Supplement to "Dynamic contracting with limited commitment and the ratchet effect"

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Proof of Lemma 3.

By contradiction, suppose there exist a PBE (σ, μ) and a history (h^t, m_t) satisfying the three properties in Lemma 3. First, consider the history $(h^t, m_t, (x_H, q_H))$. The firm's belief will be equal to zero and, in equilibrium, the menu offered by the firm in period $t+1, t+2, \ldots,$ will contain the contract $(\theta_H q_H^* + \alpha, q_H^*)$. Furthermore, the high type will select this contract in every period. We conclude that following $(h^t, m_t, (x_H, q_H))$, the high type's continuation payoff (evaluated at the beginning of period $t+1$) will be equal to zero. Furthermore, if the low type deviates and accepts the contract (x_H, q_H), then his continuation payoff will be at least $\Delta\theta q_H^*$ (in fact, the low type can mimic the high type and accept the contract $(\theta_H q_H^* + \alpha, q_H^*)$ in period $t+1, t+2, \ldots$).

Consider now the history $(h^t, m_t, (x_L, q_L))$. The firm's belief will be equal to one and, in equilibrium, the low type will accept the contract $(\theta_L q_L^* + \alpha, q_L^*)$ in period $t+1, t+2, \ldots$. We conclude that after the history $(h^t, m_t, (x_L, q_L))$ the equilibrium continuation payoff of both types (again, evaluated at the beginning of period $t+1$) is equal to zero.

Clearly, in equilibrium, the worker's decision must be sequentially rational. Therefore, the contracts (x_H, q_H) and (x_L, q_L) must satisfy the following IC constraints:

$$x_H - \theta_H q_H - \alpha \geqslant x_L - \theta_H q_L - \alpha$$
$$(1-\delta)(x_L - \theta_L q_L - \alpha) \geqslant (1-\delta)(x_H - \theta_L q_H - \alpha) + \delta \Delta\theta q_H^*$$

Combining the two constraints we obtain

$$\theta_H (q_L - q_H) \geqslant x_L - x_H \geqslant \theta_L (q_L - q_H) + \frac{\delta}{1-\delta} \Delta\theta q_H^*$$

which implies

$$\Delta\theta \geqslant \Delta\theta (q_L - q_H) \geqslant \frac{\delta}{1-\delta} \Delta\theta q_H^*$$

Clearly, the second inequality cannot be satisfied if $\delta > \hat{\delta}$. ■

Proof of Proposition 4.

We now describe a strategy profile and a system of beliefs which yield the payoffs $(V_{F,H}, V_{F,L}, W_H, W_L)$ (we divide the description into different phases). Then we show that unilateral deviations are not profitable when the discount factor δ is sufficiently large.

Screening Phase: In the first period, the firm offers the menu $\{(x_H, q_H), (x_L, q_L)\}$ (recall that (x_i, q_i), $i \in \{H, L\}$, is a contract yielding the payoff $V_{F,i}$ to the firm and the payoff W_i to type i). If both contracts are rejected, the firm does not update its belief and insists on the same menu until a contract (x_i, q_i), $i \in \{H, L\}$, is accepted. In this case, the firm's belief assigns probability one to type i. Furthermore, the firm does not revise its belief in future periods and the continuation equilibrium consistent with the automaton described below starting at the state $(i, 0)$ follows.

Suppose that during the screening phase the firm deviates and offers a menu m different from $\{(x_H, q_H), (x_L, q_L)\}$. Let $(x^*(m), q^*(m)) \in m$ denote the optimal contract for the high type in m. Formally:

$$x^*(m) - \theta_H q^*(m) - \alpha \geqslant x_j - \theta_H q_j - \alpha$$

for all $(x_j, q_j) \in m$.[23]

If $x^*(m) < \alpha + \theta_H + v(1)$, every type of the worker rejects all the contracts and the screening phase continues in the next period with the firm insisting on the menu $\{(x_L, q_L), (x_H, q_H)\}$. If any contract $(x_k, q_k) \in m$ is selected, the firm's belief assigns probability one to the low type and the continuation equilibrium consistent with the automaton described below starting at the state $(L, 2)$ follows.

If $x^*(m) \geqslant \alpha + \theta_H + v(1)$, every type of the worker accepts the contract $(x^*(m), q^*(m))$ and the screening phase continues in the next period. If any other contract $(x_k, q_k) \in m$ is accepted or if all the contracts are rejected, the firm's belief assigns probability one to the low type and the continuation equilibrium consistent with the automaton described below starting at the state $(L, 2)$ follows.

Post-Screening Phase: According to the description above, a post-screening phase can be reached in a state $(i, r) \in \{H, L\} \times \{0, 1, 2\}$. The transition function among the states and the action prescription for the firm and for type $i \in \{H, L\}$ in state (i, r) are the

[23] If there are several optimal contracts for type H, we select the contract with the smallest index.

same as the ones in the automaton for type i presented in Section 6. The action prescription for type $j \neq i$ in a state (i, r) are defined below.

Actions of type L in the state $(H, 0)$: If the firm offers the menu $\{(x_H, q_H)\}$, the low type accepts (x_H, q_H). If the firm deviates and offers a different menu, then type L accepts the contract that yields the largest current payoff, provided that this is positive (if it is negative, the worker rejects all the contracts).[24]

Actions of type L in the state $(H, 1)$: If the firm offers the menu $\{(\bar{x}_H, q_H^*)\}$, the low type accepts $(\bar{x}_H, q_H^*)$. Consider a deviation by the firm. The low type rejects all the contracts (x, q) with $x < v(1) + \alpha$. Among the remaining contracts, the low type selects the contract which yields the largest current payoff, provided that this is positive (if it is negative, the worker rejects all the contracts).

