Technical Appendix for "Trade Clustering and Power Laws in Financial Markets"

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This technical appendix provides derivations omitted in "Trade Clustering and Power Laws in Financial Markets."

Verification for the signal examples

Example 1: A linear distribution

Consider a signal X that follows

$$f_n^H(x) = \frac{1}{2} + \epsilon_n x \quad \text{and} \quad f^L(x) = \frac{1}{2}, \qquad -1 \leq x \leq 1,$$

where $\epsilon_n = n^{-\xi}/3$ and $0 < \xi < 1$. In this section, we show that this signal satisfies Assumptions 1, 2 and 3.

Clearly, the densities are strictly positive and continuously differentiable. The likelihood ratio satisfies MLRP, because

$$\ell_n(x) = \frac{f_n^H(x)}{f^L(x)} = 1 + 2\epsilon_n x$$

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is strictly increasing in x. Moreover, $\ell_n(x) \to 1$ as $n \to \infty$ uniformly in $x \in [-1,1]$, satisfying Assumption 2. The cumulative distributions are

$$F_n^H(x) = \int_{-1}^{x} \frac{1}{2} + \epsilon_n z \, dz = \frac{x+1}{2} + \frac{x^2-1}{2}\epsilon_n,$$
$$F^L(x) = \int_{-1}^{x} \frac{1}{2} \, dz = \frac{x+1}{2}.$$

Hence, $\lambda_n(x) = 1 + (x-1)\epsilon_n$. This implies $\lambda_n''(x) = 0$, satisfying Assumption 3.

Finally, we investigate Assumption 1. We note that

$$\lim_{x\to 1} \log\left(\frac{\Lambda_n(x)}{\lambda_n(x)}\right) = \log\left(\frac{f_n^H(1)}{f^L(1)}\right) = \log(1+2\epsilon_n) = O(\epsilon_n).$$

Thus, in order to show that the signal satisfies Assumption 1, it suffices to show that $\log\left(\Lambda_n(x)/\lambda_n(x)\right)$ is decreasing in $x \in [-1,1]$. We have

$$\begin{aligned}
\frac{d}{dx}\log\left(\frac{\Lambda_n(x)}{\lambda_n(x)}\right) &= \frac{d}{dx}\left[\log\left(\frac{1}{F_n^H(x)} - 1\right) - \log\left(\frac{1}{F^L(x)} - 1\right)\right] \\
&= \frac{1}{1-F^L(x)}\frac{f^L(x)}{F^L(x)} - \frac{1}{1-F_n^H(x)}\frac{f_n^H(x)}{F_n^H(x)} \\
&= \frac{(1-F_n^H(x))F_n^H(x)f^L(x) - (1-F^L(x))F^L(x)f_n^H(x)}{(1-F^L(x))F^L(x)(1-F_n^H(x))F_n^H(x)}.
\end{aligned}$$

The denominator is positive. We inspect the numerator to find it negative:

$$\begin{aligned}
&\left(1 - \frac{x+1}{2} - \frac{x^2-1}{2}\epsilon_n\right)\left(\frac{x+1}{2} + \frac{x^2-1}{2}\epsilon_n\right)\left(\frac{1}{2}\right) - \left(1 - \frac{x+1}{2}\right)\left(\frac{x+1}{2}\right)\left(\frac{1}{2} + x\epsilon_n\right) \\
&= \frac{\left(1 - x - (x^2-1)\epsilon_n\right)\left((x+1) + (x^2-1)\epsilon_n\right) - (1-x)\,(x+1)(1+2x\epsilon_n)}{8} \\
&= \frac{x+1}{8}\left[\left(1 - x - (x^2-1)\epsilon_n\right)\left(1 + (x-1)\epsilon_n\right) - (1-x)\left(1+2x\epsilon_n\right)\right] \\
&= \frac{(x+1)(1-x)}{8}\left[\left(1 + (1+x)\epsilon_n\right)\left(1 + (x-1)\epsilon_n\right) - \left(1+2x\epsilon_n\right)\right] \\
&= \frac{(x+1)(1-x)}{8}\left[(1+x)\epsilon_n\left(1 + (x-1)\epsilon_n\right) - (x+1)\epsilon_n\right] \\
&= -\frac{(x+1)^2(1-x)^2\epsilon_n^2}{8} < 0.
\end{aligned}$$

Hence, $\log(\Lambda_n(x)/\lambda_n(x))$ is bounded from below by $\log(1+2\epsilon_n)$. Thus Assumption 1 is satisfied.

