Supplement to "On the Optimal Design of Biased Contests"

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We assume in the main text that contestants are endowed with the same contest technology $h(\cdot)$ and effort cost function $c(\cdot)$. In Section 2, we demonstrate that our baseline analysis would be immune to a variation in which each contestant bears an effort cost $c(x_i)/d_i$. In this online appendix, we show that many of our results do not depend on this modeling specification.

We now allow the heterogeneity in contestants' contest technologies and effort cost functions to be more generally modeled. Let one's impact function take the form

$$f_i\left(x_i;\alpha_i,\beta_i\right)=\alpha_i\cdot h_i\left(x_i\right)+\beta_i,$$

and effort cost function be $c_i(x_i)$, where $h_i(\cdot)$ and $c_i(\cdot)$ satisfy the following standard regularity conditions.

Assumption A1 ***(Concave Contest Technology and Convex Effort Cost Function)*** *The contest technology $h_i(\cdot)$ and effort cost function $c_i(\cdot)$ are assumed to have the following properties:*

i. $h_i(\cdot)$ is twice differentiable, with $h_i(0)=0$, $h_i'(x)>0$, and $h_i''(x)\leq 0$ for all $x>0$;

ii. $c_i(\cdot)$ is twice differentiable, with $c_i(0)=0$, $c_i'(x)>0$, and $c_i''(x)\geq 0$ for all $x>0$.

Theorem 1 in our baseline analysis proves the existence and uniqueness of pure-strategy equilibrium in a regular concave contest, which is established assuming a general concave

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impact function $f_i(\cdot)$. This result obviously would not vary when the cost function is heterogeneous. To see this, define $\widehat{c}_i := c_i(x_i)$, $\widehat{\boldsymbol{c}} := (\widehat{c}_1 \ldots, \widehat{c}_n)$, and $\widehat{f}_i(\widehat{c}_i) := f_i(c_i^{-1}(\widehat{c}_i))$; each contestant equivalently maximizes an expected payoff

$$\widehat{\pi}_i(\widehat{\boldsymbol{c}}) := \frac{\widehat{f}_i(\widehat{c}_i)}{\sum_{j=1}^n \widehat{f}_j(\widehat{c}_j)} v_i - \widehat{c}_i.$$

The transformation leads to a regular concave contest that satisfies the requirements of Definition 1, and Theorem 1 naturally extends.

Next, we show that Theorems 2-3 and Propositions 1-2 would also remain qualitatively unchanged. We first obtain the following.

Theorem A1 ***(Suboptimality of Headstart with Heterogeneous Contest Technologies and Cost Functions)*** *Suppose that Assumptions* 2 *and A1 are satisfied. The optimum can always be achieved by choosing multiplicative biases* $\boldsymbol{\alpha}$ *only and setting headstarts* $\boldsymbol{\beta}$ *to zero.*

Proof. We follow the notation in the main text and denote the optimal contest rule that maximizes $\Lambda(\boldsymbol{x}, \boldsymbol{p}, \boldsymbol{v})$ by $(\boldsymbol{\alpha}^*, \boldsymbol{\beta}^*) \equiv \big((\alpha_1^*, \ldots, \alpha_n^*), (\beta_1^*, \ldots, \beta_n^*)\big)$; denote the corresponding equilibrium effort profile and winning probabilities by $\boldsymbol{x}^* \equiv (x_1^*, \ldots, x_n^*)$ and $\boldsymbol{p}^* \equiv (p_1^*, \ldots, p_n^*)$, respectively.

