Supplementary Material

**Supplement to “Optimal contracts with a risk-taking agent”**


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These proofs refer to results in the text and the Appendices.

**D.4 Proof of existence for \( u = -\infty \)**

Suppose that \( u = -\infty \), which corresponds to the agent having no limited liability constraint. This section gives conditions under which a unique solution to (P) exists and satisfies certain properties. Say that \( u(\cdot) \) is regular if \( w = -\infty \) and \( \frac{u''(w)u'''(w)}{|u''(w)|^2} < 3 \) for all \( w \in \mathbb{R} \). These conditions are quite mild; in particular, the second condition means that \( u'(\cdot) \) is not excessively convex, in the sense that it has local concavity everywhere greater than \(-2\). See Prékopa (1973) and Borell (1975) for details.

**Proposition 10.** Suppose \( \pi(y) \equiv y, u(\cdot) \) is strictly concave and regular, and \( u = -\infty \). Then for any \( a \geq 0 \), there exists a unique contract \( v(\cdot) \) that implements \( a \) at maximum profit. Furthermore, there exists \( \overline{u} < \infty \) independent of \( u \) such that \( v(y) < \overline{u} \) and \( v(y) > -\overline{u} \).

**Proof.** Given Lemma 6, it is enough to show that for some \( u \), \( v_a(y) > u \). Assume not, so that, in particular, for all \( u \), \( v_a(y) = u \). We show that this leads to a contradiction. We henceforth restrict attention to \( u \leq 0 \). For \( y \) sufficiently negative, it cannot be the case that \( v_a \) is linear. In particular, if \( v_a \) is linear, then since \( v_a(y) > u_0 + c(a) \), we have that

\[
\int v_a(x)f_a(x|a)dx = \int v'_a(x)(-F_a(x|a))dx \geq \frac{u_0 + c(a) - u}{y - y_0},
\]

which diverges in \( u \), contradicting that \( v_a \) must satisfy (IC-FOC) with equality. Hence, for each \( u \), we can take a point \( x_u \in C_{v_u} \), and derive \( \lambda_u \) and \( \mu_u \) as in the proof of Proposition 3.

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Let \( z_u(\cdot) = \rho(\lambda_u + \mu_u l(|\cdot|a)) \), where we follow the convention that \( \rho(s) = -\infty \) for \( s \leq 0 \). The contract \( v_u \) will, in general, differ from \( z_u \), since \( z_u \) need not be neither concave nor satisfy the limited liability constraint. Note that \( n_u(\cdot) = \rho^{-1}(v_u(\cdot)) - (\lambda_u + \mu_u l(|\cdot|a)) = v_u(\cdot) - z_u(\cdot) \).

**STEP 1.** There is \( \overline{\mu} < \infty \) such that \( \mu_u \leq \mu \) for all \( u \).

**Proof.** Applying a small positive amount of \( t_{x_u, y} \) adds cost at rate at most \( \rho^{-1}(\overline{u}) \times \int_{x_u}^{y} (x - x_u) f(x|a) \ dx \), adds incentives at rate \( \int_{x_u}^{y} (x - x_u) f_a(x|a) \ dx \), and relaxes (IR). It follows that

\[
\mu_u \leq \rho^{-1}(\overline{u}) \frac{\int_{x_u}^{y} (x - y) f(x|a) \ dx}{\int_{x_u}^{y} f_a(x|a) \ dx}.
\]

But, as in the proof that \( |Q(0)| > 0 \),

\[
\frac{\partial}{\partial x_u} \int_{x_u}^{y} (x - x_u) f(x|a) \ dx = \frac{\int_{x_u}^{y} (x - x_u) f_a(x|a) \ dx}{\int_{x_u}^{y} f_a(x|a) \ dx} - \frac{\int_{x_u}^{y} f_a(x|a) \ dx}{\int_{x_u}^{y} f(x|a) \ dx} \leq 0,
\]

and so we can take

\[
\overline{\mu} = \rho^{-1}(\overline{u}) \frac{\int_{x_u}^{y} (x - y) f(x|a) \ dx}{\int_{x_u}^{y} f_a(x|a) \ dx} < \infty.
\]

**STEP 2.** There is \( \underline{\mu} > 0 \) and \( u^* < -\infty \) such that \( \mu_u \geq \underline{\mu} \) for all \( u < u^* \).

