}$ such that $g_{k_m}(x) \to \nu$. As $\tilde{F}(x, \cdot)$ is continuous, $\tilde{F}(x, g_{k_m}(x)) \to \tilde{F}(x, \nu)$. That is,

$$z \in \overline{\bigcup_{k\geq 1} \tilde{F}(x, g_k(x))} = \overline{\bigcup_{k\geq 1} \Pi^k(x)}.$$

Therefore, $\overline{\bigcup_{k\geq 1} \Pi^k(x)} = \Pi(x)$ for any $x \in X$. Lemma 2 (1) and (2) imply that Π is measurable.

(3) Define a correspondence $\hat{F} \colon \mathcal{M}(X \times Y) \to Z$ as follows:

$$\hat{F}(\tau) = \left\{ \int_{X\times Y} f(x,y)\tau(\mathrm{d}(x,y)) \colon f \text{ is a Borel measurable selection of } F \right\}.$$

By (1), $\hat{F}$ is continuous. Define a correspondence $\hat{G} \colon X \to \mathcal{M}(X \times Y)$ as $\hat{G}(x) = \{\delta_x \otimes \nu : \nu \in G(x)\}$. Since $\hat{G}$ and $\hat{F}$ are both nonempty valued, $\Pi(x) = \hat{F}(\hat{G}(x))$ is nonempty. As

$\hat{G}$ is compact valued and $\hat{F}$ is continuous, Π is compact valued by Lemma 2. As $\hat{G}$ and $\hat{F}$ are both continuous, Π is continuous by Lemma 1 (7). □

The following lemma shows that a measurable and sectionally continuous correspondence on a product space is approximately continuous on the product space.

Lemma B.3. *Let S, X and Y be Polish spaces endowed with the Borel σ-algebras, and λ a Borel probability measure on S. Denote $\mathcal{S}$ as the completion of the Borel σ-algebra $\mathcal{B}(S)$ of S under the probability measure λ. Suppose that D is a $\mathcal{B}(S) \otimes \mathcal{B}(Y)$-measurable subset of $S \times Y$, where $D(s)$ is nonempty and compact for all $s \in S$. Let A be a nonempty and compact valued correspondence from D to X, which is sectionally continuous on Y and has a $\mathcal{B}(S \times Y \times X)$-measurable graph. Then*

(i) $\tilde{A}(s) = Gr(A(s, \cdot))$ is an $\mathcal{S}$-measurable mapping from S to the set of nonempty and compact subsets $\mathcal{K}_{Y \times X}$ of $Y \times X$;

(ii) there exist countably many disjoint compact subsets $\{S_m\}_{m \geq 1}$ of S such that (1) $\lambda(\cup_{m \geq 1} S_m) = 1$, and (2) for each $m \geq 1$, $D_m = D \cap (S_m \times Y)$ is compact, and A is nonempty and compact valued, and continuous on each D_m.

Proof. (i) $A(s, \cdot)$ is continuous and $D(s)$ is compact, $\mathrm{Gr}(A(s, \cdot)) \subseteq Y \times X$ is compact by Lemma 2. Thus, $\tilde{A}$ is nonempty and compact valued. Since A has a measurable graph, $\tilde{A}$ is an $\mathcal{S}$-measurable mapping from S to the set of nonempty and compact subsets $\mathcal{K}_{Y \times X}$ of $Y \times X$ by Lemma 1 (4).

(ii) Define a correspondence $\tilde{D}$ from S to Y such that $\tilde{D}(s) = \{y \in Y : (s, y) \in D\}$. Then $\tilde{D}$ is nonempty and compact valued. As in (i), $\tilde{D}$ is $\mathcal{S}$-measurable. By Lemma 3 (Lusin's Theorem), there exists a compact subset $S_1 \subseteq S$ such that $\lambda(S \setminus S_1) < \frac{1}{2}$, $\tilde{D}$ and $\tilde{A}$ are continuous functions on S_1. By Lemma 1 (3), $\tilde{D}$ and $\tilde{A}$ are continuous correspondences on S_1. Let $D_1 = \{(s, y) \in D : s \in S_1, y \in \tilde{D}(s)\}$. Since S_1 is compact and $\tilde{D}$ is continuous, D_1 is compact (see Lemma 2 (6)).

Following the same procedure, for any $m \geq 1$, there exists a compact subset $S_m \subseteq S$ such that (1) $S_m \cap (\cup_{1 \leq k \leq m-1} S_k) = \emptyset$ and $D_m = D \cap (S_m \times Y)$ is compact, (2) $\lambda(S_m) > 0$ and $\lambda(S \setminus (\cup_{1 \leq k \leq m} S_m)) < \frac{1}{2m}$, and (3) A is nonempty and compact valued, and continuous on D_m. This completes the proof. □

The lemma below states an equivalence property for the weak convergence of Borel probability measures obtained from the product of transition probabilities.

Lemma B.4. *Let S and X be Polish spaces, and λ a Borel probability measure on S. Suppose that $\{S_k\}_{k\geq 1}$ is a sequence of disjoint compact subsets of S such that $\lambda(\cup_{k\geq 1}S_k) = 1$. For each k, define a probability measure on S_k as $\lambda_k(D) = \frac{\lambda(D)}{\lambda(S_k)}$ for any measurable subset $D \subseteq S_k$. Let $\{\nu_m\}_{m\geq 0}$ be a sequence of transition probabilities from S to $\mathcal{M}(X)$, and $\tau_m = \lambda \diamond \nu_m$ for any $m \geq 0$. Then τ_m weakly converges to τ_0 if and only if $\lambda_k \diamond \nu_m$ weakly converges to $\lambda_k \diamond \nu_0$ for each $k \geq 1$.*

Proof. First, we assume that τ_m weakly converges to τ_0. For any closed subset $E \subseteq S_k \times X$, we have $\limsup_{m\to\infty} \tau_m(E) \leq \tau_0(E)$. That is, $\limsup_{m\to\infty} \lambda \diamond \nu_m(E) \leq \lambda \diamond \nu_0(E)$. For any k, $\frac{1}{\lambda(S_k)} \limsup_{m\to\infty} \lambda \diamond \nu_m(E) \leq \frac{1}{\lambda(S_k)} \lambda \diamond \nu_0(E)$, which implies that $\limsup_{m\to\infty} \lambda_k \diamond \nu_m(E) \leq \lambda_k \diamond \nu_0(E)$. Thus, $\lambda_k \diamond \nu_m$ weakly converges to $\lambda_k \diamond \nu_0$ for each $k \geq 1$.

Second, we consider the case that $\lambda_k \diamond \nu_m$ weakly converges to $\lambda_k \diamond \nu_0$ for each $k \geq 1$. For any closed subset $E \subseteq S \times X$, let $E_k = E \cap (S_k \times X)$ for each $k \geq 1$. Then $\{E_k\}$ are disjoint closed subsets and $\limsup_{m\to\infty} \lambda_k \diamond \nu_m(E_k) \leq \lambda_k \diamond \nu_0(E_k)$. Since $\lambda_k \diamond \nu_m(E') = \frac{1}{\lambda(S_k)} \lambda \diamond \nu_m(E')$ for any k, m and measurable subset $E' \subseteq S_k \times X$, we have that $\limsup_{m\to\infty} \lambda \diamond \nu_m(E_k) \leq \lambda \diamond \nu_0(E_k)$. Thus,

$$\sum_{k\geq 1} \limsup_{m\to\infty} \lambda \diamond \nu_m(E_k) \leq \sum_{k\geq 1} \lambda \diamond \nu_0(E_k) = \lambda \diamond \nu_0(E).$$

Since the limit superior is subadditive, we have

$$\sum_{k\geq 1} \limsup_{m\to\infty} \lambda \diamond \nu_m(E_k) \geq \limsup_{m\to\infty} \sum_{k\geq 1} \lambda \diamond \nu_m(E_k) = \limsup_{m\to\infty} \lambda \diamond \nu_m(E).$$

Therefore, $\limsup_{m\to\infty} \lambda \diamond \nu_m(E) \leq \lambda \diamond \nu_0(E)$, which implies that τ_m weakly converges to τ_0. □

The following is a key lemma that allows one to drop the continuity condition on the state variables through a reference measure in Theorem 3.

Lemma B.5. *Suppose that X, Y and S are Polish spaces and Z is a compact metric space. Let λ be a Borel probability measure on S, and A a nonempty and compact valued correspondence from $Z \times S$ to X which is sectionally upper hemicontinuous on Z and has a $\mathcal{B}(Z \times S \times X)$-measurable graph. Let G be a nonempty and compact valued, continuous correspondence from Z to $\mathcal{M}(X \times S)$. We assume that for any $z \in Z$ and $\tau \in G(z)$, the marginal of τ on S is λ and $\tau(Gr(A(z, \cdot))) = 1$. Let F be a measurable, nonempty, convex and compact valued correspondence from $Gr(A) \to \mathcal{M}(Y)$ such that F*

is sectionally continuous on $Z \times X$*. Define a correspondence* Π *from* Z *to* $\mathcal{M}(X \times S \times Y)$ *by letting*

$$\Pi(z) = \{g(z) \diamond f(z, \cdot) \colon g \text{ is a Borel measurable selection of } G, \\ f \text{ is a Borel measurable selection of } F\}.$$

Then the correspondence Π *is nonempty and compact valued, and continuous.*

Proof. Let $\mathcal{S}$ be the completion of $\mathcal{B}(S)$ under the probability measure λ. By Lemma B.3, $\tilde{A}(s) = \text{Gr}(A(s, \cdot))$ can be viewed as an $\mathcal{S}$-measurable mapping from S to the set of nonempty and compact subsets $\mathcal{K}_{Z \times X}$ of $Z \times X$. For any $s \in S$, the correspondence $F_s = F(\cdot, s)$ is continuous on $\tilde{A}(s)$. By Lemma 3, there exists a measurable, nonempty and compact valued correspondence $\tilde{F}$ from $Z \times X \times S$ to $\mathcal{M}(Y)$ and a Borel measurable subset S' of S with $\lambda(S') = 1$ such that for each $s \in S'$, $\tilde{F}_s$ is continuous on $Z \times X$, and the restriction of $\tilde{F}_s$ to $\tilde{A}(s)$ is F_s.

By Lemma 3 (Lusin's theorem), there exists a compact subset $S_1 \subseteq S'$ such that $\tilde{A}$ is continuous on S_1 and $\lambda(S_1) > \frac{1}{2}$. Let $K_1 = \tilde{A}(S_1)$. Then $K_1 \subseteq Z \times X$ is compact.

Let $C(K_1, \mathcal{K}_{\mathcal{M}(Y)})$ be the space of continuous functions from K_1 to $\mathcal{K}_{\mathcal{M}(Y)}$, where $\mathcal{K}_{\mathcal{M}(Y)}$ is the set of nonempty and compact subsets of $\mathcal{M}(Y)$. Suppose that the restriction of $\mathcal{S}$ on S_1 is $\mathcal{S}_1$. Let $\tilde{F}_1$ be the restriction of $\tilde{F}$ to $K_1 \times S_1$. Then $\tilde{F}_1$ can be viewed as an $\mathcal{S}_1$-measurable function from S_1 to $C(K_1, \mathcal{K}_{\mathcal{M}(Y)})$ (see Theorem 4.55 in [1]). Again by Lemma 3 (Lusin's theorem), there exists a compact subset of S_1, say itself, such that $\lambda(S_1) > \frac{1}{2}$ and $\tilde{F}_1$ is continuous on S_1. As a result, $\tilde{F}_1$ is a continuous correspondence on $\text{Gr}(A) \cap (S_1 \times Z \times X)$, so is F. Let λ_1 be a probability measure on S_1 such that $\lambda_1(D) = \frac{\lambda(D)}{\lambda(S_1)}$ for any measurable subset $D \subseteq S_1$.

For any $z \in Z$ and $\tau \in G(z)$, the definition of G implies that there exists a transition probability ν from S to X such that $\lambda \diamond \nu = \tau$. Define a correspondence G_1 from Z to $\mathcal{M}(X \times S)$ as follows: for any $z \in Z$, $G_1(z)$ is the set of all $\tau_1 = \lambda_1 \diamond \nu$ such that $\tau = \lambda \diamond \nu \in G(z)$. It can be easily checked that G_1 is also a nonempty and compact valued, and continuous correspondence. Let

$$\Pi_1(z) = \{\tau_1 \diamond f(z, \cdot) \colon \tau_1 = \lambda_1 \diamond \nu \in G_1(z), \\ f \text{ is a Borel measurable selection of } \tilde{F}\}.$$

By Lemma 9, Π_1 is nonempty and compact valued, and continuous. Furthermore, it is easy to see that for any z, $\Pi_1(z)$ coincides with the set

$$\{(\lambda_1 \diamond \nu) \diamond f(z,\cdot)\colon \lambda \diamond \nu \in G(z), f \text{ is a Borel measurable selection of } F\}.$$

Repeat this procedure, one can find a sequence of compact subsets $\{S_t\}$ such that (1) for any $t \geq 1$, $S_t \subseteq S'$, $S_t \cap (S_1 \cup \ldots S_{t-1}) = \emptyset$ and $\lambda(S_1 \cup \ldots \cup S_t) \geq \frac{t}{t+1}$, (2) F is continuous on $\mathrm{Gr}(A) \cap (S_t \times Z \times X)$, λ_t is a probability measure on S_t such that $\lambda_t(D) = \frac{\lambda(D)}{\lambda(S_t)}$ for any measurable subset $D \subseteq S_t$, and (3) the correspondence

$$\Pi_t(z) = \{(\lambda_t \diamond \nu) \diamond f(z,\cdot)\colon \lambda \diamond \nu \in G(z),$$
$$f \text{ is a Borel measurable selection of } F\}.$$

is nonempty and compact valued, and continuous.

Pick a sequence $\{z_k\}$, $\{\nu_k\}$ and $\{f_k\}$ such that $(\lambda \diamond \nu_k) \diamond f_k(z_k,\cdot) \in \Pi(z_k)$, $z_k \to z_0$ and $(\lambda \diamond \nu_k) \diamond f_k(z_k,\cdot)$ weakly converges to some κ. It is easy to see that $(\lambda_t \diamond \nu_k) \diamond f_k(z_k,\cdot) \in \Pi_t(z_k)$ for each t. As Π_1 is compact valued and continuous, it has a subsequence, say itself, such that z_k converges to some $z_0 \in Z$ and $(\lambda_1 \diamond \nu_k) \diamond f_k(z_k,\cdot)$ weakly converges to some $(\lambda_1 \diamond \mu^1) \diamond f^1(z_0,\cdot) \in \Pi_1(z_0)$. Repeat this procedure, one can get a sequence of $\{\mu^m\}$ and f^m. Let $\mu(s) = \mu^m(s)$ and $f(z_0,s,x) = f^m(z_0,s,x)$ for any $x \in A(z_0,s)$ when $s \in S_m$. By Lemma B.4, $(\lambda \diamond \mu) \diamond f(z_0,\cdot) = \kappa$, which implies that Π is upper hemicontinuous.

Similarly, the compactness and lower hemicontinuity of Π follow from the compactness and lower hemicontinuity of Π_t for each t. □

The next lemma presents the convergence property for the integrals of a sequence of functions and probability measures.

Lemma B.6. *Let S and X be Polish spaces, and A a measurable, nonempty and compact valued correspondence from S to X. Suppose that λ is a Borel probability measure on S and $\{\nu_n\}_{1\leq n\leq\infty}$ is a sequence of transition probabilities from S to $\mathcal{M}(X)$ such that $\nu_n(A(s)|s) = 1$ for each s and n. For each $n \geq 1$, let $\tau_n = \lambda \diamond \nu_n$. Assume that the sequence $\{\tau_n\}$ of Borel probability measures on $S \times X$ converges weakly to a Borel probability measure τ_∞ on $S \times X$. Let $\{g_n\}_{1\leq n\leq\infty}$ be a sequence of functions satisfying the following three properties.*

1. *For each n between 1 and ∞, $g_n\colon S \times X \to \mathbb{R}_+$ is measurable and sectionally continuous on X.*

2. *For any $s \in S$ and any sequence $x_n \to x_\infty$ in X, $g_n(s, x_n) \to g_\infty(s, x_\infty)$ as $n \to \infty$.*

3. *The sequence $\{g_n\}_{1 \le n \le \infty}$ is integrably bounded in the sense that there exists a λ-integrable function $\psi\colon S \to \mathbb{R}_+$ such that for any n, s and x, $g_n(s, x) \le \psi(s)$.*

Then we have

$$\int_{S \times X} g_n(s, x)\tau_n(\mathrm{d}(s, x)) \to \int_{S \times X} g_\infty(s, x)\tau_\infty(\mathrm{d}(s, x)).$$

Proof. By Theorem 2.1.3 in [2], for any integrably bounded function $g\colon S \times X \to \mathbb{R}_+$ which is sectionally continuous on X, we have

$$\int_{S \times X} g(s, x)\tau_n(\mathrm{d}(s, x)) \to \int_{S \times X} g(s, x)\tau_\infty(\mathrm{d}(s, x)). \tag{1}$$

Let $\{y_n\}_{1 \le n \le \infty}$ be a sequence such that $y_n = \frac{1}{n}$ and $y_\infty = 0$. Then $y_n \to y_\infty$. Define a mapping $\tilde{g}$ from $S \times X \times \{y_1, \ldots, y_\infty\}$ such that $\tilde{g}(s, x, y_n) = g_n(s, x)$. Then $\tilde{g}$ is measurable on S and continuous on $X \times \{y_1, \ldots, y_\infty\}$. Define a correspondence G from S to $X \times \{y_1, \ldots, y_\infty\} \times \mathbb{R}_+$ such that

$$G(s) = \{(x, y_n, c)\colon c \in \tilde{g}(s, x, y_n), x \in A(s), 1 \le n \le \infty\}.$$

For any s, $A(s) \times \{y_1, \ldots, y_\infty\}$ is compact and $\tilde{g}(s, \cdot, \cdot)$ is continuous. By Lemma 2 (6), $G(s)$ is compact. By Lemma 1 (2), G can be viewed as a measurable mapping from S to the space of nonempty compact subsets of $X \times \{y_1, \ldots, y_\infty\} \times \mathbb{R}_+$. Similarly, A can be viewed as a measurable mapping from S to the space of nonempty compact subsets of X.