Suppose to the contrary that $\beta_t^* > 0$ for some $t \in \mathcal{N}$ in the optimum. Let us focus on the case of an active contestant t (i.e., $x_t^* > 0$). The equilibrium condition is given by

$$p_t^*(1 - p_t^*) v_t = c_t'(x_t^*) \cdot \frac{\alpha_t^* h_t(x_t^*) + \beta_t^*}{\alpha_t^* h_t'(x_t^*)}.$$

Denote by $x^\dagger$ the unique solution to the following equation:

$$c_t'(x_t^*) \cdot \frac{\alpha_t^* h_t(x_t^*) + \beta_t^*}{\alpha_t^* h_t'(x_t^*)} = c_t'(x^\dagger) \cdot \frac{h_t(x^\dagger)}{h_t'(x^\dagger)}. \tag{A1}$$

Simple analysis would verify that $x^\dagger > x_t^*$, given $\beta_t^* > 0$. Consider an alternative contest rule with $\widetilde{\boldsymbol{\alpha}} \equiv (\widetilde{\alpha}_1, \ldots, \widetilde{\alpha}_n)$ and $\widetilde{\boldsymbol{\beta}} \equiv (\widetilde{\beta}_1, \ldots, \widetilde{\beta}_n)$, such that

$$\left(\widetilde{\alpha}_i, \widetilde{\beta}_i\right) := \begin{cases} \left(\frac{\alpha_t^* h_t(x_t^*) + \beta_t^*}{h_t(x^\dagger)}, 0\right) & \text{for} \quad i = t, \\ (\alpha_i^*, \beta_i^*) & \text{for} \quad i \neq t. \end{cases}$$

In words, all contestants are awarded the same identity-dependent treatment as before except for contestant t. The new contest rule removes the headstart for contestant t. Simple algebra verifies that the equilibrium effort profile under the new contest rule $(\widetilde{\boldsymbol{\alpha}}, \widetilde{\boldsymbol{\beta}})$—which we denote by $\widetilde{\boldsymbol{x}}^* \equiv (\widetilde{x}_1^*, \ldots, \widetilde{x}_n^*)$—is given by

$$\widetilde{x}_i^* = \begin{cases} x^\dagger & \text{for} \quad i = t, \\ x_i^* & \text{for} \quad i \neq t. \end{cases}$$

The new contest rule outperforms under Assumption 2. It induces the same winning probability distribution, because $\widetilde{\alpha}_t \cdot h_t(x^\dagger) + \widetilde{\beta}_t = \alpha_t^* \cdot h_t(x_t^*) + \beta_t^*$ by our construction, while the effort of contestant t strictly increases because $x^\dagger > x_t^*$ by Equation (A1).

The proof for the case of inactive contestant t (i.e., $x_t^* = 0$) is similar and is omitted for brevity. This completes the proof. ■

We thus verify the robustness of Theorem 2 in the extended setting, which allows us to simplify the optimization problem by focusing on only the optimal choice of $\boldsymbol{\alpha}$. By Theorem A1, the following must hold in an equilibrium:

$$p_i(1-p_i)v_i = c_i'(x_i) \cdot \frac{h_i(x_i)}{h_i'(x_i)}, \forall\, i \in \mathcal{N}. \tag{A2}$$

Define the inverse of $\log(c_i'(x) \cdot h_i(x)/h_i'(x))$ as $g_i(\cdot)$. Then the correspondence (A2) can be rewritten as

$$x_i = g_i\Big(\log(p_i\left(1-p_i\right)) + \log(v_i)\Big), \forall\, i \in \mathcal{N}. \tag{A3}$$

We further obtain the following, which, together with the correspondence, reinstates our optimization approach.

Theorem A2 ***(Implementing Winning Probabilities by Setting Biases with Heterogeneous Contest Technologies and Cost Functions)*** *Fix any equilibrium winning probability distribution $\boldsymbol{p} \equiv (p_1, \ldots, p_n) \in \Delta^{n-1}$.*

i. If $p_j = 1$ for some $j \in \mathcal{N}$, then $\boldsymbol{p} \equiv (p_1, \ldots, p_n)$ can be induced by the following set of biases $\boldsymbol{\alpha}(\boldsymbol{p}) \equiv \big(\alpha_1(\boldsymbol{p}), \ldots, \alpha_n(\boldsymbol{p})\big)$:

$$\alpha_i(\boldsymbol{p}) = \begin{cases} 1 & \textit{if} \quad i = j, \\ 0 & \textit{if} \quad i \neq j. \end{cases}$$