**Proof.** Choose \( -\infty < u^* \leq 0 \) such that

\[
\rho^{-1}(u^*) < \frac{1}{2} \rho^{-1}(u_0 + c(a)),
\]

\[
c'(a) < \frac{u_0 + c(a) - u^*}{\overline{y} - \underline{y}},
\]

where such a \( u^* \) exists since by assumption \( \lim_{w \to -\infty} \frac{1}{u(w)} = 0 \). Let

\[
r \equiv \sup_{\tau \in [\frac{1}{2} \rho^{-1}(u_0 + c(a)), \infty)} \rho'(\tau).
\]

Since \( \rho(1/\mu'(w)) = u(w) \), we have that

\[
\rho'(\frac{1}{u'(w)}) = \frac{(u')^3}{-u''(w)},
\]
from which

\[
\frac{\rho''\left(\frac{1}{u'(w)}\right)}{\rho'(\frac{1}{u'(w)})} = u'(w)\left(\frac{\rho''(w)u'(w)}{(u''(w))^2} - 3\right).
\]  

(17)

Since \(u\) is regular, it follows that \(\rho'' < 0\) and so \(r < \infty\). Let \(I_x = \max_x l_x(x\mid a)\) and choose \(\mu > 0\) such that

\[
\mu < \frac{1}{2} \frac{\rho^{-1}(u_0 + c(a))}{l(y\mid a) - l(y\mid a)} - \frac{1}{2} \frac{\rho^{-1}(u_0 + c(a))}{l(y\mid a) - l(y\mid a)} (l(y\mid a) - l(y\mid a))
\]

(18)

\[
\mu < \frac{1}{rI_x} \frac{u_0 + c(a)}{y - \bar{y}}.
\]

(19)

Assume that for some \(u < u^*\), \(\mu_u < \mu\). We show that this leads to a contradiction, establishing the result.

Using Corollary 2 (which depends only on the necessity part of the proof of Proposition 3, which is proved in Appendix B) and the fact that \(\bar{y}\) is free, \(n(\bar{y}) \leq 0\), and so \(\lambda_u + \mu_u l(y\mid a) \geq \rho^{-1}(v_u(y)) \geq \rho^{-1}(u_0 + c(a))\). Thus,

\[
\lambda_u + \mu u l(y\mid a) = \lambda_u + \mu u l(y\mid a) - \mu u \left(l(y\mid a) - l(y\mid a)\right)
\]

\[
\geq \rho^{-1}(u_0 + c(a)) - \frac{1}{2} \frac{\rho^{-1}(u_0 + c(a))}{l(y\mid a) - l(y\mid a)} (l(y\mid a) - l(y\mid a))
\]

\[
= \frac{1}{2} \rho^{-1}(u_0 + c(a)),
\]

(20)

where the inequality follows from \(\mu_u < \mu\) and (18).

Since \(u < u^*\), and by (15), \(\rho^{-1}(v_u(y)) = \rho^{-1}(u) < \frac{1}{r} \rho^{-1}(u_0 + c(a))\). Thus, using (20), \(n(y)\) is strictly positive and it follows by Corollary 2 that \(v_u\) begins with a linear segment, the slope of which (by concavity) is at least

\[
\frac{u_0 + c(a) - \bar{u}}{\bar{y} - \bar{y}} \geq \frac{u_0 + c(a)}{\bar{y} - \bar{y}}.
\]

But using (20) and the definition of \(r\), we have that for all \(x\),

\[
z_u'(x) = \rho'\left(\lambda_u + \mu_u l(x\mid a)\right) \mu u l_x(x\mid a)
\]

\[
\leq r \mu u l_x
\]

\[
< \frac{u_0 + c(a)}{\bar{y} - \bar{y}},
\]

where the strict inequality follows from (19). Hence, the initial linear segment of \(v_u\) crosses \(z_u\) at most once (from below). This implies that the entire contract is, in fact, linear with slope at least \((u_0 + c(a) - \bar{u})/(\bar{y} - \bar{y})\). In particular, let \(x_H\) be the right end
of the linear segment. If \( x_H \) is at or before the crossing point, then \( v_u \) violates (2) and so cannot be optimal by part (i) of Proposition 3. If \( x_H < y \) is after the crossing, then we violate Corollary 2. It follows that \( v_u \) generates incentives at least

\[
\frac{u_0 + c(a) - u^*}{\bar{y} - y} > c'(a)
\]

using (16). But we have shown that (IC-FOC) binds at \( v_u \), leading to the desired contradiction.

\( \triangleright \)

**Step 3.** There is \( u_0 + c(a) > u_e > -\infty \) such that if \( u < u^* \) and \( \rho(\lambda u + \mu_u(l(x|a))) < u_e \), then \( z_u(y) \) is concave at \( x \).