Actions of type L in the state $(H, 2)$: If the firm offers the menu $\{(\underline{x}_H, q_H^*)\}$, the low type accepts $(\underline{x}_H, q_H^*)$. If the firm deviates and offers a different menu, then type L accepts the contract that yields the largest current payoff, provided that this is positive.

Actions of type H in the state $(L, 0)$: If the firm offers the menu $\{(x_L, q_L)\}$, the high type accepts (x_L, q_L) if and only if $x_L - \theta_H q_L - \alpha \geqslant 0$. If the firm deviates and offers a different menu, then type H accepts the contract that yields the largest current payoff, provided that this is positive (if it is negative, the worker rejects all the contracts).

Actions of type H in the state $(L, 1)$: We distinguish between two cases. First, assume that $\bar{x}_L - \theta_H q_L^* - \alpha > 0$. In this case, if the firm offers the menu $\{(\bar{x}_L, q_L^*)\}$, the high type accepts $(\bar{x}_L, q_L^*)$. Consider a deviation by the firm. The high type rejects all the contracts (x, q) with $x < v(1) + \alpha$. Among the remaining contracts, the high type selects the contract which yields the largest current payoff, provided that this is positive.

Suppose now that $\bar{x}_L - \theta_H q_L^* - \alpha \leqslant 0$. In this case, the high type selects the contract which yields the largest current payoff, provided that this is positive.

Actions of type H in the state $(L, 2)$: If the firm offers the menu $\{(\underline{x}_L, q_L^*)\}$, the high type accepts $(\underline{x}_L, q_L^*)$ provided that it yields a positive current payoff. If the firm deviates and offers a different menu, then type H accepts the contract that yields the largest current payoff, provided that this is positive.

Optimality of the Proposed Strategies. We now analyze the parties' incentives and show deviations are not profitable for δ sufficiently large. Let h^t be an arbitrary history

[24] As usual, the worker selects the contract with the smallest index among those who yield the largest current payoff.

in the screening phase. We let $V_F(S)$ denote the firm's continuation payoff at h^t (the payoff is computed before the firm offers the menu). Recall that the firm's belief at h^t is equal to the prior p_0. We also let $W_i(S)$, $i \in \{H, L\}$, denote the continuation payoff of type i at h^t. We have:

$$V_F(S) = (1 - p_0) V_{F,H} + p_0 V_{F,L} \quad W_L(S) = W_L \quad W_H(S) = W_H$$

We now turn to the post-screening phase. For $i \in \{H, L\}$ and $r \in \{0, 1, 2\}$, let $V_F(i, r)$ and $W_i(i, r)$ denote the firm and type i's continuation payoff, respectively, in the state (i, r).[25] These payoffs are equal to:

$$\begin{array}{llll} V_F(0,H) = V_{F,H} & V_F(0,L) = V_{F,L} & W_H(0,H) = W_H & W_L(0,L) = W_L \\ V_F(1,H) = \frac{\varepsilon}{2} & V_F(1,L) = \frac{\varepsilon}{2} & W_H(1,H) = \pi_H(q_H^*) - \frac{\varepsilon}{2} & W_L(1,L) = \pi_L(q_L^*) - \frac{\varepsilon}{2} \\ V_F(2,H) = \pi_H(q_H^*) - \frac{\varepsilon}{2} & V_F(2,L) = \pi_L(q_L^*) - \frac{\varepsilon}{2} & W_H(2,H) = \frac{\varepsilon}{2} & W_L(2,L) = \frac{\varepsilon}{2} \end{array}$$

Next, we specify the continuation payoff of type $i \in \{H, L\}$ in the state (j, r), $j \neq i$ and $r \in \{0, 1, 2\}$. We have:

$$\begin{array}{ll} W_L(0,H) = W_H + \Delta\theta q_H & W_H(0,L) = \max\{W_L - \Delta\theta q_L, 0\} \\ W_L(1,H) = \pi_H(q_H^*) + \Delta\theta q_H^* - \frac{\varepsilon}{2} & W_H(1,L) = \max\left\{\pi_L(q_L^*) - \Delta\theta q_L^* - \frac{\varepsilon}{2}, 0\right\} \\ W_L(2,H) = \Delta\theta q_H^* + \frac{\varepsilon}{2} & W_H(2,L) = \max\left\{-\Delta\theta q_L^* + \frac{\varepsilon}{2}, 0\right\} \end{array}$$

To show that unilateral deviations from the proposed strategy profile are not profitable, it is enough to verify that finitely many inequalities are satisfied. For every inequality there exists a critical threshold of the discount factor (smaller than one) above which the inequality is satisfied. Thus, there exists $\delta^\dagger \in (0, 1)$ such that for $\delta \geqslant \delta^\dagger$ no unilateral deviation is profitable.

Belief Update. After each menu posted by the firm, the proposed system of beliefs satisfies Bayes's rule for every action taken by the worker with positive probability.

We conclude that the strategy profile and the system of beliefs presented above constitute a PBE when $\delta \geqslant \delta^\dagger$. ■

Proof of Lemma 8.

We develop an iterative procedure which will deliver the pair (V, Φ) with the desired properties.

[25] The action prescription for type i in the state (i, r) is specified in Section 6.