Fix an arbitrary $\epsilon > 0$. By Lemma 3 (Lusin's theorem), there exists a compact subset $S_1 \subseteq S$ such that A and G are continuous on S_1 and $\lambda(S \setminus S_1) < \epsilon$. Without loss of generality, we can assume that $\lambda(S \setminus S_1)$ is sufficiently small such that $\int_{S \setminus S_1} \psi(s)\lambda(\mathrm{d}s) < \frac{\epsilon}{6}$. Thus, for any n,

$$\int_{(S \setminus S_1) \times X} \psi(s)\tau_n(\mathrm{d}(s, x)) = \int_{(S \setminus S_1)} \psi(s)\nu_n(X)\lambda(\mathrm{d}s) < \frac{\epsilon}{6}.$$

By Lemma 2 (6), the set $E = \{(s, x)\colon s \in S_1, x \in A(s)\}$ is compact. Since G is continuous on S_1, $\tilde{g}$ is continuous on $E \times \{y_1, \ldots, y_\infty\}$. Since $E \times \{y_1, \ldots, y_\infty\}$ is compact, $\tilde{g}$ is uniformly continuous on $E \times \{y_1, \ldots, y_\infty\}$. Thus, there exists a positive integer $N_1 > 0$ such that for any $n \ge N_1$, $|g_n(s, x) - g_\infty(s, x)| < \frac{\epsilon}{3}$ for any $(s, x) \in E$.

By Equation (1), there exists a positive integer N_2 such that for any $n \geq N_2$,

$$\left|\int_{S\times X} g_\infty(s,x)\tau_n(\mathrm{d}(s,x)) - \int_{S\times X} g_\infty(s,x)\tau_\infty(\mathrm{d}(s,x))\right| < \frac{\epsilon}{3}.$$

Let $N_0 = \max\{N_1, N_2\}$. For any $n \geq N_0$,

$$\begin{aligned}
&\left|\int_{S\times X} g_n(s,x)\tau_n(\mathrm{d}(s,x)) - \int_{S\times X} g_\infty(s,x)\tau_\infty(\mathrm{d}(s,x))\right| \\
\leq &\left|\int_{S\times X} g_n(s,x)\tau_n(\mathrm{d}(s,x)) - \int_{S\times X} g_\infty(s,x)\tau_n(\mathrm{d}(s,x))\right| \\
+ &\left|\int_{S\times X} g_\infty(s,x)\tau_n(\mathrm{d}(s,x)) - \int_{S\times X} g_\infty(s,x)\tau_\infty(\mathrm{d}(s,x))\right| \\
\leq &\left|\int_{S_1\times X} g_n(s,x)\tau_n(\mathrm{d}(s,x)) - \int_{S_1\times X} g_\infty(s,x)\tau_n(\mathrm{d}(s,x))\right| \\
+ &\left|\int_{(S\setminus S_1)\times X} g_n(s,x)\tau_n(\mathrm{d}(s,x)) - \int_{(S\setminus S_1)\times X} g_\infty(s,x)\tau_n(\mathrm{d}(s,x))\right| \\
+ &\left|\int_{S\times X} g_\infty(s,x)\tau_n(\mathrm{d}(s,x)) - \int_{S\times X} g_\infty(s,x)\tau_\infty(\mathrm{d}(s,x))\right| \\
\leq &\int_E \left|g_n(s,x) - g_\infty(s,x)\right| \tau_n(\mathrm{d}(s,x)) + 2\cdot\int_{(S\setminus S_1)\times X} \psi(s)\tau_n(\mathrm{d}(s,x)) \\
+ &\left|\int_{S\times X} g_\infty(s,x)\tau_n(\mathrm{d}(s,x)) - \int_{S\times X} g_\infty(s,x)\tau_\infty(\mathrm{d}(s,x))\right| \\
< &\frac{\epsilon}{3} + 2\cdot\frac{\epsilon}{6} + \frac{\epsilon}{3} \\
= &\epsilon.
\end{aligned}$$

This completes the proof. □

B.3 Discontinuous games with endogenous stochastic sharing rules

It was proved in [7] that a Nash equilibrium exists in discontinuous games with endogenous sharing rules. In particular, they considered a static game with a payoff correspondence P that is bounded, nonempty, convex and compact valued, and upper hemicontinuous. They showed that there exists a Borel measurable selection p of the payoff correspondence, namely the endogenous sharing rule, and a mixed strategy profile α such that α is a Nash equilibrium when players take p as the payoff function (see Lemma 10).

In this section, we shall consider discontinuous games with endogenous stochastic sharing rules. That is, we allow the payoff correspondence to depend on some state variable in a measurable way as follows:

1. let S be a Borel subset of a Polish space, Y a Polish space, and λ a Borel probability measure on S;
2. D is a $\mathcal{B}(S) \otimes \mathcal{B}(Y)$-measurable subset of $S \times Y$, where $D(s)$ is compact for all $s \in S$ and $\lambda(\{s \in S \colon D(s) \neq \emptyset\}) > 0$;
3. $X = \prod_{1 \leq i \leq n} X_i$, where each X_i is a Polish space;
4. for each i, A_i is a measurable, nonempty and compact valued correspondence from D to X_i, which is sectionally continuous on Y;
5. $A = \prod_{1 \leq i \leq n} A_i$ and $E = \mathrm{Gr}(A)$;
6. P is a bounded, measurable, nonempty, convex and compact valued correspondence from E to $\mathbb{R}^n$ which is essentially sectionally upper hemicontinuous on $Y \times X$.

A stochastic sharing rule at $(s, y) \in D$ is a Borel measurable selection of the correspondence $P(s, y, \cdot)$; i.e., a Borel measurable function $p \colon A(s, y) \to \mathbb{R}^n$ such that $p(x) \in P(s, y, x)$ for all $x \in A(s, y)$. Given $(s, y) \in D$, $P(s, y, \cdot)$ represents the set of all possible payoff profiles, and a sharing rule p is a particular choice of the payoff profile.

Now we shall prove the following proposition.

Proposition B.2. *There exists a $\mathcal{B}(D)$-measurable, nonempty and compact valued correspondence Φ from D to $\mathbb{R}^n \times \mathcal{M}(X) \times \triangle(X)$ such that Φ is essentially sectionally upper hemicontinuous on Y, and for λ-almost all $s \in S$ with $D(s) \neq \emptyset$ and $y \in D(s)$, $\Phi(s, y)$ is the set of points (v, α, μ) that*

1. *$v = \int_X p(s, y, x)\alpha(\mathrm{d}x)$ such that $p(s, y, \cdot)$ is a Borel measurable selection of $P(s, y, \cdot)$;*[2]
2. *$\alpha \in \otimes_{i \in I} \mathcal{M}(A_i(s, y))$ is a Nash equilibrium in the subgame (s, y) with payoff profile $p(s, y, \cdot)$, and action space $A_i(s, y)$ for each player i;*
3. *$\mu = p(s, y, \cdot) \circ \alpha$.*[3]

[2] Note that we require $p(s, y, \cdot)$ to be measurable for each (s, y), but p may not be jointly measurable.
[3] The finite measure $\mu = p(s, y, \cdot) \circ \alpha$ if $\mu(B) = \int_B p(s, y, x)\alpha(\mathrm{d}x)$ for any Borel subset $B \subseteq X$.

In addition, denote the restriction of Φ on the first component $\mathbb{R}^n$ as $\Phi|_{\mathbb{R}^n}$, which is a correspondence from D to $\mathbb{R}^n$. Then $\Phi|_{\mathbb{R}^n}$ is bounded, measurable, nonempty and compact valued, and essentially sectionally upper hemicontinuous on Y.

Proof. There exists a Borel subset $\hat{S} \subseteq S$ with $\lambda(\hat{S}) = 1$ such that $D(s) \neq \emptyset$ for each $s \in \hat{S}$, and P is sectionally upper hemicontinuous on Y when it is restricted on $D \cap (\hat{S} \times Y)$. Without loss of generality, we assume that $\hat{S} = S$.

Suppose that $\mathcal{S}$ is the completion of $\mathcal{B}(S)$ under the probability measure λ. Let $\mathcal{D}$ and $\mathscr{E}$ be the restrictions of $\mathcal{S} \otimes \mathcal{B}(Y)$ and $\mathcal{S} \otimes \mathcal{B}(Y) \otimes \mathcal{B}(X)$ on D and E, respectively.

Define a correspondence $\tilde{D}$ from S to Y such that $\tilde{D}(s) = \{y \in Y : (s, y) \in D\}$. Then $\tilde{D}$ is nonempty and compact valued. By Lemma 1 (4), $\tilde{D}$ is $\mathcal{S}$-measurable.

Since $\tilde{D}(s)$ is compact and $A(s, \cdot)$ is upper hemicontinuous for any $s \in S$, $E(s)$ is compact by Lemma 2 (6). Define a correspondence Γ from S to $Y \times X \times \mathbb{R}^n$ as $\Gamma(s) = \mathrm{Gr}(P(s, \cdot, \cdot))$. For all s, $P(s, \cdot, \cdot)$ is bounded, upper hemicontinuous and compact valued on $E(s)$, hence it has a compact graph. As a result, Γ is compact valued. By Lemma 1 (1), P has an $\mathcal{S} \otimes \mathcal{B}(Y \times X \times \mathbb{R}^n)$-measurable graph. Since $\mathrm{Gr}(\Gamma) = \mathrm{Gr}(P)$, $\mathrm{Gr}(\Gamma)$ is $\mathcal{S} \otimes \mathcal{B}(Y \times X \times \mathbb{R}^n)$-measurable. Due to Lemma 1 (4), the correspondence Γ is $\mathcal{S}$-measurable. We can view Γ as a function from S into the space $\mathcal{K}$ of nonempty compact subsets of $Y \times X \times \mathbb{R}^n$. By Lemma B.1, $\mathcal{K}$ is a Polish space endowed with the Hausdorff metric topology. Then by Lemma 1 (2), Γ is an $\mathcal{S}$-measurable function from S to $\mathcal{K}$. One can also define a correspondence $\tilde{A}_i$ from S to $Y \times X$ as $\tilde{A}_i(s) = \mathrm{Gr}(A_i(s, \cdot))$. It is easy to show that $\tilde{A}_i$ can be viewed as an $\mathcal{S}$-measurable function from S to the space of nonempty compact subsets of $Y \times X$, which is endowed with the Hausdorff metric topology. By a similar argument, $\tilde{D}$ can be viewed as an $\mathcal{S}$-measurable function from S to the space of nonempty compact subsets of Y.

By Lemma 3 (Lusin's Theorem), there exists a compact subset $S_1 \subseteq S$ such that $\lambda(S \setminus S_1) < \frac{1}{2}$, Γ, $\tilde{D}$ and $\{\tilde{A}_i\}_{1 \leq i \leq n}$ are continuous functions on S_1. By Lemma 1 (3), Γ, $\tilde{D}$ and $\tilde{A}_i$ are continuous correspondences on S_1. Let $D_1 = \{(s, y) \in D : s \in S_1, y \in \tilde{D}(s)\}$. Since S_1 is compact and $\tilde{D}$ is continuous, D_1 is compact (see Lemma 2 (6)). Similarly, $E_1 = E \cap (S_1 \times Y \times X)$ is also compact. Thus, P is an upper hemicontinuous correspondence on E_1. Define a correspondence Φ_1 from D_1 to $\mathbb{R}^n \times \mathcal{M}(X) \times \triangle(X)$ as in Lemma 10, then it is nonempty and compact valued, and upper hemicontinuous on D_1.

Following the same procedure, for any $m \geq 1$, there exists a compact subset $S_m \subseteq S$ such that (1) $S_m \cap (\cup_{1 \leq k \leq m-1} S_k) = \emptyset$ and $D_m = D \cap (S_m \times Y)$ is compact, (2) $\lambda(S_m) > 0$ and $\lambda\left(S \setminus (\cup_{1 \leq k \leq m} S_m)\right) < \frac{1}{2m}$, and (3) there is a nonempty and compact valued, upper hemicontinuous correspondence Φ_m from D_m to $\mathbb{R}^n \times \mathcal{M}(X) \times \triangle(X)$, which satisfies conditions (1)-(3) in Lemma 10. Thus, we have countably many disjoint sets $\{S_m\}_{m\geq 1}$ such that (1) $\lambda(\cup_{m\geq 1} S_m) = 1$, (2) Φ_m is nonempty and compact valued, and upper hemicontinuous on each D_m, $m \geq 1$.

Since A_i is a $\mathcal{B}(S)\otimes\mathcal{B}(Y)$-measurable, nonempty and compact valued correspondence, it has a Borel measurable selection a_i by Lemma 2 (3). Fix a Borel measurable selection p of P (such a selection exists also due to Lemma 2 (3)). Define a mapping (v_0, α_0, μ_0) from D to $\mathbb{R}^n \times \mathcal{M}(X) \times \triangle(X)$ such that (1) $\alpha_i(s, y) = \delta_{a_i(s,y)}$ and $\alpha_0(s, y) = \otimes_{i\in I}\alpha_i(s, y)$; (2) $v_0(s, y) = p(s, y, a_1(s, y). \ldots, a_n(s, y))$ and (3) $\mu_0(s, y) = p(s, y, \cdot) \circ \alpha_0$. Let $D_0 = D \setminus (\cup_{m\geq 1} D_m)$ and $\Phi_0(s, y) = \{(v_0(s, y), \alpha_0(s, y), \mu_0(s, y))\}$ for $(s, y) \in D_0$. Then, Φ_0 is $\mathcal{B}(S) \otimes \mathcal{B}(Y)$-measurable, nonempty and compact valued.

Let $\Phi(s, y) = \Phi_m(s, y)$ if $(s, y) \in D_m$ for some $m \geq 0$. Then, $\Phi(s, y)$ satisfies conditions (1)-(3) if $(s, y) \in D_m$ for $m \geq 1$. That is, Φ is $\mathcal{B}(D)$-measurable, nonempty and compact valued, and essentially sectionally upper hemicontinuous on Y, and satisfies conditions (1)-(3) for λ-almost all $s \in S$.

Then consider $\Phi|_{\mathbb{R}^n}$, which is the restriction of Φ on the first component $\mathbb{R}^n$. Let $\Phi_m|_{\mathbb{R}^n}$ be the restriction of Φ_m on the first component $\mathbb{R}^n$ with the domain D_m for each $m \geq 0$. It is obvious that $\Phi_0|_{\mathbb{R}^n}$ is measurable, nonempty and compact valued. For each $m \geq 1$, D_m is compact and Φ_m is upper hemicontinuous and compact valued. By Lemma 2 (6), $\mathrm{Gr}(\Phi_m)$ is compact. Thus, $\mathrm{Gr}(\Phi_m|_{\mathbb{R}^n})$ is also compact. By Lemma 2 (4), $\Phi_m|_{\mathbb{R}^n}$ is measurable. In addition, $\Phi_m|_{\mathbb{R}^n}$ is nonempty and compact valued, and upper hemicontinuous on D_m. Notice that $\Phi|_{\mathbb{R}^n}(s, y) = \Phi_m|_{\mathbb{R}^n}(s, y)$ if $(s, y) \in D_m$ for some $m \geq 0$. Thus, $\Phi|_{\mathbb{R}^n}$ is measurable, nonempty and compact valued, and essentially sectionally upper hemicontinuous on Y.

The proof is complete. □

B.4 Proof of Theorem 3

B.4.1 Backward induction

For any $t \geq 1$, suppose that the correspondence Q_{t+1} from H_t to $\mathbb{R}^n$ is bounded, measurable, nonempty and compact valued, and essentially sectionally upper hemicontinuous on X^t. For any $h_{t-1} \in H_{t-1}$ and $x_t \in A_t(h_{t-1})$, let

$$P_t(h_{t-1}, x_t) = \int_{S_t} Q_{t+1}(h_{t-1}, x_t, s_t) f_{t0}(\mathrm{d}s_t|h_{t-1})$$

$$= \int_{S_t} Q_{t+1}(h_{t-1}, x_t, s_t)\varphi_{t0}(h_{t-1}, s_t)\lambda_t(\mathrm{d}s_t).$$

It is obvious that the correspondence P_t is measurable and nonempty valued. Since Q_{t+1} is bounded, P_t is bounded. For λ^t-almost all $s^t \in S^t$, $Q_{t+1}(\cdot, s^t)$ is bounded and upper hemicontinuous on $H_t(s^t)$, and $\varphi_{t0}(s^t, \cdot)$ is continuous on $\mathrm{Gr}(A_0^t)(s^t)$. As φ_{t0} is integrably bounded, $P_t(s^{t-1}, \cdot)$ is also upper hemicontinuous on $\mathrm{Gr}(A^t)(s^{t-1})$ for λ^{t-1}-almost all $s^{t-1} \in S^{t-1}$ (see Lemma 4); that is, the correspondence P_t is essentially sectionally upper hemicontinuous on X^t. Again by Lemma 4, P_t is convex and compact valued since λ_t is an atomless probability measure. That is, $P_t \colon \mathrm{Gr}(A^t) \to \mathbb{R}^n$ is a bounded, measurable, nonempty, convex and compact valued correspondence which is essentially sectionally upper hemicontinuous on X^t.