**Proof.** Note first that \( \rho \) is trivially concave anywhere that it is equal to \(-\infty\) and that, by assumption, \( \lim_{s \to 0} \rho(s) = -\infty \). Hence, it is enough to prove concavity where \( \rho(\lambda u + \mu_u(l(x|a))) \) is finite. But it follows from (17) and the fact that \( u \) is regular that \( \lim_{t \to 0} \rho'(t)/\rho'(t) = -\infty \) and so \( \rho''(t)/\rho'(t) \) is negative for \( t \) below some \( t' \). Assume \( \lambda u + \mu_u(l(x|a)) < t' \). Then

\[
\frac{\partial^2}{\partial x^2} \rho(\lambda u + \mu_u(l(x|a))) = \frac{\partial}{\partial x} \left( \rho'(\lambda u + \mu_u(l(x|a))) \mu_u l_x(x|a) \right)
\]

\[
= \rho''(\lambda u + \mu_u(l(x|a))) (\mu_u l_x(x|a))^2
\]

\[
+ \rho'(\lambda u + \mu_u(l(x|a))) \mu_u l_{xx}(x|a)
\]

\[
= \frac{\rho''}{\rho}(\lambda u + \mu_u(l(x|a))) \mu_u + \frac{l_{xx}(x|a)}{l_x^2}(x|a)
\]

\[
\leq \frac{\rho''}{\rho}(\lambda u + \mu_u(l(x|a))) \mu_u + \frac{l_{xx}(x|a)}{l_x^2}.
\]

The second term is bounded by assumption. The first term diverges to \(-\infty\) as \( \lambda u + \mu_u(l(x|a)) \to 0 \). Hence, since \( \rho \) is monotone and since \( \lim_{w \to -\infty} u'(w) = \infty \), the result follows.

\( \triangleright \)

**Step 4.** As in the derivation of \( r \) in **Step 2**, let \( \hat{r} \) be such that for all \( t \geq \rho^{-1}(u_e), \rho'(t) \leq \hat{r} \). Let \(-\infty < \hat{u} \leq u^* \) satisfy

\[
\hat{s} \equiv \frac{u_0 + c(a) - \hat{u}}{\bar{y} - y} \geq \max\{c'(a), \bar{l}_x \hat{r}\}
\]

and assume that \( u < \hat{u} \). Then \( z_u(y) \leq u \).

**Proof.** Assume that \( z_u(y) > u \). Then, since \( v_u(y) = u \), \( v_u \) begins with a linear segment of positive length of slope at least \( \hat{s} \), and so by Proposition 3 and part (i) of Definition 2, crosses \( z_u \) from below and is strictly above \( z_u \) for an interval of positive length as well. Let \( x_{u,c} \) be defined by \( z_u(x_{u,c}) = u_c \). If \( v_u \) has its initial crossing of \( z_u \) at or before \( x_{u,c} \), then
since $z_u$ is concave until $x_{u,c}$, $v_u$ remains above $z_u$ until $x_{u,c}$. But then, since for $x > x_{u,c}$, $\hat{s} \geq \hat{s}_u$, $v_u$ in fact never re-crosses $z_u$. Alternatively, if the initial crossing of $z_u$ by $v_u$ is after $x_{u,c}$, then again, since $v_u$ has slope greater than $\hat{s}_u$ for $x > x_{u,c}$, $v_u$ never re-crosses $z_u$. In either case, by Corollary 2, $v_u$ is thus linear on all of $[y, y]$, a contradiction.

**Step 5.** Let $u_{y_0} = u_0 + c(a) - c'(a) (\bar{y} - y_0) > -\infty$. Then $v_u(y_0) \geq u_{y_0}$.

**Proof.** Since $v_u(\bar{y}) \geq u_0 + c(a)$, it follows that everywhere on $[y, y_0)$, $v_u(\cdot)$ is below the line $L(\cdot)$ that goes through $(y_0, v_u(y_0))$ and $(\bar{y}, u_0 + c(a))$, and everywhere on $(y_0, \bar{y})$, $v_u(\cdot)$ is above $L(\cdot)$. Hence, since $f_u < 0$ on $[y, y_0)$ and $f_u > 0$ on $(y_0, \bar{y})$,

$$c'(a) = \int v_u(x) f_u(x|a) dx \geq \int L(x) f_u(x|a) dx = \frac{u_0 + c(a) - v_u(y_0)}{\bar{y} - y_0}.$$

Rearranging yields the desired result.