Step 1

First, we allow the firm to propose a menu which separates the two types (with employment). Specifically, for every belief p we consider the following optimization problem:

$$\begin{aligned} V^1(p) := & \max_{(q_H,q_L)\in[0,1]^2, x\in\mathbb{R}} (1-p)\left[(1-\delta)\pi_H(q_H)+\delta\pi_H(q_H^*)\right]+ \\ & \quad p\left[(1-\delta)(v(q_L)-x)+\delta\pi_L(q_L^*)\right] \\ \text{s.t.} \quad & x-\theta_H q_L-\alpha \leqslant 0 \\ & (1-\delta)(x-\theta_L q_L-\alpha) \geqslant (1-\delta)\Delta\theta q_H+\delta\Delta\theta q_H^* \end{aligned}$$

The firm offers the contracts $(\theta_H q_H+\alpha, q_H)$ to the high type and the contract (x, q_L) to the low type. Clearly, at the optimum the low type's IC constraint is binding. Thus, we can rewrite the problem as

$$\begin{aligned} V^1(p) = & \max_{(q_H,q_L)\in[0,1]^2, x\in\mathbb{R}} (1-p)\left[(1-\delta)\pi_H(q_H)+\delta\pi_H(q_H^*)\right]+ \\ & p\left[(1-\delta)\pi_L(q_L)+\delta\pi_L(q_L^*)-(1-\delta)\Delta\theta q_H-\delta\Delta\theta q_H^*\right] \end{aligned} \tag{33}$$

$$\text{s.t.} \quad q_H-q_L+\tfrac{\delta}{1-\delta}q_H^* \leqslant 0 \tag{34}$$

We let $(q_H^1(p), q_L^1(p))$ denote the solution to the above problem. It follows from the concavity of the functions π_H and π_L that $q_H^1(p)$ is uniquely defined for $p\in[0,1)$, and $q_L^1(p)$ is uniquely defined for $p\in(0,1]$. Furthermore $q_H^1(\cdot)$ and $q_L^1(\cdot)$ are upper hemicontinuous (theorem of the maximum), and $V^1(\cdot)$ is continuous (again, theorem of the maximum) and convex (notice that the pairs (q_H, q_L) satisfying constraint (34) do not vary with p). Finally, $q_H^1(p)\leqslant q_H^* < q_L^* \leqslant q_L^1(p)$ for any p, and $q_H^1(\cdot)$ is decreasing in p.

We now distinguish among different cases.

Case 1.1. For every $p\in[0,1]$,

$$V^1(p)\leqslant \max\left\{\pi_H(q_H^*), p\pi_L(q_L^*)\right\}$$

In this case, we let V and Φ be defined as in equations (16) and (17), respectively.

Case 1.2. There exists $p\in(0,1)$ such that

$$V^1(p) > \max\left\{\pi_H(q_H^*), p\pi_L(q_L^*)\right\} \tag{35}$$

Notice that

$$\frac{\partial V^1(p)}{\partial p} = (1-\delta)\pi_L\left(q_L^1(p)\right)+\delta\pi_L(q_L^*)-(1-\delta)\Delta\theta q_H^1(p)-\delta\Delta\theta q_H^*-(1-\delta)\pi_H\left(q_H^1(p)\right)-\delta\pi_H(q_H^*)$$

If $V^1(p) > \pi_H(q_H^*)$ it must be that

$$(1-\delta)\,\pi_L\left(q_L^1(p)\right) + \delta\pi_L(q_L^*) - (1-\delta)\,\Delta\theta q_H - \delta\Delta\theta q_H^* > \pi_H(q_H^*)$$

and, therefore, $\partial V^1(p)/\partial p$ must be strictly positive at any point p which satisfies inequality (35).

Also, recall that V^1 is convex and $V^1(p) \leqslant p\pi_L(q_L^*)$ for every $p \geqslant p^C$. We conclude that the set of beliefs for which inequality (35) holds is an interval $\left(\underline{p}_1, \bar{p}_1\right)$, with $\underline{p}_1 \in [0, \hat{p})$ and $\bar{p}_1 \in \left(\hat{p}, p^C\right]$.

Case 1.2.1. $\underline{p}_1 = 0$.

In this case, $q_H^1(0) = q_H^*$. We point out that the case $\underline{p}_1 = 0$ can arise only if $\delta \leqslant 1 - q_H^*$ (if $\delta > 1 - q_H^*$ it is impossible to find q_L such that the pair (q_H^*, q_L) satisfies constraint (34)).

We claim that for generic values of δ, if $\underline{p}_1 = 0$, then

$$\partial_+ V^1(0) = \lim_{p\downarrow 0}(1-\delta)\,\pi_L\left(q_L^1(p)\right) + \delta\pi_L(q_L^*) - \Delta\theta q_H^* - \pi_H(q_H^*)$$

is strictly positive.

First, for $\delta \leqslant 1 - q_H^*/q_L^*$, $q_L^1(p) = q_L^*$ for every $p > 0$, and thus

$$\partial_+ V^1(0) = \pi_L(q_L^*) - \Delta\theta q_H^* - \pi_H(q_H^*) = \pi_L(q_L^*) - \pi_L(q_H^*) > 0$$

Suppose now that $\delta \in (1 - q_H^*/q_L^*, 1 - q_H^*]$ and $q_H^1(0) = q_H^*$. Then for each δ, there exists ε such that

$$q_L^1(p) = q_H^1(p) + \frac{\delta}{1-\delta}q_H^*$$

for every $p \in [0, \varepsilon]$. Therefore, we have

$$\partial_+ V^1(0) = (1-\delta)\,\pi_L\left(\frac{q_H^*}{1-\delta}\right) + \delta\pi_L(q_L^*) - \Delta\theta q_H^* - \pi_H(q_H^*)$$

Notice that the function $g(\cdot)$ defined by

$$g(\delta) = (1-\delta)\,\pi_L\left(\frac{q_H^*}{1-\delta}\right) + \delta\pi_L(q_L^*) - \Delta\theta q_H^* - \pi_H(q_H^*)$$

is strictly concave and, therefore, there can be at most two distinct values of δ for which $g(\delta)$ is equal to zero. This shows that generically, if $\underline{p}_1 = 0$, then $\partial_+ V^1(p) > 0$. In what follows, we say that the value of δ is generic if $g(\delta) \neq 0$.