By Proposition B.2, there exists a bounded, measurable, nonempty and compact valued correspondence Φ_t from H_{t-1} to $\mathbb{R}^n \times \mathcal{M}(X_t) \times \triangle(X_t)$ such that Φ_t is essentially sectionally upper hemicontinuous on X^{t-1}, and for λ^{t-1}-almost all $h_{t-1} \in H_{t-1}$, $(v, \alpha, \mu) \in \Phi_t(h_{t-1})$ if

1. $v = \int_{A_t(h_{t-1})} p_t(h_{t-1}, x)\alpha(\mathrm{d}x)$ such that $p_t(h_{t-1}, \cdot)$ is a Borel measurable selection of $P_t(h_{t-1}, \cdot)$;
2. $\alpha \in \otimes_{i\in I}\mathcal{M}(A_{ti}(h_{t-1}))$ is a Nash equilibrium in the subgame h_{t-1} with payoff $p_t(h_{t-1}, \cdot)$ and action space $\prod_{i\in I} A_{ti}(h_{t-1})$;
3. $\mu = p_t(h_{t-1}, \cdot) \circ \alpha$.

Denote the restriction of Φ_t on the first component $\mathbb{R}^n$ as $\Phi(Q_{t+1})$, which is a correspondence from H_{t-1} to $\mathbb{R}^n$. By Proposition B.2, $\Phi(Q_{t+1})$ is bounded, measurable, nonempty and compact valued, and essentially sectionally upper hemicontinuous on X^{t-1}.

B.4.2 Forward induction

The following proposition presents the result on the step of forward induction.

Proposition B.3. *For any $t \geq 1$ and any Borel measurable selection q_t of $\Phi(Q_{t+1})$, there exists a Borel measurable selection q_{t+1} of Q_{t+1} and a Borel measurable mapping $f_t \colon H_{t-1} \to \otimes_{i\in I}\mathcal{M}(X_{ti})$ such that for λ^{t-1}-almost all $h_{t-1} \in H_{t-1}$,*

1. *$f_t(h_{t-1}) \in \otimes_{i\in I}\mathcal{M}(A_{ti}(h_{t-1}))$;*

2. $q_t(h_{t-1}) = \int_{A_t(h_{t-1})} \int_{S_t} q_{t+1}(h_{t-1}, x_t, s_t) f_{t0}(\mathrm{d}s_t | h_{t-1}) f_t(\mathrm{d}x_t | h_{t-1})$;

3. *$f_t(\cdot | h_{t-1})$ is a Nash equilibrium in the subgame h_{t-1} with action spaces $A_{ti}(h_{t-1}), i \in I$ and the payoff functions*

$$\int_{S_t} q_{t+1}(h_{t-1}, \cdot, s_t) f_{t0}(\mathrm{d}s_t | h_{t-1}).$$

Proof. We divide the proof into three steps. In step 1, we show that there exist Borel measurable mappings $f_t \colon H_{t-1} \to \otimes_{i\in I} \mathcal{M}(X_{ti})$ and $\mu_t \colon H_{t-1} \to \triangle(X_t)$ such that (q_t, f_t, μ_t) is a selection of Φ_t. In step 2, we obtain a Borel measurable selection g_t of P_t such that for λ^{t-1}-almost all $h_{t-1} \in H_{t-1}$,

1. $q_t(h_{t-1}) = \int_{A_t(h_{t-1})} g_t(h_{t-1}, x) f_t(\mathrm{d}x | h_{t-1})$;

2. $f_t(h_{t-1})$ is a Nash equilibrium in the subgame h_{t-1} with payoff $g_t(h_{t-1}, \cdot)$ and action space $A_t(h_{t-1})$;

In step 3, we show that there exists a Borel measurable selection q_{t+1} of Q_{t+1} such that for all $h_{t-1} \in H_{t-1}$ and $x_t \in A_t(h_{t-1})$,

$$g_t(h_{t-1}, x_t) = \int_{S_t} q_{t+1}(h_{t-1}, x_t, s_t) f_{t0}(\mathrm{d}s_t | h_{t-1}).$$

Combining Steps 1-3, the proof is complete.

Step 1. Let $\Psi_t \colon \mathrm{Gr}(\Phi_t(Q_{t+1})) \to \mathcal{M}(X_t) \times \triangle(X_t)$ be

$$\Psi_t(h_{t-1}, v) = \{(\alpha, \mu) \colon (v, \alpha, \mu) \in \Phi_t(h_{t-1})\}.$$

Recall the construction of Φ_t and the proof of Proposition B.2, H_{t-1} can be divided into countably many Borel subsets $\{H^m_{t-1}\}_{m\ge 0}$ such that

1. $H_{t-1} = \cup_{m\ge 0} H^m_{t-1}$ and $\frac{\lambda^{t-1}(\cup_{m\ge 1} \mathrm{proj}_{S^{t-1}}(H^m_{t-1}))}{\lambda^{t-1}(\mathrm{proj}_{S^{t-1}}(H_{t-1}))} = 1$, where $\mathrm{proj}_{S^{t-1}}(H^m_{t-1})$ and $\mathrm{proj}_{S^{t-1}}(H_{t-1})$ are projections of H^m_{t-1} and H_{t-1} on S^{t-1};

2. for $m \ge 1$, H^m_{t-1} is compact, Φ_t is upper hemicontinuous on H^m_{t-1}, and P_t is upper hemicontinuous on

$$\{(h_{t-1}, x_t) \colon h_{t-1} \in H^m_{t-1}, x_t \in A_t(h_{t-1})\};$$

3. there exists a Borel measurable mapping (v_0, α_0, μ_0) from H_{t-1}^0 to $\mathbb{R}^n \times \mathcal{M}(X_t) \times \triangle(X_t)$ such that $\Phi_t(h_{t-1}) \equiv \{(v_0(h_{t-1}), \alpha_0(h_{t-1}), \mu_0(h_{t-1}))\}$ for any $h_{t-1} \in H_{t-1}^0$.

Denote the restriction of Φ_t on H_{t-1}^m as Φ_t^m. For $m \geq 1$, $\text{Gr}(\Phi_t^m)$ is compact, and hence the correspondence $\Psi_t^m(h_{t-1}, v) = \{(\alpha, \mu) \colon (v, \alpha, \mu) \in \Phi_t^m(h_{t-1})\}$ has a compact graph. For $m \geq 1$, Ψ_t^m is measurable by Lemma 2 (4), and has a Borel measurable selection ψ_t^m due to Lemma 2 (3). Define $\psi_t^0(h_{t-1}, v_0(h_{t-1})) = (\alpha_0(h_{t-1}), \mu_0(h_{t-1}))$ for $h_{t-1} \in H_{t-1}^0$. For $(h_{t-1}, v) \in \text{Gr}(\Phi(Q_{t+1}))$, let $\psi_t(h_{t-1}, v) = \psi_t^m(h_{t-1}, v)$ if $h_{t-1} \in H_{t-1}^m$. Then ψ_t is a Borel measurable selection of Ψ_t.

Given a Borel measurable selection q_t of $\Phi(Q_{t+1})$, let

$$\phi_t(h_{t-1}) = (q_t(h_{t-1}), \psi_t(h_{t-1}, q_t(h_{t-1}))).$$

Then ϕ_t is a Borel measurable selection of Φ_t. Denote $\tilde{H}_{t-1} = \cup_{m \geq 1} H_{t-1}^m$. By the construction of Φ_t, there exists Borel measurable mappings $f_t \colon H_{t-1} \to \otimes_{i \in I} \mathcal{M}(X_{ti})$ and $\mu_t \colon H_{t-1} \to \triangle(X_t)$ such that for all $h_{t-1} \in \tilde{H}_{t-1}$,

1. $q_t(h_{t-1}) = \int_{A_t(h_{t-1})} p_t(h_{t-1}, x) f_t(\mathrm{d}x|h_{t-1})$ such that $p_t(h_{t-1}, \cdot)$ is a Borel measurable selection of $P_t(h_{t-1}, \cdot)$;

2. $f_t(h_{t-1}) \in \otimes_{i \in I} \mathcal{M}(A_{ti}(h_{t-1}))$ is a Nash equilibrium in the subgame h_{t-1} with payoff $p_t(h_{t-1}, \cdot)$ and action space $\prod_{i \in I} A_{ti}(h_{t-1})$;

3. $\mu_t(\cdot|h_{t-1}) = p_t(h_{t-1}, \cdot) \circ f_t(\cdot|h_{t-1})$.

Step 2. Since P_t is upper hemicontinuous on $\{(h_{t-1}, x_t) \colon h_{t-1} \in H_{t-1}^m, x_t \in A_t(h_{t-1})\}$, due to Lemma 6, there exists a Borel measurable mapping g^m such that (1) $g^m(h_{t-1}, x_t) \in P_t(h_{t-1}, x_t)$ for any $h_{t-1} \in H_{t-1}^m$ and $x_t \in A_t(h_{t-1})$, and (2) $g^m(h_{t-1}, x_t) = p_t(h_{t-1}, x_t)$ for $f_t(\cdot|h_{t-1})$-almost all x_t. Fix an arbitrary Borel measurable selection g' of P_t. Define a Borel measurable mapping from $\text{Gr}(A_t)$ to $\mathbb{R}^n$ as

$$g(h_{t-1}, x_t) = \begin{cases} g^m(h_{t-1}, x_t) & \text{if } h_{t-1} \in H_{t-1}^m \text{ for } m \geq 1; \\ g'(h_{t-1}, x_t) & \text{otherwise.} \end{cases}$$

Then g is a Borel measurable selection of P_t.

In a subgame $h_{t-1} \in \tilde{H}_{t-1}$, let

$$B_{ti}(h_{t-1}) = \{y_i \in A_{ti}(h_{t-1}) \colon$$

$$\int_{A_{t(-i)}(h_{t-1})} g_i(h_{t-1}, y_i, x_{t(-i)}) f_{t(-i)}(\mathrm{d}x_{t(-i)}|h_{t-1}) > \int_{A_t(h_{t-1})} p_{ti}(h_{t-1}, x_t) f_t(\mathrm{d}x_t|h_{t-1})\}.$$

Since $g(h_{t-1}, x_t) = p_t(h_{t-1}, x_t)$ for $f_t(\cdot|h_{t-1})$-almost all x_t,

$$\int_{A_t(h_{t-1})} g(h_{t-1}, x_t) f_t(\mathrm{d}x_t|h_{t-1}) = \int_{A_t(h_{t-1})} p_t(h_{t-1}, x_t) f_t(\mathrm{d}x_t|h_{t-1}).$$

Thus, B_{ti} is a measurable correspondence from $\tilde{H}_{t-1}$ to $A_{ti}(h_{t-1})$. Let $B_{ti}^c(h_{t-1}) = A_{ti}(h_{t-1}) \setminus B_{ti}(h_{t-1})$ for each $h_{t-1} \in H_{t-1}$. Then B_{ti}^c is a measurable and closed valued correspondence, which has a Borel measurable graph by Lemma 1. As a result, B_{ti} also has a Borel measurable graph. As $f_t(h_{t-1})$ is a Nash equilibrium in the subgame $h_{t-1} \in \tilde{H}_{t-1}$ with payoff $p_t(h_{t-1}, \cdot)$, $f_{ti}(B_{ti}(h_{t-1})|h_{t-1}) = 0$.

Denote $\beta_i(h_{t-1}, x_t) = \min P_{ti}(h_{t-1}, x_t)$, where $P_{ti}(h_{t-1}, x_t)$ is the projection of $P_t(h_{t-1}, x_t)$ on the i-th dimension. Then the correspondence P_{ti} is measurable and compact valued, and β_i is Borel measurable. Let $\Lambda_i(h_{t-1}, x_t) = \{\beta_i(h_{t-1}, x_t)\} \times [0, \gamma]^{n-1}$, where $\gamma > 0$ is the upper bound of P_t. Denote $\Lambda_i'(h_{t-1}, x_t) = \Lambda_i(h_{t-1}, x_t) \cap P_t(h_{t-1}, x_t)$. Then Λ_i' is a measurable and compact valued correspondence, and hence has a Borel measurable selection β_i'. Note that β_i' is a Borel measurable selection of P_t. Let

$$g_t(h_{t-1}, x_t) =$$

$$\begin{cases} \beta_i'(h_{t-1}, x_t) & \text{if } h_{t-1} \in \tilde{H}_{t-1}, x_{ti} \in B_{ti}(h_{t-1}) \text{ and } x_{tj} \notin B_{tj}(h_{t-1}), \forall j \neq i; \\ g(h_{t-1}, x_t) & \text{otherwise.} \end{cases}$$

Notice that

$$\begin{aligned} &\{(h_{t-1}, x_t) \in \mathrm{Gr}(A_t)\colon h_{t-1} \in \tilde{H}_{t-1}, x_{ti} \in B_{ti}(h_{t-1}) \text{ and } x_{tj} \notin B_{tj}(h_{t-1}), \forall j \neq i;\} \\ &= \mathrm{Gr}(A_t) \cap \cup_{i \in I} \left((\mathrm{Gr}(B_{ti}) \times \prod_{j \neq i} X_{tj}) \setminus (\cup_{j \neq i} (\mathrm{Gr}(B_{tj}) \times \prod_{k \neq j} X_{tk})) \right), \end{aligned}$$

which is a Borel set. As a result, g_t is a Borel measurable selection of P_t. Moreover, $g_t(h_{t-1}, x_t) = p_t(h_{t-1}, x_t)$ for all $h_{t-1} \in \tilde{H}_{t-1}$ and $f_t(\cdot|h_{t-1})$-almost all x_t.

Fix a subgame $h_{t-1} \in \tilde{H}_{t-1}$. We will show that $f_t(\cdot|h_{t-1})$ is a Nash equilibrium given the payoff $g_t(h_{t-1}, \cdot)$ in the subgame h_{t-1}. Suppose that player i deviates to some action $\tilde{x}_{ti}$.

If $\tilde{x}_{ti} \in B_{ti}(h_{t-1})$, then player i's expected payoff is

$$\begin{aligned}
&\int_{A_{t(-i)}(h_{t-1})} g_{ti}(h_{t-1}, \tilde{x}_{ti}, x_{t(-i)}) f_{t(-i)}(\mathrm{d}x_{t(-i)}|h_{t-1}) \\
&= \int_{\prod_{j\neq i} B^c_{tj}(h_{t-1})} g_{ti}(h_{t-1}, \tilde{x}_{ti}, x_{t(-i)}) f_{t(-i)}(\mathrm{d}x_{t(-i)}|h_{t-1}) \\
&= \int_{\prod_{j\neq i} B^c_{tj}(h_{t-1})} \beta_{i}(h_{t-1}, \tilde{x}_{ti}, x_{t(-i)}) f_{t(-i)}(\mathrm{d}x_{t(-i)}|h_{t-1}) \\
&\leq \int_{\prod_{j\neq i} B^c_{tj}(h_{t-1})} p_{ti}(h_{t-1}, \tilde{x}_{ti}, x_{t(-i)}) f_{t(-i)}(\mathrm{d}x_{t(-i)}|h_{t-1}) \\
&= \int_{A_{t(-i)}(h_{t-1})} p_{ti}(h_{t-1}, \tilde{x}_{ti}, x_{t(-i)}) f_{t(-i)}(\mathrm{d}x_{t(-i)}|h_{t-1}) \\
&\leq \int_{A_{t}(h_{t-1})} p_{ti}(h_{t-1}, x_{t}) f_{t}(\mathrm{d}x_{t}|h_{t-1}) \\
&= \int_{A_{t}(h_{t-1})} g_{ti}(h_{t-1}, x_{t}) f_{t}(\mathrm{d}x_{t}|h_{t-1}).
\end{aligned}$$

The first and the third equalities hold since $f_{tj}(B_{tj}(h_{t-1})|h_{t-1}) = 0$ for each j, and hence $f_{t(-i)}(\prod_{j\neq i} B^c_{tj}(h_{t-1})|h_{t-1}) = f_{t(-i)}(A_{t(-i)}(h_{t-1})|h_{t-1})$. The second equality and the first inequality are due to the fact that $g_{ti}(h_{t-1}, \tilde{x}_{ti}, x_{t(-i)}) = \beta_i(h_{t-1}, \tilde{x}_{ti}, x_{t(-i)}) = \min P_{ti}(h_{t-1}, \tilde{x}_{ti}, x_{t(-i)}) \leq p_{ti}(h_{t-1}, \tilde{x}_{ti}, x_{t(-i)})$ for $x_{t(-i)} \in \prod_{j\neq i} B^c_{tj}(h_{t-1})$. The second inequality holds since $f_t(\cdot|h_{t-1})$ is a Nash equilibrium given the payoff $p_t(h_{t-1}, \cdot)$ in the subgame h_{t-1}. The fourth equality follows from the fact that $g_t(h_{t-1}, x_t) = p_t(h_{t-1}, x_t)$ for $f_t(\cdot|h_{t-1})$-almost all x_t.