**Step 6.** Choose $\infty < u_s < \min(u_{y_0}, u_c, \rho(-\bar{my}(y|a)), \hat{u})$ small enough that for all $t \leq u_s$,

$$\rho\left(\frac{1}{u'(-1(t))}\right) > \rho\left(\frac{1}{u'(-1(t))}\right) \geq \hat{s}, \quad (21)$$

where $l_s = \min_x l_s(x|a) > 0$. Since $\rho'(\tau)$ diverges to $\infty$ as $\tau \downarrow 0$, and since $1/u'(u^{-1}(t))$ goes to $0$ as $t \downarrow -\infty$, such a $u_s$ is guaranteed to exist.

**Step 7.** Choose $u < u_s$. Let $\bar{z}_u(\cdot) = \rho(\bar{\lambda}_u + \bar{ml}(y|a))$, where $\bar{\lambda}_u$ solves $\rho(\bar{\lambda}_u + \bar{ml}(y|a)) = u$. By Step 4, $z_u(y) \leq \bar{u}$, and so, since $\mu_u \leq \bar{u}$, $z_u(\cdot) \leq \bar{z}_u(\cdot)$. Let $x_{u,s}$ be defined by $\bar{z}_u(x_{u,s}) = u_s$. Since $\bar{\lambda}_u + \bar{ml}(y|a) = 1/u'(u^{-1}(u)) > 0$, it follows that $\bar{\lambda}_u + \bar{ml}(y_0|a) \geq -\bar{ml}(y|a)$ and, hence,

$$\rho(\bar{\lambda}_u + \bar{ml}(y_0|a)) > \rho(-\bar{ml}(y|a)) > u_s,$$

where the last inequality is by definition of $u_s$ in Step 6. Thus, $x_{u,s} < y_0$.

**Step 8.** For all $x < x_{u,s}$, $v_u(x) \leq \bar{z}_u(x)$.

**Proof.** Let $x_{u,c}$ be defined by $z_u(x_{u,c}) = u_c$. By construction, $\bar{z}_u(\cdot)$ is concave where $x \leq x_{u,c}$. Using (21), $\bar{z}_u(\cdot) > \hat{s}$ for $x < x_{u,s}$, and $\bar{z}_u(\cdot) < \hat{s}$ for $x \geq x_{u,c}$. Assume that for some $\hat{x} < x_{u,s}$, $v_u(\hat{x}) > \bar{z}_u(\hat{x}) \geq z_u(\hat{x})$. By Corollary 2, $v_u$ is linear at $\hat{x}$. If $v'_u(\hat{x}) \leq \bar{z}'_u(\hat{x})$, then, since $\bar{z}_u$ is concave on $[y, x_{u,s}]$ and again using Corollary 2, $v_u$ is also above $\bar{z}_u$ and, hence, is linear, for all $x$ in $[y, \hat{x}]$. But then

$$v_u(y) - \bar{z}_u(y) \geq v_u(\hat{x}) - \bar{z}_u(\hat{x}) > 0,$$
contradicting that \( v_u(y) = u \). Thus, \( v_u'(\tilde{x}) > \sigma_u'(\tilde{x}) > \hat{s} \). But then \( v_u \) remains linear and, hence, strictly above the concave function \( z_u \) at least until \( x_u \). For \( x \geq x_u \), \( \sigma_u'(\tilde{x}) \leq \hat{s} \), and so as before \( v \) can never re-cross \( \sigma_u \), and so a fortiori can never re-cross \( z_u \). Hence, \( v_u \) is linear on \( [\tilde{x}, \bar{y}] \), with slope at least \( \hat{s} \). Let \( L \) be the line that agrees with \( v_u \) on \( [\tilde{x}, \bar{y}] \).

To the left of \( \tilde{x} \), \( v_u \), being concave, lies below \( L \). But \( \tilde{x} < x_u \) and so, since \( f_a(\cdot | a) \) is negative on \( [\bar{y}, \tilde{x}] \),

\[
\int v_u(x) f_a(x|a) \, dx \geq \int L(x) f_a(x|a) \, dx \geq \hat{s} > c'(a),
\]

again a contradiction.

\( \triangleright \)

**Step 9.** We show that \( \lim_{u \to -\infty} \int v_u(x) f_a(x|a) \, dx = \infty \). For \( u \) sufficiently negative, this provides the necessary contradiction to the original supposition that \( v_u(y) = u \) for all \( u \), proving the result.