When $\underline{p}_1 = 0$ we define $V(\cdot;1)$ and $\Phi(\cdot;1)$ as follows:

$$V(p;1) = \begin{cases} V^1(p) & \text{for } p \leqslant \bar{p}_1 \\ p\pi_L(q_L^*) & \text{for } p > \bar{p}_1 \end{cases}$$

$$\Phi(p;1) = \begin{cases} (1-\delta)\,\Delta\theta q_H^1(p) + \delta\Delta\theta q_H^* & \text{for } p < \bar{p}_1 \\ [0, (1-\delta)\,\Delta\theta q_H^1(\bar{p}_1) + \delta\Delta\theta q_H^*] & \text{for } p = \bar{p}_1 \\ 0 & \text{for } p > \bar{p}_1 \end{cases}$$

Case 1.2.2. $\underline{p}_1 > 0$.

We claim that for every δ we have $\partial_+ V^1\left(\underline{p}_1\right) > 0$. Notice that $V^1(\cdot)$ cannot be constant and equal to $\pi_H(q_H^*)$ in the interval $\left[0, \underline{p}_1\right)$. In fact, if $V^1(0) = \pi_H(q_H^*)$, then we have $q_H^1(0) = q_H^*$. This and the firm's optimality condition imply that $q_H^1(p)$ is strictly decreasing in p in a neighborhood of zero, which, in turn, implies the strict convexity of $V^1(\cdot)$ near zero. Therefore, we conclude that either $V^1(0) < \pi_H(q_H^*)$ or $V^1(0) = \pi_H(q_H^*)$ and $V^1(\cdot)$ is strictly convex in a neighborhood of zero. In either case, $V^1(\cdot)$ achieves a minimum at $p_\dagger \in \left[0,\ \underline{p}_1\right)$ and $V^1(p_\dagger) < \pi_H(q_H^*) = V^1\left(\underline{p}_1\right)$. This and the convexity of $V^1(\cdot)$ imply $\partial_+ V^1\left(\underline{p}_1\right) > 0$.

In this case ($\underline{p}_1 > 0$), we define $V(\cdot;1)$ and $\Phi(\cdot;1)$ as follows:

$$V(p;1) = \begin{cases} \pi_H(q_H^*) & \text{for } p \leqslant \underline{p}_1 \\ V^1(p) & \text{for } p \in \left(\underline{p}_1, \bar{p}_1\right) \\ p\pi_L(q_L^*) & \text{for } p \geqslant \bar{p}_1 \end{cases}$$

$$\Phi(p;1) = \begin{cases} \Delta\theta q_H^* & p < \underline{p}_1 \\ \left[(1-\delta)\,\Delta\theta q_H^1\left(\underline{p}_1\right) + \delta\Delta\theta q_H^*, \Delta\theta q_H^*\right] & p = \underline{p}_1 \\ (1-\delta)\,\Delta\theta q_H^1(p) + \delta\Delta\theta q_H^* & p \in \left(\underline{p}_1, \bar{p}_1\right) \\ [0, (1-\delta)\,\Delta\theta q_H^1(\bar{p}_1) + \delta\Delta\theta q_H^*] & p = \bar{p}_1 \\ 0 & p > \bar{p}_1 \end{cases}$$

Step 2

We now consider the case of probabilistic separation. That is, the firm offers two contracts. The high type chooses the first contract, while the low type randomizes between the two contracts.

For every $p \geqslant \underline{p}_1$, we consider the following optimization problem

$$\begin{array}{ll} V^2(p) := & \max_{(q_H,q_L)\in[0,1]^2, x\in\mathbb{R}, \tilde{p}\in\left[\underline{p}_1, \min\{p,\bar{p}_1\}\right]} \frac{1-p}{1-\tilde{p}}\left[(1-\delta)\pi_H(q_H)+\delta V(\tilde{p};1)\right]+ \\ & \frac{p-\tilde{p}}{1-\tilde{p}}\left[(1-\delta)(v(q_L)-x)+\delta\pi_L(q_L^*)\right] \\ \text{s.t.} & x-\theta_H q_L-\alpha \leqslant 0 \\ & (1-\delta)(x-\theta_L q_L-\alpha) \geqslant (1-\delta)\Delta\theta q_H + \delta \min\Phi(\tilde{p};1) \end{array}$$

The second constraint must bind and we can rewrite the problems as

$$\begin{array}{c} V^2(p) = \max_{(q_H,q_L)\in[0,1]^2, \tilde{p}\in\left[\underline{p}_1, \min\{p,\bar{p}_1\}\right]} \frac{1-p}{1-\tilde{p}}\left[(1-\delta)\pi_H(q_H)+\delta V(\tilde{p};1)\right]+ \\ \frac{p-\tilde{p}}{1-\tilde{p}}\left[(1-\delta)\pi_L(q_L)+\delta\pi_L(q_L^*)-(1-\delta)\Delta\theta q_H-\delta\min\Phi(\tilde{p};1)\right] \\ \\ \text{s.t.} \quad (q_H-q_L)\Delta\theta+\frac{\delta}{1-\delta}\min\Phi(\tilde{p};1)\leqslant 0 \end{array} \tag{36}$$

If $V^2(p) \leqslant V(p;1)$ for every $p \in [0,1]$, then we set $V(\cdot)$ equal to $V(\cdot;1)$, and $\Phi(\cdot)$ equal to $\Phi(\cdot;1)$. On the other hand, if $V^2(p) > V(p;1)$ for some p, we distinguish among different cases.