If $\tilde{x}_{ti} \notin B_{ti}(h_{t-1})$, then player i's expected payoff is

$$\begin{aligned}
&\int_{A_{t(-i)}(h_{t-1})} g_{ti}(h_{t-1}, \tilde{x}_{ti}, x_{t(-i)}) f_{t(-i)}(\mathrm{d}x_{t(-i)}|h_{t-1}) \\
&= \int_{\prod_{j\neq i} B^c_{tj}(h_{t-1})} g_{ti}(h_{t-1}, \tilde{x}_{ti}, x_{t(-i)}) f_{t(-i)}(\mathrm{d}x_{t(-i)}|h_{t-1}) \\
&= \int_{\prod_{j\neq i} B^c_{tj}(h_{t-1})} g_{i}(h_{t-1}, \tilde{x}_{ti}, x_{t(-i)}) f_{t(-i)}(\mathrm{d}x_{t(-i)}|h_{t-1}) \\
&= \int_{A_{t(-i)}(h_{t-1})} g_{i}(h_{t-1}, \tilde{x}_{ti}, x_{t(-i)}) f_{t(-i)}(\mathrm{d}x_{t(-i)}|h_{t-1}) \\
&\leq \int_{A_{t}(h_{t-1})} p_{ti}(h_{t-1}, x_{t}) f_{t}(\mathrm{d}x_{t}|h_{t-1}) \\
&= \int_{A_{t}(h_{t-1})} g_{ti}(h_{t-1}, x_{t}) f_{t}(\mathrm{d}x_{t}|h_{t-1}).
\end{aligned}$$

The first and the third equalities hold since

$$f_{t(-i)}\left(\prod_{j\neq i} B_{tj}^c(h_{t-1})|h_{t-1}\right) = f_{t(-i)}(A_{t(-i)}(h_{t-1})|h_{t-1}).$$

The second equality is due to the fact that $g_{ti}(h_{t-1}, \tilde{x}_{ti}, x_{t(-i)}) = g_i(h_{t-1}, \tilde{x}_{ti}, x_{t(-i)})$ for $x_{t(-i)} \in \prod_{j\neq i} B_{tj}^c(h_{t-1})$. The first inequality follows from the definition of B_{ti}, and the fourth equality holds since $g_t(h_{t-1}, x_t) = p_t(h_{t-1}, x_t)$ for $f_t(\cdot|h_{t-1})$-almost all x_t.

Thus, player i cannot improve his payoff in the subgame h_t by a unilateral change in his strategy for any $i \in I$, which implies that $f_t(\cdot|h_{t-1})$ is a Nash equilibrium given the payoff $g_t(h_{t-1}, \cdot)$ in the subgame h_{t-1}.

Step 3. For any $(h_{t-1}, x_t) \in \mathrm{Gr}(A_t)$,

$$P_t(h_{t-1}, x_t) = \int_{S_t} Q_{t+1}(h_{t-1}, x_t, s_t) f_{t0}(\mathrm{d}s_t|h_{t-1}).$$

By Lemma 5, there exists a Borel measurable mapping q from $\mathrm{Gr}(P_t) \times S_t$ to $\mathbb{R}^n$ such that

1. $q(h_{t-1}, x_t, e, s_t) \in Q_{t+1}(h_{t-1}, x_t, s_t)$ for any $(h_{t-1}, x_t, e, s_t) \in \mathrm{Gr}(P_t) \times S_t$;
2. $e = \int_{S_t} q(h_{t-1}, x_t, e, s_t) f_{t0}(\mathrm{d}s_t|h_{t-1})$ for any $(h_{t-1}, x_t, e) \in \mathrm{Gr}(P_t)$, where $(h_{t-1}, x_t) \in \mathrm{Gr}(A_t)$.

Let

$$q_{t+1}(h_{t-1}, x_t, s_t) = q(h_{t-1}, x_t, g_t(h_{t-1}, x_t), s_t)$$

for any $(h_{t-1}, x_t, s_t) \in H_t$. Then q_{t+1} is a Borel measurable selection of Q_{t+1}.

For $(h_{t-1}, x_t) \in \mathrm{Gr}(A_t)$,

$$\begin{aligned} g_t(h_{t-1}, x_t) &= \int_{S_t} q(h_{t-1}, x_t, g_t(h_{t-1}, x_t), s_t) f_{t0}(\mathrm{d}s_t|h_{t-1}) \\ &= \int_{S_t} q_{t+1}(h_{t-1}, x_t, s_t) f_{t0}(\mathrm{d}s_t|h_{t-1}). \end{aligned}$$

Therefore, we have a Borel measurable selection q_{t+1} of Q_{t+1}, and a Borel measurable mapping $f_t\colon H_{t-1} \to \otimes_{i\in I}\mathcal{M}(X_{ti})$ such that for all $h_{t-1} \in \tilde{H}_{t-1}$, properties (1)-(3) are satisfied. The proof is complete. □

If a dynamic game has only T stages for some positive integer $T \geq 1$, then let $Q_{T+1}(h_T) = \{u(h_T)\}$ for any $h_T \in H_T$, and $Q_t = \Phi(Q_{t+1})$ for $1 \leq t \leq T-1$. We can start with the backward induction from the last period and stop at the initial period, then run the forward induction from the initial period to the last period. Thus, the following result is immediate.

Proposition B.4. *Any finite-horizon dynamic game with the ARM condition has a subgame-perfect equilibrium.*

B.4.3 Infinite horizon case

Pick a sequence $\xi = (\xi_1, \xi_2, \dots)$ such that (1) ξ_m is a transition probability from H_{m-1} to $\mathcal{M}(X_m)$ for any $m \geq 1$, and (2) $\xi_m(A_m(h_{m-1})|h_{m-1}) = 1$ for any $m \geq 1$ and $h_{m-1} \in H_{m-1}$. Denote the set of all such ξ as Υ.

Fix any $t \geq 1$, define correspondences Ξ_t^t and Δ_t^t as follows: in the subgame h_{t-1},

$$\Xi_t^t(h_{t-1}) = \mathcal{M}(A_t(h_{t-1})) \otimes \lambda_t,$$

and

$$\Delta_t^t(h_{t-1}) = \mathcal{M}(A_t(h_{t-1})) \otimes f_{t0}(h_{t-1}).$$

For any $m_1 > t$, suppose that the correspondences $\Xi_t^{m_1-1}$ and $\Delta_t^{m_1-1}$ have been defined. Then we can define correspondences $\Xi_t^{m_1} \colon H_{t-1} \to \mathcal{M}\left(\prod_{t \leq m \leq m_1}(X_m \times S_m)\right)$ and $\Delta_t^{m_1} \colon H_{t-1} \to \mathcal{M}\left(\prod_{t \leq m \leq m_1}(X_m \times S_m)\right)$ as follows:

$$\begin{aligned}\Xi_t^{m_1}(h_{t-1}) = \{ & g(h_{t-1}) \diamond (\xi_{m_1}(h_{t-1}, \cdot) \otimes \lambda_{m_1}) \colon \\ & g \text{ is a Borel measurable selection of } \Xi_t^{m_1-1}, \\ & \xi_{m_1} \text{ is a Borel measurable selection of } \mathcal{M}(A_{m_1})\},\end{aligned}$$

and

$$\begin{aligned}\Delta_t^{m_1}(h_{t-1}) = \{ & g(h_{t-1}) \diamond (\xi_{m_1}(h_{t-1}, \cdot) \otimes f_{m_1 0}(h_{t-1}, \cdot)) \colon \\ & g \text{ is a Borel measurable selection of } \Delta_t^{m_1-1}, \\ & \xi_{m_1} \text{ is a Borel measurable selection of } \mathcal{M}(A_{m_1})\},\end{aligned}$$

where $\mathcal{M}(A_{m_1})$ is regarded as a correspondence from H_{m_1-1} to the space of Borel probability measures on X_{m_1}. For any $m_1 \geq t$, let $\rho_{(h_{t-1},\xi)}^{m_1} \in \Xi_t^{m_1}$ be the probability measure on $\prod_{t \leq m \leq m_1}(X_m \times S_m)$ induced by $\{\lambda_m\}_{t \leq m \leq m_1}$ and $\{\xi_m\}_{t \leq m \leq m_1}$, and $\varrho_{(h_{t-1},\xi)}^{m_1} \in$

$\Delta_t^{m_1}$ be the probability measure on $\prod_{t\le m\le m_1}(X_m\times S_m)$ induced by $\{f_{m0}\}_{t\le m\le m_1}$ and $\{\xi_m\}_{t\le m\le m_1}$. Then, $\Xi_t^{m_1}(h_{t-1})$ is the set of all such $\rho^{m_1}_{(h_{t-1},\xi)}$, while $\Delta_t^{m_1}(h_{t-1})$ is the set of all such $\varrho^{m_1}_{(h_{t-1},\xi)}$. Note that $\varrho^{m_1}_{(h_{t-1},\xi)}\in\Delta_t^{m_1}(h_{t-1})$ if and only if $\rho^{m_1}_{(h_{t-1},\xi)}\in\Xi_t^{m_1}(h_{t-1})$. Both $\varrho^{m_1}_{(h_{t-1},\xi)}$ and $\rho^{m_1}_{(h_{t-1},\xi)}$ can be regarded as probability measures on $H_{m_1}(h_{t-1})$.

Similarly, let $\rho_{(h_{t-1},\xi)}$ be the probability measure on $\prod_{m\ge t}(X_m\times S_m)$ induced by $\{\lambda_m\}_{m\ge t}$ and $\{\xi_m\}_{m\ge t}$, and $\varrho_{(h_{t-1},\xi)}$ the probability measure on $\prod_{m\ge t}(X_m\times S_m)$ induced by $\{f_{m0}\}_{m\ge t}$ and $\{\xi_m\}_{m\ge t}$. Denote the correspondence

$$\Xi_t\colon H_{t-1}\to\mathcal{M}(\prod_{m\ge t}(X_m\times S_m))$$

as the set of all such $\rho_{(h_{t-1},\xi)}$, and

$$\Delta_t\colon H_{t-1}\to\mathcal{M}(\prod_{m\ge t}(X_m\times S_m))$$

as the set of all such $\varrho_{(h_{t-1},\xi)}$.

The following lemma demonstrates the relationship between $\varrho^{m_1}_{(h_{t-1},\xi)}$ and $\rho^{m_1}_{(h_{t-1},\xi)}$.

Lemma B.7. *For any $m_1\ge t$ and $h_{t-1}\in H_{t-1}$,*

$$\varrho^{m_1}_{(h_{t-1},\xi)}=\left(\prod_{t\le m\le m_1}\varphi_{m0}(h_{t-1},\cdot)\right)\circ\rho^{m_1}_{(h_{t-1},\xi)}.$$ [4]

Proof. Fix $\xi\in\Upsilon$, and Borel subsets $C_m\subseteq X_m$ and $D_m\subseteq S_m$ for $m\ge t$. First, we have

$$\begin{aligned}&\varrho^{t}_{(h_{t-1},\xi)}(C_t\times D_t)=\xi_t(C_t|h_{t-1})\cdot f_{t0}(D_t|h_{t-1})\\&=\int_{X_t\times S_t}\mathbf{1}_{C_t\times D_t}(x_t,s_t)\varphi_{t0}(h_{t-1},s_t)(\xi_t(h_{t-1})\otimes\lambda_t)(\mathrm{d}(x_t,s_t)),\end{aligned}$$

which implies that $\varrho^{t}_{(h_{t-1},\xi)}=\varphi_{t0}(h_{t-1},\cdot)\circ\rho^{t}_{(h_{t-1},\xi)}$.[5]

[4] For $m\ge t\ge 1$ and $h_{t-1}\in H_{t-1}$, the function $\varphi_{m0}(h_{t-1},\cdot)$ is defined on $H_{m-1}(h_{t-1})\times S_m$, which is measurable and sectionally continuous on $\prod_{t\le k\le m-1}X_k$. By Lemma 3, $\varphi_{m0}(h_{t-1},\cdot)$ can be extended to be a measurable function $\acute{\varphi}_{m0}(h_{t-1},\cdot)$ on the product space $\left(\prod_{t\le k\le m-1}X_k\right)\times\left(\prod_{t\le k\le m}S_k\right)$, which is also sectionally continuous on $\prod_{t\le k\le m-1}X_k$. Given any $\xi\in\Upsilon$, since $\rho^{m}_{(h_{t-1},\xi)}$ concentrates on $H_m(h_{t-1})$, $\varphi_{m0}(h_{t-1},\cdot)\circ\rho^{m}_{(h_{t-1},\xi)}=\acute{\varphi}_{m0}(h_{t-1},\cdot)\circ\rho^{m}_{(h_{t-1},\xi)}$. For notational simplicity, we still use $\varphi_{m0}(h_{t-1},\cdot)$, instead of $\acute{\varphi}_{m0}(h_{t-1},\cdot)$, to denote the above extension. Similarly, we can work with a suitable extension of the payoff function u as needed.

[5] For a set A in a space X, $\mathbf{1}_A$ is the indicator function of A, which is one on A and zero on $X\setminus A$.

Suppose that $\varrho^{m_2}_{(h_{t-1},\xi)} = \left(\prod_{t\le m\le m_2} \varphi_{m0}(h_{t-1},\cdot)\right) \circ \rho^{m_2}_{(h_{t-1},\xi)}$ for some $m_2 \ge t$. Then

$$
\begin{aligned}
&\varrho^{m_2+1}_{(h_{t-1},\xi)}\left(\prod_{t\le m\le m_2+1}(C_m\times D_m)\right)\\
&= \varrho^{m_2}_{(h_{t-1},\xi)} \diamond \left(\xi_{m_2+1}(h_{t-1},\cdot)\otimes f_{(m_2+1)0}(h_{t-1},\cdot)\right)\left(\prod_{t\le m\le m_2+1}(C_m\times D_m)\right)\\
&= \int_{\prod_{t\le m\le m_2}(X_m\times S_m)}\int_{X_{m_2+1}\times S_{m_2+1}} \mathbf{1}_{\prod_{t\le m\le m_2+1}(C_m\times D_m)}(x_t,\dots,x_{m_2+1},s_t,\dots,s_{m_2+1})\cdot\\
&\quad \xi_{m_2+1}\otimes f_{(m_2+1)0}(\mathrm{d}(x_{m_2+1},s_{m_2+1})|h_{t-1},x_t,\dots,x_{m_2},s_t,\dots,s_{m_2})\\
&\quad \varrho^{m_2}_{(h_{t-1},\xi)}(\mathrm{d}(x_t,\dots,x_{m_2},s_t,\dots,s_{m_2})|h_{t-1})\\
&= \int_{\prod_{t\le m\le m_2}(X_m\times S_m)}\int_{S_{m_2+1}}\int_{X_{m_2+1}} \mathbf{1}_{\prod_{t\le m\le m_2+1}(C_m\times D_m)}(x_t,\dots,x_{m_2+1},s_t,\dots,s_{m_2+1})\cdot\\
&\quad \varphi_{(m_2+1)0}(h_{t-1},x_t,\dots,x_{m_2},s_t,\dots,s_{m_2+1})\xi_{m_2+1}(\mathrm{d}x_{m_2+1}|h_{t-1},x_t,\dots,x_{m_2},s_t,\dots,s_{m_2})\\
&\quad \lambda_{(m_2+1)0}(\mathrm{d}s_{m_2+1})\prod_{t\le m\le m_2}\varphi_{m0}(h_{t-1},x_t,\dots,x_{m-1},s_t,\dots,s_m)\\
&\quad \rho^{m_2}_{(h_{t-1},\xi)}(\mathrm{d}(x_t,\dots,x_{m_2},s_t,\dots,s_{m_2})|h_{t-1})\\
&= \int_{\prod_{t\le m\le m_2+1}(X_m\times S_m)} \mathbf{1}_{\prod_{t\le m\le m_2+1}(C_m\times D_m)}(x_t,\dots,x_{m_2+1},s_t,\dots,s_{m_2+1})\cdot\\
&\quad \prod_{t\le m\le m_2+1}\varphi_{m0}(h_{t-1},x_t,\dots,x_{m-1},s_t,\dots,s_m)\rho^{m_2+1}_{(h_{t-1},\xi)}(\mathrm{d}(x_t,\dots,x_{m_2},s_t,\dots,s_{m_2})|h_{t-1}),
\end{aligned}
$$

which implies that

$$\varrho^{m_2+1}_{(h_{t-1},\xi)} = \left(\prod_{t\le m\le m_2+1}\varphi_{m0}(h_{t-1},\cdot)\right)\circ \rho^{m_2+1}_{(h_{t-1},\xi)}.$$

The proof is thus complete. □

The next lemma shows that the correspondences $\Delta^{m_1}_t$ and Δ_t are nonempty and compact valued, and sectionally continuous.

Lemma B.8. *1. For any $t \ge 1$, the correspondence $\Delta^{m_1}_t$ is nonempty and compact valued, and sectionally continuous on X^{t-1} for any $m_1 \ge t$.*

2. For any $t \ge 1$, the correspondence Δ_t is nonempty and compact valued, and sectionally continuous on X^{t-1}.

Proof. (1) We first show that the correspondence $\Xi^{m_1}_t$ is nonempty and compact valued, and sectionally continuous on X^{t-1} for any $m_1 \geq t$.

Consider the case $m_1 = t \geq 1$, where

$$\Xi_t^t(h_{t-1}) = \mathcal{M}(A_t(h_{t-1})) \otimes \lambda_t.$$

Since A_{ti} is nonempty and compact valued, and sectionally continuous on X^{t-1}, Ξ_t^t is nonempty and compact valued, and sectionally continuous on X^{t-1}.