**Proof.** By Step 8, for \( u \) sufficiently negative, \( v_u(x) \leq \sigma_u(x) \) for all \( x \leq x_u \). Let \( v^T_u \) truncate \( v_u \) to never pay more than \( u_s \). Since \( \max(0, v_u(x) - u_s) f_a(x|a) \) is an increasing function,

\[
\int \max(0, v_u(x) - u_s) f_a(x|a) \, dx \geq 0 \quad \text{and, hence,} \quad \int v_u(x) f_a(x|a) \, dx \geq \int v^T_u(x) f_a(x|a) \, dx.
\]

Note also that since \( v_u(y_0) > u_s \), \( v^T_u(x) = u_s \) for all \( x \geq y_0 \). Let \( \sigma^T_u \) similarly truncate \( \sigma_u \) to pay \( u_s \) to the right of \( x_u \). Then \( \sigma^T_u \) is everywhere at least as large as \( v^T_u \), but equal to \( v^T_u \) everywhere to right of \( y_0 \). Hence, since \( f_a \) is negative to the left of \( y_0 \), we have

\[
\int v_u(x) f_a(x|a) \, dx \geq \int v^T_u(x) f_a(x|a) \, dx \geq \int \sigma^T_u(x) f_a(x|a) \, dx.
\]

To arrive at a contradiction, it would thus be enough to show that \( \int \sigma^T_u(x) f_a(x|a) \, dx \) diverges as \( u \to -\infty \). But by Moroni and Swinkels (2014, Lemma 4), under our regularity conditions, \( \int \sigma_u(x) f_a(x|a) \, dx \) does diverge as \( u \to -\infty \).

Let

\[
u^{**} = \rho \left( 1 + \frac{1}{u^*(\bar{y})} \right) < \infty.
\]

Then, for all \( u \) sufficiently negative that \( \frac{1}{u^*(\bar{y})} \leq 1 \), \( \sigma_u(\bar{y}) \leq \nu^{**} \). Hence,

\[
\int \sigma_u(x) f_a(x|a) \, dx - \int \sigma^T_u(x) f_a(x|a) \, dx = \int \left( \sigma_u(x) - \frac{\sigma^T_u(x)}{\nu^{**}} \right) f_a(x|a) \, dx
\leq \int_{y_0}^{\bar{y}} \left( \sigma_u(x) - \sigma^T_u(x) \right) f_a(x|a) \, dx
\leq (\nu^{**} - u_s) \int_{y_0}^{\bar{y}} f_a(x|a) \, dx
< \infty,
\]

where the first inequality follows because \( \sigma_u(x) - \sigma^T_u \) is weakly positive, and the second inequality follows because it is bounded above by \( \nu^{**} - u_s \).
D.5 Agent reports $x$

In this section, we allow the agent to send a contractible message $\tilde{x}$ after he observes $x$ but before $y$ is realized. Payments can therefore depend on both $\tilde{x}$ and $y$, which allows the principal to discipline the agent from engaging in risk-taking. Restricting attention to the case where both parties are risk-neutral, we show that a linear contract is optimal in this setting.

Since the principal does not benefit from risk-taking, it is without loss to restrict attention to mechanisms that punish the agent as much as possible whenever his report does not match the final output: $s(y)^\lambda_{y=\tilde{y}} - M^\lambda_{y=\tilde{y}}$ for some upper semicontinuous function $s(\cdot)$. Then the principal’s problem is

$$\max_{a,s(\cdot)} \mathbb{E}_{F(\cdot|a),G}[y - s(y)^{\lambda_{y=\tilde{y}}} - M^{\lambda_{y=\tilde{y}}} + M]$$

subject to $a, G, \tilde{x} \in \arg\max_{\hat{a}, \hat{G} \in \mathcal{G}} \left\{ \mathbb{E}_{F(\cdot|\hat{a}),G}[s(y)^{\lambda_{y=\tilde{y}}} - M^{\lambda_{y=\tilde{y}}} + c(\hat{a})] \right\},$

$$\mathbb{E}_{F(\cdot|a),G}[s(y)^{\lambda_{y=\tilde{y}}} - M^{\lambda_{y=\tilde{y}}} + c(\hat{a})] - c(a) \geq u_0,$$

$$s(\cdot) \geq -M,$$

where $\tilde{x}$ maps $x$ to a report made to the principal.