ii. If there exist at least two active contestants, then $\boldsymbol{p} \equiv (p_1, \ldots, p_n)$ can be induced by the following set of biases $\boldsymbol{\alpha}(\boldsymbol{p}) \equiv \big(\alpha_1(\boldsymbol{p}), \ldots, \alpha_n(\boldsymbol{p})\big)$:

$$\alpha_i(\boldsymbol{p}) = \begin{cases} \dfrac{p_i}{h_i\Big(g_i\big(\log(p_i(1-p_i)) + \log(v_i)\big)\Big)} & \textit{if} \quad p_i > 0, \\ 0 & \textit{if} \quad p_i = 0. \end{cases} \tag{A4}$$

Proof. Part (i) of the theorem is trivial, and it remains to show part (ii). It is clear that $x_i = 0$ is a strictly dominant strategy if $\alpha_i = 0$. For $(p_i, p_j) > (0, 0)$, we must have $(x_i, x_j) > (0, 0)$. Therefore, the following first-order conditions must be satisfied by Equation (A3):

$$\begin{aligned} x_i &= g_i\left(\log(p_i\left(1-p_i\right)) + \log(v_i)\right), \\ x_j &= g_i\left(\log(p_j\left(1-p_j\right)) + \log(v_j)\right). \end{aligned}$$

Note that Equation (1) implies that

$$\frac{p_i}{p_j} = \frac{\frac{\alpha_i \cdot h_i(x_i)}{\sum_{k=1}^{n} \alpha_k \cdot h_k(x_k)}}{\frac{\alpha_j \cdot h_i(x_j)}{\sum_{k=1}^{n} \alpha_k \cdot h_k(x_k)}} = \frac{\alpha_i \cdot h_i(x_i)}{\alpha_j \cdot h_j(x_j)}.$$

Combining the above conditions, we can obtain that

$$\frac{\alpha_i}{\alpha_j} = \frac{p_i/h_i(x_i)}{p_j/h_j(x_j)} = \frac{\frac{p_i}{h_i\left(g_i\left(\log(p_i\left(1-p_i\right))+\log(v_i)\right)\right)}}{\frac{p_j}{h_j\left(g_j\left(\log(p_j\left(1-p_j\right))+\log(v_j)\right)\right)}}.$$

The last equation clearly holds for the set of weights specified in Equation (A4). This completes the proof. ■

This restores Theorem 3 in our baseline setting, which states that any winning probability distribution can be induced in equilibrium by an $\boldsymbol{\alpha}$. We then proceed to apply our approach to optimal design for the maximization of total effort and the expected winner's effort.

Proposition A1 ***(Total-effort-maximizing Contests with Heterogeneous Contest Technologies and Cost Functions)*** *Suppose that $n \geq 2$, Assumption A1 is satisfied, and the designer aims to maximize total effort. Then the following statements hold:*

i. The optimal contest allows for at least three active players if possible.

ii. The optimal contest does not allow any contestant to win with a probability more than 1/2, i.e., $p_i^* \leq 1/2$, $\forall\, i \in \mathcal{N}$, with equality if and only $n = 2$.

Proof. The same logic as that in the main text would reveal $p_1^* = p_2^* = \frac{1}{2}$ in the optimum when $n = 2$. We now verify the claim for the case of $n \geq 3$. We first prove part (i) of the proposition. Suppose, to the contrary, that only two players remain active in the optimal contest. It is clear that $p_1^* = p_2^* = \frac{1}{2}$ in the optimum. Without loss of generality, assume that contestants 1 and 2 are active. Now consider the following profile of equilibrium winning probabilities $\boldsymbol{p} = (\frac{1}{2}, \frac{1}{2} - \epsilon, \epsilon, 0, \ldots, 0)$. It can be verified that the total effort under $\boldsymbol{p}$ is equal to

$$\Lambda(\boldsymbol{x}, \boldsymbol{p}, \boldsymbol{v}) = g_1 \left(\log(\frac{1}{4}) + \log(v_1) \right) + g_2 \left(\log(\frac{1}{4} - \epsilon^2) + \log(v_2) \right) + g_3 \left(\left(\log(\epsilon(1-\epsilon)\right) + \log(v_3) \right).$$

Simple algebra shows that $\partial\Lambda/\partial\epsilon > 0$ when ϵ is sufficiently small. Therefore, at least three players will remain active in the optimum.