Now suppose that $\Xi_t^{m_2}$ is nonempty and compact valued, and sectionally continuous on X^{t-1} for some $m_2 \geq t \geq 1$. Notice that

$$\begin{aligned}\Xi_t^{m_2+1}(h_{t-1}) = &\{g(h_{t-1}) \diamond (\xi_{m_2+1}(h_{t-1}, \cdot) \otimes \lambda_{(m_2+1)}) \colon \\ &g \text{ is a Borel measurable selection of } \Xi_t^{m_2}, \\ &\xi_{m_2+1} \text{ is a Borel measurable selection of } \mathcal{M}(A_{m_2+1})\}.\end{aligned}$$

First, we claim that $H_t(s_0, s_1, \ldots, s_t)$ is compact for any $(s_0, s_1, \ldots, s_t) \in S^t$. We prove this claim by induction.

1. Notice that $H_0(s_0) = X_0$ for any $s_0 \in S_0$, which is compact.
2. Suppose that $H_{m'}(s_0, s_1, \ldots, s_{m'})$ is compact for some $0 \leq m' \leq t-1$ and any $(s_0, s_1, \ldots, s_{m'}) \in S^{m'}$.
3. Since $A_{m'+1}(\cdot, s_0, s_1, \ldots, s_{m'})$ is continuous and compact valued, it has a compact graph by Lemma 2 (6), which is $H_{m'+1}(s_0, s_1, \ldots, s_{m'+1})$ for any $(s_0, s_1, \ldots, s_{m'+1}) \in S^{m'+1}$.

Thus, we prove the claim.

Define a correspondence A_t^t from $H_{t-1} \times S_t$ to X_t as $A_t^t(h_{t-1}, s_t) = A_t(h_{t-1})$. Then A_t^t is nonempty and compact valued, sectionally continuous on X_{t-1}, and has a $\mathcal{B}(X^t \times S^t)$-measurable graph. Since the graph of $A_t^t(\cdot, s_0, s_1, \ldots, s_t)$ is $H_t(s_0, s_1, \ldots, s_t)$ and $H_t(s_0, s_1, \ldots, s_t)$ is compact, $A_t^t(\cdot, s_0, s_1, \ldots, s_t)$ has a compact graph. For any $h_{t-1} \in H_{t-1}$ and $\tau \in \Xi_t^t(h_{t-1})$, the marginal of τ on S_t is λ_t and $\tau(\mathrm{Gr}(A_t^t(h_{t-1}, \cdot))) = 1$.

For any $m_1 > t$, suppose that the correspondence

$$A_t^{m_1-1} \colon H_{t-1} \times \prod_{t \leq m \leq m_1-1} S_m \to \prod_{t \leq m \leq m_1-1} X_m$$

has been defined such that

1. it is nonempty and compact valued, sectionally upper hemicontinuous on X_{t-1}, and has a $\mathcal{B}(X^{m_1-1} \times S^{m_1-1})$-measurable graph;

2. for any $(s_0, s_1, \ldots s_{m_1-1})$, $A_t^{m_1-1}(\cdot, s_0, s_1, \ldots s_{m_1-1})$ has a compact graph;

3. for any $h_{t-1} \in H_{t-1}$ and $\tau \in \Xi_t^{m_1-1}(h_{t-1})$, the marginal of τ on $\prod_{t \le m \le m_1-1} S_m$ is $\otimes_{t \le m \le m_1-1} \lambda_m$ and $\tau(\mathrm{Gr}(A_t^{m_1-1}(h_{t-1}, \cdot))) = 1$.

We define a correspondence $A_t^{m_1} \colon H_{t-1} \times \prod_{t \le m \le m_1} S_m \to \prod_{t \le m \le m_1} X_m$ as follows:

$$
\begin{aligned}
A_t^{m_1}(h_{t-1}, s_t, \ldots, s_{m_1}) = \{ (x_t, \ldots, x_{m_1}) \colon & \\
& x_{m_1} \in A_{m_1}(h_{t-1}, x_t, \ldots, x_{m_1-1}, s_t, \ldots, s_{m_1-1}), \\
& (x_t, \ldots, x_{m_1-1}) \in A_t^{m_1-1}(h_{t-1}, s_t, \ldots, s_{m_1-1}) \}.
\end{aligned}
$$

It is obvious that $A_t^{m_1}$ is nonempty valued. For any $(s_0, s_1, \ldots, s_{m_1})$, since $A_t^{m_1-1}(\cdot, s_0, s_1, \ldots s_{m_1-1})$ has a compact graph and $A_{m_1}(\cdot, s_0, s_1, \ldots, s_{m_1-1})$ is continuous and compact valued, $A_t^{m_1}(\cdot, s_0, s_1, \ldots s_{m_1})$ has a compact graph by Lemma 2 (6), which implies that $A_t^{m_1}$ is compact valued and sectionally upper hemicontinuous on X_{t-1}. In addition, $\mathrm{Gr}(A_t^{m_1}) = \mathrm{Gr}(A_{m_1}) \times S_{m_1}$, which is $\mathcal{B}(X^{m_1} \times S^{m_1})$-measurable. For any $h_{t-1} \in H_{t-1}$ and $\tau \in \Xi_t^{m_1}(h_{t-1})$, it is obvious that the marginal of τ on $\prod_{t \le m \le m_1} S_m$ is $\otimes_{t \le m \le m_1} \lambda_m$ and $\tau(\mathrm{Gr}(A_t^{m_1}(h_{t-1}, \cdot))) = 1$.

By Lemma B.5, $\Xi_t^{m_2+1}$ is nonempty and compact valued, and sectionally continuous on X^{t-1}.

Now we show that the correspondence $\Delta_t^{m_1}$ is nonempty and compact valued, and sectionally continuous on X^{t-1} for any $m_1 \ge t$.

Given s^{t-1} and a sequence $\{x_0^k, x_1^k, \ldots, x_{t-1}^k\} \in H_{t-1}(s^{t-1})$ for $1 \le k \le \infty$. Let $h_{t-1}^k = (s^{t-1}, (x_0^k, x_1^k, \ldots, x_{t-1}^k))$. It is obvious that $\Delta_t^{m_1}$ is nonempty valued, we first show that $\Delta_t^{m_1}$ is sectionally upper hemicontinuous on X^{t-1}. Suppose that $\varrho_{(h_{t-1}^k, \xi^k)}^{m_1} \in \Delta_t^{m_1}(h_{t-1}^k)$ for $1 \le k < \infty$ and $(x_0^k, x_1^k, \ldots, x_{t-1}^k) \to (x_0^\infty, x_1^\infty, \ldots, x_{t-1}^\infty)$, we need to show that there exists some ξ^∞ such that a subsequence of $\varrho_{(h_{t-1}^k, \xi^k)}^{m_1}$ weakly converges to $\varrho_{(h_{t-1}^\infty, \xi^\infty)}^{m_1}$ and $\varrho_{(h_{t-1}^\infty, \xi^\infty)}^{m_1} \in \Delta_t^{m_1}(h_{t-1}^\infty)$.

Since $\Xi_t^{m_1}$ is sectionally upper hemicontinuous on X^{t-1}, there exists some ξ^∞ such that a subsequence of $\rho_{(h_{t-1}^k, \xi^k)}^{m_1}$, say itself, weakly converges to $\rho_{(h_{t-1}^\infty, \xi^\infty)}^{m_1}$ and $\rho_{(h_{t-1}^\infty, \xi^\infty)}^{m_1} \in \Xi_t^{m_1}(h_{t-1}^\infty)$. Then $\varrho_{(h_{t-1}^\infty, \xi^\infty)}^{m_1} \in \Delta_t^{m_1}(h_{t-1}^\infty)$.

For any bounded continuous function ψ on $\prod_{t\le m\le m_1}(X_m \times S_m)$, let

$$\chi_k(x_t, \ldots, x_{m_1}, s_t, \ldots, s_{m_1}) =$$

$$\psi(x_t, \ldots, x_{m_1}, s_t, \ldots, s_{m_1}) \cdot \prod_{t\le m\le m_1} \varphi_{m0}(h_{t-1}^k, x_t, \ldots, x_{m-1}, s_t, \ldots, s_m).$$

Then $\{\chi_k\}$ is a sequence of functions satisfying the following three properties.

1. For each k, χ_k is jointly measurable and sectionally continuous on $\prod_{t\le m\le m_1} X_m$.
2. For any $(s_t, \ldots, s_{m_1})$ and any sequence $(x_t^k, \ldots, x_{m_1}^k) \to (x_t^\infty, \ldots, x_{m_1}^\infty)$ in $\prod_{t\le m\le m_1} X_m$, $\chi_k(x_t^k, \ldots, x_{m_1}^k, s_t, \ldots, s_{m_1}) \to \chi_\infty(x_t^\infty, \ldots, x_{m_1}^\infty, s_t, \ldots, s_{m_1})$ as $k \to \infty$.
3. The sequence $\{\chi_k\}_{1\le k\le\infty}$ is integrably bounded in the sense that there exists a function $\chi'\colon \prod_{t\le m\le m_1} S_m \to \mathbb{R}_+$ such that χ' is $\otimes_{t\le m\le m_1}\lambda_m$-integrable and for any k and $(x_t, \ldots, x_{m_1}, s_t, \ldots, s_{m_1})$, $\chi_k(x_t, \ldots, x_{m_1}, s_t, \ldots, s_{m_1}) \le \chi'(s_t, \ldots, s_{m_1})$.

By Lemma B.6, as $k \to \infty$,

$$\int_{\prod_{t\le m\le m_1}(X_m\times S_m)} \chi_k(x_t, \ldots, x_{m_1}, s_t, \ldots, s_{m_1}) \rho^{m_1}_{(h_{t-1}^k, \xi^k)}(\mathrm{d}(x_t, \ldots, x_{m_1}, s_t, \ldots, s_{m_1}))$$

$$\to \int_{\prod_{t\le m\le m_1}(X_m\times S_m)} \chi_\infty(x_t, \ldots, x_{m_1}, s_t, \ldots, s_{m_1}) \rho^{m_1}_{(h_{t-1}^\infty, \xi^\infty)}(\mathrm{d}(x_t, \ldots, x_{m_1}, s_t, \ldots, s_{m_1})).$$

Then by Lemma B.7,

$$\int_{\prod_{t\le m\le m_1}(X_m\times S_m)} \psi(x_t, \ldots, x_{m_1}, s_t, \ldots, s_{m_1}) \varrho^{m_1}_{(h_{t-1}^k, \xi^k)}(\mathrm{d}(x_t, \ldots, x_{m_1}, s_t, \ldots, s_{m_1}))$$

$$\to \int_{\prod_{t\le m\le m_1}(X_m\times S_m)} \psi(x_t, \ldots, x_{m_1}, s_t, \ldots, s_{m_1}) \varrho^{m_1}_{(h_{t-1}^\infty, \xi^\infty)}(\mathrm{d}(x_t, \ldots, x_{m_1}, s_t, \ldots, s_{m_1})),$$

which implies that $\varrho^{m_1}_{(h_{t-1}^k, \xi^k)}$ weakly converges to $\varrho^{m_1}_{(h_{t-1}^\infty, \xi^\infty)}$. Therefore, $\Delta_t^{m_1}$ is sectionally upper hemicontinuous on X^{t-1}. If one chooses $h_{t-1}^1 = h_{t-1}^2 = \cdots = h_{t-1}^\infty$, then we indeed show that $\Delta_t^{m_1}$ is compact valued.

In the argument above, we indeed proved that if $\rho^{m_1}_{(h_{t-1}^k, \xi^k)}$ weakly converges to $\rho^{m_1}_{(h_{t-1}^\infty, \xi^\infty)}$, then $\varrho^{m_1}_{(h_{t-1}^k, \xi^k)}$ weakly converges to $\varrho^{m_1}_{(h_{t-1}^\infty, \xi^\infty)}$.

The left is to show that $\Delta_t^{m_1}$ is sectionally lower hemicontinuous on X^{t-1}. Suppose that $(x_0^k, x_1^k, \ldots, x_{t-1}^k) \to (x_0^\infty, x_1^\infty, \ldots, x_{t-1}^\infty)$ and $\varrho^{m_1}_{(h_{t-1}^\infty, \xi^\infty)} \in \Delta_t^{m_1}(h_{t-1}^\infty)$, we need to show that there exists a subsequence $\{(x_0^{k_m}, x_1^{k_m}, \ldots, x_{t-1}^{k_m})\}$ of $\{(x_0^k, x_1^k, \ldots, x_{t-1}^k)\}$ and $\varrho^{m_1}_{(h^{k_m}_{t-1}, \xi^{k_m})} \in \Delta_t^{m_1}(h^{k_m}_{t-1})$ for each k_m such that $\varrho^{m_1}_{(h^{k_m}_{t-1}, \xi^{k_m})}$ weakly converges to $\varrho^{m_1}_{(h^{\infty}_{t-1}, \xi^{\infty})}$.

Since $\varrho^{m_1}_{(h^{\infty}_{t-1}, \xi^{\infty})} \in \Delta_t^{m_1}(h^{\infty}_{t-1})$, we have $\rho^{m_1}_{(h^{\infty}_{t-1}, \xi^{\infty})} \in \Xi_t^{m_1}(h^{\infty}_{t-1})$. Because $\Xi_t^{m_1}$ is sectionally lower hemicontinuous on X^{t-1}, there exists a subsequence of $\{(x_0^k, x_1^k, \ldots, x_{t-1}^k)\}$, say itself, and $\rho^{m_1}_{(h^k_{t-1}, \xi^k)} \in \Xi_t^{m_1}(h^k_{t-1})$ for each k such that $\rho^{m_1}_{(h^k_{t-1}, \xi^k)}$ weakly converges to $\rho^{m_1}_{(h^{\infty}_{t-1}, \xi^{\infty})}$. As a result, $\varrho^{m_1}_{(h^k_{t-1}, \xi^k)}$ weakly converges to $\varrho^{m_1}_{(h^{\infty}_{t-1}, \xi^{\infty})}$, which implies that $\Delta_t^{m_1}$ is sectionally lower hemicontinuous on X^{t-1}.

Therefore, $\Delta_t^{m_1}$ is nonempty and compact valued, and sectionally continuous on X^{t-1} for any $m_1 \geq t$.

(2) We show that Δ_t is nonempty and compact valued, and sectionally continuous on X^{t-1}.

It is obvious that Δ_t is nonempty valued, we first prove that it is compact valued.

Given h_{t-1} and a sequence $\{\tau^k\} \subseteq \Delta_t(h_{t-1})$, there exists a sequence of $\{\xi^k\}_{k \geq 1}$ such that $\xi^k = (\xi_1^k, \xi_2^k, \ldots) \in \Upsilon$ and $\tau^k = \varrho_{(h_{t-1}, \xi^k)}$ for each k.

By (1), Ξ_t^t is compact. Then there exists a measurable mapping g_t such that (1) $g^t = (\xi_1^1, \ldots, \xi_{t-1}^1, g_t, \xi_{t+1}^1, \ldots) \in \Upsilon$, and (2) a subsequence of $\{\rho^t_{(h_{t-1}, \xi^k)}\}$, say $\{\rho^t_{(h_{t-1}, \xi^{k_{1l}})}\}_{l \geq 1}$, which weakly converges to $\rho^t_{(h_{t-1}, g^t)}$. Note that $\{\xi^k_{t+1}\}$ is a Borel measurable selection of $\mathcal{M}(A_{t+1})$. By Lemma B.5, there is a Borel measurable selection g_{t+1} of $\mathcal{M}(A_{t+1})$ such that there is a subsequence of $\{\rho^{t+1}_{(h_{t-1}, \xi^{k_{1l}})}\}_{l \geq 1}$, say $\{\rho^{t+1}_{(h_{t-1}, \xi^{k_{2l}})}\}_{l \geq 1}$, which weakly converges to $\rho^{t+1}_{(h_{t-1}, g^{t+1})}$, where $g^{t+1} = (\xi_1^1, \ldots, \xi_{t-1}^1, g_t, g_{t+1}, \xi_{t+2}^1, \ldots) \in \Upsilon$.

Repeat this procedure, one can construct a Borel measurable mapping g such that $\rho_{(h_{t-1}, \xi^{k_{11}})}, \rho_{(h_{t-1}, \xi^{k_{22}})}, \rho_{(h_{t-1}, \xi^{k_{33}})}, \ldots$ weakly converges to $\rho_{(h_{t-1}, g)}$. That is, $\rho_{(h_{t-1}, g)}$ is a convergent point of $\{\rho_{(h_{t-1}, \xi^k)}\}$, which implies that $\varrho_{(h_{t-1}, g)}$ is a convergent point of $\{\varrho_{(h_{t-1}, \xi^k)}\}$.

The sectional upper hemicontinuity of Δ_t follows a similar argument as above. In particular, given s^{t-1} and a sequence $\{x_0^k, x_1^k, \ldots, x_{t-1}^k\} \subseteq H_{t-1}(s^{t-1})$ for $k \geq 0$. Let $h^k_{t-1} = (s^{t-1}, (x_0^k, x_1^k, \ldots, x_{t-1}^k))$. Suppose that $(x_0^k, x_1^k, \ldots, x_{t-1}^k) \to (x_0^0, x_1^0, \ldots, x_{t-1}^0)$. If $\{\tau^k\} \subseteq \Delta_t(h^k_{t-1})$ for $k \geq 1$ and $\tau^k \to \tau^0$, then one can show that $\tau^0 \in \Delta_t(h^0_{t-1})$ by repeating a similar argument as in the proof above.