Fix $s(\cdot)$, and consider the agent’s choice of $G_x$ and $\tilde{x}$ following any intermediate output $x > y$. Define

$$\lambda_s(x) = \max \{ \lambda : \lambda(y - y) - M = s(y) \text{ for some } y \geq x \}.$$

Intuitively, $\lambda_s(x)$ is the smallest slope such that $\lambda_s(x)(y - y) - M \geq s(y)$ for all $y \geq x$. We show that following intermediate output $x > y$, the agent optimally chooses $G_x$ and $\tilde{x}$ so that his expected payoff is $\lambda_s(x)(x - y) - M$.\(^{25}\)

**Lemma 7.** For any $s(\cdot)$ and $x \in \mathcal{Y}$, the principal’s expected payment to the agent equals

$$\sigma_s(x) \equiv \max_{G_x, \tilde{x}} \mathbb{E}_{G_x}[s(y)^{\lambda_{y=\tilde{y}}} - M^{\lambda_{y=\tilde{y}}} + M] = \begin{cases} 0 & \text{if } x = y, \\ s(y) - \lambda(x)(x - y) - M & \text{if } x > y. \end{cases} \quad (22)$$

**Proof.** Fix $s(\cdot)$ and $x > y$. First, we show that there exists some $G_x$ and $\tilde{x}$ such that

$$\mathbb{E}_{G_x}[s(y)^{\lambda_{y=\tilde{y}}} - M^{\lambda_{y=\tilde{y}}} + M] = \lambda_s(x)(x - y) - M. \quad (23)$$

By definition of $\lambda_s(\cdot)$, there exists a $\hat{y} \geq x$ such that $\lambda_s(x)(\hat{y} - y) - M = s(\hat{y})$. Let $\tilde{x} = \hat{y}$ and $G_x(y) = (1 - p_{\hat{y}}) + p_{\hat{y}}^\lambda_{y=\hat{y}}$, where $p_{\hat{y}} = \frac{x - y}{\hat{y} - y}$, i.e., $y = \hat{y}$ with probability $1 - p_{\hat{y}}$ and $y = \hat{y}$ with probability $p_{\hat{y}}$. Then the agent’s expected payoff is

$$p_{\hat{y}}s(\hat{y}) - (1 - p_{\hat{y}})M = \frac{x - y}{\hat{y} - y} - \frac{\hat{y} - x}{\hat{y} - y}M = \frac{x - y}{\hat{y} - y} \left[ \lambda_s(x)(\hat{y} - y) - M \right] - \frac{\hat{y} - x}{\hat{y} - y}M = \lambda_s(x)(x - y) - M.$$

\(^{25}\)If $x = y$, then the agent is compelled to choose $G_y(y) = 1$, so his expected payoff is equal to $s(y)$.\}
Next we show that the agent cannot earn more than $\lambda_s(x)(x - y) - M$ following intermediate output $x$. For any report $\tilde{x}$, the agent earns more than $-M$ only if $y = \tilde{x}$, so his optimal distribution $G_x$ maximizes the probability that $y = \tilde{x}$ subject to the constraint that $E_{G_x}[y] = x$. This is accomplished by choosing $G_x(\cdot)$ such that $y = \tilde{x}$ with some probability $p_\tilde{x}$ and $y = y$ with probability $1 - p_\tilde{x}$, where $p_\tilde{x}(1 - p_\tilde{x})y = x$. It suffices to show that the agent’s expected payoff under this distribution is maximized if $\tilde{x} = \hat{y}$.

Suppose that there exists some $\tilde{x} \neq \hat{y}$ such that $p_\tilde{x}s(\tilde{x}) - (1 - p_\tilde{x})M > p_\hat{y}s(\hat{y}) - (1 - p_\hat{y})M = \lambda_s(x)(x - y) - M$. Then there must exist some $\tilde{\lambda} > \lambda_s(x)$ such that $\tilde{\lambda}(\tilde{x} - y) - M = s(\tilde{x})$, which contradicts the definition of $\lambda_s(x)$. Therefore, for all $x$, the agent’s expected payoff equals $\lambda_s(x)(x - y) - M$. \hfill \qed

To see this result, recall that the agent earns $-M$ whenever his report does not equal the realized output. Therefore, if he misreports $\tilde{x} \neq x$, then he chooses $G_x$ to maximize the probability that $y = \tilde{x}$. In particular, it is optimal for $G_x$ to put weight on only two points, $\tilde{x}$ and $y$. Given this $\tilde{x}$, the agent’s payoff can be written as $p_\tilde{x}s(\tilde{x}) - (1 - p_\tilde{x})M$, where $p_\tilde{x}(1 - p_\tilde{x})y = x$. It can be shown that the agent’s payoff can be rewritten as $\lambda(x - y) - M$, where $\lambda \leq \lambda_s(x)$. There exists some report $\hat{x}$ that sets $\lambda = \lambda_s(x)$, proving the result.