Case 2.1. $\underline{p}_1 = 0$

First, we assume that $\underline{p}_1 = 0$ and consider the generic values of δ for which $\partial_+ V(0;1) > 0$. We show that when the belief is sufficiently low the firm does not benefit from an additional possibility of screening the worker.

Claim 2 *Assume that* $\underline{p}_1 = 0$. *There exists* $\varepsilon > 0$ *such that* $V^2(p) = V(p;1)$ *for every* $p \in [0,\varepsilon]$.

Proof of Claim 2.

For every p and $\tilde{p} \leqslant p$ define $V^2(p,\tilde{p})$ as follows:

$$\begin{array}{c} V^2(p,\tilde{p}) = \max_{(q_H,q_L)\in[0,1]^2} \frac{1-p}{1-\tilde{p}}\left[(1-\delta)\pi_H(q_H)+\delta V(\tilde{p};1)\right]+ \\ \frac{p-\tilde{p}}{1-\tilde{p}}\left[(1-\delta)\pi_L(q_L)+\delta\pi_L(q_L^*)-(1-\delta)\Delta\theta q_H-\delta\min\Phi(\tilde{p};1)\right] \\ \\ \text{s.t.} \quad (q_H-q_L)\Delta\theta+\frac{\delta}{1-\delta}\min\Phi(\tilde{p};1)\leqslant 0 \end{array} \tag{37}$$

and notice that $V^2(p,0) = V(p;1)$ (recall that $\Phi(p;1) = \Delta\theta q_H^*$).

We show that for p close to zero, the function $V^2(p,\cdot)$ is decreasing in $\tilde{p}$. This will prove our claim.

We let $q_H(p,\tilde{p})$ and $q_L(p,\tilde{p})$ denote the solution to the above problem and let $\gamma(p,\tilde{p})$ denote the Lagrangian multiplier. From the first order conditions with respect to q_L we have

$$\frac{p-\tilde{p}}{1-\tilde{p}}(1-\delta)\frac{\partial \pi_L(q_L(p,\tilde{p}))}{q_L}=\gamma(p,\tilde{p})$$

We apply the envelope theorem and obtain[26]

$$\begin{gathered}\tfrac{\partial V^2(p,\tilde{p})}{\partial \tilde{p}}=\tfrac{1-p}{(1-\tilde{p})^2}\left[(1-\delta)\pi_H(q_H(p,\tilde{p}))+\delta V(\tilde{p};1)\right]-\\ \tfrac{1-p}{(1-\tilde{p})^2}\left[(1-\delta)\pi_L(q_L(p,\tilde{p}))+\delta\pi_L(q_L^*)-(1-\delta)\Delta\theta q_H(\tilde{p})-\delta\min\Phi(\tilde{p};1)\right]+\\ \left(\tfrac{1-p}{1-\tilde{p}}\right)\delta\tfrac{\partial V(\tilde{p};1)}{\partial\tilde{p}}-\left(\tfrac{p-\tilde{p}}{1-\tilde{p}}\right)\delta\tfrac{\partial\min\Phi(\tilde{p};1)}{\partial\tilde{p}}+\gamma(p,\tilde{p})\tfrac{\delta}{1-\delta}\tfrac{\partial\min\Phi(\tilde{p};1)}{\partial\tilde{p}}=\\ \tfrac{1-p}{(1-\tilde{p})^2}\left[(1-\delta)\pi_H(q_H(p,\tilde{p}))+\delta V(\tilde{p};1)\right]-\\ \tfrac{1-p}{(1-\tilde{p})^2}\left[(1-\delta)\pi_L(q_L(p,\tilde{p}))+\delta\pi_L(q_L^*)-(1-\delta)\Delta\theta q_H(p,\tilde{p})-\delta\min\Phi(\tilde{p};1)\right]+\\ \left(\tfrac{1-p}{1-\tilde{p}}\right)\delta\tfrac{\partial V(\tilde{p};1)}{\partial\tilde{p}}-\left(\tfrac{p-\tilde{p}}{1-\tilde{p}}\right)\delta\tfrac{\partial\min\Phi(\tilde{p};1)}{\partial\tilde{p}}+\delta\tfrac{p-\tilde{p}}{1-\tilde{p}}\tfrac{\partial\pi_L(q_L(p,\tilde{p}))}{q_L}\tfrac{\partial\min\Phi(\tilde{p};1)}{\partial\tilde{p}}\end{gathered}$$

Recall that we are considering the case in which $\underline{p}_1=0$. Therefore, as p converges to zero $\min_{\tilde{p}\leqslant p} q_H(p,\tilde{p})$ must converge to q_H^*. Also, as $\tilde{p}$ shrinks to zero, $V(\tilde{p};1)$ and $\min\Phi(\tilde{p};1)$ converge to $\pi_H(q_H^*)$ and $\Delta\theta q_H^*$, respectively, and the derivative of $\min\Phi(\tilde{p};1)$ (with respect to $\tilde{p}$) is bounded. Therefore, we have

$$\begin{gathered}\lim_{p\downarrow 0}\max_{\tilde{p}\leqslant p}\tfrac{\partial V^2(p,\tilde{p})}{\partial\tilde{p}}=\pi_H(q_H^*)-\left[(1-\delta)\pi_L(\max\left\{q_L^*,\tfrac{q_H^*}{1-\delta}\right\})+\delta\pi_L(q_L^*)-\Delta\theta q_H^*\right]+\\ \delta\partial_+V(0;1)=-(1-\delta)\partial_+V(0;1)<0\end{gathered}$$

where the inequality follows from our genericity assumption.