Finally, we consider the sectional lower hemicontinuity of Δ_t. Suppose that $\tau^0 \in \Delta_t(h^0_{t-1})$. Then there exists some $\xi \in \Upsilon$ such that $\tau^0 = \varrho_{(h^0_{t-1}, \xi)}$. Denote $\tilde{\tau}^m = \varrho^m_{(h^0_{t-1}, \xi)} \in \Delta_t^m(h^0_{t-1})$ for $m \geq t$. As Δ_t^m is continuous, for each m, there exists some $\xi^m \in \Upsilon$ such that $d(\varrho^m_{(h^{k_m}_{t-1}, \xi^m)}, \tilde{\tau}^m) \leq \frac{1}{m}$ for k_m sufficiently large, where d is the Prokhorov metric. Let

$\tau^m = \varrho_{(h_{t-1}^{k_m},\xi^m)}$. Then τ^m weakly converges to τ^0, which implies that Δ_t is sectionally lower hemicontinuous. $\square$

Define a correspondence $Q_t^\tau \colon H_{t-1} \to \mathbb{R}^n_{++}$ as follows:

$$Q_t^\tau(h_{t-1}) =$$

$$\begin{cases} \{\int_{\prod_{m\geq t}(X_m\times S_m)} u(h_{t-1},x,s)\varrho_{(h_{t-1},\xi)}(\mathrm{d}(x,s)) \colon \varrho_{(h_{t-1},\xi)} \in \Delta_t(h_{t-1})\}; & t > \tau; \\ \Phi(Q_{t+1}^\tau)(h_{t-1}) & t \leq \tau. \end{cases}$$

The lemma below presents several properties of the correspondence Q_t^τ.

Lemma B.9. *For any $t,\tau \geq 1$, Q_t^τ is bounded, measurable, nonempty and compact valued, and essentially sectionally upper hemicontinuous on X^{t-1}.*

Proof. We prove the lemma in three steps.

Step 1. Fix $t > \tau$. We will show that Q_t^τ is bounded, nonempty and compact valued, and sectionally upper hemicontinuous on X^{t-1}.

The boundedness and nonemptiness of Q_t^τ are obvious. We shall prove that Q_t^τ is sectionally upper hemicontinuous on X^{t-1}. Given s^{t-1} and a sequence $\{x_0^k, x_1^k, \ldots, x_{t-1}^k\} \subseteq H_{t-1}(s^{t-1})$ for $k \geq 0$. Let $h_{t-1}^k = (s^{t-1}, (x_0^k, x_1^k, \ldots, x_{t-1}^k))$. Suppose that $a^k \in Q_t^\tau(h_{t-1}^k)$ for $k \geq 1$, $(x_0^k, x_1^k, \ldots, x_{t-1}^k) \to (x_0^0, x_1^0, \ldots, x_{t-1}^0)$ and $a^k \to a^0$, we need to show that $a^0 \in Q_t^\tau(h_{t-1}^0)$.

By the definition, there exists a sequence $\{\xi^k\}_{k\geq 1}$ such that

$$a^k = \int_{\prod_{m\geq t}(X_m\times S_m)} u(h_{t-1}^k,x,s)\varrho_{(h_{t-1}^k,\xi^k)}(\mathrm{d}(x,s)),$$

where $\xi^k = (\xi_1^k, \xi_2^k, \ldots) \in \Upsilon$ for each k. As Δ_t is compact valued and sectionally continuous on X^{t-1}, there exist some $\varrho_{(h_{t-1}^0,\xi^0)} \in \Delta_t(h_{t-1}^0)$ and a subsequence of $\varrho_{(h_{t-1}^k,\xi^k)}$, say itself, which weakly converges to $\varrho_{(h_{t-1}^0,\xi^0)}$ for $\xi^0 = (\xi_1^0, \xi_2^0, \ldots) \in \Upsilon$.

We shall show that

$$a^0 = \int_{\prod_{m\geq t}(X_m\times S_m)} u(h_{t-1}^0,x,s)\varrho_{(h_{t-1}^0,\xi^0)}(\mathrm{d}(x,s)).$$

For this aim, we only need to show that for any $\delta > 0$,

$$\left| a^0 - \int_{\prod_{m\geq t}(X_m\times S_m)} u(h_{t-1}^0,x,s)\varrho_{(h_{t-1}^0,\xi^0)}(\mathrm{d}(x,s)) \right| < \delta. \tag{2}$$

Since the game is continuous at infinity, there exists a positive integer $\tilde{M} \geq t$ such that $w^m < \frac{1}{5}\delta$ for any $m > \tilde{M}$.

For each $j > \tilde{M}$, by Lemma 3, there exists a measurable selection ξ'_j of $\mathcal{M}(A_j)$ such that ξ'_j is sectionally continuous on X^{j-1}. Let $\mu \colon H_{\tilde{M}} \to \prod_{m>\tilde{M}}(X_m \times S_m)$ be the transition probability which is induced by $(\xi'_{\tilde{M}+1}, \xi'_{\tilde{M}+2}, \ldots)$ and $\{f_{(\tilde{M}+1)0}, f_{(\tilde{M}+2)0}, \ldots\}$. By Lemma 9, μ is measurable and sectionally continuous on $X^{\tilde{M}}$. Let

$$V_{\tilde{M}}(h_{t-1}, x_t, \ldots, x_{\tilde{M}}, s_t, \ldots, s_{\tilde{M}}) =$$

$$\int_{\prod_{m>\tilde{M}}(X_m \times S_m)} u(h_{t-1}, x_t, \ldots, x_{\tilde{M}}, s_t, \ldots, s_{\tilde{M}}, x, s)\, \mathrm{d}\mu(x, s | h_{t-1}, x_t, \ldots, x_{\tilde{M}}, s_t, \ldots, s_{\tilde{M}}).$$

Then $V_{\tilde{M}}$ is bounded and measurable. In addition, $V_{\tilde{M}}$ is sectionally continuous on $X^{\tilde{M}}$ by Lemma B.6.

For any $k \geq 0$, we have

$$\begin{aligned}
&\Big|\int_{\prod_{m\geq t}(X_m \times S_m)} u(h^k_{t-1}, x, s)\varrho_{(h^k_{t-1}, \xi^k)}(\mathrm{d}(x, s)) \\
&- \int_{\prod_{t\leq m\leq\tilde{M}}(X_m \times S_m)} V_{\tilde{M}}(h^k_{t-1}, x_t, \ldots, x_{\tilde{M}}, s_t, \ldots, s_{\tilde{M}})\varrho^{\tilde{M}}_{(h^k_{t-1}, \xi^k)}(\mathrm{d}(x_t, \ldots, x_{\tilde{M}}, s_t, \ldots, s_{\tilde{M}}))\Big| \\
&\leq w^{\tilde{M}+1} \\
&< \frac{1}{5}\delta.
\end{aligned}$$

Since $\varrho_{(h^k_{t-1}, \xi^k)}$ weakly converges to $\varrho_{(h^0_{t-1}, \xi^0)}$ and $\varrho^{\tilde{M}}_{(h^k_{t-1}, \xi^k)}$ is the marginal of $\varrho_{(h^k_{t-1}, \xi^k)}$ on $\prod_{t\leq m\leq\tilde{M}}(X_m \times S_m)$ for any $k \geq 0$, the sequence $\varrho^{\tilde{M}}_{(h^k_{t-1}, \xi^k)}$ also weakly converges to $\varrho^{\tilde{M}}_{(h^0_{t-1}, \xi^0)}$. By Lemma B.6, we have

$$\begin{aligned}
&\Big|\int_{\prod_{t\leq m\leq\tilde{M}}(X_m \times S_m)} V_{\tilde{M}}(h^k_{t-1}, x_t, \ldots, x_{\tilde{M}}, s_t, \ldots, s_{\tilde{M}})\varrho^{\tilde{M}}_{(h^k_{t-1}, \xi^k)}(\mathrm{d}(x_t, \ldots, x_{\tilde{M}}, s_t, \ldots, s_{\tilde{M}})) \\
&- \int_{\prod_{t\leq m\leq\tilde{M}}(X_m \times S_m)} V_{\tilde{M}}(h^0_{t-1}, x_t, \ldots, x_{\tilde{M}}, s_t, \ldots, s_{\tilde{M}})\varrho^{\tilde{M}}_{(h^0_{t-1}, \xi^0)}(\mathrm{d}(x_t, \ldots, x_{\tilde{M}}, s_t, \ldots, s_{\tilde{M}}))\Big| \\
&< \frac{1}{5}\delta
\end{aligned}$$

for $k \geq K_1$, where K_1 is a sufficiently large positive integer. In addition, there exists a positive integer K_2 such that $|a^k - a^0| < \frac{1}{5}\delta$ for $k \geq K_2$.

Fix $k > \max\{K_1, K_2\}$. Combining the inequalities above, we have

$$\begin{aligned}
&\left|\int_{\prod_{m\geq t}(X_m\times S_m)} u(h^0_{t-1},x,s)\varrho_{(h^0_{t-1},\xi^0)}(\mathrm{d}(x,s)) - a^0\right| \\
&\leq \Big|\int_{\prod_{m\geq t}(X_m\times S_m)} u(h^0_{t-1},x,s)\varrho_{(h^0_{t-1},\xi^0)}(\mathrm{d}(x,s)) \\
&- \int_{\prod_{t\leq m\leq \tilde{M}}(X_m\times S_m)} V_{\tilde{M}}(h^0_{t-1},x_t,\ldots,x_{\tilde{M}},s_t,\ldots,s_{\tilde{M}})\varrho^{\tilde{M}}_{(h^0_{t-1},\xi^0)}(\mathrm{d}(x_t,\ldots,x_{\tilde{M}},s_t,\ldots,s_{\tilde{M}}))\Big| \\
&+ \Big|\int_{\prod_{t\leq m\leq \tilde{M}}(X_m\times S_m)} V_{\tilde{M}}(h^0_{t-1},x_t,\ldots,x_{\tilde{M}},s_t,\ldots,s_{\tilde{M}})\varrho^{\tilde{M}}_{(h^0_{t-1},\xi^0)}(\mathrm{d}(x_t,\ldots,x_{\tilde{M}},s_t,\ldots,s_{\tilde{M}})) \\
&- \int_{\prod_{t\leq m\leq \tilde{M}}(X_m\times S_m)} V_{\tilde{M}}(h^k_{t-1},x_t,\ldots,x_{\tilde{M}},s_t,\ldots,s_{\tilde{M}})\varrho^{\tilde{M}}_{(h^k_{t-1},\xi^k)}(\mathrm{d}(x_t,\ldots,x_{\tilde{M}},s_t,\ldots,s_{\tilde{M}}))\Big| \\
&+ \Big|\int_{\prod_{t\leq m\leq \tilde{M}}(X_m\times S_m)} V_{\tilde{M}}(h^k_{t-1},x_t,\ldots,x_{\tilde{M}},s_t,\ldots,s_{\tilde{M}})\varrho^{\tilde{M}}_{(h^k_{t-1},\xi^k)}(\mathrm{d}(x_t,\ldots,x_{\tilde{M}},s_t,\ldots,s_{\tilde{M}})) \\
&- \int_{\prod_{m\geq t}(X_m\times S_m)} u(h^k_{t-1},x,s)\varrho_{(h^k_{t-1},\xi^k)}(\mathrm{d}(x,s))\Big| \\
&+ \Big|\int_{\prod_{m\geq t}(X_m\times S_m)} u(h^k_{t-1},x,s)\varrho_{(h^k_{t-1},\xi^k)}(\mathrm{d}(x,s)) - a^0\Big| \\
&< \delta.
\end{aligned}$$

Thus, we proved inequality (2), which implies that Q^τ_t is sectionally upper hemicontinuous on X^{t-1} for $t > \tau$.

Furthermore, to prove that Q^τ_t is compact valued, we only need to consider the case that $\{x^k_0, x^k_1, \ldots, x^k_{t-1}\} = \{x^0_0, x^0_1, \ldots, x^0_{t-1}\}$ for any $k \geq 0$, and repeat the above proof.

Step 2. Fix $t > \tau$, we will show that Q^τ_t is measurable.

Fix a sequence $(\xi'_1, \xi'_2, \ldots)$, where ξ'_j is a selection of $\mathcal{M}(A_j)$ measurable in s^{j-1} and continuous in x^{j-1} for each j. For any $M \geq t$, let

$$W^M_M(h_{t-1},x_t,\ldots,x_M,s_t,\ldots,s_M) =$$

$$\left\{\int_{\prod_{m>M}(X_m\times S_m)} u(h_{t-1},x_t,\ldots,x_M,s_t,\ldots,s_M,x,s)\varrho_{(h_{t-1},x_t,\ldots,x_M,s_t,\ldots,s_M,\xi')}(\mathrm{d}(x,s))\right\}.$$

By Lemma 9, $\varrho_{(h_{t-1},x_t,\ldots,x_M,s_t,\ldots,s_M,\xi')}$ is measurable from H_M to $\mathcal{M}\left(\prod_{m>M}(X_m\times S_m)\right)$, and sectionally continuous on X^M. Thus, W^M_M is bounded, measurable, nonempty, convex and compact valued. By Lemma B.6, W^M_M is sectionally continuous on X^M.

Suppose that for some $t \le j \le M$, W_M^j has been defined such that it is bounded, measurable, nonempty, convex and compact valued, and sectionally continuous on X^j. Let

$$\begin{aligned}
& W_M^{j-1}(h_{t-1}, x_t, \ldots, x_{j-1}, s_t, \ldots, s_{j-1}) = \\
\{ & \int_{X_j \times S_j} w_M^j(h_{t-1}, x_t, \ldots, x_j, s_t, \ldots, s_j) \varrho^j_{(h_{t-1}, x_t, \ldots, x_{j-1}, s_t, \ldots, s_{j-1}, \xi)}(\mathrm{d}(x_j, s_j)) \colon \\
& \varrho^j_{(h_{t-1}, x_t, \ldots, x_{j-1}, s_t, \ldots, s_{j-1}, \xi)} \in \Delta_j^j(h_{t-1}, x_t, \ldots, x_{j-1}, s_t, \ldots, s_{j-1}), \\
& w_M^j \text{ is a Borel measurable selection of } W_M^j \}.
\end{aligned}$$

Let $\check{S}_j = S_j$.[6] Since

$$\begin{aligned}
& \int_{X_j \times S_j} W_M^j(h_{t-1}, x_t, \ldots, x_j, s_t, \ldots, s_j) \varrho^j_{(h_{t-1}, x_t, \ldots, x_{j-1}, s_t, \ldots, s_{j-1}, \xi)}(\mathrm{d}(x_j, s_j)) \\
= & \int_{S_j} \int_{X_j \times \check{S}_j} W_M^j(h_{t-1}, x_t, \ldots, x_j, s_t, \ldots, s_j) \rho^j_{(h_{t-1}, x_t, \ldots, x_{j-1}, s_t, \ldots, s_{j-1}, \xi)}(\mathrm{d}(x_j, \check{s}_j)) \\
& \cdot \varphi_{j0}(h_{t-1}, x_t, \ldots, x_{j-1}, s_t, \ldots, s_j) \lambda_j(\mathrm{d}s_j),
\end{aligned}$$

we have

$$\begin{aligned}
& W_M^{j-1}(h_{t-1}, x_t, \ldots, x_{j-1}, s_t, \ldots, s_{j-1}) = \\
\{ & \int_{S_j} \int_{X_j \times \check{S}_j} w_M^j(h_{t-1}, x_t, \ldots, x_j, s_t, \ldots, s_j) \rho^j_{(h_{t-1}, x_t, \ldots, x_{j-1}, s_t, \ldots, s_{j-1}, \xi)}(\mathrm{d}(x_j, \check{s}_j)) \\
& \cdot \varphi_{j0}(h_{t-1}, x_t, \ldots, x_{j-1}, s_t, \ldots, s_j) \lambda_j(\mathrm{d}s_j) \colon \\
& \rho^j_{(h_{t-1}, x_t, \ldots, x_{j-1}, s_t, \ldots, s_{j-1}, \xi)} \in \Xi_j^j(h_{t-1}, x_t, \ldots, x_{j-1}, s_t, \ldots, s_{j-1}), \\
& w_M^j \text{ is a Borel measurable selection of } W_M^j \}.
\end{aligned}$$

Let

$$\begin{aligned}
& \check{W}_M^j(h_{t-1}, x_t, \ldots, x_{j-1}, s_t, \ldots, s_j) = \\
\{ & \int_{X_j \times \check{S}_j} w_M^j(h_{t-1}, x_t, \ldots, x_j, s_t, \ldots, s_j) \cdot \rho^j_{(h_{t-1}, x_t, \ldots, x_{j-1}, s_t, \ldots, s_{j-1}, \xi)}(\mathrm{d}(x_j, \check{s}_j)) \colon \\
& \rho^j_{(h_{t-1}, x_t, \ldots, x_{j-1}, s_t, \ldots, s_{j-1}, \xi)} \in \Xi_j^j(h_{t-1}, x_t, \ldots, x_{j-1}, s_t, \ldots, s_{j-1}),
\end{aligned}$$

[6] We will need to use Lemma B.2 below, which requires the continuity of the correspondences in terms of the integrated variables. Since W_M^j is only measurable, but not continuous, in s_j, we add a dummy variable $\tilde{s}_j$ so that W_M^j is trivially continuous in such a variable.

w_M^j is a Borel measurable selection of $W_M^j\}$.