Using Lemma 7, we can rewrite the principal’s problem as

$$\max_{a, \tilde{x}(\cdot)} \mathbb{E}_{F(\cdot|a)}[x - \sigma_s(x)]$$

subject to

$$a \in \arg\max_{\tilde{a}} \left\{ \mathbb{E}_{F(\cdot|\tilde{a})} [\sigma_s(x)] - c(\tilde{a}) \right\},$$

$$\mathbb{E}_{F(\cdot|a)}[\sigma_s(x)] - c(a) \geq u_0,$$

$$s(\cdot) \geq -M,$$

where, for any contract $s(\cdot)$, $\sigma_s(\cdot)$ is given by (22).

Recall the definition of $s^L_a(\cdot)$ from Section 4. We show that if $a \geq 0$ is such that (LL) holds with equality after $y$ under $s^L_a(\cdot)$, then $s^L_a(\cdot)$ implements $a$ at maximum profit in this setting. Consequently, if (LL) binds for the optimal $a \geq 0$, then a linear contract is optimal as in Proposition 2.

**Proposition 11.** Fix any effort $a \geq 0$. If $s^L_a(y) = -M$, then $s^L_a(\cdot)$ implements $a$ at maximum profit.

**Proof.** Note that $\lambda_s(\cdot)$ is decreasing for any $s(\cdot)$ and, moreover, is constant for all $x \in \mathcal{Y}$ if $s(\cdot)$ is affine. Let $\tilde{s}(\cdot)$ implement $a$ at maximum profit and suppose there exists $x_L < x_H$ such that $\lambda_{\tilde{s}}(x_L) > \lambda_s(x_H)$.

Define $s_L(y) = \beta(y - y) - M$, where $\beta$ is chosen such that $\mathbb{E}_{F(\cdot|a)}[s_L(y) - \lambda_{\tilde{s}}(y) \times (y - y) + M] = 0$. Such a $\beta$ exists by the intermediate value theorem because $\lambda_{\tilde{s}}(y) \geq 0$ is finite. Since $\lambda_{\tilde{s}}(\cdot)$ is strictly decreasing over some interval, there exists some $y^* \in (y, \mathcal{Y})$ such that $\lambda_{\tilde{s}}(y) \geq \beta$ if and only if $y \leq y^*$. Then $\beta - \lambda_{\tilde{s}}(y)$ is first negative and then
positive, \( \int [\beta - \lambda_3(y)](y - y)f(y|a)\,dy = 0 \) by construction, and \( f_2(a|\alpha) \) is strictly increasing, so Beesack’s inequality implies that

\[
\int [\beta - \lambda_3(y)](y - y)f_a(y|a)\,dy > 0.
\]

Therefore, \( s_L(\cdot) \) implements some effort level \( a' > a \), which implies that \( \beta > c'(a) \).

Observe that \( s^*_L(y) < s_L(y) \) for all \( y > y_\lambda \), because \( s^*_L(y) = -M \) by assumption and \( c'(a) < \beta \). Moreover, \( s^*_L(\cdot) \) implements \( a \) and satisfies both the individual rationality and the limited liability constraints. Therefore, \( s^*_L(\cdot) \) implements effort \( a \) at strictly higher profit than \( \hat{s}(\cdot) \). So \( \lambda_3(\cdot) \) must be constant and \( \sigma_3(y) = -M \), in which case \( s^*_L(\cdot) \) is also optimal.

\[ \square \]

D.6 Comparative static of optimal contract with respect to \( y \)

This appendix considers how \( a^* \) changes with the lower bound \( y \) on output. A decrease in \( y \) implies that the agent can take on more severe left-tail risk by gambling over worse outcomes. We prove that a lower \( y \) makes it costlier for the principal to induce any nonzero effort level. As \( y \) approaches \(-\infty\), inducing any positive effort becomes arbitrarily expensive and so the agent exerts no effort in the optimal contract.