Example 2: An exponential signal

Consider a signal X that follows an exponential distribution with

$$f^H(x) = \frac{\mu e^{-\mu x}}{1-e^{-\mu}} \quad \text{and} \quad f_n^L(x) = \frac{(\mu+\epsilon_n)e^{-(\mu+\epsilon_n)x}}{1-e^{-(\mu+\epsilon_n)}}, \qquad 0 \le x \le 1,$$

where $\epsilon_n = \delta_\epsilon n^{-\xi}$ is a positive sequence, and $\delta_\epsilon > 0$, $\mu > 2$, $\xi \in (0,1)$ are constants. In this section, we show that this signal satisfies Assumptions 1, 2 and 3.

The signal has the monotone increasing likelihood ratio: $\ell_n(x) = (\mu/(1-e^{-\mu})((1-e^{-(\mu+\epsilon_n)})/(\mu+\epsilon_n))e^{\epsilon_n x}$. Thus, the signal satisfies all the properties assumed in Section 2.2. In particular, f_n^s is continuously differentiable and strictly positive over common bounded support $\mathcal{X}$ and satisfies MLRP ($\ell_n'(x) > 0$) for any $x \in \mathcal{X}$. Moreover, ℓ_n converges to 1 uniformly on $\mathcal{X}$, and therefore satisfies Assumption 2.

Next we show that the signal satisfies Assumption 1. We have

$$\begin{aligned} F^H(x) &= \frac{1-e^{-\mu x}}{1-e^{-\mu}}, \quad 1-F^H(x) = \frac{e^{-\mu x}-e^{-\mu}}{1-e^{-\mu}}, \\ F_n^L(x) &= \frac{1-e^{-(\mu+\epsilon_n)x}}{1-e^{-(\mu+\epsilon_n)}}, \quad 1-F_n^L(x) = \frac{e^{-(\mu+\epsilon_n)x}-e^{-(\mu+\epsilon_n)}}{1-e^{-(\mu+\epsilon_n)}} \end{aligned}$$

and $\Lambda_n = (1-F^H)/(1-F_n^L)$ and $\lambda_n = F^H/F_n^L$. Let $\delta_n := \log(\Lambda_n/\lambda_n)$. Then,

$$\begin{aligned} \delta_n(x,\epsilon_n) &= \log\left(\frac{e^{-\mu x}-e^{-\mu}}{e^{-(\mu+\epsilon_n)x}-e^{-(\mu+\epsilon_n)}}\frac{1-e^{-(\mu+\epsilon_n)x}}{1-e^{-\mu x}}\right) \\ &= \log\left(\frac{e^{(\mu+\epsilon_n)x}-1}{e^{\mu x}-1}\right) - \log\left(\frac{e^{(\mu+\epsilon_n)(x-1)}-1}{e^{\mu(x-1)}-1}\right). \end{aligned}$$

Note that δ_n is an analytic function of ϵ_n and converges to 0 as $\epsilon_n \to 0$ for any $x \in \mathcal{X}$. Thus, the first-order Taylor expansion of δ_n around $\epsilon_n = 0$ yields

$$\begin{aligned} \delta_n(x,\epsilon_n) &= \left(\frac{xe^{\mu x}}{e^{\mu x}-1} - \frac{(x-1)e^{\mu(x-1)}}{e^{\mu(x-1)}-1}\right)\epsilon_n + O(\epsilon_n^2) \\ &= (h(x)-h(x-1))\epsilon_n + O(\epsilon_n^2) \qquad (*) \end{aligned}$$

where $h(x) := x/(1-e^{-\mu x})$. We note that $h(x)$ is strictly increasing in x:

$$h'(x) = \frac{1-e^{-\mu x}-\mu x e^{-\mu x}}{(1-e^{-\mu x})^2} > 0.$$

The inequality holds since $1-e^{-y}-ye^{-y} > 0$ for any $y \neq 0$, and also since $h'(0) = 1/2$ by l'Hôpital's rule. Hence $h(x) - h(x-1)$ is bounded below by a positive number uniformly on $\mathcal{X}$.