We conclude that when $\underline{p}_1=0$ (and δ is generic), there exists $\varepsilon>0$ such that for $p\leqslant\varepsilon$ the function $V^2(p,\cdot)$ is decreasing in the interval $[0,p]$. Thus, for $p\leqslant\varepsilon$, $V^2(p)=V^2(p,0)=V(p;1)$.

In general, the value of ε above depends on δ. However, there exists ε such that for any (generic) $\delta\leqslant 1-q_H^*/q_L^*$ and for any $p\leqslant\varepsilon$, $V^2(p)=V^2(p,0)=V(p;1)$. ■

We define $\underline{p}_2>0$ as

$$\underline{p}_2=\inf\left\{p:V^2(p)>V(p;1)\right\}$$

We now show that the function $V^2(\cdot)$ is convex. Clearly, the restriction of $V^2(\cdot)$ to the interval $\left[0,\underline{p}_2\right]$ is convex since, in this interval, $V^2(\cdot)$ is equal to $V(\cdot;1)$.

[26] The function $\min\Phi(\cdot;1)$ is differentiable in a neighborhood of zero.

We now consider the interval $\left[\underline{p}_2, 1\right]$ and observe that there exists $\eta > 0$ such that

$$V^2(p) \geqslant V(p;1) > \pi_H(q_H^*) + \eta$$

for every $p \in \left[\underline{p}_2, 1\right]$. Thus, for $p \geqslant \underline{p}_2$ we have

$$V^2(p,p) \leqslant (1-\delta)\pi_H(q_H^*) + \delta V(p;1) < V(p;1) - (1-\delta)\eta$$

This, together with the continuity of $V^2(p,\tilde{p})$ with respect to $\tilde{p}$, imply that for every $p' \geqslant \underline{p}_2$, there exists $\varepsilon > 0$ such that for any $p \in (p'-\varepsilon, p'+\varepsilon)$ the optimal value of $\tilde{p}$ (in the optimization problem (36)) is below $p'-\varepsilon$. This means that the restriction $V^2(\cdot)$ to the interval $(p'-\varepsilon, p'+\varepsilon)$ is the upper envelope of a fixed family of affine functions. Thus, the function $V^2(\cdot)$ is locally convex in $[0,1]$, and, therefore, convex.

It follows from the convexity of $V^2(\cdot)$ that there exists a point $\bar{p}_2 \in \left(\bar{p}_1, p^C\right]$ such that $V^2(\cdot) < p\pi_L(q_L^*)$ if $p < \bar{p}_2$, and $V^2(\cdot) > p\pi_L(q_L^*)$ if $p > \bar{p}_2$.

We conclude Step 2.1 by defining $V(\cdot;2)$ and $\Phi(\cdot;2)$ as follows:

$$V(p;2) = \begin{cases} V(p;1) & \text{for } p \leqslant \underline{p}_2 \\ V^2(p) & \text{for } p \in \left(\underline{p}_2, \bar{p}_2\right) \\ p\pi_L(q_L^*) & \text{for } p \geqslant \bar{p}_2 \end{cases} \tag{38}$$

$$\Phi(p;2) = \begin{cases} \Phi(p;1) & p < \underline{p}_2 \\ Conv\left(\left\{(1-\delta)\Delta\theta q_H^2\left(\underline{p}_2\right) + \delta \min \Phi\left(\tilde{p}^2\left(\underline{p}_2\right);1\right)\right\} \cup \Phi\left(\underline{p}_2;1\right)\right) & p = \underline{p}_2 \\ (1-\delta)\Delta\theta q_H^2(p) + \delta \min \Phi\left(\tilde{p}^2(p);1\right) & p \in \left(\underline{p}_2, \bar{p}_2\right) \\ \left[0, (1-\delta)\Delta\theta q_H^2(\bar{p}_2) + \delta \min \Phi\left(\tilde{p}^2(p);1\right)\right] & p = \bar{p}_2 \\ 0 & p > \bar{p}_2 \end{cases} \tag{39}$$

where $q_H^2(p)$ and $\tilde{p}^2(p)$ denote the optimal values of q_H and $\tilde{p}$, respectively, in the optimization problem (36), and $Conv(\cdot)$ denotes the convex hull of a given set.

Case 2.2. $\underline{p}_1 > 0$.

We distinguish between two cases.

Case 2.2.1. There exists $\varepsilon > 0$ such that $V^2(p) = V(p;1)$ for every $p \in \left[\underline{p}_1, \underline{p}_1 + \varepsilon\right]$. We let $\underline{p}_2$ denote

$$\inf\left\{p : V^2(p) > V(p;1)\right\}$$

Similarly to the previous case, the function $V^2(\cdot)$ is convex and we let $\bar{p}_2 \in \left(\bar{p}_1, p^C\right]$

denote the point at which $V^2(\cdot)$ intersects the function $p\pi_L(q_L^*)$.

We define $V(\cdot;2)$ and $\Phi(\cdot;2)$ as in (38) and (39), respectively.

Case 2.2.2. For every $\varepsilon > 0$, there exists $p \in \left(\underline{p}_1,\ \underline{p}_1 + \varepsilon\right)$ such that $V^2(p) > V(p;1)$.