Next, we prove part (ii). Suppose, to the contrary, that $p_i^* \geq \frac{1}{2}$ for some $i \in \mathcal{N}$. If $p_i^* > \frac{1}{2}$, then the contest designer can assign probability $1 - p_i^*$ to contestant i and probability $p_j^* + (2p_i^* - 1)$ to an arbitrary contestant $j \neq i$. Because at least three players remain active in the optimum, we must have $p_i^* + p_j^* < 1$. This in turn implies that $|p_j^* + (2p_i^* - 1) - \frac{1}{2}| < |p_j^* - \frac{1}{2}|$, and thus contestant j's effort strictly increases. Furthermore, it follows from Equation (A3) that contestant i's effort remains the same. Therefore, the total effort strictly increases after the adjustment. If $p_i^* = \frac{1}{2}$, then there exists an active player $j \in \mathcal{N}$ such that $p_j \in (0, \frac{1}{2})$, because at least three players remain active in the optimum. In such a scenario, the designer can increase the total effort by reducing p_i^* by a sufficiently small amount and increasing p_j^* by the same amount. This completes the proof. ■

The result of Proposition 1 in the baseline analysis is perfectly preserved. We then examine the case of maximizing the expected winner's effort.

Proposition A2 ***(Optimal Contest that Maximizes the Expected Winner's Effort with Heterogeneous Contest Technologies and Cost Functions)*** *Suppose that Assumption A1 is satisfied and the designer aims to maximize the expected winner's effort. Then only two contestants would remain active in the optimal contest.*

Proof. It is useful to first prove the following intermediate result.

Lemma A1 *Consider a contest with three players who are indexed by i, j, and k. Suppose that the contest designer aims to maximize the expected winner's effort. Then setting $p_i = p_j = p_k = \frac{1}{3}$ is suboptimal.*

Proof. Without loss of generality, we assume that

$$g_i \left(\log\left(\frac{2}{9}\right) + \log(v_i) \right) \geq g_j \left(\log\left(\frac{2}{9}\right) + \log(v_j) \right) \geq g_k \left(\log\left(\frac{2}{9}\right) + \log(v_k) \right).$$

The difference between the expected winner's effort under $(p_i, p_j, p_k) = (\frac{1}{2}, \frac{1}{2}, 0)$ and that under $(p_i, p_j, p_k) = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ can be derived as

$$
\begin{aligned}
&\left[\frac{1}{2}g_i\left(\log\left(\frac{1}{4}\right)+\log(v_i)\right)+\frac{1}{2}g_j\left(\log\left(\frac{1}{4}\right)+\log(v_j)\right)\right]\\
&\quad-\left[\frac{1}{3}g_i\left(\log\left(\frac{2}{9}\right)+\log(v_i)\right)+\frac{1}{3}g_j\left(\log\left(\frac{2}{9}\right)+\log(v_j)\right)+\frac{1}{3}g_k\left(\log\left(\frac{2}{9}\right)+\log(v_k)\right)\right]\\
&>\frac{1}{6}\left[g_i\left(\log\left(\frac{2}{9}\right)+\log(v_i)\right)-g_j\left(\log\left(\frac{2}{9}\right)+\log(v_j)\right)\right]\\
&\geq 0,
\end{aligned}
$$

where the strict inequality follows from $\frac{1}{4} > \frac{2}{9}$, $g_j\left(\log\left(\frac{2}{9}\right)+\log(v_j)\right) \geq g_k\left(\log\left(\frac{2}{9}\right)+\log(v_k)\right)$, and the monotonicity of $g_i(\cdot)$, $g_j(\cdot)$, and $g_k(\cdot)$. Therefore, setting $p_i = p_j = p_k = \frac{1}{3}$ is suboptimal. This completes the proof. ■