**Corollary 3.** Consider a decreasing sequence \( \{y_k\}_{k=0}^\infty \) with \( \lim_{k \to \infty} y_k = -\infty \). For each \( k \geq 0 \), consider \( \mathcal{Y} = [y_k, \overline{y}] \) and some output distribution \( F_k(\cdot | a) \) that satisfies our assumptions (i.e., has full support on \([y_k, \overline{y}]\), satisfies \( \mathbb{E}_{F_k(\cdot | a)}[x] = a \), etc.), and let \( a_k^* \) be the corresponding optimal effort. Then \( \lim_{k \to \infty} a_k^* = 0 \), and if \( \pi(y) \equiv y \), then \( a_k^* \) is decreasing in \( k \).

Proposition 2 implies that the principal’s expected payment from inducing \( a^* \geq 0 \) equals \( \mathbb{E}_{F(\cdot | a^*)}[\pi(y - c'(a^*)(y - y) + w)] \). For small enough \( y \), \( s^*_L(y) = -M \). But then implementing \( a^* > 0 \) becomes arbitrarily costly as \( y \to -\infty \), in which case the principal is better off not motivating the agent at all. If the principal is risk-neutral, then we can show that the principal’s profit under \( s^*_L(\cdot) \) is supermodular in \( a^* \) and \( y \), so that \( a^* \) is increasing in \( y \).

**Proof of Corollary 3.** Fix \( \hat{a} > 0 \). Define

\[
y_1 \equiv \min_{a \in [\hat{a}, a^*F]} \left\{ a - \frac{c(a) + u_0 + M}{c'(a)} \right\}
\]

and

\[
y_2 \equiv \min_{a \in [\hat{a}, a^*F]} \left\{ \frac{u^{-1}(u_0) - (1 - c'(a)a) - M}{c'(\hat{a})} \right\},
\]

and note that since \( c'(a) > c'(\hat{a}) \), \( a \geq \hat{a} \), \( y_{\min} \equiv \min(0, y_1, y_2) > -\infty \).

Let \( y < y_{\min} \) and suppose toward a contradiction that there exists a distribution \( F(\cdot | a) \) on \([y, \overline{y}]\) such that effort \( a^* \geq \hat{a} \) is optimal under \( F(\cdot | a) \). Note first that Proposition 2 implies that the principal’s expected payoff equals

\[
\mathbb{E}_{F(\cdot | a^*)}[\pi(y - s^*_L(y))] = \mathbb{E}_{F(\cdot | a^*)}[\pi(y - c'(a^*)(y - y) + \min\{M, c'(a^*)(a^* - y) - c(a^*) - u_0\})].
\]
Since \( y < y_1 \), \( c'(a^*)(a^* - y) - c(a^*) - u_0 > M \). Furthermore, the principal’s payoff is bounded above by
\[
\pi \left( (1 - c'(a^*))a^* + c'(a^*)y + M \right)
\]
by Jensen’s inequality. Since \( y < \min\{0, y_2\} \), \((1 - c'(a))a + c'(a)y + M < u^{-1}(u_0)\) for any \( a \in [\hat{a}, a_{FB}] \). But then \( a^* \geq \hat{a} \) cannot be optimal because it is strictly dominated by \( a^* = 0 \) and \( s(\cdot) = u^{-1}(u_0) \), a contradiction. Hence, for \( y < y_{\min} \), any distribution \( F(\cdot|a) \), and any optimal \( a^* \), it must be that \( a^* < \hat{a} \). Since \( \hat{a} > 0 \) is arbitrary, \( \lim_{y \to -\infty} a^* = 0 \).

Suppose \( \pi(y) \equiv y \). To prove that \( a^* \) is increasing in \( y \), it suffices to show that the principal’s payoff from implementing \( a \) in an optimal contract, \( \Pi(a, y) = a - c'(a)(a - y) + w \), is supermodular in \( a \) and \( y \).

Recall that \( w = \min\{M, c'(a)(a - y) - c(a) - u_0\} \) is a function of \((a, y)\). Therefore,
\[
\frac{\partial \Pi}{\partial a} = 1 - c''(a)(a - y) - c'(a) + \frac{\partial w}{\partial a}
\]
and so
\[
\frac{\partial^2 \Pi}{\partial y \partial a} = c''(a) + \frac{\partial^2 w}{\partial y \partial a}.
\]
But \( \frac{\partial^2 w}{\partial y \partial a} = 0 \) if \( M < c'(a)(a - y) - c(a) - u_0 \) and \( \frac{\partial^2 w}{\partial y \partial a} = -c''(a) \) otherwise. In either case, \( \frac{\partial^2 \Pi}{\partial y \partial a} \geq 0 \) and so optimal effort \( a^* \) is increasing in \( y \), as desired. \( \square \)

**References**


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