The term $O(\epsilon_n^2)$ can be made arbitrarily small (say, a half of the lower bound of $(h(x) - h(x-1))\epsilon_n$) for large enough n. Therefore, applying $\epsilon_n = \delta_\epsilon n^{-\xi}$ to Equation (*) above, we see that there exist constants $\delta > 0$ and n_1 such that $\delta_n(x) > \delta n^{-\xi}$ for any $x \in \mathcal{X}$ and for all $n > n_1$. This confirms that the signal satisfies Assumption 1.

Finally, we show that the signal satisfies Assumption 3. Let us write $\mu_L := \mu + \epsilon_n$. For this particular signal, we have

$$\begin{aligned}
\lambda_n(x) &= \frac{1-e^{-\mu_L}}{1-e^{-\mu}}\frac{1-e^{-\mu x}}{1-e^{-\mu_L x}} \\
\lambda_n'(x) &= \frac{1-e^{-\mu_L}}{1-e^{-\mu}}\frac{\mu e^{-\mu x}(1-e^{-\mu_L x}) - \mu_L e^{-\mu_L x}(1-e^{-\mu x})}{(1-e^{-\mu_L x})^2}.
\end{aligned}$$

Thus, we have

$$\lambda_n''(x) = \frac{1-e^{-\mu_L}}{1-e^{-\mu}}\frac{\left[-\frac{\mu^2}{e^{\mu x}-1} + \frac{\mu_L^2}{e^{\mu_L x}-1}\right](1-e^{-\mu_L x}) - 2\mu_L e^{-\mu_L x}\left[\frac{\mu}{e^{\mu x}-1} - \frac{\mu_L}{e^{\mu_L x}-1}\right]}{(1-e^{-\mu_L x})^2(1-e^{-\mu x})^{-1}}. \qquad (**)$$

Now, we have

$$\frac{d}{d\mu}\left(\frac{\mu}{e^{\mu x}-1}\right) = \frac{e^{\mu x}-1-\mu x e^{\mu x}}{(e^{\mu x}-1)^2}$$

is negative for $\mu x > 0$, because $y - 1 < y \log y$ for any $y > 1$. Hence, the term

$$-2\mu_L e^{-\mu_L x}\left[\frac{\mu}{e^{\mu x}-1} - \frac{\mu_L}{e^{\mu_L x}-1}\right]$$

in Equation (**) is negative since $\mu_L > \mu$. Also, we have

$$\frac{d}{d\mu}\left(\frac{\mu^2}{e^{\mu x}-1}\right) = \frac{2\mu e^{\mu x}\left(1-e^{-\mu x}-\mu x/2\right)}{(e^{\mu x}-1)^2}.$$

Note that $1 - e^{-y} - y/2$ is strictly negative at $y = 2$ and decreasing in y for $y > 2$. Hence, for any fixed $\mu > 2$, there exists an $x_c < 1$ such that the above derivative is negative for any $x \in [x_c, 1]$. Thus, $\left[-\frac{\mu^2}{e^{\mu x}-1} + \frac{\mu_L^2}{e^{\mu_L x}-1}\right](1 - e^{-\mu_L x})$ in Equation (**) is negative in $x \in [x_c, 1]$ for any n, since $\mu_L > \mu$. Hence, there exists an x_c such that, for every n, $\lambda_n''(x) \leq 0$ holds for any $x \in [x_c, 1]$. Thus we verify that the signal satisfies Assumption 3.

Derivation of $\lambda_n'(x_a) = \ell_n'(x_a)/2$ and $\Lambda_n'(x_b) = \ell_n'(x_b)/2$ for Equations (8,9)

Using (8), we obtain

$$\begin{aligned}
\lim_{x \to x_a} \lambda_n'(x) &= f_n^L(x_a) \lim_{x \to x_a} \frac{\ell_n(x) - \lambda_n(x)}{F_n^L(x)} \\
&= f_n^L(x_a) \frac{\ell_n'(x_a) - \lambda_n'(x_a)}{f_n^L(x_a)} \\
&= \ell_n'(x_a) - \lambda_n'(x_a)
\end{aligned}$$

which implies $\lambda_n'(x_a) = \ell_n'(x_a)/2$.