Since $W_M^j(h_{t-1}, x_t, \ldots, x_j, s_t, \ldots, s_j)$ is continuous in x_j and does not depend on $\check{s}_j$, it is continuous in $(x_j, \check{s}_j)$. In addition, W_M^j is bounded, measurable, nonempty, convex and compact valued. By Lemma B.2, $\check{W}_M^j$ is bounded, measurable, nonempty and compact valued, and sectionally continuous on X^{j-1}.

It is easy to see that

$$W_M^{j-1}(h_{t-1}, x_t, \ldots, x_{j-1}, s_t, \ldots, s_{j-1}) =$$

$$\int_{S_j} \check{W}_M^j(h_{t-1}, x_t, \ldots, x_{j-1}, s_t, \ldots, s_j)\varphi_{j0}(h_{t-1}, x_t, \ldots, x_{j-1}, s_t, \ldots, s_j)\lambda_j(\mathrm{d}s_j).$$

By Lemma 4, it is bounded, measurable, nonempty and compact valued, and sectionally continuous on X^{j-1}. By induction, one can show that W_M^{t-1} is bounded, measurable, nonempty and compact valued, and sectionally continuous on X^{t-1}.

Let $W^{t-1} = \overline{\cup_{M \geq t} W_M^{t-1}}$. That is, W^{t-1} is the closure of $\cup_{M \geq t} W_M^{t-1}$, which is measurable due to Lemma 2.

First, $W^{t-1} \subseteq Q_t^\tau$ because $W_M^{t-1} \subseteq Q_t^\tau$ for each $M \geq t$ and Q_t^τ is compact valued. Second, fix h_{t-1} and $q \in Q_t^\tau(h_{t-1})$. Then there exists a mapping $\xi \in \Upsilon$ such that

$$q = \int_{\prod_{m \geq t}(X_m \times S_m)} u(h_{t-1}, x, s)\varrho_{(h_{t-1}, \xi)}(\mathrm{d}(x, s)).$$

For $M \geq t$, let

$$V_M(h_{t-1}, x_t, \ldots, x_M, s_t, \ldots, s_M) =$$

$$\int_{\prod_{m > M}(X_m \times S_m)} u(h_{t-1}, x_t, \ldots, x_M, s_t, \ldots, s_M, x, s)\varrho_{(h_{t-1}, x_t, \ldots, x_M, s_t, \ldots, s_M, \xi)}(x, s)$$

and

$$q_M = \int_{\prod_{t \leq m \leq M}(X_m \times S_m)} V_M(h_{t-1}, x, s)\varrho_{(h_{t-1}, \xi)}^M(\mathrm{d}(x, s)).$$

Hence, $q_M \in W_M^{t-1}$. Because the dynamic game is continuous at infinity, $q_M \to q$, which implies that $q \in W^{t-1}(h_{t-1})$ and $Q_t^\tau \subseteq W^{t-1}$.

Therefore, $W^{t-1} = Q_t^\tau$, and hence Q_t^τ is measurable for $t > \tau$.

Step 3. For $t \leq \tau$, we can start with $Q_{\tau+1}^\tau$. Repeating the backward induction in Subsection B.4.1, we have that Q_t^τ is also bounded, measurable, nonempty and compact valued, and essentially sectionally upper hemicontinuous on X^{t-1}. $\square$

Denote

$$Q_t^\infty = \begin{cases} Q_t^{t-1}, & \text{if } \cap_{\tau\geq 1} Q_t^\tau = \emptyset; \\ \cap_{\tau\geq 1} Q_t^\tau, & \text{otherwise.} \end{cases}$$

The following three lemmas show that $Q_t^\infty(h_{t-1}) = \Phi(Q_{t+1}^\infty)(h_{t-1}) = E_t(h_{t-1})$ for λ^{t-1}-almost all $h_{t-1} \in H_{t-1}$.[7]

Lemma B.10. *1. The correspondence Q_t^∞ is bounded, measurable, nonempty and compact valued, and essentially sectionally upper hemicontinuous on X^{t-1}.*

2. For any $t \geq 1$, $Q_t^\infty(h_{t-1}) = \Phi(Q_{t+1}^\infty)(h_{t-1})$ for λ^{t-1}-almost all $h_{t-1} \in H_{t-1}$.

Proof. (1) It is obvious that Q_t^∞ is bounded. By the definition of Q_t^τ, for λ^{t-1}-almost all $h_{t-1} \in H_{t-1}$, $Q_t^{\tau_1}(h_{t-1}) \subseteq Q_t^{\tau_2}(h_{t-1})$ for $\tau_1 \geq \tau_2$. Since Q_t^τ is nonempty and compact valued, $Q_t^\infty = \cap_{\tau\geq 1} Q_t^\tau$ is nonempty and compact valued for λ^{t-1}-almost all $h_{t-1} \in H_{t-1}$. If $\cap_{\tau\geq 1} Q_t^\tau = \emptyset$, then $Q_t^\infty = Q_t^{t-1}$. Thus, $Q_t^\infty(h_{t-1})$ is nonempty and compact valued for all $h_{t-1} \in H_{t-1}$. By Lemma 2 (2), $\cap_{\tau\geq 1} Q_t^\tau$ is measurable, which implies that Q_t^∞ is measurable.

Fix any $s^{t-1} \in S^{t-1}$ such that $Q_t^\tau(\cdot, s^{t-1})$ is upper hemicontinuous on $H_{t-1}(s^{t-1})$ for any τ. By Lemma 2 (7), $Q_t^\tau(\cdot, s^{t-1})$ has a closed graph for each τ, which implies that $Q_t^\infty(\cdot, s^{t-1})$ has a closed graph. Referring to Lemma 2 (7) again, $Q_t^\infty(\cdot, s^{t-1})$ is upper hemicontinuous on $H_{t-1}(s^{t-1})$. Since Q_t^τ is essentially upper hemicontinuous on X^{t-1} for each τ, Q_t^∞ is essentially upper upper hemicontinuous on X^{t-1}.

(2) For any $\tau \geq 1$ and λ^{t-1}-almost all $h_{t-1} \in H_{t-1}$, $\Phi(Q_{t+1}^\infty)(h_{t-1}) \subseteq \Phi(Q_{t+1}^\tau)(h_{t-1}) \subseteq Q_t^\tau(h_{t-1})$, and hence $\Phi(Q_{t+1}^\infty)(h_{t-1}) \subseteq Q_t^\infty(h_{t-1})$.

The space $\{1, 2, \ldots \infty\}$ is a countable compact set endowed with the following metric: $d(k, m) = |\frac{1}{k} - \frac{1}{m}|$ for any $1 \leq k, m \leq \infty$. The sequence $\{Q_{t+1}^\tau\}_{1\leq\tau\leq\infty}$ can be regarded as a correspondence Q_{t+1} from $H_t \times \{1, 2, \ldots, \infty\}$ to $\mathbb{R}^n$, which is measurable, nonempty and compact valued, and essentially sectionally upper hemicontinuous on $X^t \times \{1, 2, \ldots, \infty\}$. The backward induction in Subsection B.4.1 shows that $\Phi(Q_{t+1})$ is measurable, nonempty and compact valued, and essentially sectionally upper hemicontinuous on $X^t \times \{1, 2, \ldots, \infty\}$.

[7] The proofs for Lemmas B.10 and B.12 follow the standard ideas with various modifications; see, for example, [3], [4] and [5].

Since $\Phi(Q_{t+1})$ is essentially sectionally upper hemicontinuous on $X^t \times \{1, 2, \ldots, \infty\}$, there exists a measurable subset $\check{S}^{t-1} \subseteq S^{t-1}$ such that $\lambda^{t-1}(\check{S}^{t-1}) = 1$, and $\Phi(Q_{t+1})(\cdot, \cdot, \check{s}^{t-1})$ is upper hemicontinuous for any $\check{s}^{t-1} \in \check{S}^{t-1}$. Fix $\check{s}^{t-1} \in \check{S}^{t-1}$. For $h_{t-1} = (x^{t-1}, \check{s}^{t-1}) \in H_{t-1}$ and $a \in Q_t^{\infty}(h_{t-1})$, by its definition, $a \in Q_t^{\tau}(h_{t-1}) = \Phi(Q_{t+1}^{\tau})(h_{t-1})$ for $\tau \geq t$. Thus, $a \in \Phi(Q_{t+1}^{\infty})(h_{t-1})$.

In summary, $Q_t^{\infty}(h_{t-1}) = \Phi(Q_{t+1}^{\infty})(h_{t-1})$ for λ^{t-1}-almost all $h_{t-1} \in H_{t-1}$. □

Though the definition of Q_t^{τ} involves correlated strategies for $\tau < t$, the following lemma shows that one can work with mixed strategies in terms of equilibrium payoffs via the combination of backward and forward inductions in multiple steps.

Lemma B.11. *If c_t is a measurable selection of $\Phi(Q_{t+1}^{\infty})$, then $c_t(h_{t-1})$ is a subgame-perfect equilibrium payoff vector for λ^{t-1}-almost all $h_{t-1} \in H_{t-1}$.*

Proof. Without loss of generality, we only prove the case $t = 1$.

Suppose that c_1 is a measurable selection of $\Phi(Q_2^{\infty})$. Apply Proposition B.3 recursively to obtain Borel measurable mappings $\{f_{ki}\}_{i \in I}$ for $k \geq 1$. That is, for any $k \geq 1$, there exists a Borel measurable selection c_k of Q_k^{∞} such that for λ_{k-1}-almost all $h_{k-1} \in H_{k-1}$,

1. $f_k(h_{k-1})$ is a Nash equilibrium in the subgame h_{k-1}, where the action space is $A_{ki}(h_{k-1})$ for player $i \in I$, and the payoff function is given by

$$\int_{S_k} c_{k+1}(h_{k-1}, \cdot, s_k) f_{k0}(\mathrm{d}s_k | h_{k-1}).$$

2. $$c_k(h_{k-1}) = \int_{A_k(h_{k-1})} \int_{S_k} c_{k+1}(h_{k-1}, x_k, s_k) f_{k0}(\mathrm{d}s_k | h_{k-1}) f_k(\mathrm{d}x_k | h_{k-1}).$$

We need to show that $c_1(h_0)$ is a subgame-perfect equilibrium payoff vector for λ_0-almost all $h_0 \in H_0$.

Step 1. We show that for any $k \geq 1$ and λ_{k-1}-almost all $h_{k-1} \in H_{k-1}$,

$$c_k(h_{k-1}) = \int_{\prod_{m \geq k}(X_m \times S_m)} u(h_{k-1}, x, s) \varrho_{(h_{k-1}, f)}(\mathrm{d}(x, s)).$$

Since the game is continuous at infinity, there exists some positive integer $M > k$ such that w^M is sufficiently small. By Lemma B.10, $c_k(h_{k-1}) \in Q_k^{\infty}(h_{k-1}) = \cap_{\tau \geq 1} Q_k^{\tau}(h_{k-1})$

for λ_{k-1}-almost all $h_{k-1} \in H_{k-1}$. Since $Q_k^\tau = \Phi^{\tau-k+1}(Q_{\tau+1}^\tau)$ for $k \le \tau$, $c_k(h_{k-1}) \in \cap_{\tau \ge k} \Phi^{\tau-k+1}(Q_{\tau+1}^\tau)(h_{k-1}) \subseteq \Phi^{M-k+1}(Q_{M+1}^M)(h_{k-1})$ for λ_{k-1}-almost all $h_{k-1} \in H_{k-1}$. Thus, there exists a Borel measurable selection w of Q_{M+1}^M and some $\xi \in \Upsilon$ such that for λ_{M-1}-almost all $h_{M-1} \in H_{M-1}$,

i. $f_M(h_{M-1})$ is a Nash equilibrium in the subgame h_{M-1}, where the action space is $A_{Mi}(h_{M-1})$ for player $i \in I$, and the payoff function is given by

$$\int_{S_M} w(h_{M-1}, \cdot, s_M) f_{M0}(\mathrm{d}s_M | h_{M-1});$$

ii.

$$c_M(h_{M-1}) = \int_{A_M(h_{M-1})} \int_{S_M} w(h_{M-1}, x_M, s_M) f_{M0}(\mathrm{d}s_M | h_{M-1}) f_M(\mathrm{d}x_M | h_{M-1});$$

iii. $w(h_M) = \int_{\prod_{m \ge M+1}(X_m \times S_m)} u(h_M, x, s) \varrho_{(h_M, \xi)}(\mathrm{d}(x, s))$.

Then for λ_{k-1}-almost all $h_{k-1} \in H_{k-1}$,

$$c_k(h_{k-1}) = \int_{\prod_{m \ge k}(X_m \times S_m)} u(h_{k-1}, x, s) \varrho_{(h_{k-1}, f^M)}(\mathrm{d}(x, s)),$$

where f_k^M is f_k if $k \le M$, and ξ_k if $k \ge M+1$. Since the game is continuous at infinity,

$$\int_{\prod_{m \ge k}(X_m \times S_m)} u(h_{k-1}, x, s) \varrho_{(h_{k-1}, f^M)}(\mathrm{d}(x, s))$$

converges to

$$\int_{\prod_{m \ge k}(X_m \times S_m)} u(h_{k-1}, x, s) \varrho_{(h_{k-1}, f)}(\mathrm{d}(x, s))$$

when M goes to infinity. Thus, for λ_{k-1}-almost all $h_{k-1} \in H_{k-1}$,

$$c_k(h_{t-1}) = \int_{\prod_{m \ge k}(X_m \times S_m)} u(h_{k-1}, x, s) \varrho_{(h_{k-1}, f)}(\mathrm{d}(x, s)). \tag{3}$$

Step 2. Below, we show that $\{f_{ki}\}_{i \in I}$ is a subgame-perfect equilibrium.

Fix a player i and a strategy $g_i = \{g_{ki}\}_{k \ge 1}$. For each $k \ge 1$, define a new strategy $\tilde{f}_i^k$ as follows: $\tilde{f}_i^k = (g_{1i}, \ldots, g_{ki}, f_{(k+1)i}, f_{(k+2)i}, \ldots)$. That is, we simply replace the initial k stages of f_i by g_i. Denote $\tilde{f}^k = (\tilde{f}_i^k, f_{-i})$.

Fix $k \geq 1$ and a measurable subset $D^k \subseteq S^k$ such that (1) and (2) of step 1 and Equation (3) hold for all $s_k \in D^k$ and $x^k \in H_k(s^k)$, and $\lambda^k(D^k) = 1$. For each $\tilde{M} > k$, by the Fubini property, there exists a measurable subset $E_k^{\tilde{M}} \subseteq S^k$ such that $\lambda^k(E_k^{\tilde{M}}) = 1$ and $\otimes_{k+1 \leq j \leq \tilde{M}} \lambda_j(D^{\tilde{M}}(s^k)) = 1$ for all $s^k \in E_k^{\tilde{M}}$, where

$$D^{\tilde{M}}(s^k) = \{(s_{k+1}, \ldots, s_{\tilde{M}}) \colon (s^k, s_{k+1}, \ldots, s_{\tilde{M}}) \in D^{\tilde{M}}\}.$$

Let $\hat{D}^k = (\cap_{\tilde{M} > k} E_k^{\tilde{M}}) \cap D^k$. Then $\lambda^k(\hat{D}^k) = 1$.

For any $h_k = (x^k, s^k)$ such that $s^k \in \hat{D}^k$ and $x^k \in H_k(s^k)$, we have

$$\begin{aligned}
&\int_{\prod_{m \geq k+1}(X_m \times S_m)} u(h_k, x, s) \varrho_{(h_k, f)}(\mathrm{d}(x, s)) \\
= &\int_{A_{k+1}(h_k)} \int_{S_{k+1}} c_{(k+2)i}(h_k, x_{k+1}, s_{k+1}) f_{(k+1)0}(\mathrm{d}s_{k+1}|h_k) f_{k+1}(\mathrm{d}x_{k+1}|h_k) \\
\geq &\int_{A_{k+1}(h_k)} \int_{S_{k+1}} c_{(k+2)i}(h_k, x_{k+1}, s_{k+1}) f_{(k+1)0}(\mathrm{d}s_{k+1}|h_k) \left(f_{(k+1)(-i)} \otimes g_{(k+1)i}\right) (\mathrm{d}x_{k+1}|h_k) \\
= &\int_{A_{k+1}(h_k)} \int_{S_{k+1}} \int_{A_{k+2}(h_k, x_{k+1}, s_{k+1})} \int_{S_{k+2}} c_{(k+3)i}(h_k, x_{k+1}, s_{k+1}, x_{k+2}, s_{k+2}) \\
&\quad f_{(k+2)0}(\mathrm{d}s_{k+2}|h_k, x_{k+1}, s_{k+1}) f_{(k+2)(-i)} \otimes f_{(k+2)i}(\mathrm{d}x_{k+2}|h_k, x_{k+1}, s_{k+1}) \\
&\quad f_{(k+1)0}(\mathrm{d}s_{k+1}|h_k) f_{(k+1)(-i)} \otimes g_{(k+1)i}(\mathrm{d}x_{k+1}|h_k) \\
\geq &\int_{A_{k+1}(h_k)} \int_{S_{k+1}} \int_{A_{k+2}(h_k, x_{k+1}, s_{k+1})} \int_{S_{k+2}} c_{(k+3)i}(h_k, x_{k+1}, s_{k+1}, x_{k+2}, s_{k+2}) \\
&\quad f_{(k+2)0}(\mathrm{d}s_{k+2}|h_k, x_{k+1}, s_{k+1}) f_{(k+2)(-i)} \otimes g_{(k+2)i}(\mathrm{d}x_{k+2}|h_k, x_{k+1}, s_{k+1}) \\
&\quad f_{(k+1)0}(\mathrm{d}s_{k+1}|h_k) f_{(k+1)(-i)} \otimes g_{(k+1)i}(\mathrm{d}x_{k+1}|h_k) \\
= &\int_{\prod_{m \geq k+1}(X_m \times S_m)} u(h_k, x, s) \varrho_{(h_k, \tilde{f}^{k+2})}(\mathrm{d}(x, s)).
\end{aligned}$$

The first and the last equalities follow from Equation (3) in the end of step 1. The second equality is due to (2) in step 1. The first inequality is based on (1) in step 1. The second inequality holds by the following arguments:

i. by the choice of h_k and (1) in step 1, for λ_{k+1}-almost all $s_{k+1} \in S_{k+1}$ and all $x_{k+1} \in X_{k+1}$ such that $(h_k, x_{k+1}, s_{k+1}) \in H_{k+1}$, we have

$$\begin{aligned}
&\int_{A_{k+2}(h_k, x_{k+1}, s_{k+1})} \int_{S_{k+2}} c_{(k+3)i}(h_k, x_{k+1}, s_{k+1}, x_{k+2}, s_{k+2}) \\
&f_{(k+2)0}(\mathrm{d}s_{k+2}|h_k, x_{k+1}, s_{k+1}) f_{(k+2)(-i)} \otimes f_{(k+2)i}(\mathrm{d}x_{k+2}|h_k, x_{k+1}, s_{k+1})
\end{aligned}$$

$$\geq \int_{A_{k+2}(h_k,x_{k+1},s_{k+1})} \int_{S_{k+2}} c_{(k+3)i}(h_k, x_{k+1}, s_{k+1}, x_{k+2}, s_{k+2})$$
$$f_{(k+2)0}(\mathrm{d}s_{k+2}|h_k, x_{k+1}, s_{k+1}) f_{(k+2)(-i)} \otimes g_{(k+2)i}(\mathrm{d}x_{k+2}|h_k, x_{k+1}, s_{k+1});$$

ii. since $f_{(k+1)0}$ is absolutely continuous with respect to λ_{k+1}, the above inequality also holds for $f_{(k+1)0}(h_k)$-almost all $s_{k+1} \in S_{k+1}$ and all $x_{k+1} \in X_{k+1}$ such that $(h_k, x_{k+1}, s_{k+1}) \in H_{k+1}$.