Now we can prove the proposition. Suppose, to the contrary, that three or more players remain active in the optimal contest. Then there exist $i, j, k \in \mathcal{N}$ such that $p_i^{**} \geq p_j^{**} > 0$ and $p_i^{**} \geq p_k^{**} > 0$. Lemma A1 implies that $\min\{2p_j^{**} + p_k^{**}, p_j^{**} + 2p_k^{**}\} < 1$. Let $p_{jk}^{**} := p_j^{**} + p_k^{**}$. Without loss of generality, suppose that

$$
g_j\left(\log\left(p_{jk}^{**}(1-p_{jk}^{**})\right)+\log(v_j)\right) \geq g_k\left(\log\left(p_{jk}^{**}(1-p_{jk}^{**})\right)+\log(v_k)\right).
$$

It follows immediately that

$$
\begin{aligned}
g_j\left(\log\left(p_{jk}^{**}(1-p_{jk}^{**})\right)+\log(v_j)\right) &\geq g_k\left(\log\left(p_{jk}^{**}(1-p_{jk}^{**})\right)+\log(v_k)\right)\\
&> g_k\left(\log\left(p_k^{**}(1-p_k^{**})\right)+\log(v_k)\right),
\end{aligned}
\tag{A5}
$$

where the strict inequality follows from $\min\{2p_j^{**} + p_k^{**}, p_j^{**} + 2p_k^{**}\} < 1$ and the monotonicity of $g_k(\cdot)$. Suppose that the contest designer assigns probability $p_{jk}^{**} := p_j^{**} + p_k^{**}$ to player j and 0 to player k, and does not change the equilibrium winning probability of all other players. Then the difference between the expected winner's effort under the new profile of winning probabilities and that under $\boldsymbol{p}^{**} \equiv (p_1^{**}, \ldots, p_n^{**})$ can be derived as

$$
\begin{aligned}
&(p_j^{**} + p_k^{**}) g_j \left(\log \left(p_{jk}^{**}(1 - p_{jk}^{**}) \right) + \log(v_j) \right) \\
&\quad - \left[p_j^{**} g_j \left(\log \left(p_j^{**}(1 - p_j^{**}) \right) + \log(v_j) \right) + p_k^{**} g_k \left(\log \left(p_k^{**}(1 - p_k^{**}) \right) + \log(v_k) \right) \right] \\
&= p_j^{**} \left[g_j \left(\log \left(p_{jk}^{**}(1 - p_{jk}^{**}) \right) + \log(v_j) \right) - g_j \left(\log \left(p_j^{**}(1 - p_j^{**}) \right) + \log(v_j) \right) \right] \\
&\quad + p_k^{**} \left[g_j \left(\log \left(p_{jk}^{**}(1 - p_{jk}^{**}) \right) + \log(v_j) \right) - g_k \left(\log \left(p_k^{**}(1 - p_k^{**}) \right) + \log(v_k) \right) \right] \\
&> 0,
\end{aligned}
$$

where the strict inequality follows from $\min\{2p_j^{**} + p_k^{**}, p_j^{**} + 2p_k^{**}\} < 1$, the monotonicity of $g_j(\cdot)$, and (A5). A contradiction. Therefore, only two contestants would remain active in the optimal contest. This completes the proof. ■

Proposition 2 states that only two active contestants remain in the optimum. This continues to hold in the extended setting.

In conclusion, most of our results would qualitatively hold when the heterogeneity in contest technologies and effort cost functions are generally modeled. However, encapsulating contestants' heterogeneity into the difference in their prize valuations—or the cost parameter d_i as in the isomorphic setting—provides a convenient measure or definition of contestants' strength and allows for handy and lucid comparative statics, which gives rise to Theorem 4 (the general exclusion principle), one part of Proposition 2 (winning probability ranking for the maximization of the expected winner's effort), and Proposition 3 (winning probability ranking under total-effort maximization). All of these results provide comparative statics of winning probability rankings with respect to the difference in contestants' prize valuations.