Similarly, using (9) we obtain

$$\begin{aligned}
\lim_{x \to x_b} \Lambda_n'(x) &= f_n^L(x_b) \lim_{x \to x_b} \frac{\Lambda_n(x) - \ell_n(x)}{1 - F_n^L(x)} \\
&= f_n^L(x_b) \frac{\Lambda_n'(x_b) - \ell_n'(x_b)}{-f_n^L(x_b)} \\
&= -(\Lambda_n'(x_b) - \ell_n'(x_b))
\end{aligned}$$

which implies $\Lambda_n'(x_b) = \ell_n'(x_b)/2$.

Supplement on Proof of Lemma 2

In this section, we show that the probability of $\Gamma(t)/n$ in (15) exceeding $n^{-\nu_0}$ for some $\nu_0 > 0$ converges to 0 as $n \to \infty$.

From Lemma 1, $K_t \equiv \Gamma(t+1) - \Gamma(1)$ asymptotically follows a Poisson distribution with mean t. Combining with inequalities $\sqrt{2\pi} e^{-k} k^{k+0.5} \leq k! \leq e^{1-k} k^{k+0.5}$ for any integer k, we obtain

$$
\begin{aligned}
\Pr(K_t \geq k) &= \sum_{K_t=k}^{\infty} t^{K_t} e^{-t}/K_t! \\
&= \sum_{s=0}^{\infty} t^{k+s} e^{-t}/(k+s)! \\
&= t^k e^{-t} \sum_{s=0}^{\infty} \frac{t^s}{s!} \frac{s!}{(k+s)!} \\
&\leq t^k e^{-t} \sum_{s=0}^{\infty} \frac{t^s}{s!} \frac{e^{1-s} s^{s+0.5}}{\sqrt{2\pi} e^{-(k+s)} (k+s)^{k+s+0.5}} \\
&= t^k e^{-t} \sum_{s=0}^{\infty} \frac{t^s}{s!} \frac{e^{k+1}}{\sqrt{2\pi}(k+s)^k} \left(\frac{s}{k+s} \right)^{s+0.5} \\
&\leq t^k e^{-t} \sum_{s=0}^{\infty} \frac{t^s}{s!} \frac{e^{k+1}}{\sqrt{2\pi} k^k} \\
&= \frac{e}{\sqrt{2\pi}} \left(\frac{te}{k} \right)^k .
\end{aligned}
$$

Now we consider a region $t \in [0, T]$ and let $k = n^{1-\nu_0}$ for some $\nu_0 \in (0,1)$. The upper bound of $\Pr(K_T \geq k)$ becomes $(e/\sqrt{2\pi})(n^{\nu_0 - 1} T e)^{n^{1-\nu_0}}$, which converges to 0 from above as $n \to \infty$. Also note that $\Gamma(t)$ is non-decreasing in t. Thus, the probability of events in which $\Gamma(t)$ exceeds $k = n^{1-\nu_0}$ declines to 0 as $n \to \infty$.

Derivation of (13)

This section derives the asymptotic expression (13) from (12) by applying Stirling's formula $m! \sim \sqrt{2\pi m}(m/e)^m$ as $m \to \infty$.

Substituting Stirling's formula into (12), we obtain

$$
\begin{aligned}
\frac{b_o}{m}\frac{e^{-\phi m}(\phi m)^{m-b_o}}{(m-b_o)!} &\sim \frac{b_o}{m}\frac{e^{-\phi m+m-b_o}(\phi m)^{m-b_o}}{\sqrt{2\pi(m-b_o)}(m-b_o)^{m-b_o}} \\
&= \frac{b_o}{m\sqrt{2\pi(m-b_o)}}e^{-\phi m+m-b_o+(m-b_o)\log\phi}\left(1-\frac{b_o}{m}\right)^{-m+b_o} \\
&\sim \frac{b_o(\phi e)^{-b_o}}{m\sqrt{2\pi(m-b_o)}}e^{-(\phi-1-\log\phi)m}e^{b_o} \\
&\sim \frac{b_o\phi^{-b_o}}{\sqrt{2\pi}}\frac{e^{-(\phi-1-\log\phi)m}}{m^{1.5}}.
\end{aligned}
$$