In this case we have $V^2\left(\underline{p}_1\right) = V\left(\underline{p}_1;1\right) = \pi_H(q_H^*)$, $q_H^2\left(\underline{p}_1\right) = q_H^*$ and

$$0 < \partial_+ V^1\left(\underline{p}_1\right) < \partial_+ V^2\left(\underline{p}_1\right) = \lim_{p \downarrow \underline{p}_1} \frac{\partial V^2(p)}{\partial p} =$$
$$\frac{1}{1-\underline{p}_1}\left[(1-\delta)\pi_L\left(q_L^2\left(\underline{p}_1\right)\right) + \delta\pi_L(q_L^*) - (1-\delta)\Delta\theta q_H^* - \delta\min\Phi\left(\underline{p}_1;1\right) - \pi_H(q_H^*)\right] \tag{40}$$

where $q_L^2(p)$ denotes the optimal value of q_L (given the belief p) in the optimization problem (36).

Recall the definition of $V^2(p,\tilde{p})$ in the optimization problem (36). It follows from inequality (40) that there exists $\varepsilon > 0$ such that $V^2\left(p,\underline{p}_1\right) > V(p;1)$ for every $p \in \left(\underline{p}_1, \underline{p}_1 + \varepsilon\right)$.

The function $V^2\left(p,\underline{p}_1\right)$ is convex is p. Thus, there exists $\bar{p}_2 \in \left(\bar{p}_1, p^C\right]$ at which the function $V^2\left(p,\underline{p}_1\right)$ and the function $p\pi_L(q_L^*)$ intersect. We define $V(\cdot;2)$ and $\Phi(\cdot;2)$ as follows:

$$V(p;2) = \begin{cases} V(p;1) & \text{for } p \leqslant \underline{p}_1 \\ V^2\left(p,\underline{p}_1\right) & \text{for } p \in \left(\underline{p}_1, \bar{p}_2\right) \\ p\pi_L(q_L^*) & \text{for } p \geqslant \bar{p}_2 \end{cases}$$

$$\Phi(p;2) = \begin{cases} \Phi(p;1) & p < \underline{p}_1 \\ Conv\left(\left\{(1-\delta)\Delta\theta q_H\left(\underline{p}_1,\underline{p}_1\right) + \delta\min\Phi\left(\underline{p}_1;1\right)\right\} \cup \Phi\left(\underline{p}_1;1\right)\right) & p = \underline{p}_1 \\ (1-\delta)\Delta\theta q_H\left(p,\underline{p}_1\right) + \delta\min\Phi\left(\underline{p}_1;1\right) & p \in \left(\underline{p}_1,\bar{p}_2\right) \\ \left[0, (1-\delta)\Delta\theta q_H\left(\bar{p}_2,\underline{p}_1\right) + \delta\min\Phi(\bar{p}_1;1)\right] & p = \bar{p}_2 \\ 0 & p > \bar{p}_2 \end{cases}$$

Then for every $p \geqslant \underline{p}_1$, we consider the following optimization problem

$$V^3(p) = \max_{(q_H,q_L)\in[0,1]^2,\tilde{p}\in\left[\underline{p}_1,\min\{p,\bar{p}_2\}\right]} \frac{1-p}{1-\tilde{p}}\left[(1-\delta)\pi_H(q_H) + \delta V(\tilde{p};1)\right] +$$
$$\frac{p-\tilde{p}}{1-\tilde{p}}\left[(1-\delta)\pi_L(q_L) + \delta\pi_L(q_L^*) - (1-\delta)\Delta\theta q_H - \delta\min\Phi(\tilde{p};1)\right]$$
$$\text{s.t.}\quad (q_H - q_L)\Delta\theta + \frac{\delta}{1-\delta}\min\Phi(\tilde{p};1) \leqslant 0$$

We claim that there exists $\varepsilon > 0$ such that $V^3(p) = V(p;2)$ for every $p \in \left[\underline{p}_1, \underline{p}_1 + \varepsilon\right]$ (the proof of this fact is similar to the proof of Claim 2 and we omit it).

If $V^3(p) \leqslant V(p;2)$ for every p, then we set $V(\cdot)$ equal to $V(\cdot;2)$, and $\Phi(\cdot)$ equal to $\Phi(\cdot;2)$. Otherwise we define $\underline{p}_3 > \underline{p}_1$ as

$$\underline{p}_3 = \inf\left\{p : V^3(p) > V(p;2)\right\}$$

and let $\bar{p}_3 > \underline{p}_3$ denote the point at which the function $V^2\left(p, \underline{p}_1\right)$ and the function $p\pi_L(q_L^*)$ intersect (observe that the function $V^3(\cdot)$ is convex).

Finally, we define the function $V(\cdot;3)$, and $\Phi(\cdot;3)$ as follows:

$$V(p;3) = \begin{cases} V(p;2) & \text{for } p \leqslant \underline{p}_3 \\ V^3(p) & \text{for } p \in \left(\underline{p}_3, \bar{p}_3\right) \\ p\pi_L(q_L^*) & \text{for } p \geqslant \bar{p}_2 \end{cases}$$

$$\Phi(p;3) = \begin{cases} \Phi(p;2) & p < \underline{p}_3 \\ Conv\left(\left\{(1-\delta)\Delta\theta q_H^3\left(\underline{p}_3\right) + \delta \min \Phi\left(\tilde{p}\left(\underline{p}_3\right);2\right)\right\} \cup \Phi\left(\underline{p}_3;2\right)\right) & p = \underline{p}_3 \\ (1-\delta)\Delta\theta q_H^3(p) + \delta \min \Phi\left(\tilde{p}^3(p);2\right) & p \in \left(\underline{p}_3, \bar{p}_3\right) \\ \left[0, (1-\delta)\Delta\theta q_H^3(\bar{p}_3) + \delta \min \Phi\left(\tilde{p}^3(\bar{p}_3);2\right)\right] & p = \bar{p}_3 \\ 0 & p > \bar{p}_3 \end{cases}$$

This concludes Step 2.