Repeating the above argument, one can show that

$$\int_{\prod_{m\geq k+1}(X_m\times S_m)} u(h_k, x, s) \varrho_{(h_k,f)}(\mathrm{d}(x, s))$$
$$\geq \int_{\prod_{m\geq k+1}(X_m\times S_m)} u(h_k, x, s) \varrho_{(h_k,\tilde{f}^{\tilde{M}+1})}(\mathrm{d}(x, s))$$

for any $\tilde{M} > k$. Since

$$\int_{\prod_{m\geq k+1}(X_m\times S_m)} u(h_k, x, s) \varrho_{(h_k,\tilde{f}^{\tilde{M}+1})}(\mathrm{d}(x, s))$$

converges to

$$\int_{\prod_{m\geq k+1}(X_m\times S_m)} u(h_k, x, s) \varrho_{(h_k,(g_i,f_{-i}))}(\mathrm{d}(x, s))$$

as $\tilde{M}$ goes to infinity, we have

$$\int_{\prod_{m\geq k+1}(X_m\times S_m)} u(h_k, x, s) \varrho_{(h_k,f)}(\mathrm{d}(x, s))$$
$$\geq \int_{\prod_{m\geq k+1}(X_m\times S_m)} u(h_k, x, s) \varrho_{(h_k,(g_i,f_{-i}))}(\mathrm{d}(x, s)).$$

Therefore, $\{f_{ki}\}_{i\in I}$ is a subgame-perfect equilibrium. □

By Lemma B.10 and Proposition B.2, the correspondence $\Phi(Q_{t+1}^{\infty})$ is measurable, nonempty and compact valued. By Lemma 2 (3), it has a measurable selection. Then Theorem 3 follows from the above lemma.

For $t \geq 1$ and $h_{t-1} \in H_{t-1}$, recall that $E_t(h_{t-1})$ is the set of payoff vectors of subgame-perfect equilibria in the subgame h_{t-1}. The following lemma shows that $E_t(h_{t-1})$ is essentially the same as $Q_t^{\infty}(h_{t-1})$.

Lemma B.12. *For any $t \geq 1$, $E_t(h_{t-1}) = Q_t^{\infty}(h_{t-1})$ for λ^{t-1}-almost all $h_{t-1} \in H_{t-1}$.*

Proof. (1) We will first prove the following claim: for any t and τ, if $E_{t+1}(h_t) \subseteq Q^\tau_{t+1}(h_t)$ for λ^t-almost all $h_t \in H_t$, then $E_t(h_{t-1}) \subseteq Q^\tau_t(h_{t-1})$ for λ^{t-1}-almost all $h_{t-1} \in H_{t-1}$. We only need to consider the case that $t \le \tau$.

By the construction of $\Phi(Q^\tau_{t+1})$ in Subsection B.4.1, there exists a measurable subset $\acute{S}^{t-1} \subseteq S^{t-1}$ with $\lambda^{t-1}(\acute{S}^{t-1}) = 1$ such that for any c_t and $h_{t-1} = (x^{t-1}, \acute{s}^{t-1}) \in H_{t-1}$ with $\acute{s}^{t-1} \in \acute{S}^{t-1}$, if

1. $c_t = \int_{A_t(h_{t-1})} \int_{S_t} q_{t+1}(h_{t-1}, x_t, s_t) f_{t0}(\mathrm{d}s_t|h_{t-1}) \alpha(\mathrm{d}x_t)$, where $q_{t+1}(h_{t-1}, \cdot)$ is measurable and $q_{t+1}(h_{t-1}, x_t, s_t) \in Q^\tau_{t+1}(h_{t-1}, x_t, s_t)$ for λ_t-almost all $s_t \in S_t$ and $x_t \in A_t(h_{t-1})$;

2. $\alpha \in \otimes_{i \in I} \mathcal{M}(A_{ti}(h_{t-1}))$ is a Nash equilibrium in the subgame h_{t-1} with payoff $\int_{S_t} q_{t+1}(h_{t-1}, \cdot, s_t) f_{t0}(\mathrm{d}s_t|h_{t-1})$ and action space $\prod_{i \in I} A_{ti}(h_{t-1})$,

then $c_t \in \Phi(Q^\tau_{t+1})(h_{t-1})$.

Fix a subgame $h_{t-1} = (x^{t-1}, \acute{s}^{t-1})$ such that $\acute{s}^{t-1} \in \acute{S}^{t-1}$. Pick a point $c_t \in E_t(\acute{s}^{t-1})$. There exists a strategy profile f such that f is a subgame-perfect equilibrium in the subgame h_{t-1} and the payoff is c_t. Let $c_{t+1}(h_{t-1}, x_t, s_t)$ be the payoff vector induced by $\{f_{ti}\}_{i \in I}$ in the subgame $(h_t, x_t, s_t) \in \mathrm{Gr}(A_t) \times S_t$. Then we have

1. $c_t = \int_{A_t(h_{t-1})} \int_{S_t} c_{t+1}(h_{t-1}, x_t, s_t) f_{t0}(\mathrm{d}s_t|h_{t-1}) f_t(\mathrm{d}x_t|h_{t-1})$;

2. $f_t(\cdot|h_{t-1})$ is a Nash equilibrium in the subgame h_{t-1} with action space $A_t(h_{t-1})$ and payoff $\int_{S_t} c_{t+1}(h_{t-1}, \cdot, s_t) f_{t0}(\mathrm{d}s_t|h_{t-1})$.

Since f is a subgame-perfect equilibrium in the subgame h_{t-1}, $c_{t+1}(h_{t-1}, x_t, s_t) \in E_{t+1}(h_{t-1}, x_t, s_t) \subseteq Q^\tau_{t+1}(h_{t-1}, x_t, s_t)$ for λ_t-almost all $s_t \in S_t$ and $x_t \in A_t(h_{t-1})$, which implies that $c_t \in \Phi(Q^\tau_{t+1})(h_{t-1}) = Q^\tau_t(h_{t-1})$.

Therefore, $E_t(h_{t-1}) \subseteq Q^\tau_t(h_{t-1})$ for λ^{t-1}-almost all $h_{t-1} \in H_{t-1}$.

(2) For any $t > \tau$, $E_t \subseteq Q^\tau_t$. If $t \le \tau$, we can start with $E_{\tau+1} \subseteq Q^\tau_{\tau+1}$ and repeat the argument in (1), then we can show that $E_t(h_{t-1}) \subseteq Q^\tau_t(h_{t-1})$ for λ^{t-1}-almost all $h_{t-1} \in H_{t-1}$. Thus, $E_t(h_{t-1}) \subseteq Q^\infty_t(h_{t-1})$ for λ^{t-1}-almost all $h_{t-1} \in H_{t-1}$.

(3) Suppose that c_t is a measurable selection from $\Phi(Q^\infty_{t+1})$. Apply Proposition B.3 recursively to obtain Borel measurable mappings $\{f_{ki}\}_{i \in I}$ for $k \ge t$. By Lemma B.11, $c_t(h_{t-1})$ is a subgame-perfect equilibrium payoff vector for λ^{t-1}-almost all $h_{t-1} \in H_{t-1}$. Consequently, $\Phi(Q^\infty_{t+1})(h_{t-1}) \subseteq E_t(h_{t-1})$ for λ^{t-1}-almost all $h_{t-1} \in H_{t-1}$.

By Lemma B.10, $E_t(h_{t-1}) = Q^\infty_t(h_{t-1}) = \Phi(Q^\infty_{t+1})(h_{t-1})$ for λ^{t-1}-almost all $h_{t-1} \in H_{t-1}$. □

B.5 Proof of Proposition B.1

We will highlight the needed changes in comparison with the proofs presented in Subsections B.4.1-B.4.3.

1. Backward induction. We first consider stage t with $N_t = 1$.

If $N_t = 1$, then $S_t = \{\acute{s_t}\}$. Thus, $P_t(h_{t-1}, x_t) = Q_{t+1}(h_{t-1}, x_t, \acute{s_t})$, which is nonempty and compact valued, and essentially sectionally upper hemicontinuous on $X^t \times \hat{S}^{t-1}$. Notice that P_t may not be convex valued.

We first assume that P_t is upper hemicontinuous. Suppose that j is the player who is active in this period. Consider the correspondence $\Phi_t \colon H_{t-1} \to \mathbb{R}^n \times \mathcal{M}(X_t) \times \triangle(X_t)$ defined as follows: $(v, \alpha, \mu) \in \Phi_t(h_{t-1})$ if

1. $v = p_t(h_{t-1}, A_{t(-j)}(h_{t-1}), x^*_{tj})$ such that $p_t(h_{t-1}, \cdot)$ is a measurable selection of $P_t(h_{t-1}, \cdot)$;[8]
2. $x^*_{tj} \in A_{tj}(h_{t-1})$ is a maximization point of player j given the payoff function $p_{tj}(h_{t-1}, A_{t(-j)}(h_{t-1}), \cdot)$ and the action space $A_{tj}(h_{t-1})$, $\alpha_i = \delta_{A_{ti}(h_{t-1})}$ for $i \neq j$ and $\alpha_j = \delta_{x^*_{tj}}$;
3. $\mu = \delta_{p_t(h_{t-1}, A_{t(-j)}(h_{t-1}), x^*_{tj})}$.

This is a single agent problem. We need to show that Φ_t is nonempty and compact valued, and upper hemicontinuous.

If P_t is nonempty, convex and compact valued, and upper hemicontinuous, then we can use Lemma 10, the main result of [7], to prove the nonemptiness, compactness, and upper hemicontinuity of Φ_t. In [7], the only step they need the convexity of P_t for the proof of their main theorem is Lemma 2 therein. However, the one-player pure-strategy version of their Lemma 2, stated in the following, directly follows from the upper hemicontinuity of P_t without requiring the convexity.

> Let Z be a compact metric space, and $\{z_n\}_{n \geq 0} \subseteq Z$. Let $P \colon Z \to \mathbb{R}_+$ be a bounded, upper hemicontinuous correspondence with nonempty and compact values. For each $n \geq 1$, let q_n be a Borel measurable selection of P such that $q_n(z_n) = d_n$. If z_n converges to z_0 and d_n converges to some d_0, then $d_0 \in P(z_0)$.

Repeat the argument in the proof of the main theorem of [7], one can show that Φ_t is nonempty and compact valued, and upper hemicontinuous.

[8] Note that $A_{t(-j)}$ is point valued since all players other than j are inactive.

Then we go back to the case that P_t is nonempty and compact valued, and essentially sectionally upper hemicontinuous on $X^t \times \hat{S}^{t-1}$. Recall that we proved Proposition B.2 based on Lemma 10. If P_t is essentially sectionally upper hemicontinuous on $X^t \times \hat{S}^{t-1}$, we can show the following result based on a similar argument as in Sections B.3: there exists a bounded, measurable, nonempty and compact valued correspondence Φ_t from H_{t-1} to $\mathbb{R}^n \times \mathcal{M}(X_t) \times \triangle(X_t)$ such that Φ_t is essentially sectionally upper hemicontinuous on $X^{t-1} \times \hat{S}^{t-1}$, and for λ^{t-1}-almost all $h_{t-1} \in H_{t-1}$, $(v, \alpha, \mu) \in \Phi_t(h_{t-1})$ if

1. $v = p_t(h_{t-1}, A_{t(-j)}(h_{t-1}), x_{tj}^*)$ such that $p_t(h_{t-1}, \cdot)$ is a measurable selection of $P_t(h_{t-1}, \cdot)$;
2. $x_{tj}^* \in A_{tj}(h_{t-1})$ is a maximization point of player j given the payoff function $p_{tj}(h_{t-1}, A_{t(-j)}(h_{t-1}), \cdot)$ and the action space $A_{tj}(h_{t-1})$, $\alpha_i = \delta_{A_{ti}(h_{t-1})}$ for $i \neq j$ and $\alpha_j = \delta_{x_{tj}^*}$;
3. $\mu = \delta_{p_t(h_{t-1}, A_{t(-j)}(h_{t-1}), x_{tj}^*)}$.

Next we consider the case that $N_t = 0$. Suppose that the correspondence Q_{t+1} from H_t to $\mathbb{R}^n$ is bounded, measurable, nonempty and compact valued, and essentially sectionally upper hemicontinuous on $X^t \times \hat{S}^t$. For any $(h_{t-1}, x_t, \hat{s}_t) \in \mathrm{Gr}(\hat{A}_t)$, let

$$\begin{aligned} R_t(h_{t-1}, x_t, \hat{s}_t) &= \int_{\tilde{S}_t} Q_{t+1}(h_{t-1}, x_t, \hat{s}_t, \tilde{s}_t) \tilde{f}_{t0}(\mathrm{d}\tilde{s}_t | h_{t-1}, x_t, \hat{s}_t) \\ &= \int_{\tilde{S}_t} Q_{t+1}(h_{t-1}, x_t, \hat{s}_t, \tilde{s}_t) \varphi_{t0}(h_{t-1}, x_t, \hat{s}_t, \tilde{s}_t) \lambda_t(\mathrm{d}\tilde{s}_t). \end{aligned}$$

Then following the same argument as in Subsection B.4.1, one can show that R_t is a nonempty, convex and compact valued, and essentially sectionally upper hemicontinuous correspondence on $X^t \times \hat{S}^t$.

For any $h_{t-1} \in H_{t-1}$ and $x_t \in A_t(h_{t-1})$, let

$$P_t(h_{t-1}, x_t) = \int_{\hat{A}_{t0}(h_{t-1}, x_t)} R_t(h_{t-1}, x_t, \hat{s}_t) \hat{f}_{t0}(\mathrm{d}\hat{s}_t | h_{t-1}, x_t).$$

By Lemma 7, P_t is nonempty, convex and compact valued, and essentially sectionally upper hemicontinuous on $X^t \times \hat{S}^{t-1}$. The rest of the step remains the same as in Subsection B.4.1.

2. Forward induction: unchanged.

3. Infinite horizon: we need to slightly modify the definition of $\Xi_t^{m_1}$ for any $m_1 \geq t \geq 1$. Fix any $t \geq 1$. Define a correspondence Ξ_t^t as follows: in the subgame h_{t-1},

$$\Xi_t^t(h_{t-1}) = (\mathcal{M}(A_t(h_{t-1})) \diamond \hat{f}_{t0}(h_{t-1}, \cdot)) \otimes \lambda_t.$$

For any $m_1 > t$, suppose that the correspondence $\Xi_t^{m_1-1}$ has been defined. Then we can define a correspondence $\Xi_t^{m_1}: H_{t-1} \to \mathcal{M}\left(\prod_{t \leq m \leq m_1}(X_m \times S_m)\right)$ as follows:

$$\begin{aligned}\Xi_t^{m_1}(h_{t-1}) = &\Big\{g(h_{t-1}) \diamond \Big((\xi_{m_1}(h_{t-1}, \cdot) \diamond \hat{f}_{m_1 0}(h_{t-1}, \cdot)) \otimes \lambda_{m_1}\Big) : \\ &g \text{ is a Borel measurable selection of } \Xi_t^{m_1-1}, \\ &\xi_{m_1} \text{ is a Borel measurable selection of } \mathcal{M}(A_{m_1})\Big\}.\end{aligned}$$

Then the result in Subsection B.4.3 is true with the above $\Xi_t^{m_1}$.

Consequently, a subgame-perfect equilibrium exists.

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