Step 3.

The analysis in Step 2 shows that there exists $\hat{k} = 2, 3$ such that $V\left(\cdot;\hat{k}\right)$ and $V\left(\cdot;\hat{k}-1\right)$ coincides in the interval $\left[0, \underline{p}_{\hat{k}}\right]$, $\underline{p}_{\hat{k}} > 0$, and $V\left(\underline{p}_{\hat{k}};\hat{k}\right)$ is strictly larger than $\pi_H(q_H^*)$.

We now proceed by induction. For any $k = \hat{k}, \hat{k}+1, \ldots$, we take as given the pair $(V(\cdot;k), \Phi(\cdot;k))$ and construct the pair $(V(\cdot;k+1), \Phi(\cdot;k+1))$ using the same procedure described in Step 2 (see the optimization problem (36)).

For any k, the function $V(\cdot;k)$ is increasing and convex. Also, by construction, there exists $\hat{\eta} > 0$ such that for any k, and any $p \geqslant \underline{p}_k$ the following inequality holds:

$$V(p;k) > \pi_H(q_H^*) + \hat{\eta}$$

We use this fact to show that the iterative procedure ends after finitely many rounds. Recall that p^C is the belief above which the unique optimal mechanism with commitment is to offer the menu $\{(\theta_L q_L^* + \alpha, q_L^*)\}$. Therefore, $\underline{p}_k \leqslant p^C$ for any k.

Claim 3 *For any $k = \hat{k}, \hat{k}+1, \ldots,$*

$$\underline{p}_{k+1} - \underline{p}_k > \frac{(1-\delta)\hat{\eta}\left(1-p^C\right)}{2\pi_L(q_L^*)} \tag{41}$$

Proof of Claim 3.

Fix k and consider the optimization problem which defines the pair $(V(\cdot;k+1),\Phi(\cdot;k+1))$. Consider $p \geqslant \underline{p}_k$ and let $\tilde{p}^{k+1}(p)$ denote the optimal value of $\tilde{p}$.

Suppose that inequality (41) does not hold. Thus, there exists $p \in \left[\underline{p}_k, \underline{p}_k + \frac{(1-\delta)\hat{\eta}\left(1-p^C\right)}{\pi_L\left(q_L^*\right)}\right]$ such that $V^{k+1}(p) > V(p;k)$. Clearly, the last inequality holds only if $\tilde{p}^{k+1}(p) \geqslant \underline{p}_k$. However, this implies the following contradiction:

$$
\begin{aligned}
V^{k+1}(p) &\leqslant \tfrac{1-p}{1-\tilde{p}^{k+1}(p)}\left[(1-\delta)\pi_H(q_H^*) + \delta V\left(\tilde{p}^{k+1}(p);k\right)\right] + \tfrac{p-\tilde{p}^{k+1}(p)}{1-\tilde{p}^{k+1}(p)}\pi_L(q_L^*) \leqslant \\
&\tfrac{1-p}{1-\tilde{p}^{k+1}(p)}\left[(1-\delta)\pi_H(q_H^*) + \delta V(p;k)\right] + \tfrac{p-\tilde{p}^{k+1}(p)}{1-\tilde{p}^{k+1}(p)}\pi_L(q_L^*) \leqslant \\
&\tfrac{1-p}{1-\underline{p}_k}\left[(1-\delta)\pi_H(q_H^*) + \delta V(p;k)\right] + \tfrac{p-\underline{p}_k}{1-\underline{p}_k}\pi_L(q_L^*) \leqslant \\
&\left[(1-\delta)\pi_H(q_H^*) + \delta V(p;k)\right] + \tfrac{p-\underline{p}_k}{1-p^C}\pi_L(q_L^*) < \\
&V(p;k) - (1-\delta)\hat{\eta} + \tfrac{p-\underline{p}_k}{1-p^C}\pi_L(q_L^*) \leqslant V(p;k)
\end{aligned}
$$

where the second inequality follows from the monotonicity of $V(\cdot,k)$. This concludes the proof of Claim 3. ■

This shows that there exists k^* for which the pairs $(V(\cdot;k^*),\Phi(\cdot;k^*))$ and $(V(\cdot;k^*+1),\Phi(\cdot;k^*+1))$ coincide on the entire unit interval. We set $(V(\cdot),\Phi(\cdot))$ equal to $(V(\cdot;k^*),\Phi(\cdot;k^*))$. By construction, $(V(\cdot),\Phi(\cdot))$ satisfies all the properties in Lemma 8.

The number of iterations k^* necessary to get the fixed point $(V(\cdot),\Phi(\cdot))$ generally depends on the value of the discount factor. However, there exists $\breve{k}$ such that for generic values of δ in $(0, 1-q_H^*/q_L^*]$, the number of iterations necessary to get the fixed point $(V(\cdot),\Phi(\cdot))$ is bounded by $\breve{k}$. This is because there exists $\breve{\eta} > 0$ such that for any generic value of $\delta \leqslant 1-q_H^*/q_L^*$, for any k, and any $p \geqslant \underline{p}_k$, we have $V(p;k) > \pi_H(q_H^*) + \breve{\eta}$ (this, in turn, follows from the convexity of the function $V(\cdot;k)$ and our discussion at the end of the proof of Claim 2. ■